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MOTIVIC GALOIS GROUPS

What are they good for?
How are they constructed?
How can one compute them?

Very roughly: generalization of Galois theory to systems of polynomial equations in several variables...
Introduced by Grothendieck.

k : base field, $\bar{k}$: separable closure

$$\begin{array}{ccc} \{\text{finite } Gal(\bar{k}/k)\text{-sets}\} & \simeq & \{\text{0-dim}^{l} \text{ etale varieties}\} \\ \Big\downarrow \omega & & Z(\bar{k}) \leftarrow\!\shortmid Z \\ \{\text{finite sets}\} & & \end{array}$$

$Gal(\bar{k}/k) = \mathrm{Aut}\,\omega$.

Linearization.

F : field of char. 0 (field of coefficients)

$$\begin{array}{ccc} \{\text{f.d.}^l \text{ (cont.) } Gal(\bar{k}/k)\text{-rep's}/_F\} & \simeq & \{F\text{-motives attached to 0-dim! étale k-varieties}\} \\ \Big\downarrow{\scriptstyle\omega} & & (= \text{Artin motives}) \\ Vec_F & & \end{array}$$

$$Gal(\bar{k}/k) = Aut^{\otimes}\omega .$$

"Higher dimensional" generalization ?

{ f.d.^{l} rep's /F of some "motivic Galois group" } $\overset{\sim}{=}$? { motives (in some appropriate sense) } $\otimes$

→ <u>Questions</u> : How to recognize when a given $\otimes$-category $\mathcal{T}$ is equivalent to the category $Rep_F\, G$ of f.d.^{l} rep's$/_F$ of some affine F-group-scheme G, and how to recover G from $\mathcal{T}$?

(Questions handled by the tannakian theory initiated by Grothendieck for the purpose of motivic Galois theory).

<u>Main purpose</u> : to translate some problems about algebraic cycles, notably existence problems, into the framework of classical representation theory.

(1) Elements of tannakian theory.

$\mathcal{T}$: F-linear rigid symmetric $\otimes$-category

$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$

assoc. constraint $\phi_{LMN} : L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N$

comm. " $\psi_{LM} : L \otimes M \xrightarrow{\sim} M \otimes L$ $\qquad \psi_{LM}^{-1} = \psi_{ML}$

unit $\mathbb{1}$ $\qquad \nu_L : L \otimes \mathbb{1} \xrightarrow{\sim} L$

invol. self-duality $^\vee : \mathcal{T} \to \mathcal{T}^{op}$

$? \otimes M^\vee$ left adjoint of $? \otimes M$

$M^\vee \otimes ?$ right adjoint of $M \otimes ?$

f $\quad \mathrm{Tr}\, f \in \mathrm{End}\, \mathbb{1}$, $\qquad \mathrm{rk}\, M = \mathrm{Tr}\, \mathrm{id}_M$.

(comm. F-alg.)

$\otimes$-functor $\qquad \omega : \mathcal{T} \to \mathcal{T}'$

$\nu_{MN} : \omega(M \otimes N) \xrightarrow{\sim} \omega(M) \otimes \omega(N)$

$\nu_{\mathbb{1}} : \omega(\mathbb{1}) \simeq \mathbb{1}$

compatible with ϕ, ψ, ν (autom. comp. with $^\vee$).

Assume $\mathrm{End}\, \mathbb{1} = F$. $\qquad$ Assume $\mathcal{T}$ abelian.

Def: • a fibre functor on $\mathcal{T}$ is an exact faithful $\otimes$-functor $\omega : \mathcal{T} \to \mathrm{Vec}_K$ for some K/F.

• If such a fibre functor exists, $\mathcal{T}$ is said to be tannakian over F.

Thm (Deligne) $\mathcal{T}$: abelian F-linear rigid symm. $\otimes$-category, with $\mathrm{End}\,\mathbb{1} = F$. Then

$\mathcal{T}$ tannakian $\iff$ the dimension of any object is a natural integer

$\iff$ for any object M, $\exists n$ s.t. $\overset{n}{\Lambda} M = 0$.

$\omega : \mathcal{T} \longrightarrow \mathrm{Vec}_K$ fibre functor

$\rightsquigarrow G = \underline{\mathrm{Aut}}^{\otimes} \omega$ affine K-gp scheme

(contravariant for exact $\mathcal{T}' \to \mathcal{T}$)

$\underline{\mathrm{Aut}}^{\otimes} \omega$ f. d! $\iff$ $\mathcal{T}$ has a $\otimes$-generator

$\underline{\mathrm{Aut}}^{\otimes} \omega$ (pro)reductive $\iff$ $\mathcal{T}$ semi-simple.

"Neutral case" : $F = K$

Then
$$\begin{tikzcd} \mathcal{T} \arrow[r, "\sim"] \arrow[rr, bend right, "\omega"'] & \mathrm{Rep}_F G \arrow[r, "\text{forget}"'] & \mathrm{Vec}_F \end{tikzcd}$$

$\mathcal{T} = Gr_F$ or $s\mathrm{Vec}_F$ $\quad\psi$ defined via Koszul's sign rule

not tannakian

even objects: $\exists n$ s.t $\overset{n}{\Lambda} M = 0$

odd objects: " $\overset{n}{S} M = 0$

and in general : M killed by a Schur functor.

Thm (Deligne) $\mathcal{T}$: abelian F-linear rigid symm. $\otimes$-category, with $\mathrm{End}\,\mathbb{1} = F$. Then

$\mathcal{T}$ "supertannakian" $\iff$ any object is killed by some Schur functor.

($\omega : \mathcal{T} \to sVec_K$ super fiber functor
$\rightsquigarrow \underline{Aut}^{\otimes}\omega$ super affine K-gp scheme)

More generally, $\mathcal{T}$: abelian F-linear rigid ~~symm~~ $\otimes$-category, with $\mathrm{End}\,\mathbb{1} = F$, and such that all objects and all spaces of morphisms are of finite length. Deligne attaches to $\mathcal{T}$ an avatar of tannakian gp, the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ (independent of any ω).

Actually, he defines $\mathcal{O}(\pi(\mathcal{T}))$ as a commutative Hopf algebra in $\mathrm{Ind}\,\mathcal{T}$.

An exact $\otimes$-functor
$\phi : \mathcal{T}' \to \mathcal{T}$ gives rise to
$\pi(\mathcal{T}) \to \phi\pi(\mathcal{T}')$.

PURE MOTIVIC GALOIS GROUPS

② Review of the standard conjectures.

X proj. sm $/k$, $\dim X = d$ H: "classical" cohomology

$\eta \in H^2(X)(1)$ class of an ample line bundle

$? \cup \eta^{d-i} : H^i(X) \xrightarrow{\sim} H^{2d-i}(X)(d-i)$ "strong Lefschetz"

$\rightsquigarrow$ Lefschetz dec. $\underset{H^i}{x} = \sum_j \underset{H^{i-2j}(X)(-j)}{x_{i-2j}} \cup \eta^j$ $(0 \leq i-2j \leq d)$

primitif: $x_{i-2j} \cdot \eta^{d+1-i+2j} = 0$

"Hodge operator" $* = *_{X,\eta}$

$$*x = \sum_j (-)^{\frac{(i-2j)(i-2j+1)}{2}} \frac{j!}{(d+j-i)!} x_{i-2j} \cup \eta^{d+j-i} \in H^{2d-i}(X)(d-i)$$

$*^2 = id$

if $k = \mathbb{C}$, $*$ is the usual Hodge operator in cplex geometry on the $(0,0)$-part of $H^{2p}(X, \mathbb{R})(p)$. One has $\langle \alpha, *\alpha \rangle > 0$ for any $\alpha \neq 0$ in this $(0,0)$-part.

Grothendieck's std. cj.

I. the Künneth projectors $\pi_X^i : H(X) \to H^i(X) \hookrightarrow H(X)$ are algebraic (ie induced by alg. corr.)

II. $*_{X,\eta}$ is algebraic

III. For any alg. class $\alpha \neq 0$ on any power of X, $\langle \alpha, *\alpha \rangle \in \mathbb{Q}_{>0}$

IV. $\sim_{hom} = \sim_{num}$ for alg. classes on powers of X.

Prop (Grothendieck) II $\Rightarrow$ I, II + III $\Rightarrow$ IV $\Rightarrow$ II.

by Hodge theory, if $\underline{\text{char } k = 0}$, cj III is a thm.

Thus the $\underline{\text{std. cj amount to II alone}}$,

I holds for finite k (Katz-Messing) and $\underline{\mathrm{II} \Leftrightarrow \mathrm{IV}}$.

3) $\underline{\text{Construction of Pure motivic Galois groups}}$.

One would like to apply tannakian theory to some category $\mathcal{M}_\sim(k)_F$ of pure motives.

The natural choice of $\sim$ is $\sim_{num}$, due to

__Thm__ (Jannsen) $\mathcal{M}_{num}(k)_F$ is abelian semi-simple.

a) $\underline{\text{Construction under st. cj. IV}}$:

$$\begin{array}{ccc} \mathcal{M}_{hom}(k)_F & \xrightarrow[\text{faithful, exact}]{H} & \left\{\begin{array}{l} Gr_K \\ s\mathrm{Vec}_K \end{array}\right. \\ \text{hyp.} \to \quad \| & & \\ \mathcal{M}_{num}(k)_F & & \end{array}$$

$$\begin{array}{ccc} \mathrm{IV} \Rightarrow \mathrm{I} & \Rightarrow \forall X,\ \pi_X^+ = \sum \pi_X^{2i} & \text{algebraic} \\ \Downarrow & \Downarrow & \\ \mathcal{M}_{hom} \text{ is } \mathbb{Z}\text{-graded} & \Rightarrow \mathcal{M}_{hom} \text{ is } \mathbb{Z}/2\text{-graded} & \\ & \text{and } H \text{ is compatible with grading.} & \end{array}$$

$\underline{\text{Trick}}$: change comm. constr. φ in $\mathcal{M}_{hom}(k)_F$ according to sign rule $\rightsquigarrow$ $\dot{\mathcal{M}}_{hom}(k) \xrightarrow{\dot{H}} \mathrm{Vec}_K$ $\underline{\text{fiber functor}}$

$G_{mot} = \underline{\mathrm{Aut}}^{\otimes} \dot{H}$ affine gp-scheme$/_K$: "pure absolute motivic Galois gp attached to H".

$\mathcal{M}_{num}$ semi-simple $\Rightarrow G_{mot}$ (pro-)reductive.

Actually, g I $\to$ $\dot{\mathcal{M}}_{hom}(k) \xrightarrow{\dot{H}} \mathrm{Rep}_{K}\mathbb{G}_m$

$\rightsquigarrow$ $\mathbb{G}_m \longrightarrow G_{mot}$ "weight cocharacter" (central).

- For any $M \in \dot{\mathcal{M}}_{hom}(k) = \dot{\mathcal{M}}_{num}(k)$, one can consider the sub-tannakian cat. $\langle M \rangle_\otimes$ gen. by M, and form $\underset{mot}{\underline{G}}(M) = \underline{\mathrm{Aut}}^{\otimes} H|_{\langle M\rangle_\otimes}$ (linear alg. gp)

and $G_{mot} = \varprojlim G_{mot}(M)$.

Ex. $X \in \mathcal{P}(k)$, $M = h(X)$ — $\dim X = 0$ $\to$ $G_{mot}(M) =$ q^t of $\mathrm{Gal}(\bar{k}/k)$ which acts effectively on $H(X)$.

$d = \dim X > 0$ $\Rightarrow$ $\mathbb{1}(1) \in \langle M\rangle_\otimes$ and $\dot{M} = M(d)$.

$G_{mot}(h(X)) \cong$ largest closed alg. subgp of $GL(H(X)) \times \mathbb{G}_m$ which fixes alg. classes in twisted coh. spaces $\underbrace{H(X)^{\otimes n}}_{\simeq H(X^n)}(m)$, n, m

Principle: a few alg classes of low degree on small powers of X suffice to determine $G_{mot}(h(X))$, which determines in turn — with help of classical invariant theory — all algebraic classes on all powers of X.

- st. cj III (a thm. in char. 0) implies strong restrictions on $G_{mot}(M)$. Assume for simplicity $k \subseteq \mathbb{C}$, ~~H=B~~
  - Any rep. of $G_{mot}(M)$ is self-dual up to a twist
  - the reductive gp $G_{mot}(M)$ "splits" over $\mathbb{Q}^{cm}$
  - there is an involution in $G_{mot}(M)^{ad}(\mathbb{R})$ whose centralizer is a maxi. cpct. subgp ...

) Construction under algebraicity of the even Künneth proj π^+ only

One can still do the trick of changing the comm. constraint φ, so that the rank of any motive $\in \mathbb{N}$. By Jannsen + Deligne's theorems, $\dot{M}_{num}(k)$ is tannakian (i.e. $\exists$ abstract fiber functor).

Relation to cohomology?

$$\begin{tikzcd} \dot{M}_{hom}(k) \arrow[r, "\dot{H}"] \arrow[d, "\pi"'] & Vec_K \\ \dot{M}_{num}(k) & \end{tikzcd}$$

Thm (A., Kahn) π admits a $\otimes$-section σ, unique up to isomorphism.

$$\begin{tikzcd} \dot{M}_{num} \arrow[r, "\sigma"] \arrow[rr, bend right, "\text{fiber functor}"'] & \dot{M}_{hom} \arrow[r, "\dot{H}"] & Vec_K \end{tikzcd}$$

$$G_{mot} = \underline{Aut}^{\otimes}(\dot{H} \circ \sigma).$$

c) Construction without "trick"?

Changing the comm. constr. φ is a rather artificial trick: it would not apply to other $\otimes$ cat. of motives $M_\sim(k)$, although one expect the π^+_X (and in fact the π^i_X) to lift to idempotent — but not central — correspondences modulo any $\sim$ (Murre).

$\mathcal{M}_{hom}(k)_F \xrightarrow[\text{faithf.}]{} sVec_K \Rightarrow$ any object of $\mathcal{M}_{hom}$, hence of $\mathcal{M}_{num}$, is killed by some Schur functor

$\xrightarrow{\text{Jannsen + Deligne}}$ $\mathcal{M}_{num}(k)_F$ is supertannakian.

Up to extending F, one may assume it is $\simeq$ a category of rep's of a <u>super affine gp scheme</u>$/_F$

(a usual affine gp scheme $\Leftrightarrow$ the π^+ are alg.)

<u>Rmk</u>: $\mathcal{M}_{num}(k)_F$ is semisimple (Jannsen). But Cat. of rep's of non-split super algebraic gps are rarely semi-simple !

In fact, in the setting of super Lie algebras:

<u>Thm</u> (Djokovich, Hochschild) the only simple non-split super-Lie algebras such that Rep. cat. is semi-simple are the "symplectic ones".

(Constructed as follows: $V, \langle\ \rangle$ symplectic space, $L = \underset{\underset{sp(V)}{\text{ss}}}{\overset{+}{S^2 V}} \oplus \overset{-}{V}$ with "obvious" superbracket)

In many cases of concrete motives M, one can rule out the appearance of such an L by simple representation-theoretic arguments $\rightsquigarrow$ repr.-theoretic proof of alg. of π_X^+ for some X.

) ~~How~~ to bypass the standard conjectures.

Without any cj, one defines $\left\lfloor k \subset \mathbb{C},\ H = H_B \atop F = \mathbb{Q}\right.$

$G_{alg}(X)$ = largest closed alg. subgp of $GL(H(X)) \times \mathbb{G}_m$ which fixes alg. classes in twisted coh. spaces $H(X)^{\otimes n}(m)$, n, m.

Under st^d cj IV:

- invariants of $G_{alg}(X)$ in $H(X)^{\otimes n}(m)$ are precisely the alg. classes.
- $G_{alg}(X \sqcup Y) \to G_{alg}(X)$ surjective.

Without cj, one is led to fix a subcat. $\mathcal{V} \subset \mathcal{P}(k)$, stable under $\times$, $\sqcup$, and to consider all invariants under $\mathrm{Im}(G_{alg}(X \sqcup Y) \to G_{alg}(X))$ in $H(X)^{\otimes n}(m)$ for various n, m, and $Y \in \mathcal{V}$.

How to describe these classes?

(n=1) : they are exactly the classes of the form (motivated classes)

$$\left(pr^{X \times Y}_{X}\right)_*\left(\alpha \cup *_{X \times Y, \eta}(\beta)\right)$$

$Y \in \mathcal{V}$, α, β algebraic classes on $X \times Y$

Hodge op. on $X \times Y$ w.r.t a product polarization η.

Rmks: • motivated = algebraic $\Leftrightarrow$ st^d cj II holds.
...correspondences...

Using motivated classes (modelled on $\mathcal{V}$), one can construct a $\otimes$-category of motives $\mathcal{M}_{\mathcal{V}}$

For $\mathcal{V} = \mathcal{P}(k)$, $\mathcal{M}_{hom}(k) \subseteq \mathcal{M}_{\mathcal{P}(k)}$
full?

char k = 0

Thm. $\mathcal{M}_{\mathcal{V}}$ is a (neutral) tannakian semi-simple category.

$$\mathcal{M}_{\mathcal{V}} \xrightarrow{H} Gr_{\mathbb{Q}}$$

or

$$\dot{\mathcal{M}}_{\mathcal{V}} \longrightarrow Rep_{\mathbb{Q}} \mathbb{G}_m$$

$\rightsquigarrow$ Motivic Galois gp $G_{mot,\mathcal{V}}$

with weight cocharacter $\mathbb{G}_m \nearrow$

$$\dot{\mathcal{M}}_{\mathcal{V}} \simeq Rep_{\mathbb{Q}} G_{mot,\mathcal{V}}.$$

What if char $k = p$?

. Can still define $\dot{\mathcal{M}}_{\mathcal{V}}$.

$$\dot{\mathcal{M}}_{\mathcal{V}} \xrightarrow{\dot{H}} Vec_K \qquad \text{(e.g. etale cohomology)}$$

$$\pi \downarrow \; \uparrow \sigma$$

$$\bar{\mathcal{M}}_{\mathcal{V}} = \dot{\mathcal{M}}_{\mathcal{V}} / \text{maxi } \otimes\text{-ideal} \qquad \text{ab. semi simple}$$

$\dot{H} \circ \sigma$: fibre functor $\rightsquigarrow$ motivic Galois gp.

$\bar{\mathcal{M}}_{\mathcal{V}}$ tannakian over $F = \mathbb{Q}(\langle \alpha, *\alpha \rangle)$
all alg. cycles α on all $Y \in \mathcal{V}$

(st! cj II $\Rightarrow$ $F = \mathbb{Q}$).