A Limiting Version of a Theorem in Cohomology

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Dedicated to Professor Martin Kneser

June 27, 2004

Abstract

This paper completes the study started in [1]. Scheme-theoretic methods are used to classify line-bundle-valued rank 3 quadratic bundles. The classification is done in terms of schematic specialisations of rank 4 Azumaya algebra bundles in the sense of Part A, [2]. For a quadratic form q on a rank 3 vector bundle V with values in a line bundle I over a scheme X, the degree zero subalgebra $C_0(V,q,I)$ of the generalised Clifford algebra $\widetilde{C}(V,q,I)$ of the triple (V,q,I) in the sense of Bichsel-Knus [3], is seen to be such a specialised algebra by results in Part A, [2]. The Witt-invariant of (V,q,I), which may be defined as the the isomorphism class (as algebra bundle) of $C_0(V,q,I)$, is shown to determine (V,q,I) upto tensoring by a twisted discriminant line bundle. Further, each specialised algebra arises in this way upto isomorphism, so that the association $(V,q,I) \mapsto C_0(V,q,I)$ induces a natural bijection from the set of equivalence classes of line-bundle-valued quadratic forms on rank 3 vector bundles upto tensoring by a twisted discriminant bundle and the set of isomorphism classes of schematic specialisations of rank 4 Azumaya bundles over X. This statement may be viewed as a limiting version of the natural bijection involving cohomology given by $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X,\mathsf{O}_3)/\check{\mathrm{H}}^1_{\mathrm{fppf}}(X,\mu_2) \cong \check{\mathrm{H}}^1_{\text{étale}}(X,\mathsf{PGL}_2)$. The special, usual and the general orthogonal groups of (V,q,I) are computed and canonically determined in terms of $\mathrm{Aut}(C_0(V,q,I))$, and it is shown that the general orthogonal group is always a semidirect product. Any element of $\mathrm{Aut}(C_0(V,q,I))$ can be lifted to a self-similarity, and in fact to an element of the orthogonal group provided the determinant of the automorphism is a square. The special orthogonal group and the group of determinant 1 automorphisms of $C_0(V,q,I)$ are naturally isomorphic. A specialised algebra bundle A arises as $C_0(V,q,I)$ with $I \cong \mathcal{O}_X$ iff $\det(A) \in 2.\mathrm{Pic}(X)$; and arises with q induced from a global bilinear form iff the line subbundle generated by 1_A is a direct summand of A. The use of the nice technical notion of semiregularity introduced by Kneser in [4] allows working with an arbitrary X, some (or even all) of whose points may have residue fields of characteristic two.

Keywords: semiregular form, quadratic bundle, Azumaya bundle, Witt-invariant, line-bundle-valued form, Clifford algebra, discriminant bundle, orthogonal group, similarity, similitude.

MSC Subject Classification Numbers: 11Exx, 11E12, 11E20, 11E88, 11R52, 14L15, 15A63, 16S, 16H05.

Contents

1 Overview of the Main Results

THEOREM 1.1 *Over each scheme X the natural map $(V, q, I) \mapsto C_0(V, q, I)$ that associates to a quadratic bundle the degree zero subalgebra of its generalised Clifford algebra, induces a bijection from the set of orbits*

$$\left\{ \begin{array}{c} \textit{isomorphism classes } (V,q,I), \textit{ with} \\ \textit{rank}_X(V)=3, \textit{rank}_X(I)=1 \textit{ and} \\ q \textit{ a quadratic form on the bundle } V \\ \textit{with values in the line bundle } I \end{array} \right\} \textit{modulo the group} \left\{ \begin{array}{c} \textit{isomorphism classes } (L,h,J), \\ \textit{with rank}_X(L)=\textit{rank}_X(J)=1 \\ \textit{and } h: L\otimes L \cong J \textit{ a linear} \\ \textit{isomorphism; product induced by } \otimes \end{array} \right\}$$

to the set of algebra-bundle-isomorphism classes of associative unital algebra structures A, on vector bundles of rank 4, that are Zariski-locally isomorphic to even Clifford algebras of rank 3 quadratic bundles. This bijection maps the subset

$$\left\{ \begin{array}{c} \textit{isomorphism classes } (V,q,I), \textit{ with} \\ q \textit{ semiregular} \end{array} \right\} \textit{modulo the group} \left\{ \begin{array}{c} \textit{isomorphism classes } (L,h,J), \\ \textit{as above} \end{array} \right\}$$

surjectively onto the subset of isomorphism classes of rank 4 Azumaya algebra bundles on X.

It was shown in Part A, [2] that algebra bundles such as A in the statement above are precisely the scheme-theoretic specialisations (or limits) of rank 4 Azumaya algebra bundles on X. The main result there was the smoothness of the schematic closure of Azumaya algebra structures on a fixed vector bundle of rank 4 over any scheme. Part B of [2] had applied this result to obtain desingularisations (with good specialisation properties) of certain moduli spaces over fairly general base schemes, and applications to the study of degenerations of quadratic forms on rank 3 vector bundles was initiated in [1]. Though [1] only treats forms with values in the structure sheaf, in the following we consider line-bundle-valued quadratic forms. Such a viewpoint is affordable thanks to the construction of Bichsel-Knus [3] of the generalised Clifford algebra $\widetilde{C}(V, q, I)$. This is an $\mathbb{Z}$-graded algebra and we let $C_0(V, q, I)$ denote its degree zero subalgebra. The other difference with [1] is to consider $C_0(V, q, I)$ instead of the usual even Clifford algebra of a quadratic form with values in the structure sheaf. We shall see later ((b1), Theorem 1.6) that it is necessary to consider line-bundle-valued quadratic forms to obtain the surjectivity of Theorem 1.1 in those cases for which $\det(A) \notin 2.\mathrm{Pic}(X)$.

Remark 1.2 Assume for simplicity that X is an affine scheme. Following Sec.3, Chap.V, of the book of Knus [5], it can be seen that the association of a semiregular quadratic bundle to its even Clifford algebra induces a bijection from the set of orbits

$$\left\{ \begin{array}{c} \text{isomorphism classes } (V,q,\mathcal{O}_X), \\ \text{with } \mathrm{rank}_X(V)=3, \text{ and} \\ q \text{ semiregular} \end{array} \right\} \text{modulo the group} \left\{ \begin{array}{c} \text{isomorphism classes } (L,h), \text{ with} \\ \mathrm{rank}_X(L)=1 \text{ and } h: L\otimes L \cong \mathcal{O}_X \\ \mathcal{O}_X\text{-linear; product under } \otimes \end{array} \right\}$$

with the set of isomorphism classes of rank 4 Azumaya bundles on X. The group above is also called the group of discriminant bundles on X and is denoted by $\mathrm{Disc}(X)$. By Prop.3.2.2, Sec.3, Chap.III, [5], it is naturally isomorphic to the cohomology (abelian group) $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mu_2)$. We may thus call the triples (L, h, J) occurring in the statement of Theorem 1.1 as twisted discriminant bundles. Further, by Lemma 3.2.1, Sec.3, Chap.IV, [5] the cohomology $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)$ classifies the set of isomorphism classes of semiregular rank 3 quadratic bundles with values in the trivial line bundle. On the other hand, the set of isomorphism classes of rank 4 Azumaya algebras on X may be interpreted as the cohomology $\check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$ (see page 145, Sec.5, Chap.III, [5]). Thus we have the bijection $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)/\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mu_2) \cong \check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$ and Theorem 1.1 may be considered as the limiting version of this bijection.

The following results lead to a proof of Theorem 1.1. Their proofs in turn follow from techniques of [1] adapted to include the case of line-bundle-valued forms. The proof of the injectivity essentially follows from the next couple of theorems, which describe the general, usual and special orthogonal groups of (V, q, I) in terms of the (algebra) automorphism group of $C_0(V, q, I)$.

A bilinear form b, with values in a line bundle I, defined on a vector bundle V over the scheme X induces an I-valued quadratic form q_b given on sections by $x \mapsto b(x, x)$. Further, b also naturally defines an $\mathbb{Z}$-graded linear isomorphism $\psi_b : \widetilde{C}(V, q_b, I) \cong \Lambda(V) \otimes L[I]$ (see (2d), Theorem 2.1). Here $L[I] := \mathcal{O}_X \oplus \left(\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1}))\right)$ is the Laurent-Rees algebra of I, where elements of V (resp. of I) are declared to be of degree one (resp. of degree two). In fact, $\widetilde{C}(V, 0, I) = \Lambda(V) \otimes L[I]$. Since in general a quadratic bundle (V, q, I) on a non-affine scheme X may not be induced from a global I-valued bilinear form, one is unable to identify the $\mathbb{Z}$–graded vector bundle underlying its generalised Clifford algebra bundle with $\Lambda(V) \otimes L[I]$. The following result overcomes this problem.

PROPOSITION 1.3 *Every isomorphism of algebra-bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$ is naturally associated to an isomorphism of bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$ which induces a map*

$$\zeta_{\Lambda^2} : \mathrm{Iso}[C_0(V, q, I), C_0(V', q', I')] \longrightarrow \mathrm{Iso}[\Lambda^2(V) \otimes I^{-1}, \Lambda^2(V') \otimes (I')^{-1}] : \phi \mapsto \phi_{\Lambda^2}$$

where $\mathrm{Iso}[C_0(V, q, I), C_0(V', q', I')]$ *is the set of algebra bundle isomorphisms. When $V = V'$ and $I = I'$, we may thus denote the subset of those ϕ for which* $\det(\phi_{\Lambda^2}) \in \mathrm{Aut}[\det(\Lambda^2(V) \otimes I^{-1})] \equiv \Gamma(X, \mathcal{O}_X^*)$ *is a square by* $\mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)]$ *and those for which* $\det(\phi_{\Lambda^2}) = 1$ *by the smaller subset* $\text{S-Iso}[C_0(V, q, I), C_0(V, q', I)]$*. Taking $q = q'$ in these sets and replacing " Iso" by "Aut" in their notations respectively defines the groups* $\mathrm{Aut}(C_0(V, q, I)) \supset \mathrm{Aut}'(C_0(V, q, I)) \supset \text{S-Aut}(C_0(V, q, I))$.

Denote by $\mathrm{Sim}[(V, q, I), (V', q', I')]$ the set of isomorphisms from (V, q, I) to (V', q', I'). These are generalised similarities i.e., pairs (g, m) such that $g : V \cong V'$ and $m : I \cong I'$ are linear isomorphisms and the following diagram commutes (where q and q' are considered as morphisms of sheaves of *sets*):

$$\begin{tikzcd} \mathcal{V} \arrow[r, "g", "\cong"'] \arrow[d, "q"'] & \mathcal{V}' \arrow[d, "q'"] \\ I \arrow[r, "\cong", "m"'] & I' \end{tikzcd}$$

When $I = I'$, since an $m \in \mathrm{Aut}(I)$ may be thought of as multiplication by a scalar $l \in \Gamma(X, \mathcal{O}_X^*) \equiv \mathrm{Aut}(I)$, we may call the isomorphism (g, m) as an I-similarity with multiplier l. In such a case we may as well denote (g, m) by the pair (g, l) and we often write $g : (V, q, I) \cong_l (V', q', I)$. Let $\mathrm{Iso}[(V, q, I), (V', q', I)]$ be the subset of isometries (i.e., those pairs (g, m) with $m =$ Identity or I-similarities with trivial multipliers). When $V = V'$, the subset of isometries with trivial determinant is denoted $\text{S-Iso}[(V, q, I), (V, q', I)]$. On taking $q = q'$ these sets naturally become subgroups of $\mathrm{Aut}(V) \times \Gamma(X, \mathcal{O}_X^*) = \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$ and we define

$$\begin{array}{rcl} \mathrm{Sim}[(V, q, I), (V, q, I)] =: \mathrm{GO}(V, q, I) & \supset & \mathrm{Iso}[(V, q, I), (V, q, I)] =: \mathrm{O}(V, q, I) \\ & \supset & \text{S-Iso}[(V, q, I), (V, q, I)] =: \mathrm{SO}(V, q, I). \end{array}$$

Of course, $\mathrm{O}(V, q, I)$ and $\mathrm{SO}(V, q, I)$ may be thought of as subgroups of $\mathrm{GL}(V) \equiv \mathrm{GL}(V) \times \{1\}$.

THEOREM 1.4 *For I-valued quadratic forms q and q' on a rank 3 vector bundle V over a scheme X, we have the following commuting diagram of natural maps of sets with the downward arrows being the canonical inclusions, the horizontal arrows being surjective and the top horizontal arrow being bijective:*

$$\begin{tikzcd} \text{S-Iso}[(V, q, I), (V, q', I)] \arrow[r, "\cong"] \arrow[d, "\mathrm{inj}"'] & \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)] \arrow[d, "\mathrm{inj}"] \\ \mathrm{Iso}[(V, q, I), (V, q', I)] \arrow[r, "\mathrm{onto}"] \arrow[d, "\mathrm{inj}"'] & \mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)] \arrow[d, "\mathrm{inj}"] \\ \mathrm{Sim}[(V, q, I), (V, q', I)] \arrow[r, "\mathrm{onto}"] & \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)] \end{tikzcd}$$

With respect to the surjections of the horizontal arrows in the diagram above, we further have the following (where l is the function that associates a similarity to its multiplier, $\det(g, l) := \det(g)$ *for an I-similarity g with multiplier l and ζ_{Λ^2} is the map of Prop.1.3 above):*

(a) *there is a family of sections* $s_{2k+1} : \mathrm{Iso}[C_0(V,q,I), C_0(V,q',I)] \longrightarrow \mathrm{Sim}[(V,q,I),(V,q',I)]$ *indexed by the integers such that* $l \circ s_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ *and such that* $(\det^2 \circ s_{2k+1}) \times (l^{-3} \circ s_{2k+1}) = \det \circ \zeta_{\Lambda^2}$*;*

(b) *there is also a section* $s' : \mathrm{Iso}'[C_0(V,q,I), C_0(V,q',I)] \longrightarrow \mathrm{Iso}[(V,q,I),(V,q',I)]$ *such that* $\det^2 \circ s' = \det \circ \zeta_{\Lambda^2}$*;*

(c) *there is a family of sections* $s^+_{2k+1} : \mathrm{Iso}[C_0(V,q,I), C_0(V,q',I)] \longrightarrow \mathrm{Sim}[(V,q,I),(V,q',I)]$ *indexed by the integers which is multiplicative when followed by the natural inclusions into* $\mathrm{GL}(V) \times \Gamma(X, \mathcal{O}^*_X)$*, i.e., if* $\phi_i \in \mathrm{Iso}[C_0(V,q_i,I), C_0(V,q_{i+1},I)]$ *then* $s^+_{2k+1}(\phi_2 \circ \phi_1) = s^+_{2k+1}(\phi_2) \circ s^+_{2k+1}(\phi_1) \in \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}^*_X)$*. Further,* $l \circ s^+_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ *and* $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det \circ \zeta_{\Lambda^2}$*.*

(d) *The maps* s_{2k+1} *and* s' *above may not be multiplicative but are mutliplicative upto* $\mu_2(\Gamma(X, \mathcal{O}_X))$ *i.e., these maps followed by the canonical quotient map, on taking the quotient of* $\mathrm{GL}(V) \times \Gamma(X, \mathcal{O}^*_X)$ *by* $\mu_2(\Gamma(X, \mathcal{O}_X)).\mathrm{Id}_V \times \{1\}$*, become multiplicative.*

THEOREM 1.5 *For a rank 3 quadratic bundle* (V,q,I) *on a scheme* X*, one has the following natural commutative diagram of groups with exact rows, where the downward arrows are the canonical inclusions and where* l *is the function that associates to any* I*-(self)similarity its multiplier:*

$$\begin{tikzcd}
 & & \mathrm{SO}(V,q,I) \arrow[r, "\cong"] \arrow[d, "\mathrm{inj}"'] & \text{S-Aut}(C_0(V,q,I)) \arrow[d, "\mathrm{inj}"] & \\
1 \arrow[r] & \mu_2(\Gamma(X,\mathcal{O}_X)) \arrow[r] \arrow[d, "\mathrm{inj}"'] & \mathrm{O}(V,q,I) \arrow[r] \arrow[d, "\mathrm{inj}"'] & \mathrm{Aut}'(C_0(V,q,I)) \arrow[r] \arrow[d, "\mathrm{inj}"] & 1 \\
1 \arrow[r] & \Gamma(X,\mathcal{O}^*_X) \arrow[r] & \mathrm{GO}(V,q,I) \arrow[r] \arrow[d, "\det^2 \times l^{-3}"'] & \mathrm{Aut}(C_0(V,q,I)) \arrow[r] \arrow[d, "\det"] & 1 \\
 & & \Gamma(X,\mathcal{O}^*_X) \arrow[r, equal] & \Gamma(X,\mathcal{O}^*_X) &
\end{tikzcd}$$

Further, we have:

(a) *There are splitting homomorphisms* $s^+_{2k+1} : \mathrm{Aut}(C_0(V,q,I)) \longrightarrow \mathrm{GO}(V,q,I)$ *such that* $l \circ s^+_{2k+1} = \det^{2k+1}$ *and* $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det$*. The restriction of* s^+_{2k+1} *to* $\mathrm{Aut}'(C_0(V,q,I))$ *does not necessarily take values in* $\mathrm{O}(V,q,I)$*, but the further restriction to* $\text{S-Aut}(C_0(V,q,I))$ *does take values in* $\mathrm{SO}(V,q,I)$*. In particular,* $\mathrm{GO}(V,q,I)$ *is a semidirect product. The maps* s_{2k+1} *and* s' *of Theorem 1.4 above (under the current hypotheses) may not be homomorphisms but are homomorphisms upto* $\mu_2(\Gamma(X, \mathcal{O}_X))$*.*

(b) *Suppose* X *is integral and* $q \otimes \kappa(x)$ *is semiregular at some point* x *of* X *with residue field* $\kappa(x)$*. Then any automorphism of* $C_0(V,q,I)$ *has determinant 1. Hence* $\mathrm{Aut}(C_0(V,q,I)) = \mathrm{Aut}'(C_0(V,q,I)) = \text{S-Aut}(C_0(V,q,I))$ *and* $\mathrm{O}(V,q,I)$ *is the semidirect product of* $\mu_2(\Gamma(X, \mathcal{O}_X))$ *and* $\mathrm{SO}(V,q,I)$*.*

The proofs of the above results, and of the injectivity part of Theorem 1.1, will be given in §3. As for the proof of the surjectivity part of Theorem 1.1, we have

THEOREM 1.6 *Let* X *be a scheme and* A *a specialisation of rank 4 Azumaya algebra bundles on* X*. Let* $\mathcal{O}_X.1_A \hookrightarrow A$ *be the line sub-bundle generated by the nowhere-vanishing global section of* A *corresponding to the unit for algebra multiplication.*

(a) *There exist a rank 3 vector bundle* V *on* X*, a quadratic form* q *on* V *with values in the line bundle* $I := \det^{-1}(A)$*, and an isomorphism of algebra bundles* $A \cong C_0(V,q,I)$*. This gives the surjectivity in the statement of Theorem 1.1. Further, the following linear isomorphisms may be deduced:*

- **(1)** $\det(A) \otimes \Lambda^2(V) \cong A/\mathcal{O}_X.1_A$*, from which follow:*
- **(2)** $\det(\Lambda^2(V)) \cong (\det(A))^{\otimes -2}$;
- **(3)** $V \cong (A/\mathcal{O}_X.1_A)^\vee \otimes \det(V) \otimes \det(A)$;
- **(4)** $\det(A^\vee) \cong (\det(A))^{\otimes -3} \otimes (\det(V))^{\otimes -2}$ *which implies that* $\det(A) \otimes \det(A^\vee) \in 2.\mathrm{Pic}(X)$.

(b) *There exists a quadratic bundle* (V',q',I') *such that* $C_0(V',q',I') \cong A$ *and with*

- **(1)** $I' = \mathcal{O}_X$ *iff* $\det(A) \in 2.\mathrm{Pic}(X)$*;*
- **(2)** *with* q' *induced from a global* I'*-valued bilinear form iff* $\mathcal{O}_X.1_A$ *is an* $\mathcal{O}_X$*-direct summand of* A*;*

(3) *with both* $I' = \mathcal{O}_X$ *and with* q' *induced from a global bilinear form (with values in* I'*) iff* $\mathcal{O}_X.1_A$ *is an* $\mathcal{O}_X$*-direct summand of* A *and* $\det(A) \in 2.\mathrm{Pic}(X)$.

The next theorem is a key ingredient in the proof of part (a) above. It describes specialisations as bilinear forms under certain conditions. As a preparation towards its statement, we recall a few definitions and results from Part A, [2]. For a rank n^2 vector bundle W on a scheme X and $w \in \Gamma(X, W)$ a nowhere-vanishing global section, recall that if Id-w-Azu$_W$ is the open X–subscheme of Azumaya algebra structures on W with identity w then its schematic image (or the scheme of specialisations or the limiting scheme) in the bigger X–scheme Id-w-Assoc$_W$ of associative w-unital algebra structures on W is the X–scheme Id-w-Sp-Azu$_W$. By definition, the set of distinct specialised w-unital algebra structures on W corresponds precisely to the set of global sections of this last scheme over X. If $\mathrm{Stab}_w \subset \mathrm{GL}_W$ is the stabiliser subgroupscheme of w, recall from Theorems 3.4 and 3.8, Part A, [2], that there exists a canonical action of Stab_w on Id-w-Sp-Azu$_W$ such that the natural inclusions

$$(\clubsuit) \qquad \text{Id-}w\text{-Azu}_W \hookrightarrow \text{Id-}w\text{-Sp-Azu}_W \hookrightarrow \text{Id-}w\text{-Assoc-Alg}_W$$

are all Stab_w-equivariant. Now let V be a rank 3 vector bundle on the scheme X and $\mathrm{Bil}_{(V,I)}$ be the associated rank 9 vector bundle of bilinear forms on V with values in the line bundle I. Let $\mathrm{Bil}^{sr}_{(V,I)} \hookrightarrow \mathrm{Bil}_{(V,I)}$ correspond to the open subscheme of semiregular bilinear forms. We say that a bilinear form b is semiregular if there is a trivialisation $\{U_i\}$ of I, such that over each open subscheme U_i, the quadratic form q_i with values in the trivial line bundle induced from $q_b|U_i$ is semiregular (it may turn out that a semiregular bilinear form may be degenerate). This definition is independent of the choice of a trivialisation, since q_i is semiregular iff λq_i is semiregular for every $\lambda \in \Gamma(U_i, \mathcal{O}_X^*)$. Let $W := \Lambda^{even}(V, I) := \bigoplus_{n\geq 0} \Lambda^{2n}(V) \otimes I^{-\otimes n}$ and let $w \in \Gamma(X, W)$ be the nowhere-vanishing global section corresponding to the unit for the natural multiplication in the twisted even-exterior algebra bundle W. There is an obvious natural action of GL_V on $\mathrm{Bil}_{(V,I)}$. There is also a natural morphism of groupschemes $\mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$ given on valued points by $g \mapsto \bigoplus_{n\geq 0} \Lambda^{2n}(g) \otimes \mathrm{Id}$ and therefore the natural inclusions marked by $(\clubsuit)$ above are GL_V-equivariant. Finally, note that there is an obvious involution Σ on Id-w-Assoc-Alg$_W$ given by $A \mapsto \mathrm{opposite}(A)$ which leaves the open subscheme Id-w-Azu$_W$ invariant.

THEOREM 1.7

(1) *Let* V *be a rank 3 vector bundle on the scheme* X*,* $W := \Lambda^{even}(V, I)$ *and* $w \in \Gamma(X, W)$ *correspond to 1 in the twisted even-exterior algebra bundle. There is a natural* GL_V*-equivariant morphism of* X*–schemes* $\Upsilon' = \Upsilon'_X : \mathrm{Bil}_{(V,I)} \longrightarrow$ Id-w-Assoc$_W$ *whose schematic image is precisely the scheme of specialisations* Id-w-Sp-Azu$_W$*. Further if* Υ' *factors canonically through* $\Upsilon = \Upsilon_X : \mathrm{Bil}_{(V,I)} \longrightarrow$ Id-w-Sp-Azu$_W$*, then* Υ *is a* GL_V*-equivariant isomorphism and it maps the* GL_V*-stable open subscheme* $\mathrm{Bil}^{sr}_{(V,I)}$ *isomorphically onto the* GL_V*-stable open subscheme* Id-w-Azu$_W$.

(2) *The involution* Σ *of* Id-w-Assoc-Alg$_W$ *defines a unique involution (also denoted by* Σ*) on the scheme of specialisations* Id-w-Sp-Azu$_W$ *leaving the open subscheme* Id-w-Azu$_W$ *invariant, and therefore via the isomorphism* Υ*, it defines an involution on* $\mathrm{Bil}_{(V,I)}$*. This involution is none other than the one on valued points given by* $B \mapsto \mathrm{transpose}(-B)$.

(3) *For an* X*-scheme* T*, let* V_T *(resp.* W_T*, resp.* I_T*) denote the pullback of* V *(resp.* W*, resp.* I*) to* T*, and let* w_T *be the global section of* W_T *induced by* w*. Then the base-changes of* Υ'_X *and* Υ_X *to* T*, namely* $\Upsilon'_X \times_X T : \mathrm{Bil}_{(V,I)} \times_X T \longrightarrow$ Id-w-Assoc$_W \times_X T$ *and* $\Upsilon_X \times_X T : \mathrm{Bil}_{(V,I)} \times_X T \cong$ Id-w-Sp-Azu$_W \times_X T$ *may be canonically identified with the corresponding ones over* T *namely* $\Upsilon'_T : \mathrm{Bil}_{(V_T,I_T)} \longrightarrow$ Id-w_T-Assoc$_{W_T}$ *and* $\Upsilon_T : \mathrm{Bil}_{(V_T,I_T)} \cong$ Id-w_T-Sp-Azu$_{W_T}$.

The proofs of Theorems 1.6 and 1.7 will be given in §4. Some notations and preliminaries are explained in §2.

2 Notations and Preliminaries

This section collects together some definitions and results. It is essentially Section 2 of [1] redone for forms with values in line bundles. We omit the proofs which follow by localisation and the corresponding results of Sec.2, [1]. For the systematic treatment of the generalised Clifford algebra and its properties, we refer the reader to the paper of Bichsel-Knus [3].

Quadratic and Bilinear Forms with Values in a Line Bundle. Let V be a vector bundle (of constant positive rank) and I a line bundle on a scheme X. A bilinear form (resp. alternating form, resp. quadratic form) with values in I on V over an open set $U \hookrightarrow X$ is by definition a section over U of the vector bundle $\mathrm{Bil}_{(V,I)}$ (resp. of $\mathrm{Alt}^2_{(V,I)}$, resp. of $\mathrm{Quad}_{(V,I)}$), or equivalently, an element of $\Gamma\left(U, \mathrm{Bil}_{(\mathcal{V},\mathcal{I})} := (T^2_{\mathcal{O}_X}(\mathcal{V}))^\vee \otimes \mathcal{I}\right)$ (resp. of $\Gamma\left(U, \mathrm{Alt}^2_{(\mathcal{V},\mathcal{I})} := (\Lambda^2_{\mathcal{O}_X}(\mathcal{V}))^\vee \otimes \mathcal{I}\right)$, resp. of $\Gamma(U, \mathrm{Quad}_{(\mathcal{V},\mathcal{I})})$), where the sheaf $\mathrm{Quad}_{(\mathcal{V},\mathcal{I})}$—the (coherent locally-free) sheaf of $\mathcal{O}_X$-modules corresponding to the bundle $\mathrm{Quad}_{(V,I)}$ of I-valued quadratic forms on V—is defined by the exactness of the following sequence:

$$0 \longrightarrow \mathrm{Alt}^2_{(\mathcal{V},\mathcal{I})} \longrightarrow \mathrm{Bil}_{(\mathcal{V},\mathcal{I})} \longrightarrow \mathrm{Quad}_{(\mathcal{V},\mathcal{I})} \longrightarrow 0.$$

In terms of the corresponding (geometric) vector bundles over X, the above translates into the following sequence of morphisms of vector bundles, with the first one a closed immersion and the second one a Zariski locally-trivial principal $\mathrm{Alt}^2_{(V,I)}$-bundle:

$$\mathrm{Alt}^2_{(V,I)} \hookrightarrow \mathrm{Bil}_{(V,I)} \twoheadrightarrow \mathrm{Quad}_{(V,I)}.$$

Given a quadratic form $q \in \Gamma(U, \mathrm{Quad}_{(\mathcal{V},\mathcal{I})})$, recall that the usual 'associated' symmetric bilinear form $b_q \in \Gamma(U, \mathrm{Bil}_{(\mathcal{V},\mathcal{I})})$ is given on sections (over open subsets of U) by $v \otimes v' \mapsto q(v+v') - q(v) - q(v')$. Given a (not-necessarily symmetric!) bilinear form b, we also have the induced quadratic form q_b given on sections by $v \mapsto b(v \otimes v)$. A global quadratic form may not be induced from a global bilinear form, unless we assume something more, for e.g., that the scheme is affine, or more generally that the sheaf cohomology group $\mathrm{H}^1(X, \mathrm{Alt}^2_{(\mathcal{V},\mathcal{I})}) = 0$.

The Generalised Clifford Algebra of Bichsel-Knus. Let$R$ be a commutative ring (with 1), I an invertible R-module and V a projective R-module. Let $L[I] := R \oplus \left(\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1}))\right)$ be the Laurent-Rees algebra of I, and define the $\mathbb{Z}$-gradation on the tensor product of algebras $TV \otimes L[I]$ by requiring elements of V (resp. of I) to be of degree one (resp. of degree two). Let $q : V \longrightarrow I$ be an I-valued quadratic form on V. Following the definition of Bichsel-Knus [3], let $J(q,I)$ be the two-sided ideal of $TV \otimes L[I]$ generated by the set $\{(x \otimes_{TV} x) \otimes 1_{L[I]} - 1_{TV} \otimes q(x) \,|\, x \in V\}$ and let the generalised Clifford algebra of q be defined by $\widetilde{C}(V,q,I) := TV \otimes L[I]/J(q,I)$. This is an $\mathbb{Z}$-graded algebra by definition. Let C_n be the submodule of elements of degree n. Then C_0 is a subalgebra, playing the role of the even Clifford algebra in the classical situation (i.e., $I = R$) and C_1 is a C_0-bimodule. Bichsel and Knus baptize C_0 and C_1 respectively as the *even Clifford algebra* and the *Clifford module* associated to the triple (V,q,I). The generalised Clifford algebra satisfies an appropriate universal property which ensures it behaves well functorially. Since V is projective, the canonical maps $V \longrightarrow \widetilde{C}(V,q,I)$ and $L[I] \longrightarrow \widetilde{C}(V,q,I)$ are injective. For proofs of these facts, see Sec.3 of [3]. If (V,q,I) is an I-valued quadratic form on the vector bundle V over a scheme X, with I a line bundle, then the above construction may be carried out to define the generalised Clifford algebra bundle $\widetilde{C}(V,q,I)$ which is an $\mathbb{Z}$-graded algebra bundle on X.

Bourbaki's Tensor Operations with Values in a Line Bundle. Let R be a commutative ring and $L[I] := R \oplus \left(\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1}))\right)$ as above. We denote by $\otimes_T$ (resp. by $\otimes_L$) the tensor product and by 1_T (resp. 1_L) the unit element in the algebra TV (resp. in $L[I]$).

THEOREM 2.1 (with the above notations)

(1) *Let $q : V \longrightarrow I$ be an I-valued quadratic form on V and $f \in \mathrm{Hom}_R(V,I)$. Then there exists an R-linear endomorphism t_f of the algebra $TV \otimes L[I]$ which is unique with respect to the first three of the following properties it satisfies:*

- **(a)** $t_f(1_T \otimes \lambda) = 0 \,\forall\, \lambda \in L[I]$;
- **(b)** $t_f((x \otimes_T y) \otimes \lambda) = y \otimes (f(x) \otimes_L \lambda) - (x \otimes_T 1_T) t_f(y \otimes \lambda)$ *for any* $x \in V$, $y \in TV$, *and* $\lambda \in L[I]$;
- **(c)** *following the definition of Bichsel-Knus [3], let $J(q,I)$ be the two-sided ideal of $TV \otimes L[I]$ generated by the set $\{(x \otimes_T x) \otimes 1_L - 1_T \otimes q(x) \,|\, x \in V\}$; then $t_f(J(q,I)) \subset J(q,I)$;*
- **(d)** *t_f is homogeneous of degree $+1$ (except for elements which it does not annihilate);*
- **(e)** *recall from the definition of Bichsel-Knus [3] that the generalised Clifford algebra of q is $\widetilde{C}(V,q,I) := TV \otimes L[I]/J(q,I)$; by (c) above, t_f induces a $\mathbb{Z}$-graded endomorphism of degree $+1$ denoted by $d^q_f : \widetilde{C}(V,q,I) \longrightarrow \widetilde{C}(V,q,I)$;*
- **(f)** $t_f \circ t_f = 0$;
- **(g)** *if $g \in \mathrm{Hom}_R(V,I)$, then $t_f \circ t_g + t_g \circ t_f = 0$;*

(h) *if $\alpha \in \mathrm{End}_R(V)$, then $t_f \circ (T(\alpha) \otimes \mathrm{Id}_{L[I]}) = (T(\alpha) \otimes \mathrm{Id}_{L[I]}) \circ t_{\alpha^* f}$ where $\alpha^* f \in \mathrm{Hom}_R(V, I)$ is defined by $x \mapsto f(\alpha(x))$;*

(i) *$t_f \equiv 0$ on the R-subalgebra of $TV \otimes L[I]$ generated by $\mathrm{kernel}(f) \otimes L[I]$. In fact, atleast when V is a projective R-module, the smallest R-subalgebra of $TV \otimes L[I]$ containing $\mathrm{kernel}(f) \otimes R.1_L$ is $\mathrm{Image}(T(\mathrm{kernel}(f)) \otimes R.1_L$ and t_f vanishes on this R-subalgebra.*

(2) *Let $q, q' : V \longrightarrow I$ be two I-valued quadratic forms whose difference is the quadratic form q_b induced by an I-valued bilinear form $b \in \mathrm{Bil}_R(V, I) := \mathrm{Hom}_R(V \otimes_R V, I)$ i.e., $q'(x) - q(x) = q_b(x) := b(x, x) \, \forall x \in V$. Further, for any $x \in V$ denote by b_x the element of $\mathrm{Hom}_R(V, I)$ given by $y \mapsto b(x, y)$. Then there exists an R-linear automorphism Ψ_b of $TV \otimes L[I]$ which is unique with respect to the first three of the following properties it satisfies:*

(a) $\Psi_b(1_T \otimes \lambda) = (1_T \otimes \lambda) \, \forall \lambda \in L[I]$;

(b) *$\Psi_b((x \otimes_T y) \otimes \lambda) = (x \otimes 1_L).\Psi_b(y \otimes \lambda) + t_{b_x}(\Psi_b(y \otimes \lambda))$ for any $x \in V$, $y \in TV$ and $\lambda \in L[I]$;*

(c) $\Psi_b(J(q', I)) \subset J(q, I)$;

(d) *by the previous property, Ψ_b induces an isomorphism of $\mathbb{Z}$-graded R-modules*

$$\psi_b : \widetilde{C}(V, q', I) \cong \widetilde{C}(V, q, I);$$

in particular, given a quadratic form $q_1 : V \longrightarrow I$, since there always exists an I-valued bilinear form b_1 that induces q_1 (i.e., such that $q_1 = q_{b_1}$), setting $q' = q_1$, $q = 0$ and $b = b_1$ in the above gives an $\mathbb{Z}$-graded linear isomorphism $\psi_{b_1} : \widetilde{C}(V, q_1, I) \cong \widetilde{C}(V, 0, I) = \Lambda(V) \otimes L[I]$;

(e) $\Psi_b(T^{2n}V \otimes L[I]) \subset \oplus_{(i \leq n)}(T^{2i}V \otimes L[I])$, $\Psi_b(T^{2n+1}V \otimes L[I]) \subset \oplus_{(\text{odd } i \leq 2n+1)}(T^iV \otimes L[I])$, $\Psi_b(T^{2n}V \otimes I^{-n}) \subset \oplus_{(i \leq n)}(T^{2i}V \otimes I^{-i})$ *and* $\Psi_b(T^{2n+1}V \otimes I^{-n}) \subset \oplus_{(\text{odd } i \leq 2n+1)}(T^iV \otimes I^{\frac{1-i}{2}})$;

(f) *in particular, for $x, x' \in V$, $\Psi_b((x \otimes_T x') \otimes 1_L) = (x \otimes_T x') \otimes 1_L + 1_T \otimes b(x, x')$ so that for $\psi_b : C_0(V, q_b, I) \cong C_0(V, 0, I) = \oplus_{n \geq 0}(\Lambda^{2n}(V) \otimes I^{-n})$ we have $\psi_b(((x \otimes_T x') \otimes \zeta) \bmod J(q_b, I)) = (x \wedge x') \otimes \zeta + \zeta(b(x, x')).1$ for any $x, x' \in V$ and $\zeta \in I^{-1} \equiv \mathrm{Hom}_R(I, R)$;*

(g) *if $f \in \mathrm{Hom}_R(V, I)$, and t_f is given by (1) above, then $\Psi_b \circ t_f = t_f \circ \Psi_b$;*

(h) *for I-valued bilinear forms b_i on V, $\Psi_{b_1 + b_2} = \Psi_{b_1} \circ \Psi_{b_2}$ and $\Psi_0 = \mathrm{Identity}$ on $TV \otimes L[I]$;*

(i) *for any $\alpha \in \mathrm{End}_R(V)$, $\Psi_b \circ (T(\alpha) \otimes \mathrm{Id}_{L[I]}) = (T(\alpha) \otimes \mathrm{Id}_{L[I]}) \circ \Psi_{(b.\alpha)}$ where $(b.\alpha)(x, x') := b(\alpha(x), \alpha(x')) \, \forall x, x' \in V$;*

(j) *by property (h), one has a homomorphism of groups $(\mathrm{Bil}_R(V, I), +) \longrightarrow (\mathrm{Aut}_R(TV \otimes L[I]), \circ) : b \mapsto \Psi_b$; the associative unital monoid $(\mathrm{End}_R(V), \circ)$ acts on $\mathrm{Bil}_R(V, I)$ on the right by $b' \rightsquigarrow b'.\alpha$ and acts on the left (resp. on the right) of $\mathrm{End}_R(TV \otimes L[I])$ by $\alpha.\Phi := (T(\alpha) \otimes \mathrm{Id}_{L[I]}) \circ \Phi$ (resp. by $\Phi.\alpha := \Phi \circ (T(\alpha) \otimes \mathrm{Id}_{L[I]})$, and the homomorphism $b \mapsto \Psi_b$ satisfies $\alpha.\Psi_{(b.\alpha)} = \Psi_b.\alpha$; the group $\mathrm{Aut}_R(V) = \mathrm{GL}_R(V)$ acts on the left of $\mathrm{Bil}_R(V, I)$ by $g.b : (x, x') \mapsto b(g^{-1}(x), g^{-1}(x'))$ and on the left of $\mathrm{Aut}_R(TV \otimes L[I])$ by conjugation via the natural group homomorphism $\mathrm{GL}_R(V) \longrightarrow \mathrm{Aut}_R(TV \otimes L[I]) : g \mapsto T(g) \otimes \mathrm{Id}_{L[I]}$ i.e., $g.\Phi := (T(g) \otimes \mathrm{Id}_{L[I]}) \circ \Phi \circ (T(g^{-1}) \otimes \mathrm{Id}_{L[I]})$, and the homomorphism $b \mapsto \Psi_b$ is $\mathrm{GL}_R(V)$-equivariant: $\Psi_{g.b} = g.\Psi_b$.*

(3) *For a commutative R-algebra S (with 1), let $(q \otimes_R S), (q' \otimes_R S) : (V \otimes_R S =: V_S) \longrightarrow (I \otimes_R S =: I_S)$ be the I_S-valued quadratic forms induced from the quadratic forms q, q' of (2) above and $(b \otimes_R S) \in \mathrm{Bil}_S(V_S, I_S)$ the I_S-valued S-bilinear form induced from the bilinear form b of (2) above. Then as a result of the uniqueness properties (2a)–(2c) satisfied by Ψ_b and $\Psi_{(b \otimes_R S)}$, the S-linear automorphisms $(\Psi_b \otimes_R S)$ and $\Psi_{(b \otimes_R S)}$ may be canonically identified. In particular, the $\mathbb{Z}$-graded S-linear isomorphism $(\psi_b \otimes_R S) : \widetilde{C}(V_S, (q' \otimes_R S), I_S) \cong \widetilde{C}(V_S, (q \otimes_R S), I_S)$ induced from ψ_b of (2d) above may be canonically identified with $\psi_{(b \otimes_R S)}$.*

Remark 2.2 As mentioned in [3], F. van Oystaeyen has observed that $L[I]$ is a faithfully-flat splitting for I, and the generalised Clifford algebra $\widetilde{C}(V, q, I)$ is nothing but the "classical" Clifford algebra of the triple $(V \otimes_R L[I], q \otimes_R L[I], I \otimes_R L[I] \equiv L[I])$ over $L[I]$. In the same vein, I-valued forms (both multilinear and quadratic) on an R-module V can be treated as the usual ($L[I]$-valued) forms on $V \otimes_R L[I]$. With this in mind, the proof of the above proposition follows from the usual Bourbaki tensor operations with respect to $V \otimes_R L[I]$ on $L[I]$. (See §9, Chap.9, [6] or para.1.7, Chap.IV, [5], or Theorem 2.1 [1] for the "classical" Bourbaki operations). However one needs to remember that the $\mathbb{Z}$-gradation on $TV \otimes_R L[I]$ as defined above is different from the usual $\mathbb{Z}_{\geq 0}$-gradation on $T(V \otimes_R L[I])$.

Tensoring by Symmetric Bilinear Bundles and Twisted Discriminant Bundles. Let V, M be vector bundles on a scheme X and let I, J be line bundles on X. Let q be a quadratic form on V with values in I and let b be a symmetric bilinear form on M with values in J.

PROPOSITION 2.3 (with the above notations)

(1) *The tensor product of* (V, q, I) *with* (M, b, J) *gives a unique quadratic bundle* $(V \otimes M, q \otimes b, I \otimes J)$. *The quadratic form on* $V \otimes M$ *is given on sections by* $v \otimes m \mapsto q(v) \otimes b(m \otimes m)$ *and has associated bilinear form* $b_{q\otimes b} = b_q \otimes b$.

(2) *When* M *is a line bundle,* (M, b, J) *is regular (=nonsingular) iff* (M, q_b, J) *is semiregular iff* $b : M \otimes M \cong J$ *is an isomorphism (i.e., in other words iff* (M, b, J) *is a twisted discriminant bundle).*

(3) *Let* V *be of odd rank and* (M, b, J) *a twisted discriminant bundle. Then* (V, q, I) *is semiregular iff* $(V, q, I) \otimes (M, b, J) = (V \otimes M, q \otimes b, I \otimes J)$ *is semiregular.*

PROPOSITION 2.4 *Let* V *and* V' *be vector bundles of the same rank on the scheme* X, (L, h, J) *a twisted discriminant line bundle on* X *and* $\alpha : V' \cong V \otimes L$ *an isomorphism of bundles.*

(1) *Over any open subset* $U \hookrightarrow X$, *given a bilinear form* $b' \in \Gamma\left(U, \mathrm{Bil}_{(V',I)}\right)$, *we can define a bilinear form* $b \in \Gamma\left(U, \mathrm{Bil}_{(V,I\otimes J^{-1})}\right)$ *using* α *and* h *as follows: we let* $b := (b' \otimes J^{-1}) \circ \left(\zeta_{(\alpha,h)}\right)^{-1}$ *where* $\zeta_{(\alpha,h)} : V' \otimes V' \otimes J^{-1} \cong V \otimes V$ *is the linear isomorphism given by the composition of the following natural morphisms:*

$$V' \otimes V' \otimes J^{-1} \overset{\alpha\otimes\alpha\otimes\mathrm{Id}\,(\cong)}{\longrightarrow} V \otimes L \otimes V \otimes L \otimes J^{-1} \overset{\mathrm{SWAP}(2,3)\,(\equiv)}{\longrightarrow} V \otimes V \otimes L^2 \otimes J^{-1} \overset{\mathrm{Id}\otimes h\otimes\mathrm{Id}\,(\cong)}{\longrightarrow}$$
$$\overset{\mathrm{Id}\otimes h\otimes\mathrm{Id}\,(\cong)}{\longrightarrow} V \otimes V \otimes J \otimes J^{-1} \overset{\mathrm{CANON}\,(\equiv)}{\longrightarrow} V \otimes V.$$

Then the association $b' \mapsto b$ *induces linear isomorphisms shown by vertical upward arrows in the following diagram of associated locally-free sheaves (with exact rows) making it commutative:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Alt}^2_{(\mathcal{V},\mathcal{I}\otimes\mathcal{J}^{-1})} \arrow[r] & \mathrm{Bil}_{(\mathcal{V},\mathcal{I}\otimes\mathcal{J}^{-1})} \arrow[r] & \mathrm{Quad}_{(\mathcal{V},\mathcal{I}\otimes\mathcal{J}^{-1})} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Alt}^2_{(\mathcal{V}',\mathcal{I})} \arrow[r] \arrow[u, "\cong"] & \mathrm{Bil}_{(\mathcal{V}',\mathcal{I})} \arrow[r] \arrow[u, "\cong"] & \mathrm{Quad}_{(\mathcal{V}',\mathcal{I})} \arrow[r] \arrow[u, "\cong"] & 0.
\end{tikzcd}$$

Therefore one also has the following commutative diagram of vector bundle morphisms with the vertical upward arrows being isomorphisms:

$$\begin{tikzcd}
\mathrm{Alt}^2_{(V,I\otimes J^{-1})} \arrow[r, "\text{closed}", "\text{immersion}"'] & \mathrm{Bil}_{(V,I\otimes J^{-1})} \arrow[r, "\text{locally}", "\text{trivial}"'] & \mathrm{Quad}_{(V,I\otimes J^{-1})} \\
\mathrm{Alt}^2_{(V',I)} \arrow[r, "\text{closed}", "\text{immersion}"'] \arrow[u, "\cong"] & \mathrm{Bil}_{(V',I)} \arrow[r, "\text{locally}", "\text{trivial}"'] \arrow[u, "\cong"] & \mathrm{Quad}_{(V',I)} \arrow[u, "\cong"]
\end{tikzcd}$$

(2) *Let* $b' \in \Gamma\left(X, \mathrm{Bil}_{(V',I)}\right)$ *be a global bilinear form and let it induce* $b \in \Gamma\left(X, \mathrm{Bil}_{(V,I\otimes J^{-1})}\right)$ *via* α *and* h *as defined in (1) above. Let* $\Psi_{b'} \in \mathrm{Aut}_{\mathcal{O}_X}(TV' \otimes L[I])$ *(resp.* $\Psi_b \in \mathrm{Aut}_{\mathcal{O}_X}(TV \otimes L[I \otimes J^{-1}])$*) be the* $\mathbb{Z}$*-graded linear isomorphism induced by* b' *(resp. by* b*) defined locally (and hence globally) as in (2), Theorem 2.1 above. Let* $Z_{(\alpha,h)} : \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) \cong \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n)$ *be the* $\mathcal{O}_X$*-algebra isomorphism induced via the isomorphism* $\zeta_{(\alpha,h)}$ *defined in (1) above. Then, taking into account (2e), Theorem 2.1, the following diagram commutes:*

$$\begin{tikzcd}
\oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) \arrow[r, "Z_{(\alpha,h)}", "\cong"'] \arrow[d, "\Psi_{b'}"', "\cong"] & \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n) \arrow[d, "\cong"', "\Psi_b"] \\
\oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) \arrow[r, "\cong", "Z_{(\alpha,h)}"'] & \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n)
\end{tikzcd}$$

thereby inducing by (2d), Theorem 2.1 the following commutative diagram of $\mathcal{O}_X$*-linear isomorphisms*

$$\begin{tikzcd}
C_0(V', q_{b'}, I) \arrow[r, "\text{via } Z_{(\alpha,h)}", "\cong"'] \arrow[d, "\psi_{b'}"', "\cong"] & C_0(V, q_b, I \otimes J^{-1}) \arrow[d, "\cong"', "\psi_b"] \\
\oplus_{n\geq 0}(\Lambda^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) \arrow[r, "\cong", "\text{via } Z_{(\alpha,h)}"'] & \oplus_{n\geq 0}(\Lambda^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n)
\end{tikzcd}$$

(3) *Let b and b' be as in (2) above. Then $\alpha : V' \cong V \otimes L$ induces an isometry of bilinear form bundles $\alpha : (V', b', I) \cong (V, b, I \otimes J^{-1}) \otimes (L, h, J)$ and also an isometry of the induced quadratic bundles $\alpha : (V', q_{b'}, I) \cong (V, q_b, I \otimes J^{-1}) \otimes (L, h, J)$. Moreover, if we are just given a global $I \otimes J^{-1}$-valued quadratic form q on V (resp. an I-valued q' on V'), then we may define the global I-valued quadratic form q' on V' (resp. $I \otimes J^{-1}$-valued q on V) via $q' := (q \otimes h) \circ \alpha$ (resp. via $q := (q' \circ \alpha^{-1}) \otimes (h^{\vee})^{-1})$) and again $\alpha : (V', q', I) \cong (V, q, I \otimes J^{-1}) \otimes (L, h, J)$ becomes an isometry of quadratic bundles.*

PROPOSITION 2.5 *Let $g : (V, q, I) \cong_l (V', q', I)$ be an I-similarity with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$.*

(1) *There exists a unique isomorphism of $\mathcal{O}_X$-algebra bundles $C_0(g, l, I) : C_0(V, q, I) \cong C_0(V', q', I)$ such that $C_0(g, l, I)(v.v'.s) = g(v).g(v').l^{-1}s$ on sections v, v' of V and s of I^{-1}.*

(2) *There exists a unique vector bundle isomorphism $C_1(g, l, I) : C_1(V, q, I) \cong C_1(V', q', I)$ such that*

(a) *$C_1(g, l, I)(v.c) = g(v).C_0(g, l, I)(c)$ and*

(b) *$C_1(g, l, I)(c.v) = C_0(g, l, I)(c).g(v)$*

for any section v of V and any section c of $C_0(V, q)$. Thus $C_1(g, l, I)$ is $C_0(g, l, I)$-semilinear.

(3) *If $g_1 : (V', q', I) \cong_{l_1} (V'', q'', I)$ is another similarity with multiplier l_1, then the composition $g_1 \circ g : (V, q, I) \cong_{ll_1} (V'', q'', I)$ is also a similarity with multiplier given by the product of the multipliers. Further $C_i(g_1 \circ g, ll_1, I) = C_i(g_1, l_1, I) \circ C_i(g, l, I)$ for $i = 0, 1$.*

A local computation shows that tensoring by a twisted discriminant bundle amounts to (locally) applying a similarity. In this case also one gets a global isomorphism of even Clifford algebras:

PROPOSITION 2.6 *Let (V, q, I) be a quadratic bundle on a scheme X and (L, h, J) be a twisted discriminant bundle. There exists a unique isomorphism of algebra bundles*

$$\gamma_{(L,h,J)} : C_0\left((V, q, I) \otimes (L, h, J)\right) \cong C_0(V, q, I)$$

given by $\gamma_{(L,h,J)}\left((v \otimes \lambda).(v' \otimes \lambda').(s \otimes t)\right) = t(h(\lambda \otimes \lambda'))v.v'.s$ for any sections v, v' of V, λ, λ' of L, s of $I^{-1} \equiv I^{\vee}$ and t of $J^{-1} \equiv J^{\vee}$.

3 Proof of Injectivity: Theorems 1.4 & 1.5

Proof of Prop.1.3: Start with an isomorphism of algebra-bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$. Let $\{U_i\}_{i \in \mathcal{I}}$ be an affine open covering of X (which may also be chosen so as to trivialise some or any of the involved bundles if needed). Choose bilinear forms $b_i \in \Gamma(U_i, \mathrm{Bil}_{(V,I)})$ and $b'_i \in \Gamma(U_i, \mathrm{Bil}_{(V',I')})$ such that $q|U_i = q_{b_i}$ and $q'|U_i = q_{b'_i}$ for each $i \in \mathcal{I}$. By (2d), Theorem 2.1, we have isomorphisms of vector bundles ψ_{b_i} and $\psi_{b'_i}$, which preserve 1 by (2a) of the same Theorem, and we define the isomorphism of vector bundles $\phi_{\Lambda_i^{ev}}$ so as to make the following diagram commute:

$$\begin{array}{ccc} C_0(V, q, I)|U_i & \xrightarrow[\cong]{\phi|U_i} & C_0(V', q', I')|U_i \\ \psi_{b_i} \Big\downarrow \cong & & \cong \Big\downarrow \psi_{b'_i} \\ (\mathcal{O}_X \oplus \Lambda^2(V) \otimes I^{-1})|U_i & \xrightarrow[\phi_{\Lambda_i^{ev}}]{\cong} & (\mathcal{O}_X \oplus \Lambda^2(V') \otimes (I')^{-1})|U_i \end{array}$$

The linear isomorphism $\phi_{\Lambda_i^{ev}}$ preserves 1 and therefore it induces a linear isomorphism from $(\Lambda^2(V) \otimes I^{-1})|U_i$ to $(\Lambda^2(V') \otimes (I')^{-1})|U_i$, which we denote by $(\phi_{\Lambda^2})_i$. Observe that $(\phi_{\Lambda^2})_i$ is independent of the choice of the bilinear forms b_i and b'_i. For, replacing these respectively by $\widehat{b_i}$ and $\widehat{b'_i}$, it follows from (2f), Theorem 2.1, that $\psi_{b_i} \circ (\psi_{\widehat{b_i}})^{-1}$ (resp. $\psi_{b'_i} \circ (\psi_{\widehat{b'_i}})^{-1}$) followed by the canonical projection onto $(\Lambda^2(V) \otimes I^{-1})|U_i$ (resp. onto $(\Lambda^2(V') \otimes (I')^{-1})|U_i$) is the same as the projection itself. By this observation, it is also clear that the isomorphisms $\{(\phi_{\Lambda^2})_i\}_{i \in \mathcal{I}}$ agree on (any open affine subscheme of, and hence on all of) any intersection $U_i \cap U_j$. Therefore they glue to give a global isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$. **Q.E.D, Prop.1.3.**

Reduction of Proof of Injectivity of Theorem 1.1 to Theorem 1.4. We start with an isomorphism of algebra-bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$, construct the isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$ and keep the notations introduced in the proof of Prop.1.3. Firstly we deduce a linear isomorphism $\det((\phi_{\Lambda^2})^\vee)^{-1} : \det((\Lambda^2(V) \otimes I^{-1})^\vee) \cong \det((\Lambda^2(V') \otimes (I')^{-1})^\vee)$. Since V and V' are of rank 3, there are canonical isomorphisms $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ and $\eta' : \Lambda^2(V') \equiv (V')^\vee \otimes \det(V')$. It follows therefore that if we set $L := \det(V') \otimes (\det(V))^{-1}$ and $J := I' \otimes I^{-1}$ then we get a twisted discriminant line bundle $(L \otimes J^{-1}, h, J)$ and a vector bundle isomorphism $\alpha : V' \cong V \otimes (L \otimes J^{-1})$.

Now for each $i \in \mathcal{I}$, the bilinear form $b_i \in \Gamma(U_i, \mathrm{Bil}_{(V,I)})$ induces, via $\alpha|U_i$ and $(LJ^{-1}, h, J)|U_i$ and (1), Prop.2.4, a bilinear form $b_i'' \in \Gamma(U_i, \mathrm{Bil}_{(V', IJ)})$. By (3) of the same Proposition, over each U_i we get an isometry of bilinear form bundles $\alpha|U_i : (V'|U_i, b_i'', IJ|U_i) \cong (V|U_i, b_i, I|U_i) \otimes (LJ^{-1}, h, J)|U_i$ and also an isometry of quadratic bundles $\alpha|U_i : (V'|U_i, q_{b_i''}, IJ|U_i) \cong (V|U_i, q_{b_i} = q|U_i, I|U_i) \otimes (LJ^{-1}, h, J)|U_i$.

On the other hand, by an assertion in (3), Prop.2.4, we could also define the global quadratic bundle (V', q'', IJ) using (V, q, I), α and (LJ^{-1}, h, J), so that we have an isometry of quadratic bundles $\alpha : (V', q'', IJ) \cong (V, q, I) \otimes (LJ^{-1}, h, J)$. It follows therefore that the $q_{b_i''}$ glue to give q''. Notice that in general the b_i'' (resp. the b_i) need not glue to give a global bilinear form b'' (resp. b) such that $q_{b''} = q''$ (resp. $q_b = q$). By (1), Prop.2.5, there exists a unique isomorphism of algebra bundles

$$C_0(\alpha, 1, IJ) : C_0(V', q'', IJ) \cong C_0\left((V, q, I) \otimes (LJ^{-1}, h, J)\right)$$

and by Prop.2.6 we have a unique isomorphism of algebra bundles

$$\gamma_{(LJ^{-1}, h, J)} : C_0\left((V, q, I) \otimes (LJ^{-1}, h, J)\right) \cong C_0(V, q, I).$$

Therefore the composition of the following sequence of isomorphisms of algebra bundles on X

$$C_0(V', q'', I') \overset{\text{using } I' \cong IJ}{\cong} C_0(V', q'', IJ) \overset{C_0(\alpha,1,IJ)\,(\cong)}{\longrightarrow} C_0\left((V, q, I) \otimes (LJ^{-1}, h, J)\right) \overset{\gamma_{(LJ^{-1},h,J)}\,(\cong)}{\longrightarrow}$$
$$\overset{\gamma_{(LJ^{-1},h,J)}\,(\cong)}{\longrightarrow} C_0(V, q, I) \overset{\phi\,(\cong)}{\longrightarrow} C_0(V', q', I')$$

is an element of $\mathrm{Iso}[C_0(V', q'', I'), C_0(V', q', I')]$, which, granting Theorem 1.4, is induced by a similarity in $(g, l) \in \mathrm{Sim}[(V', q'', I'), (V', q', I')]$. Hence we would have

$$g : (V', q'', I') \cong (V', q', I') \otimes (\mathcal{O}_X,\ \text{mult.by } l^{-1}, \mathcal{O}_X)$$

where $l \in \Gamma(X, \mathcal{O}_X^*)$. This combined with the fact that (V, q, I) and $(V', q'', I') \cong (V', q'', IJ)$ are isomorphic upto the twisted discriminant bundle (LJ^{-1}, h, J) by the construction above would imply that (V, q, I) and (V', q', I') also differ by a twisted discriminant bundle. Therefore the proof of the injectivity asserted in Theorem 1.1 reduces to the proof of Theorem 1.4.

Reduction of Theorem 1.4 to the Case when I is free: For a similarity g with multiplier l, we have $C_0(g, l, I)$ given by (1), Prop.2.5, so that we may define the map

$$\mathrm{Sim}[(V, q, I), (V, q', I)] \longrightarrow \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)] : g \mapsto C_0(g, l, I).$$

The equality $\det(C_0(g, l, I)) = \det\left((C_0(g, l, I))_{\Lambda^2}\right) = l^{-3}\det^2(g)$ was shown to hold (locally, hence globally) (1), Lemma 3.11, [1]. Thus $\mathrm{Iso}[(V, q, I), (V, q', I)]$ and $\text{S-Iso}[(V, q, I), (V, q', I)]$ are respectively mapped into $\mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)]$ and $\text{S-Iso}[C_0(V, q, I), C_0(V, q', I)]$ as claimed. We start with an isomorphism of algebra-bundles $\phi : C_0(V, q, I) \cong C_0(V, q', I)$, which by Prop.1.3 leads to the automorphism of vector bundles $\phi_{\Lambda^2} \in \mathrm{Aut}(\Lambda^2(V) \otimes I^{-1})$. Firstly, define the global bundle automorphism $g' \in \mathrm{GL}\left(V \otimes (\det(V))^{-1} \otimes I\right)$ so that the following diagram commutes:

$$\begin{array}{ccc}
(\Lambda^2(V))^\vee \otimes I & \xrightarrow[\cong]{((\phi_{\Lambda^2})^\vee)^{-1}} & (\Lambda^2(V))^\vee \otimes I \\
{\scriptstyle (\eta^\vee)^{-1} \otimes I} \big\downarrow \equiv & & \equiv \big\downarrow {\scriptstyle (\eta^\vee)^{-1} \otimes I} \\
V \otimes (\det(V))^{-1} \otimes I & \xrightarrow[g']{\cong} & V \otimes (\det(V))^{-1} \otimes I
\end{array}$$

where $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ is the canonical isomorphism (since V is of rank 3). Now let $g \in \mathrm{GL}(V) \overset{\cong}{\longleftarrow} \mathrm{GL}(V \otimes (\det(V))^{-1} \otimes I)$ be the image of g' i.e., the image of $g' \otimes \det(V) \otimes I^{-1}$ under the canonical identification $\mathrm{GL}(V \otimes (\det(V))^{-1} \otimes I \otimes \det(V) \otimes I^{-1}) \equiv \mathrm{GL}(V)$. Next, let $l \in \Gamma(X, \mathcal{O}_X^*)$ be a global section such that $\gamma(l) := (l^3).\det(\phi_{\Lambda^2})$ has a square root in $\Gamma(X, \mathcal{O}_X^*)$. For example, we have the following special cases when this is true:

Case 1. If $\det(\phi_{\Lambda^2})$ is itself a square, set $l := 1$. If further $\det(\phi_{\Lambda^2}) = 1$, set $\sqrt{\gamma(l)} = 1$, otherwise let $\sqrt{\gamma(l)}$ denote any fixed square root of $\det(\phi_{\Lambda^2})$.

Case 2. If $\det(\phi_{\Lambda^2})$ is not a square, given an integer k, take $l = (\det(\phi_{\Lambda^2}))^{2k+1}$ and let $\sqrt{\gamma(l)}$ denote any fixed square root of $(\det(\phi_{\Lambda^2}))^{6k+4}$.

For each integer k, we now associate to ϕ the element $g_l^\phi := (l^{-1}\sqrt{\gamma(l)})g$ with g as defined above. Suppose we show the following locally for the Zariski toplogy on X (more precisely, for each open subscheme of X over which V and I are free):

(1) that g_l^ϕ is an I-similarity from (V,q,I) to (V,q',I) with multiplier l;

(2) that g_l^ϕ induces ϕ i.e., with the notations of (1), Prop.2.5, that $C_0(g_l^\phi, l, I) = \phi$;

(3) that $\det(g_l^\phi) = \sqrt{\gamma(l)}$ so that $\det^2(g_l^\phi) = \det(\phi_{\Lambda^2})$ when $\det(\phi_{\Lambda^2})$ is itself a square and

(4) that the map $\text{S-Iso}[(V,q,I),(V,q',I)] \longrightarrow \text{S-Iso}[C_0(V,q,I), C_0(V,q',I)]$ is injective.

It would follow then that these statements are also true globally. The maps $s_{2k+1} : \phi \mapsto g_l^\phi$ with l as in Case 2 and $s' : \phi \mapsto g_l^\phi$ with l as in Case 1 will then give the sections to the maps (which would imply their surjectivities) as mentioned in Theorem 1.4. But these maps are not necessarily multiplicative since a computation reveals that if $\phi_i \in \text{Iso}[C_0(V,q_i,I), C_0(V,q_{i+1},I)]$ is associated to $g_{l_i}^{\phi_i} \in \text{Sim}[(V,q_i,I),(V,q_{i+1},I)]$, and $\phi_2\circ\phi_1$ to $g_{l_{21}}^{\phi_2\circ\phi_1}$, then $g_{l_{21}}^{\phi_2\circ\phi_1} = \delta g_{l_2}^{\phi_2}\circ g_{l_1}^{\phi_1}$ for $\delta \in \mu_2(\Gamma(X,\mathcal{O}_X))$ because of the ambiguity in the initial global choices of square roots for $\gamma(l_i)$ and $\gamma(l_{21})$. However this can be remedied as follows. For any given $\phi \in \text{Iso}[C_0(V,q,I), C_0(V,q',I)]$, irrespective of whether or not $\det(\phi_{\Lambda^2})$ is a square, take

$$l = (\det(\phi_{\Lambda^2}))^{2k+1}, \gamma(l) = l^3\det(\phi_{\Lambda^2}), \sqrt{\gamma(l)} := (\det(\phi_{\Lambda^2}))^{3k+2} \text{ and } s_{2k+1}^+(\phi) := g_l^\phi = \left(l^{-1}\sqrt{\gamma(l)}\right)g.$$

Then it is clear that each s_{2k+1}^+ is multiplicative with the properties as claimed in the statement. We thus reduce the proof of Theorem 1.4 to the case when the rank 3 vector bundle V and the line bundle I are free. More generally, even for the case V not necessarily free but $I = \mathcal{O}_X$, this was treated in Theorem 1.3 of [1]. **Q.E.D., Theorem 1.4 & injectivity of Theorem 1.1.**

Proof of Theorem 1.5. (The case $I = \mathcal{O}_X$ was treated in Theorem 1.4 of [1]). Taking $q' = q$ in Theorem 1.4 gives the commutative diagram of groups and homomorphisms as asserted in the statement of the theorem. We continue with the notations above. For $g \in \text{GO}(V,q,I)$ with multiplier l, that the equality $\det(C_0(g,l,I)) = \det((C_0(g,l,I))_{\Lambda^2}) = l^{-3}\det^2(g)$ holds (locally, hence globally) was shown in (1), Lemma 3.11, [1]. Assertion (2) of the same Lemma shows the following (locally, and hence globally): if $C_0(g,l,I)$ is the identity on $C_0(V,q,I)$, then $g = l^{-1}\det(g).\text{Id}_V$, and further if $g \in \text{O}(V,q,I)$ then $g = \det(g).\text{Id}_V$ with $\det^2(g) = 1$. This gives exactness at $\text{GO}(V,q,I)$ and at $\text{O}(V,q,I)$.

We proceed to prove assertion (b). Let $\phi \in \text{Aut}(C_0(V,q,I))$, and consider the self-similarity $s_{2k+1}^+(\phi) = g_l^\phi$ with multiplier $l = \det(\phi)^{2k+1}$. For the moment, assume that V and I are trivial over X. Fix a global basis $\{e_1,e_2,e_3\}$ for V and set $e_i' = g_l^\phi(e_i)$. It follows from Kneser's definition of the half-discriminant d_0—see formula (3.1.4), Chap.IV, [5]—that $d_0(q,\{e_i\}) = d_0(q,\{e_i'\})\det^2(g_l^\phi)$. Since we have $g_l^\phi.q = l^{-1}q$, a simple computation shows that $d_0(q,\{e_i'\}) = l^3 d_0(q,\{e_i\})$. The hypothesis that $q\otimes\kappa(x)$ is semiregular means that the image of the element $d_0(q,\{e_i\}) \in \Gamma(X,\mathcal{O}_X)$ in $\kappa(x)$ is nonzero. Since X is integral, we therefore deduce that $\det^2(g_l^\phi) = l^{-3}$. On the other hand, we know that $\det^2(g_l^\phi)l^{-3} = \det(\phi)$. It follows that $\det^{12k+7}(\phi) = 1\,\forall\, k \in \mathbb{Z}$, which forces $\det(\phi) = 1$. In general, even if V and I are not necessarily trivial, since this equality holds over a covering of X which trivialises both V and I, it also holds over all of X. **Q.E.D., Theorem 1.5.**

4 Proof of Surjectivity: Theorems 1.6 & 1.7

Proof of Theorem 1.7. We adopt the notations introduced just before Theorem 1.7. Let T be an X-scheme. Given a bilinear form $b \in \text{Bil}_{(V,I)}(T) = \Gamma(T, \text{Bil}_{(V_T,I_T)})$, consider the linear isomorphism $\psi_b : C_0(V_T,q_b,I_T) \cong \mathcal{O}_T \oplus \Lambda^2(V_T)\otimes(I_T)^{-1} = W_T$ of (2d), Theorem 2.1. Let A_b denote the algebra bundle structure on W_T with unit $w_T = 1$ induced via ψ_b from the even Clifford algebra $C_0(V_T,q_b,I_T)$. By definition, $A_b \in \text{Id-}w\text{-Assoc}_W(T)$ and we get a map of T-valued points

$$\Upsilon'(T) : \text{Bil}_{(V,I)}(T) \longrightarrow \text{Id-}w\text{-Assoc}_W(T) : b \mapsto A_b.$$

This is functorial in T because of (3), Theorem 2.1, and hence defines an X-morphism $\Upsilon' : \mathrm{Bil}_{(V,I)} \longrightarrow \text{Id-}w\text{-Assoc}_W$. The morphism Υ' is GL_V-equivariant due to (2j), Theorem 2.1. Notice that the schemes $\mathrm{Bil}_{(V,I)}, \mathrm{Bil}^{sr}_{(V,I)}$ and Id-w-Assoc$_W$ are well-behaved relative to X with respect to base-change. In fact, so are Id-w-Azu$_W$ and Id-w-Sp-Azu$_W$, as may be recalled from Theorems 3.4 and 3.8, Part A, [2]. In view of these observations, by taking a trivialisation for I over X, we may reduce to the case when I is trivial. But this case was treated in Theorem 1.9 of [1]. **Q.E.D., Theorem 1.7.**

Proofs of assertions in (a), Theorem 1.6 and Surjectivity part of Theorem 1.1. Let W be the rank 4 vector bundle underlying the specialised algebra A and $w \in \Gamma(X, W)$ be the global section corresponding to 1_A. We choose an affine open covering $\{U_i\}_{i\in\mathcal{I}}$ of X such that $W|U_i$ is trivial and $w|U_i$ is part of a global basis $\forall\, i$. Therefore on the one hand, for each $i \in \mathcal{I}$, we can find a linear isomorphism $\zeta_i : \Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right) \cong W|U_i$ taking $1_{\Lambda^{even}}$ onto $w|U_i$. The $(w|U_i)$-unital algebra structure $A|U_i$ induces via ζ_i an algebra structure A_i on $\Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right)$ (so that ζ_i becomes an algebra isomorphism). Recall that A_i is also a specialised algebra structure by Theorem 3.8, Part A, [2]. Hence by Theorem 1.7 applied to $X = U_i$, $V = \mathcal{O}_{U_i}^{\oplus 3}$ and $I = \mathcal{O}_{U_i}$, we can also find an $\mathcal{O}_{U_i}$-valued quadratic form q_i on $\mathcal{O}_X^{\oplus 3}|U_i$ induced from a bilinear form b_i so that the algebra structure A_i is precisely the one induced by the linear isomorphism $\psi_{b_i} : C_0\left(\mathcal{O}_X^{\oplus 3}|U_i, q_i\right) \cong \Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right)$ given by (2d) of Theorem 2.1. For each pair of indices $(i,j) \in \mathcal{I}\times\mathcal{I}$, let ζ_{ij} and ϕ_{ij} be defined so that the following diagram commutes:

$$\begin{tikzcd}
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) \arrow[r, "\psi_{b_i}|U_{ij}", "\cong"'] \arrow[d, "\phi_{ij}"', "\cong"] & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) \arrow[r, "\zeta_i|U_{ij}", "\cong"'] \arrow[d, "\zeta_{ij}"', "\cong"] & A|U_{ij} \arrow[d, "="'] \\
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) \arrow[r, "\cong", "\psi_{b_j}|U_{ij}"'] & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) \arrow[r, "\cong", "\zeta_j|U_{ij}"'] & A|U_{ij}
\end{tikzcd}$$

The above diagram means that the algebras A_i glue along $U_{ij} := U_i \cap U_j$ via ζ_{ij} to give (an algebra bundle isomorphic to) A, and in the same vein, the even Clifford algebras $C_0(\mathcal{O}_X^{\oplus 3}|U_i, q_i)$ glue along the U_{ij} via ϕ_{ij} to give A as well. Now consider the similarity $g^{\phi_{ij}}_{l_{ij}} = s^+_{-1}(\phi_{ij}) : (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) \cong_{l_{ij}} (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij})$ with multiplier $l_{ij} := \det(\phi_{ij})^{-1}$ given by (c), Theorem 1.4. Since s^+_{-1} is multiplicative, and since ϕ_{ij} satisfy the cocycle condition, it follows that $s^+_{-1}(\phi_{ij})$ also satisfy the cocycle condition and therefore glue the $\mathcal{O}_X^{\oplus 3}|U_i$ along the U_{ij} to give a rank 3 vector bundle V on X. While the q_i do not glue to give an $\mathcal{O}_X$-valued quadratic form on V, the facts that the multipliers $\{l_{ij}\}$ form a cocycle for $I := \det^{-1}(A)$ and that $s^+_{(-1)}$ is a section together imply, taking into account the uniqueness in (1), Prop.2.5, that actually the q_i glue to give an I-valued quadratic form q on V and that $C_0(V, q, I) \cong A$. It was verified in page 28, [1] that $(\phi_{ij})_{\Lambda^2} = \det(\phi_{ij}).\Lambda^2(g^{\phi_{ij}}_{l_{ij}})$. This immediately implies part (1) of assertion (a) of Theorem 1.6, from which parts (2)—(4) can be deduced using the standard properties of the determinant and the perfect pairings between suitable exterior powers of a bundle.

Proofs of assertions in (b), Theorem 1.6. We first prove (b1). Let A be a given specialisation, and let $A \cong C_0(V, q, I)$ as in part (a) of Theorem 1.6 with $I = \det^{-1}(A)$. By the injectivity part of Theorem 1.1, we have $C_0(V, q, I) \cong A \cong C_0(V', q', \mathcal{O}_X)$ iff there exists a twisted discriminant bundle (L, h, J) and an isomorphism $(V, q, I) \cong (V', q', \mathcal{O}_X) \otimes (L, h, J)$. The latter implies that $I \cong J \cong L^2$ and hence $\det(A) \in 2.\mathrm{Pic}(X)$. On the other hand, if this last condition holds, we could take for L a square root of $J := I^{-1}$, alongwith an isomorphism $h : L^2 \cong J$ and we would have by Prop.2.6 an algebra isomorphism $\gamma_{(L,h,J)} : C_0(V \otimes L, q \otimes h, \mathcal{O}_X) \equiv C_0\left((V, q, I) \otimes (L, h, J)\right) \cong C_0(V, q, I) \cong A$.

For the proof of (b2), suppose that the line subbundle $\mathcal{O}_X.1_A \hookrightarrow A$ is a direct summand of A. We may choose a splitting $A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A)$. Using assertion (1) of (a), Theorem 1.6, we see that there exists a rank 3 vector bundle V on X such that $A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A) \cong \mathcal{O}_X.1_A \oplus (\Lambda^2(V) \otimes I^{-1}) \cong \mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1} =: W$ where $I := \det^{-1}(A)$ and the last isomorphism is chosen so as to map $\mathcal{O}_X.1_A$ isomorphically onto $\mathcal{O}_X.1$. Therefore if $(W, w) := (\mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1}, 1)$, then by the above identification A induces an element of Id-w-Sp-Azu$_W(X)$, and since $\Upsilon : \mathrm{Bil}_{(V,I)} \cong$ Id-w-Sp-Azu$_W$ is an X-isomorphism by (1), Theorem 1.7, it follows that there exists an I-valued global quadratic form $q = q_b$ induced from an I-valued global bilinear form b on V such that the algebra structure $\Upsilon(b) \cong A$. (We recall that $\Upsilon(b)$ is the algebra structure induced from the linear isomorphism $\psi_b : C_0(V, q = q_b, I) \cong \mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1} = W$ of (2d), Theorem 2.1, which preserves 1 by (2a) of the same Theorem). The proof of (b3) follows from a combination of those of (b1) and (b2). **Q.E.D., Theorem 1.6 and surjectivity part of Theorem 1.1.**

Acknowledgements

The author gratefully acknowledges the Postdoctoral Fellowship (September 2003–August 2004) of the *Graduiertenkolleg Gruppen und Geometrie* under support from the *Deutschen Forschungsgemeinschaft* and the State of Niedersachsen at the Mathematisches Institut Göttingen where this paper was written. The author also thanks the National Board for Higher Mathematics, Department of Atomic Energy, Government of India, for its Postdoctoral Fellowship (August 2002–August 2003) at the Chennai Mathematical Institute, Chennai, India, during the latter half of which certain special cases of the results of this work were obtained.

References

[1] Venkata Balaji T E, Degenerations of Rank 3 Quadratic Bundles and Rank 4 Azumaya Bundles over Schemes, Preprint No.136, Linear Algebraic Groups Preprint Server, University of Bielefeld `http://www.mathematik.uni-bielefeld.de/LAG/`

[2] Venkata Balaji T E, Limits of rank 4 Azumaya algebras and applications to desingularisation, Proc. Indian Acad. Sci. (Math. Sci.), Vol.112, No.4, November 2002, pp.485–537

[3] Bichsel W and Knus, Max-Albert, Quadratic forms with values in line bundles, Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), 293–306, Contemp. Math., 155, Amer. Math. Soc., Providence, RI, 1994.

[4] Kneser M, *Quadratische Formen, Ausarbeitung einer Vorlesung*, Gottingen: Mathematisches Institut, 1974.

[5] Knus, Max-Albert, Quadratic and Hermitian Forms over Rings, *Grundlehren der mathematischen Wissenschaften* **294** (1991) (Springer-Verlag)

[6] Bourbaki N, *Éléments de Mathématique* **XXIV** Livre 2, (1959) (Hermann, Paris)