Ω-ALGEBRAS OVER HENSELIAN DISCRETE VALUED FIELDS WITH REAL CLOSED RESIDUE FIELD

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1. INTRODUCTION

Let $\mathbb{R}((t))$ be the fraction field of the complete discrete valuation ring $\mathbb{R}[[t]]$, of formal power series over $\mathbb{R}$, and let $F = \mathbb{R}((t))(x)$ be the field of rational functions in one variable over $\mathbb{R}((t))$. Let A be a central simple algebra over F of exponent 2. The quadratic extension $\mathbb{C}((t))(x)$ of F is a C_2-field (cf. [Ser]) and therefore $A \otimes_F \mathbb{C}((t))(x)$ is an algebra of index ≤ 2, cf. [Art]. It follows that the index of A over F is less than or equal to 4. A well known theorem of Albert implies that A is Brauer equivalent to a biquaternion algebra, i.e. a tensor product of two quaternion algebras. *Can one describe the algebras of exponent 2 which are exactly of index 2, i.e. which are Brauer equivalent to quaternion division algebras?* To make the question more precise we recall that the Brauer group of a rational function field $K(x)$ over any field K, which we may assume to be of characteristic not equal to 2, is described to some extent by its ramification data. To do this, one interprets $K(x)$ as the function field of the projective line $\mathbb{P}^1_K$. The closed points y of $\mathbb{P}^1_K$ correspond to the K-discrete valuations of $K(x)$, either y is the point at infinity of $\mathbb{P}^1_K$ or y corresponds to a monic irreducible polynomial in $K[x]$. The Brauer group of $K(x)$ is described by an exact sequence of cohomology groups, due to Fadeev, cf. [Fad],[Ser]. We are only interested in algebras of exponent 2 so we only consider the sequence restricted to the 2-components of the different groups;

$$0 \to {}_2\mathrm{Br}(K) \to {}_2\mathrm{Br}(K(x)) \overset{\oplus\partial_y}{\to} \bigoplus_{y\in\mathbb{P}^1_K} H^1(K(y),\mathbb{Z}/2\mathbb{Z}) \overset{\sum \mathrm{cor}}{\to} H^1(K,\mathbb{Z}/2\mathbb{Z}) \to 0. \qquad (FES)$$

Here $K(y)$ is the residue field of the discrete valuation corresponding to the closed point y and ∂_y is the associated ramification map. The map $\sum$cor is the sum of the values of the corestriction maps $H^1(K(y),\mathbb{Z}/2\mathbb{Z}) \to H^1(K,\mathbb{Z}/2\mathbb{Z})$ induced by the inclusion of the absolute Galois groups $\mathrm{Gal}(\overline{K}/K(y)) \subset \mathrm{Gal}(\overline{K}/K)$, (cf. [Ser, Chap. II, App. sec.3]). Note that $H^1(K(y),\mathbb{Z}/2\mathbb{Z}) \cong K(y)^*/K(y)^{*2}$ and $H^1(K,\mathbb{Z}/2\mathbb{Z}) \cong K^*/K^{*2}$, moreover these isomorphisms are canonical since -1 is the only primitive 2th-root of unity. After identifying these cohomology groups with these groups of square classes, the corestriction map corresponds to the norm map:

$$N_{K(y)/K} : K(y)^*/K(y)^{*2} \to K^*/K^{*2}.$$

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The ramification map ∂_y factors through ${}_2\mathrm{Br}(K(x)_y)$, where $K(x)_y$ is the completion of $K(x)$ with respect to the valuation corresponding to y. So if a central simple algebra A over $K(x)$ is trivial over $K(x)_y$ it is unramified, i.e. $\partial_y(A) = 1 \bmod K(y)^{*2}$. The value of the ramification map can be explicitly caculated. Let v be the valuation corresponding to y and let $(f,g)_{K(x)}$ be a quaternion algebra over $K(x)$, then

$$\partial_y((f,g)_{K(x)}) = (-1)^{v(f)v(g)}\overline{\left(\frac{f^{v(g)}}{g^{v(f)}}\right)} \in K(y)^*/K(y)^{*2}.$$

Since any $A \in {}_2\mathrm{Br}(K(x))$ is Brauer equivalent to a tensor product of quaternions algebras (by Merkurjev's theorem), the ramification can be calculated by linearity. (If $K = \mathbb{R}((t))$ we saw already that the algebras of exponent 2 are Brauer equivalent to a tensor product of 2 quaternion algebras.)

The fact that $\mathrm{im}(\oplus\partial_y) \subset \ker(\sum \mathrm{cor})$ is called Faddeev's reciprocity law. The exact sequence (FES) says that the Brauer class of an algebra A of exponent 2 over a rational function field $K(x)$ is "almost" given by a finite set of local data, namely its non-trivial ramification. The ramification data of A consists of a finite set of closed points $\mathrm{Ram}(A) := \{y \in \mathbb{P}^1_K | \partial_y(A) \neq 0 \in H^1(K(y), \mathbb{Z}/2\mathbb{Z})\}$, called the *ramification locus of* A, and the set $\{\partial_y(A) | y \in \mathrm{Ram}(A)\}$. Faddeev reciprocity law implies that $\sum_{y\in\mathrm{Ram}(A)} \mathrm{cor}(\partial_y(A)) = 0$ in $H^1(K, \mathbb{Z}/2\mathbb{Z})$. The exactness of the sequence (FES) in ${}_2\mathrm{Br}(K(x))$ and in $\bigoplus_{y\in\mathbb{P}^1_K} H^1(K(y), \mathbb{Z}/2\mathbb{Z})$ implies that data consisting of a finite set of points $S = \{y \in \mathbb{P}^1_K\}$ and a set of non-trivial elements $\delta_y \in H^1(K(y), \mathbb{Z}/2\mathbb{Z}), y \in S$ satisfying Faddeev's reciprocity law are exactly the ramification data of some algebra A of exponent 2 over $K(x)$, i.e. $S = \mathrm{Ram}(A)$ and $\delta_y = \partial_y(A)$ for all $y \in S$. The Brauer class of this algebra A is defined up to a factor in the Brauer group of K, more precisely two algebras A and A' have the same ramification data if and only if $A \sim A' \otimes_{K(x)} B$ where $B \cong b \otimes_K K(x)$ with b a constant algebra (by this we mean an algebra defined over K). We can now rephrase the above question in terms of ramification data:

> *Which ramification data (of algebras of exponent 2 over* $\mathbb{R}((t))(x)$*) correspond to the ramification data of a quaternion algebra?*
>
> *Can one describe a quaternion algebra A over* $\mathbb{R}((t))(x)$ *explicitly in terms of its ramification data, i.e. can one construct explicitly a quadratic splitting field for A in terms of the ramification data of A?*

In the literature one can find different problems related to these questions. We discus two such problems

1.1. **The u-invariant of a rational function field.** The u-invariant of a non-real field (i.e. fields in which -1 is a sum of squares) is the supremum of the dimensions of anisotropic quadratic forms. For real fields this definition evidently would yield that the u-invariant is infinite (since the form represented by a sum of s squares is always anisotropic over a real field). Therefore for real fields E one defines the u-invariant to be

$$u(E) = \sup\{\dim\phi | \phi \text{ an anisotropic torsion quadratic form over } E\}.$$

Where a quadratic form ϕ over E is a torsion form if and only if $n \times \phi$ is hyperbolic for some non-zero natural number. Pfister's local global principle states that the torsion quadratic forms over a real field E are exactly the forms which are hyperbolic over every real closure of E.
In [Pfis2] Pfister studied the following conjecture concerning function fields over real closed fields:

Conjecture *Let R be a real closed field and F a field of transcendence degree m over R then the $u(F) \leq 2^m$*

In case F is purely transcendental over R the conjecture would imply that $u(F) = 2^m$.
In general the conjecture is widely open. For $m = 1$ the answer is positive and due to Elman and Lam ([EL]). The case $m = 2$ is studied by Pfister in his paper. Pfister considers division algebras of exponent 2 over F which split over all real closures of F. These are the algebras representing the elements in the kernel of the map

$$\psi : {}_2\mathrm{Br}(F) \to \prod_{\alpha \in \Omega} {}_2\mathrm{Br}(F_\alpha)$$

where Ω is the set of all orderings of F. (In case F is not real the kernel is by definition all of ${}_2\mathrm{Br}(F)$.) Pfister proves that if every non-trivial element in the kernel of ψ has index 2 the conjecture on the u-invariant of F holds, i.e. $u(F) \leq 4$. So using the following definition we can rephrase the conjecture (in the case F is real):

Definition 1.1. Let E be any real field. A central simple algebra A over E whose Brauer equivanlence class is in the kernel of the map

$$\psi : \mathrm{Br}(E) \to \prod_{\alpha \in \Omega} \mathrm{Br}(E_\alpha)$$

is called an Ω-algebra.

Remark 1.2. Since we are only interested in algebras of exponent 2 we will use in the rest of the paper the term Ω-algebra for Ω-algebras of exponent ≤ 2.

So in view of Pfister's results for $m = 2$ the conjecture becomes:

Conjecture (Pfister conjecture) *Let F be a field of transcendence degree 2 over a real closed field R. If F is real then every Ω-division algebra of exponent 2 over F is a quaternion algebra. If F is non-real then every central division algebra of exponent 2 is a quaternion algebra.*

Now consider the case where F is a purely transcendental extension of degree one over the function field of a (smooth projective) curve over the reals $\mathbb{R}$, so $F = \mathbb{R}(C)(x)$. Let v be the discrete valuation corresponding to a closed point $y \in C$ and let $\mathbb{R}(C)_v$ be the completion of $\mathbb{R}(C)$ with respect to v. It is well known that $\mathbb{R}(C)_v$ is of the form $\mathbb{C}((t))$ or of the form $\mathbb{R}((t))$. Since $\mathbb{C}((t))(x)$ is a C_2-field the algebras of exponent 2 over $\mathbb{C}((t))(x)$ are also of index 2 (cf. [Art]). So the following local version of the conjecture, proved by Karim Becher, is very natural.

Theorem 1.3 (Becher)**.** *The index of Ω-algebras (of exponent 2) over $\mathbb{R}((t))(x)$ is equal to 2.*

In section 2 we will describe the ramification data of Ω-algebras. It turns out that Ω-algebras are only ramified in points of $\mathbb{P}^1_{\mathbb{R}((t))}$ corresponding to monic irreducible polynomials which are equal to a sum of squares in $\mathbb{R}((t))[x]$. Since for any such set of local data Faddeev's reciprocity law is always satisfied, it follows from Becher's result that given such local data there exists a quaternion algebra over $\mathbb{R}((t))(x)$ having the given local data as its ramification data, moreover Fadeev's exact sequence (FES) implies that there is only one such Ω-quaternion algebra. For certain types of such local data we will give an explicit description of the associated Ω-algebra.
The authors are grateful to Karim Becher for various helpful discussions and for allowing to reproduce his proof of theorem 1.3 in this paper. In [Bech] a more general result is shown. Let K be a field with Pythagoras number ≤ 2 such that $u(K(\sqrt{-1})) = 4$ then $u(K) \leq 4$. This result implies that if $K = \mathbb{R}((t))(C)$ is the function field of a curve over $\mathbb{R}((t))$ such that $p(K) = 2$ then the u-invariant of K is 4. As we noted before this yields that Ω-algebras over K of exponent 2 are Brauer equivalent to quaternion algebras. In [TVGY] the Pythagoras number of function fields of hyperelliptic curves over $\mathbb{R}((t))$ is studied. It is shown there that if C is a curve with good reduction and if $\mathbb{R}((t))(C)$ is a real field then $p(\mathbb{R}((t))(C)) = 2$. Becher's result implies that the u-invariant of such fields is 4.

1.2. **Conic bundle surfaces.** The second problem to which the questions put forward are related, concerns certain rational surfaces over K. Let K be any field of characteristic not 2. Let $\mathbb{P}^1_K$ be the projective line over K. Its function field is a purely transcendental extension $K(x)$. We denote the generic point of $\mathbb{P}^1_K$ by x, so $K(x)$ can also be interpreted as the residue field of x. A *conic bundle surface over* K is a smooth projective geometrical integral K-variety X admitting a dominant K-morphism $\varphi : X \to \mathbb{P}^1_K$ whose generic fiber X_x is isomorphic to a smooth conic. A conic bundle surface is a rational surface (i.e. birationally equivalent to $\mathbb{P}^2$ over the algebraic closure of K).
The fibration $\varphi : X \to \mathbb{P}^1_K$ degenerates at a finite number of closed points $y \in \mathbb{P}^1_K$. And each degenerate fiber consists of a pair of smooth rational curves transversally intersecting at some point. If φ is relatively minimal, i.e. no degenerate fiber can be blown down, then each component of a degenerate fiber is defined over a quadratic extension L of the residue field $K(y)$.
Let now $\varphi : X \to \mathbb{P}^1_K$ be a relatively minimal conic bundle surface. Let S_φ be the set of points $y \in \mathbb{P}^1_K$ where the conic bundle degenerates and let $T_\varphi = \{L_y/K(y) | y \in S_\varphi\}$ be the set of quadratic extensions over which the components of the respective fibers X_y are defined. The data S_φ, T_φ are called the *local invariants of* $\varphi : X \to \mathbb{P}^1_K$. (Note that the set T_φ alone determines the local invariants.) It is very natural to ask the following, given a finite set of local invariants $\{L_y/K(y)\}_{y \in S_\varphi}$ (so a finite set of quadratic extensions of different residue fields) does there exists a (relatively minimal) conic bundle with this set as it local invariants? If so, is this conic bundle unique up to a fiber preserving isomorphism?
The following proposition tells us how this question can be translated into the language of quaternion division algebras over $K(x)$.

Proposition 1.4. *(cf.* [CS], [Isk1], [Isk2]*) There is a one-to-one correspondence between classes, with respect to fiber preserving birational isomorphisms, of relatively minimal conic bundles* $\varphi : X \to \mathbb{P}^1_K$ *and isomorphism classes of quaternion algebras over* $K(x)$*. The correspondence*

associates to $\varphi : X \to \mathbb{P}^1_K$ the quaternion algebra H_φ over $K(x)$ determined by the generic fiber X_x of X (X_x is a conic over $K(x)$ its equation corresponds to the norm form of a quaternion algebra over $K(x)$). In the closed points z where φ is not degenerated H_φ defines an element of $\mathrm{Br}(\mathscr{O}_z)$, where $\mathscr{O}_z$ is the discrete valuation ring corresponding to z. The non-degenerate fibers of φ, X_z are the conics corresponding to the residue algebras $H_\varphi \otimes_{\mathscr{O}_z} K(z)$.
Moreover let $\varphi : X \to \mathbb{P}^1_K$ be a relatively minimal conic bundle and let H_φ be the corresponding quaternion algebra over $K(x)$ then the set of local invariants $\{L_y/K(y)\}$ corresponds to the set of ramification data of H_φ, i.e. $\varphi : X \to \mathbb{P}^1_K$ is degenerate in $y \in \mathbb{P}^1_K$ if and only if H_φ is ramified in y and in that case $L_y = K(y)(\sqrt{\partial_y(H_\varphi)})$.

In view of this it is clear that the problems concerning conic bundles formulated above translate exactly into the questions we considered for algebras of exponent 2 over $K(x)$. First note that instead of giving the ramification data as elements in the (abstract) Galois cohomology groups these data can also be represented by a set of quadratic extensions $\{K(\sqrt{\delta_y})/K(y) | y \in \mathrm{Ram}(A)\}$, where $\delta_y \in K(y)$ is representing the square class of $\partial_y(A)$ under the isomorphism $H^1(K(y), \mathbb{Z}/2\mathbb{Z}) \cong K(y)^*/K(y)^{*2}$. Now given a set of local invariants we know (using the proposition) that this set has to satisfy Faddeev's reciprocity (the sum of the values of the corestriction maps has to be zero) in order to be a possible set of local invariants of a conic bundle. Moreover if this condition on the local invariants is satisfied there exists an algebra of exponent 2 whose ramification data is determined by that set, but there is no guarantee that a quaternion algebra with the given ramification data exists (also not in the case $K = \mathbb{R}((t))$ as is shown by the examples 2.4). For this to be true, further conditions on the ramification data are needed. In section 3 we explain how our results on the ramification data of Ω-algebras of exponent 2 over $K = \mathbb{R}((t))$ translate into facts on relatively minimal conic bundles over $\mathbb{R}((t))$.

Notation and terminology. Throughout the paper we will use the following terminology. We call two central simple algebras A and B over a field K equivalent if they are Brauer equivalent over K, i.e. if they define the same element in $\mathrm{Br}(K)$, we use the notation $A \sim B$. We call a central simple algebra over K trivial if its class in $\mathrm{Br}(K)$ is trivial .
Quaternion algebras over a field K with a K-basis of the form $1, i, j, k$ satisfying $i^2 = a$, $j^2 = b$ and $ij = -ji = k$ with $a, b \in K$, will be denoted by the symbol $(a, b)_K$. If there is no confusion possible we will omit in proofs and in calculations the field in the index of this symbol.

2. Ω-ALGEBRAS OVER RATIONAL FUNCTION FIELDS OVER HENSELIAN DISCRETE VALUED FIELDS WITH REAL CLOSED RESIDUE FIELD

Our results concerning Ω-algebras over $\mathbb{R}((t))(x)$ only use the fact that $\mathbb{R}((t))$ is a Henselian discrete valued field with real closed residue field. So we will formulate and prove the results in this generality.
In the sequel of the paper K will be a Henselian discrete valued field with real closed residue field denote by k. The field K is the fraction field of a Henselian discrete valuation ring $\mathscr{O}_K$ and we fix a uniformizing element π in $\mathscr{O}_K$. The algebraic closure of k is $k(i)$, i being the square root of -1. If E is any field extension of k, we write $E(i)$ for $E \otimes_k k(i)$.

We need some facts on the finite extensions of K. A finite extension L is itself a Henselian discrete valued field; we denote its valuation ring by $\mathscr{O}_L$. The residue field of L is either the real closed field k or its algebraic closure $k(i)$. Any finite extension L/K can be split in a tower $K \subset N \subset L$ where N/K is an unramified extension and L/N is a totally ramified extension of K. The extension N is either equal to K (in this case L/K is totally ramified) or $N = K(i)$. This follows from the fact that the unramified algebraic extensions of a Henselian discrete valued field are unique "lifts" of the residue field extensions of the real closed field k, of course the latter only has two extension k and $k(i)$. The totally ramified part L/N has the form $L = N(\sqrt[n]{\pi'})$ with $\pi' = u\pi$ and u a unit in $\mathscr{O}_K$. Since k is real closed the units in $\mathscr{O}_K$ are all of the form $\pm 1 \cdot c^n$ for some $c \in K$. So the totally ramified extensions of N are all of the form $N(\sqrt[n]{\pi})$ or $N(\sqrt[n]{-\pi})$. It follows that the only quadratic extensions of K are $K(i), K(\sqrt{\pi}), K(\sqrt{-\pi})$. This implies that $(-1,-1)_K, (-1,\pi)_K, (-1,-\pi)_K$ represent the only non-trivial elements in ${}_2\mathrm{Br}(K)$ (cf. [Schil]). Note that K is a hereditarily pythagorean field, i.e. all finite real extension of K are pythagorean or equivalently all non-real field extensions of K contain $K(i)$ as a subfield (cf. [Beck]). We also need the following property of finite field extensions of K

Lemma 2.1. *Let L be an odd degree extension of K and let F be a finite non-real extension of K. then $[LF : F]$ is odd.*

Proof: Let $[L : K] = d$, d odd. Note that $L = K(\sqrt[d]{\pi})$. The non-real extension $F = K(i)(\sqrt[e]{\pi})$. It follows that $LF = K(i)(\sqrt[l]{\pi})$ with l the least common multiple of d and e. So $[LF : F] = \frac{l}{e}$, which is an odd number since it divides d. □

We are interested in central simple algebras of exponent 2 (in particular in Ω-algebras) over the rational function field $K(x)$, i.e. over the function field of the projective line $\mathbb{P}^1_K$ over K. The valuation defined by the degree map on $K(x)$ corresponds to a closed point of $\mathbb{P}^1_K$ which we call the point at infinity and which we denote by ∞. The "finite" closed points of $\mathbb{P}^1_K$ are parametrized by monic irreducible polynomials of $K[x]$. The order functions corresponding to these polynomials define K-discrete valuations on $K(x)$. Throughout the rest of the paper we will identify the closed points of $\mathbb{P}^1_K$, the corresponding discrete valuations and (for finite points) the monic irreducible polynomials in $K[x]$.
We collect some facts on central simple algebras of exponent 2 over $K(x)$ and on their ramification. In the introduction we remarked that all central simple algebras of exponent 2 over $\mathbb{R}((t))(x)$ are of index less than or equal to 4 since $\mathbb{C}((t))(x)$ is a C_2-field. The same is true for central simple algebras of exponent 2 over $K(x)$, since $K(i)(x)$ is also a C_2-field, [Ser, chap. II, section 3.3]. So we have

Lemma 2.2. *Let A be a central simple algebra of exponent 2 over $K(x)$. Then A is equivalent to a biquaternion algebra.*

In general it is not so that all central simple algebras of exponent 2 over $K(x)$ are of index 2. We give two examples based on the following lemma which can be found in [KRTY] for biquaternion division algebras over $F(x)$, with F a local field of characteristic zero.

Lemma 2.3. *(cf. lemma 3.10 in* [KRTY]*.) Let* $f,g \in \mathbb{R}[[t]][x]$ *such that* $\overline{g} := g \bmod (t)$ *is not a square in* $\mathbb{R}(x)$*,* $\overline{f} := f \bmod (t)$ *is non-zero and has a root* $\overline{a}$ *of odd multiplicity in* $\mathbb{R}$ *such that* $\overline{g}(\overline{a})$ *is positive in* $\mathbb{R}$*. Then the biquaternion algebra* $(f,-1)\otimes(g,t)$ *is a division algebra.*

Proof: It is well known that the assertion is equivalent to the statement that the Albert form $\langle f,-1,f,-g,-t,tg\rangle$ is anisotropic over $\mathbb{R}((t))(x)$ (see [LLT, theorem 2.3]). We will show that $\langle f,-1,f,-g,-t,tg\rangle$ is anisotropic over the larger field $\mathbb{R}(x)((t))$. By Springer's theorem (cf. [Schar, chap. 6, 2.6]) it suffices to check that the first and second residue forms with respect to the uniformizing element t, $\langle \overline{f},-1,\overline{f},-\overline{g}\rangle$ and $\langle -1,\overline{g}\rangle$ respectively, are anisotropic over the residue field $\mathbb{R}(x)$. The hypotheses imply that the second form is anisotropic. To see that the first form is anisotropic we apply Springer's theorem again, now with respect to the discrete valuation on $\mathbb{R}(x)$ corresponding to irreducible polynomial $x-\overline{a}$. Write $\overline{f} = (x-\overline{a})^m\overline{h}$ with m odd and $\overline{h}(\overline{a}) \neq 0$. The second residue form $\langle \overline{h}(\overline{a}),\overline{h}(\overline{a})\rangle$ being definite over $\mathbb{R}$ must be anisotropic. Since $\overline{g}(\overline{a})$ is positive the first residue form $-\langle 1,-\overline{g}(\overline{a})\rangle$ is also anisotropic. □

Examples 2.4. (1) Lemma 2.3 implies that the biquaternion algebra $A = (-1,-x)\otimes_{\mathbb{R}((t))(x)}(x+1,t)$ is a division algebra. This example is the analogue of the example of Jacob and Tignol of a biquaternion division algebra over $\mathbb{Q}_p(x)$. As was noticed in [KRTY] an algebra of this form is isomorphic to the tensor product of a quaternion algebra over $\mathbb{R}((t))(x)$ and a constant quaternion algebra, i.e. a quaternion algebra defined over $\mathbb{R}((t))$. Namely

$$\begin{aligned}
&(-1,-x)\otimes(x+1,t)\otimes(-1,t)\\
&\quad\sim(-1,-x)\otimes(-(x+1),t)\\
&\quad\sim(-1,-1)\otimes(x,-1)\otimes(t,-(x+1))\\
&\quad\sim(-x,x+1)\otimes(-1,-1)\otimes(x,-1)\otimes(t,-(x+1))\\
&\quad\sim(-1,x+1)\otimes(x,x+1)\otimes(-1,-1)\otimes(x,-1)\otimes(t,-(x+1))\\
&\quad\sim(-1,-(x+1))\otimes(x,-(x+1))\otimes(t,-(x+1))\\
&\quad\sim(-xt,-(x+1),
\end{aligned}$$

where the third equation follows from the fact that $(-x,x+1)\sim 1$ since $-x+(x+1)=1$. The equations yield that $(-1,-x)\otimes(x+1,t)\sim(-xt,-(x+1))\otimes(-1,t)$, i.e. A is a quaternion algebra times a constant algebra. The next example shows that this is not always the case.
(2) The biquaternion algebra $A = ((x-1)(x+1),-1)\otimes(x,t)$ is a division algebra in view of lemma 2.3. If A is isomorphic to the tensor product of a quaternion algebra over $\mathbb{R}((t))(x)$ with a constant algebra, then for some constant quaternion algebra B, $A\otimes B$ is of index 2. However the only quaternion division algebras over $\mathbb{R}((t))$ are $(-1,-1)_{\mathbb{R}((t))},(-1,t)_{\mathbb{R}((t))},(-1,-t)_{\mathbb{R}((t))}$. Now $A\otimes(-1,-1)\sim(-(x-1)(x+1),-1)\otimes(x,t)$. The latter is a division algebra by lemma 2.3, taking $f=x$ and $g=-(x-1)(x+1)$. In the same way we see that $A\otimes(-1,t)\sim((x-1)(x+1),-1)\otimes(-x,t)$ is a division algebra, applying the lemma with $f=(x-1)(x+1)$, $g=-x$ and $a=-1$. Finally also $A\otimes(-1,-t)\sim(-(x-1)(x+1),-1)\otimes(-x,t)$ is a division algebra, applying the lemma with $f=-(x-1)(x+1)$, $g=-x$ and $a=-1$. It follows that in this example A is a biquaternion algebra over $\mathbb{R}((t))(x)$ which is not isomorphic to the tensor product of a quaternion algebra over $\mathbb{R}((t))(x)$ and a constant algebra.

The following lemma will be helpful to calculate the ramification of elements in ${}_2\mathrm{Br}(K(x))$.

Lemma 2.5. (a) *Let f be a sum of squares in $K(x)$. For all points $y \in \mathbb{P}^1_K$ such that $K(y)$ is a real field, every quaternion algebra of the form $(f,g)_{K(x)}$ is trivial over the completion $K(x)_y$. In particular this holds for the point at infinity of $\mathbb{P}^1_K$.*
(b) *Consider a quaternion algebra $(g,x)_{K(x)}$, with g a square free polynomial over K not divisible by x. Let p be a monic irreducible factor of g. Then $(g,x)_{K(x)}$ is ramified at the point $y \in \mathbb{P}^1_K$ corresponding to p if and only if p has a root θ which is not a square in $K(y)(\cong K(\theta))$.*
(c) *Consider a quaternion algebra $(g,\pi)_{K(x)}$, with g a square free polynomial over K. Let q be a monic irreducible factor of g. Then $(g,\pi)_{K(x)}$ is ramified at the point $y \in \mathbb{P}^1_K$ corresponding to q if and only if π is not a square in $K(y)$.*

Proof: (a) Let $y \in \mathbb{P}^1_K$ such that $K(y)$ is a real field. Then the completion $K(x)_y$ is also a real field. The residue field of this completion has Pythagoras number one so by Hensel's lemma the same holds for $K(x)_y$. This implies that f is a square in $K(x)_y$, so $(f,g)_{K(x)_y}$ is trivial.
(b) Since $v(x)=0$ and $v(g)=1$ (where v is the valuation corresponding to p), we calculate the ramification using the formula on page 2

$$\partial_y((g,x)_{K(x)}) = (-1)^{v(g)v(x)}\overline{\left(\frac{g^{v(x)}}{x^{v(g)}}\right)} \equiv \overline{x} \equiv \theta \bmod K(y)^{*2}.$$

So the ramification in y is non-trivial if and only if θ is not a square in $K(y)$.
(c) Since $v(\pi)=0$ and $v(g)=1$ (where v is the valuation corresponding to q), the ramification formula yields

$$\partial_y((g,\pi)_{K(x)}) = (-1)^{v(g)v(\pi)}\overline{\left(\frac{g^{v(\pi)}}{\pi^{v(g)}}\right)} \equiv \overline{\pi} \bmod K(y)^{*2}.$$

So the ramification in y is non-trivial if and only if π is not a square in $K(y)$. □

It is clear that any quaternion algebra of the form $(f,g)_{K(x)}$ with f a sum of squares in $K(x)$ is trivial over all real closures $K(x)_\alpha$, $\alpha \in \Omega$ since f is a square in $K(x)_\alpha$. So any $(f,g)_{K(x)}$ with f a sum of squares in $K(x)$ is an Ω-algebra. Becher's result (theorem 2.8) yields the converse. For the sake of completeness we will include a proof of this result. It is based on the fact that the Pythagoras number of $K(x)$ is 2, which can be seen in different ways (for instance [Beck, theorem III.1.4], where it is shown that a field F is a pythagorean if and only if $p(F(x))=2$. In [Pfis1] the following more general fact is shown. Let F be a real field and d is a nonnegative integer. Then any sum of squares in $F(x)$ may be written as a sum of at most 2^{d+1} squares if and only if one may write -1 as a sum of at most 2^d squares in any non-real finite extension of F.) We give an argument here which is much in the spirit of the rest of the paper. Although we state the theorem only for the henselian discrete valued fields we are considering in this paper, the argument works over any pythagorean field. (The same argument is used in [TVGY] to study the Pythagoras number of function fields of hyperelliptic curves over K.)

Lemma 2.6. *Let K be a Henselian discrete valued field with real closed residue field. The rational function field in one variable over K, $K(x)$, has Pythagoras number 2.*

Proof: Let $f \in K(x)$ be a sum of squares. We consider the quaternion algebra $(-1,f)_{K(x)}$, note that it is an Ω-algebra since f is a sum of squares. We claim that this algebra is trivial in ${}_2\mathrm{Br}(K(x))$. If this is true it follows that f is a norm of the quadratic extension $K(i)(x)/K(x)$ and so f will be a sum of two squares. Which is what we have to prove.
To prove the claim we determine the ramification of $(-1,f)_{K(x)}$ at all closed points of $\mathbb{P}^1_K$. Assume that $y \in \mathbb{P}^1_K$ such that $K(y)$ is a non -real field, so $-1 \in K(y)^{*2}$. We have $\partial_y((-1,f)_{K(x)}) \equiv (-1)^{v(f)} \bmod K(y)^{*2} \equiv 1 \bmod K(y)^{*2}$, i.e. the algebra $(-1,f)_{K(x)}$ is unramified in y.
If y is a point with a real residue field then $(-1,f)_{K(x)}$ is unramified by lemma 2.5 (a).
Since $(-1,f)_{K(x)}$ is unramified at all closed points of $\mathbb{P}^1_K$, the exact sequence (FES) implies that the algebra is induced by a quaternion algebra defined over K, so $(-1,f)_{K(x)} \sim (-1,\varepsilon)_{K(x)}$ with $\varepsilon \in \{1,-1,t,-t\}$. Now note that for $\varepsilon \in \{-1,t,-t\}$ there is always an ordering $\alpha \in \Omega$ such that $\varepsilon < 0$ with respect to that ordering α. This implies that $(-1,\varepsilon)_{K(x)}$, with $\varepsilon \in \{-1,t,-t\}$, is non-trivial over $K(x)_\alpha$. But $(-1,f)_{K(x)_\alpha}$ is trivial over all real closures of $K(x)$ since it is an Ω-algebra. It follows that $(-1,f)_{K(x)} \not\sim (-1,\varepsilon)\otimes_K K(x) = (-1,\varepsilon)_{K(x)}$ with $\varepsilon \in \{-1,t,-t\}$, so $(-1,f)_{K(x)} \sim (-1,1)_{K(x)}$ is trivial. □

Corollary 2.7. *Let K be as in the lemma. Any polynomial $f \in K[x]$ which is a sum of squares in $K(x)$ is a sum of two squares in $K[x]$.*

Proof: This follows from the above together with a well known result of Cassels saying that a polynomial in one variable over a field represented by a quadratic form over the rational function field is also represented by that quadratic form over the polynomial ring, cf. [Pfis3, Chap. 1, theorem 2.2]. □

Theorem 2.8 (K. Becher)**.** *Every Ω-algebra A over $K(x)$ of exponent 2 is Brauer equivalent to a quaternion division algebra of the form (f^2+g^2,h) with $f,g \in K[x]$.*

Proof: The proof is based on a quadratic form argument. Let A be a Ω-algebra of exponent 2, we know (cf. lemma 2.2) that A is equivalent to a biquaternion algebra. We have to show that this biquaternion algebra is not a division algebra. We use again that this is equivalent to the statement that a 6-dimensional quadratic form, α, of discriminant 1 is isotropic. (The quadratic form α is the Albert form associated to A). Since A is an Ω-algebra A is trivial over all real closures of $K(x)$, this means that α is hyperbolic over all real closures of $K(x)$ or equivalently (by Pfister's Local-Global Principle) that α is a torsion form in the Witt ring of $K(x)$.
Assume for the sake of contradiction that α is anisotropic. Now $K(i)(x)$ is a C_2-field (cf. page 6) so the u-invariant $u(K(i)(x)) = 4$. This implies that $I^3(K(x))$ is torsion free (here $I(K(x))$ is the fundamental ideal in the Witt ring of $K(x)$). We also obtain that $\alpha \otimes K(i)(x)$ is isotropic, which tells us that α contains up to scaling the norm form of $K(i)(x)/K(x)$ as a subform (cf. [Schar]). So we may assume that $\alpha \cong \langle 1,1\rangle \perp \beta$. Since α is anisotropic so is β. The Albert form α is in $I^2(K(x))$ so $2\times\alpha$ is a torsion form in $I^3(K(x))$ and therefore hyperbolic. It contains $2\times\beta$ as a subform, and since $\dim(2\times\beta) > \frac{1}{2}\dim\alpha$ it follows that $2\times\beta$ is isotropic. It is well known (see [EL, proposition 2.2]) that this implies that β contains a 2-dimensional form γ such that $2\times\gamma$ is hyperbolic. So γ is a torsion form, its discriminant d must therefore be a sum of squares, and because the Pythagoras number of $K(x)$ is 2, actually a sum of two squares. We have (comparing

discriminants)

$$\alpha \cong \langle 1,1\rangle \perp \langle -a,-ad\rangle \perp \beta,$$

for some $a \in K(x)$. Since α and β are torsion forms the same holds for $\langle 1,1,-a,-ad\rangle$. It follows that a is a sum of squares and therefore a sum of two squares in $K(x)$, implying that $\langle 1,1,-a,-ad\rangle$ is isotropic. This contradicts the fact that α is anisotropic.
Finally we note that an Ω-quaternion algebra over $K(x)$ is of the form $(e,f)_{K(x)}$ with e a sum of two squares in $K[x]$ and $f \in K(x)^*$. This follows from the fact that its norm form is a 2-fold torsion Pfister form φ by Pfister's Local-Global Principle. Since $I^3(K(x)) = 0$ we have $2\times\varphi$ hyperbolic. So it has to contain a 2-dimensional torsion form which implies that $\varphi \cong \langle 1,-e,-f,ef\rangle$ with e a sum of two squares in $K(x)$ and $f \in K(x)^*$. Since we may multiply e with any square in $K(x)^*$, it can be replaced by a polynomial in $K[x]$ which is a sum of two squares in $K[x]$. □

We now characterize Ω-algebras over $K(x)$ by their local data.

Proposition 2.9. *Let A be a central simple algebra of exponent 2 over $K(x)$. Then A is an Ω-algebra if and only if the following two properties hold*
(a) $A_\infty = A \otimes K(x)_\infty$ *is trivial, so A is unramified at infinity.*
(b) *The monic irreducible polynomials at which A is ramified are equal to sums of two squares.*

Proof: Let A be a Ω-algebra of exponent 2. We know (cf. theorem 2.8) that A is equivalent to a quaternion algebra $(e,f)_{K(x)}$, where e is a polynomial which is a sum of two squares in $K(x)$. Lemma 2.5 (a) implies that $(e,f)\otimes_{K(x)} K(x)_\infty$ is trivial in $\mathrm{Br}(K(x)_\infty)$, this proves point (a), that A is trivial at ∞.
The (finite) ramification points correspond to monic irreducible factors of ef. We may assume that e is a square free polynomial, its irreducible factors are equal to a sum of two squares since e is a sum of two squares. (This follows from the fact that a sum of two squares is a norm of the extension $K(i)(x)/K(x)$ and that the norm is multiplicative.) Let p_i be the monic irreducible factors of f. In view of lemma 2.5 (a) the algebra $(e,f)_{K(x)}$ is unramified at points corresponding to monic irreducible polynomials p_i having a real residue field. On the other hand if the residue field p_i is a non-real extension of k, we remarked (cf. page 6) that it contains $k(i)$ but then p_i is a norm of the field extension $K(i)(x)/K(x)$ and so p_i is a sum of two squares. This proves that (b) holds.
To prove the converse let A be a central simple $K(x)$-algebra satisfying properties (a) and (b). Write A, up to Brauer equivalence, as a tensor product of quaternion algebras $\prod_i (a_i f_i, b_i g_i)_{K(x)}$ with f_i, g_i monic polynomials over K and $a_i, b_i \in K^*$. (Lemma 2.2 implies that it is a product of 2 factors but that is not necessary for the argument.) Since A is by assumption unramified at infinity the points in the ramification locus of A correspond to irreducible factors of f_i and g_i. We can expand the product $\prod_i (a_i f_i, b_i g_i)_{K(x)}$ to a product $\prod_i (a_i,b_i)_{K(x)} \otimes \prod_i (a_i,g_i)_{K(x)} \otimes \prod_i (b_i,f_i)_{K(x)} \otimes \prod_j (p_j,q_j)_{K(x)}$ with p_j and q_j monic irreducible factors of the f_i and g_i. We collect the first three parts of this expansion together with all the factors $(p_r,q_r)_{K(x)}$ with p_r and q_r not equal to a sum of squares and call this product B. As we mentioned before the remaining factors are clearly all Ω-algebras since either p_i or q_j is a sum of squares. Therefore there product, say C, is also an Ω-algebra. So by the first part of the proof C is trivial over $K(x)_\infty$ and C is only ramified in points corresponding to monic irreducible polynomials which are sums of squares. By assumption the

same facts hold for A. So because $B \sim A \otimes_{K(x)} C$ it is an Ω-algebra. It follows that $B \otimes_{K(x)} K(x)_\infty$ is trivial and that B is unramified at all finite points (since its ramification locus is a subset of $\{p_r, q_s\}_{r,s}$ but none of these polynomials is a sum of squares). But Faddeev's exact sequence (cf. (FES)) implies that the only non-trivial unramified algebras over $K(x)$ are algebras defined over K and these are non-trivial at infinity. It follows that B is trivial and therefore we may conclude that A is equivalent to the Ω-algebra C. □

Remark 2.10. It is possible to prove the above proposition without using the fact that the Ω-algebras are quaternion algebras. It is known that A is equivalent to a tensor product of quaternion algebras $\prod_i (e_i, f_i)_{K(x)}$ where the e_i are monic polynomials over K which are equal to a sum of squares. (For fields F such that $I^3(F(\sqrt{-1})) = 0$ an argument can be found in [BP], where the statement follows from the end of the proof of proposition 2.9.). The argument in the proof of proposition 2.9 can now be applied to all monic irreducible factors of the product $\prod e_i f_i$.

Corollary 2.11. *An Ω-algebra A over $K(x)$ with empty ramification locus is trivial. Two Ω-algebras A and B over $K(x)$ with the same ramification locus are equivalent.*

Proof: The first part follows from property (a) in proposition 2.9. The exact sequence (FES) yields that the unramified algebras over $K(x)$ are of the form $c \otimes_K K(x)$ with c an algebra of exponent 2 defined over K. But all these algebras, except the trivial ones, are non-trivial at the point at infinity.
The residue fields of ramification points of Ω-algebras have a unique quadratic extension, so the group of square classes of such a residue field is of order 2. It follows that if A and B are two Ω-algebras with the same ramification locus, then $A \otimes_{K(x)} B$ does not ramify in any point. The first part of the corollary now yield that A and B are equivalent. □

We will now state our main result. We first define 3 types of monic polynomials over K.

Definition 2.12. Type (1) The monic polynomials $Q \in K[x]$ which are sums of 2 squares and whose monic irreducible factors Q_i over K are of degree $2q_i$ with q_i an odd number. If x_i is a root of Q_i (in some algebraic closure of K) then $x_i \in K(x_i)^{*2}$ (i.e. x_i is a square in its root field.)

Type (2) The monic polynomials $R \in K[x]$ which are sums of 2 squares and whose monic irreducible factors R_j over K are of degree $2r_j$ with r_j an odd number. If y_j is a root of R_j (in some algebraic closure of K) then $y_j \notin K(y_j)^{*2}$ (i.e. y_j is not a square in its root field.)

Type (3) The monic polynomials $P \in K[x]$ which are sums of 2 squares and whose monic irreducible factors P_k over K are of degree $2^{s_k} p_k$ with p_k an odd number, $s_k \in \mathbb{N}$ and $s_k > 1$. If z_k is a root of P_k (in some algebraic closure of K) then $z_k \notin K(z_k)^{*2}$ (i.e. z_k is not a square in its root field.)

Note that the monic irreducible factors of polynomials of type (1), (2) or (3) are also equal to a sum of two squares in $K[x]$ (corollary 2.7).

Theorem 2.13. *Let A be an Ω-algebra over $K(x)$ of exponent 2. Let A be ramified exactly in the points corresponding to monic irreducible polynomials $Q_i, i = 1, \ldots, a$, of type (1), $R_j, j = 1, \ldots, e$, of type (2) and $P_k, k = 1, \ldots, l$, of type (3) (where any of the three sets of polynomials may* *be empty). Then there is a polynomial $h \in K[x]$ such that A is equivalent to a quaternion algebra of the form: $(\pi h, PQR)$, with $P = \prod_k P_k$, $Q = \prod_i Q_i$ and $R = \prod_j R_j$ and with π a uniformizing element for the discrete valuation on K.*

The polynomial h occuring here will be constructed explicitly in terms of the ramification data of A.

Remark 2.14. (a) The only Ω-algebras for which theorem 2.13 does not give an explicit description are those which are ramified at some monic irreducible polynomial S of degree $2^t r$, $t > 1$, r an odd number, which is a sum of two squares and with a root which is a square in the root field of S.
(b) In previous notes the authors obtained special cases of theorem 2.13. In [BY] the cases $\mathrm{Ram}(A) = \{Q_i\}_i$, $\mathrm{Ram}(A) = \{R_j\}_j$, $\mathrm{Ram}(A) = \{P_k\}_k$ or $\mathrm{Ram}(A) = \{R_j, P_k\}_{j,k}$ where treated. In [BVY] the case $\mathrm{Ram}(A) = \{Q_i, R_j\}_{i,j}$ and in [Baz] the case $\{Q_i, P_k\}_{i,k}$ was proved. The latter, i.e. an Ω-algebras A with ramification of type (1) and (3) say in points $\{Q_i, P_k\}_{i,k}$, follows from the fact that the quaternion algebra $(QP, \pi x)$, with $Q = \prod_i Q_i$ and $P = \prod_k P_k$ has the required ramification type. This can be seen by expanding $(QP, \pi x)$ into

$$(Q, \pi) \otimes (Q, x) \otimes (P, \pi) \otimes (P, x).$$

Lemma 2.5 implies that (Q, x) and (P, π) are trivial algebras. The same lemma yields that the ramification locus of (Q, π) consists exactly of the points Q_i and that the ramification locus of (P, x) consists exactly of the points P_k. Since $(QP, \pi x)$ is an Ω-algebra with exactly the same ramification as A, corollary 2.11 implies that $A \sim (QP, \pi x)$.
We mention further that in [BTY] a result complementary to the theorem stated above is given. There an explicit description of all Ω-algebras ramified in at most two points of $\mathbb{P}^1_K$ is obtained.

The proof of theorem 2.13 is organized as follows. We start with a subsection containing some technical lemmas. In a second subsection the polynomial $h \in K[x]$ occuring in the statement of the theorem is constructed. The third subsection contains a final lemma from which the proof of the theorem follows. Throughout the notation as given in theorem 2.13 remains fixed.

2.1. **Some lemmas.**

Lemma 2.15. (1) *Let $g \in K[x]$ be a monic irreducible polynomial of non-zero degree divisible by* 4 *and such that g is a sum of squares in $K(x)$. Then the quaternion algebra $(g, \pi)_{K(x)}$ is trivial.*
(2) *Let $f, g \in K[x]$ be monic irreducible polynomials, both sums of squares in $K(x)$. Let* $\deg f = 2$ *and* $4 | \deg g$. *Let y_0 be a root of g and assume that $y_0 \notin K(y_0)^{*2}$. Then the quaternion algebra $(f, g)_{K(x)}$ is trivial.*

Proof: (1) Note that the algebra (g, π) is an Ω-algebra, g being a sum of squares. It follows that the ramification can only occur at g. Let θ be a root of g in some algebraic closure of K. Since $\deg g = 2^s m$ with $s > 1$, m an odd number, and since g is a sum of squares in $K(x)$ it follows that $K(\theta) = K(i)(\sqrt[l]{\pi})$, with $l = 2^{s-1} m$. So π is a square in $K(\theta)$. Consequently, by lemma 2.5 (b), (g, π) is unramified everywhere, hence trivial (cf. corollary 2.11).
(2) Since f is a monic quadratic polynomial, lemma 2.5 (a) implies that $(f, g) \otimes K(x)_\infty$ is trivial. Let x_0 be a root of f. Put $\deg g = 4m$, $m \in \mathbb{N} \setminus \{0\}$. Since f and g are both a sum of squares in

$K(x)$ their root fields contain $k(i)$. It follows that $K(x_0)=K(i)$ and that $K(y_0)$ is a totally ramified extension of $K(x_0)$ of degree $2m$. Hence x_0 is a square in $K(y_0)$ because the latter contains the unique quadratic extension of $K(x_0)$. Since y_0 is not a square in $K(y_0)$ and all units are squares in $K(y_0)$, the values $v(x_0)$ and $v(y_0)$ (with v the valuation on $K(y_0)$) are distinct.
First assume that $v(x_0)>v(y_0)$. We have $f(y_0)=(y_0-x_0)(y_0-x_0^\tau)$, with τ the automorphism induced by sending i to $-i$. Since $v(x_0^\tau)=v(x_0)>v(y_0)$ it follows that $f(y_0)\equiv y_0^2\equiv 1 \bmod K(y_0)^{*2}$.
Now let $v(x_0)<v(y_0)$ then $f(y_0)=(y_0-x_0)(y_0-x_0^\tau)\equiv(-x_0)(-x_0^\tau)\equiv f(0)\equiv 1 \bmod K(y_0)^{*2}$ (f is a sum of squares in $K[x]$, so its constant term is a square). It follows from the ramification formula that (f,g) is unramified at g.
Since $K(y_0)$ is the unique totally ramified extension of $K(i)$, $K(y_0)/K$ is a Galois extension. Let $G=\mathrm{Gal}(K(y_0)/K)=\{\sigma_1,\dots,\sigma_{4m}\}$. Then $g(x_0)=\prod_{i=1}^{4m}(x_0-y_0^{\sigma_i})$, where for all $i=1,\dots,4m$ the elements $y_0^{\sigma_i}$ have equal values with respect to the valuation v of $K(y_0)$.
If $v(x_0)<v(y_0)$ then $v(g(x_0))=v(\prod_{i=1}^{4m}(x_0-y_0^{\sigma_i}))=v(x_0^{4m})=4mv(x_0)$. Hence $g(x_0)\equiv 1 \bmod K(x_0)^{*2}$.
If $v(x_0)>v(y_0)$ then $v(g(x_0))=v(\prod_{i=1}^{4m}(x_0-y_0^{\sigma_i}))=v(\prod_{i=1}^{4m}(-y_0^{\sigma_i}))=v(g(0))$ so $g(x_0)\equiv g(0)\equiv 1 \bmod K(x_0)^{*2}$, because g is a sum of squares in $K[x]$ and so $g(0)$ is a square in K. Hence the formula for the ramification also yields that (f,g) does not ramify at f.
It follows that the Ω-algebra (f,g) is unramified everywhere and corollary 2.11 implies that (f,g) is trivial. □

Remark 2.16. Let E be the splitting field of the polynomials Q, R and P, it is a Galois extension of K. Let H be the 2-Sylow subgroup of $\mathrm{Gal}(E/K)$. The fixed field $L=E^H$ of H is an odd degree extension of K. Since E contains all the roots of the polynomials Q_i,R_j and P_k and since $[E:L]=2^m$ for some $m\geq 1$, it follows that all the irreducible factors over L of the polynomials Q_i,R_j and P_k have degree a power of 2 (they cannot be of degree one since the degrees of the polynomials Q_i,R_j and P_k are all even and so they cannot have a root in an odd degree extension). Moreover we have,

Corollary 2.17. *Let L be as above. The irreducible factors over L of Q_i,R_j and P_k, have degrees* $2,2$ *and* 2^s_k, $s_k>1$ *respectively. They are monic irreducible polynomials over L of type (1), (2) and (3) respectively.*

Proof: The only thing we still need to show is that the polynomials are of the given type. Let $Q_{i,L}$ be an irreducible factor of Q_i over L and let x_i be a root of $Q_{i,L}$, it is also a root of Q_i so it is a square in the root field $K(x_i)$. It follows that x_i is also a square in the larger field $L(x_i)$. So $Q_{i,L}$ is of type (1).
Let $R_{j,L}$ be an irreducible factor of R_j over L and let y_j be a root of $R_{j,L}$, it is also a root of R_j so it is not a square in the root field $K(y_j)$. Lemma 2.1 implies that the degree $[L(y_j):K(y_j)]$ is odd. It follows that y_j is not a square in $L(y_j)$. So $R_{j,L}$ is of type (2). In the same way it follows that the irreducible factors of P_k over L are of type (3). □

These observations will allows us to reduce some arguments to the case where the polynomials Q_i,R_j,P_k are of degree a power of 2.

Lemma 2.18. *Let A be the Ω-algebra over $K(x)$ as in theorem 2.13. Then:*

(1) $A \sim (QR,\pi)_{K(x)} \otimes_{K(x)} (P,x)_{K(x)}$
(2) $A \sim (xQR,\pi P)_{K(x)} \otimes_{K(x)} (\pi,x)_{K(x)}$

Proof: We first prove the lemma in the special case that the polynomials Q_i and R_j are of degree 2 and that the polynomials P_k are of degree 2^{s_k}.
(1) Consider the algebra $(QR,\pi) \otimes_{K(x)} (P,x)$. We calculate its ramification. Since QR and P are polynomials which are equal to a sum of squares in $K(x)$ lemma 2.5 (a) implies that $((QR,\pi) \otimes_{K(x)} (P,x))_{K(x)_\infty}$ is trivial.
The algebra $(QR,\pi) \otimes_{K(x)} (P,x)$ can only ramify in finite points corresponding to irreducible factors of Q,R,P and x.
Since $\deg Q_i = \deg R_j = 2$ and since Q_i and R_j are both sums of squares we have $K(x_i) = K(y_j) = K(i)$. So π is not a square in $K(x_i) = K(y_j)$. Lemma 2.5 (c) implies that (QR,π) is ramified at the points corresponding to the polynomials Q_i and R_j for all i and all j. The same is true for $(QR,\pi) \otimes_{K(x)} (P,x)$ since (P,x) is unramified at these points.
Since z_k is not a square in $K(x)$ we conclude (by lemma 2.5 (b) that (P,x) is ramified at the irreducible polynomials P_k. The same is true for $(QR,\pi) \otimes_{K(x)} (P,x)$ since (QR,π) is unramified at these points.
The polynomial P is a sum of squares in $K[x]$ so $P(0) \equiv 1 \bmod K^{*2}$. Hence, by the ramification formula, (P,x) and therefore also $(QR,\pi) \otimes_{K(x)} (P,x)$ is not ramified in x.
It follows that A and $(QR,\pi) \otimes_{K(x)} (P,x)$ have the same ramification so corollary 2.11 yields that they are equivalent. This finishes the proof of (1).
(2) Lemma 2.15 implies that the quaternion algebras (Q_i,P_k) and (R_j,P_k) are trivial. Hence $(Q,P) \sim \prod_{i,k}(Q_i,P_k)$ and $(R,P) \sim \prod_{j,k}(R_j,P_k)$ are trivial. This and part (1) of the proof implies

$$\begin{aligned} A &\sim (QR,\pi) \otimes (P,x) \\ &\sim (QR,\pi) \otimes (x,\pi) \otimes (P,x) \otimes (P,Q) \otimes (P,R) \otimes (x,\pi) \\ &\sim (xQR,\pi) \otimes (xQR,P) \otimes (\pi,x) \\ &\sim (xQR,\pi P) \otimes (\pi,x). \end{aligned}$$

We now show that the general case can be reduced to the special case above. Let L/K be the odd degree extension described in remark 2.16, say with $[L:K] = d$. Choose a uniformizing element π_L in L such that $\pi = (\pi_L)^d$ (this is possible since $L = K(\sqrt[d]{\pi})$).
Since Q,R and P are also over L of type (1), (2) and (3) respectively. And since the degree of the irreducible factors of Q and R over L is 2 and since the degree of the irreducible factors of P is a power of 2. It follows from the above that $A \otimes L \sim (QR,\pi_L)_{L(x)} \otimes_{L(x)} (P,x)_{L(x)} = ((QR,\pi) \otimes (P,x)) \otimes L(x)$. But $L(x)/K(x)$ being of odd degree implies that the natural map ${}_2\mathrm{Br}(K(x)) \to {}_2\mathrm{Br}(L(x))$ is injective. So the equivalence $A \sim (QR,\pi) \otimes_{K(x)} (P,x))$ follows. The second equivalence follows in the same way. $\square$

Lemma 2.19. *Let f,g be monic irreducible polynomials in $K[x]$ such that f is a sum of squares in $K(x)$. Let x_0 be a root of f and y_0 a root of g such that $K(y_0)$ is a non-real field.*
If $f(y_0) \not\equiv 1 \bmod K(y_0)^{*2}$ then $v(x_0) = v(y_0)$ with v the valuation of the field $K(x_0,y_0)$.

If in addition f is a quadratic polynomial then $x_0 = \pi^a u_0$ with $a \in \mathbb{Z}$ and u_0 a unit in $\mathscr{O}_{K(x_0)}$. Then $w_0 = y_0\pi^{-a}$ is a unit in $\mathscr{O}_{K(x_0,y_0)}$. Let $\overline{u}_0$ and $\overline{w}_0$ be the residues in $k(i)$ of respectively u_0 and w_0, then $\overline{u}_0$ and $\overline{w}_0$ are equal or conjugated under the automorphism τ defined by $\tau(i) = -i$.

Proof: By assumption f is of even degree, say $\deg f = 2m$. Let x_0 be a root of f and let $\sigma_1, \dots \sigma_{2m}$ be the automorphisms of the splitting field L of f. So the elements $x_0^{\sigma_1}, \dots, x_0^{\sigma_{2m}}$ are exactly the $2m$ different roots of f, and $f(x) = \prod_{i=1}^{2m}(x - x_0^{\sigma_i})$. The values of the roots $x_0^{\sigma_1}, \dots, x_0^{\sigma_{2m}}$ with respect to the valuation on L are all equal.
Suppose for the sake of contradiction that $v(x_0) \neq v(y_0)$, where v is the valuation on $K(x_0,y_0)$. If $v(x_0) > v(y_0)$ then $v(f(y_0)) = v(\prod_{i=1}^{2m}(y_0 - x_0^{\sigma_i})) = v(y_0^{2m})$. Hence $f(y_0) \equiv 1 \bmod K(y_0)^{*2}$, contradicting the hypotheses.
If $v(x_0) < v(y_0)$ then $v(f(y_0)) = v(\prod_{i=1}^{2m}(y_0 - x_0^{\sigma_i})) = v(\prod_{i=1}^{2m}(-x_0^{\sigma_i})) = v(f(0))$, hence $f(y_0) \equiv f(0) \equiv 1 \bmod K^{*2}$ (since f is a sum of squares), again contradicting the hypotheses.
So $v(x_0) = v(y_0)$. In the case $m = 1$, i.e. f is a quadratic polynomial, $f(x) = (x - x_0)(x - x_0^{\tau})$, with τ inducing the non-trivial automorphism on $L = K(x_0) = K(i)$, so that $\tau(i) = -i$.
We can put $x_0 = \pi^a u_0$ with u_0 a unit in $\mathscr{O}_{K(x_0)}$ and it follows that $w_0 = y_0\pi^{-a}$ is a unit in $\mathscr{O}_{K(y_0)}$ (note that $K(x_0) = K(y_0)$ in this case). We have $f(y_0) = (\pi^a w_0 - \pi^a u_0)(\pi^a w_0 - \pi^a u_0^{\tau}) \equiv (w_0 - u_0)(w_0 - u_0^{\tau}) \bmod K(y_0)^{*2}$. By assumption $K(y_0)$ contains $k(i)$ and therefore the units in $\mathscr{O}_{K(y_0)}$ are all squares. For the sake of contradiction assume that $\overline{w_0} \neq \overline{u_0}$ and $\overline{w_0} \neq \overline{u_0}^{\tau}$. Then $w_0 - u_0$ and $w_0 - u_0^{\tau}$ are units in $\mathscr{O}_{K(y_0)}$ so it are squares in $K(y_0)$. The above calculation implies that $f(y_0) \equiv 1 \bmod K(y_0)^{*2}$ contradicting the hypotheses. This proves the lemma. □

Lemma 2.20. *Let δ be a root of the polynomial $x^n - a^2$, where $n \equiv 1 \bmod 2$ and $a \in K^*$. If $K(\delta)$ is non-real then $R(\delta) \equiv 1 \bmod K(\delta)^{*2}$ and $P(\delta) \equiv 1 \bmod K(\delta)^{*2}$ (where R and P are products of monic irreducible polynomials of type (2) and (3), respectively).*

Proof: Since $K(\delta)$ is a non-real extension over K, it contains $K(i)$,hence $K(\delta) = K(i)(\sqrt[m]{\pi})$ for some $m \in \mathbb{N}$. So $\delta = (\sqrt[m]{\pi})^p u$, with $p \in \mathbb{Z}$ and u a unit in $\mathscr{O}_{K(i)(\sqrt[m]{\pi})}$. From $\delta^n = a^2$, it follows that $\pi^{\frac{pn}{m}} u^n = a^2$. This implies $\frac{pn}{2m} \in \mathbb{Z}$ since $a \in K^*$. Let $d = \gcd(m,n)$, put $m = dm_1, n = dn_1$, with $\gcd(m_1,n_1) = 1$, note that d is odd since n is odd. Since $\frac{pn_1}{2m_1} = \frac{pn}{2m} \in \mathbb{Z}$, $n \equiv 1 \bmod 2$ and $\gcd(m_1,n_1) = 1$, it follows that $2m_1$ is a divisor of p, i.e. $p = 2m_1p_1$, with $p_1 \in \mathbb{Z}$. Since u is a unit in $\mathscr{O}_{K(i)(\sqrt[m]{\pi})}$ it is a square in $K(i)(\sqrt[m]{\pi}) = K(\delta)$. Hence, $\delta = (\sqrt[m]{\pi})^p u = (\sqrt[m]{\pi})^{2m_1p_1} u \equiv 1 \bmod K(\delta)^{*2}$.
Let w be a root of an irreducible factor R_j of R or of an irreducible factor P_k of P. Then $i \in K(w)$ so $K(w) = K(i)(\sqrt[g]{\pi})$, with $g = 2^{s-1}r, s \in \mathbb{N}, s > 1$, and r an odd number.
We first show that the values of the elements δ, w with respect to the valuation of the field $K(\delta,w)$ are distinct. To do this assume $v(\delta) = v(w)$. Now $K(\delta) = K(i)(\sqrt[m]{\pi})$ and $K(w) = K(i)(\sqrt[g]{\pi})$ implies that $K(w,\delta) = K(i)(\sqrt[h]{\pi}))$, with $h = \operatorname{lcm}(g,m)$. Since $w \not\equiv 1 \bmod K(w)^{*2}$, we have $w = \pi^{(2q+1)/g}\varepsilon$, where $q \in \mathbb{Z}$ and ε is a unit in $\mathscr{O}_{K(w)}$. Hence, from $w = (\sqrt[h]{\pi})^{(2q+1)h/g}\varepsilon$ and $\delta = (\sqrt[h]{\pi})^{ph/m}u = (\sqrt[h]{\pi})^{2p_1m_1h/m}u$ it follows that $\frac{(2q+1)h}{g} = \frac{2p_1m_1h}{m}$, since we assumed that $v(w) = v(\delta)$. So we have $\frac{2q+1}{g} = \frac{2p_1}{d}$, implying that $(2q+1)d = 2p_1g$. This is impossible, since $(2q+1)d$ is odd and $2p_1g$ is even.

Since the values of the elements δ and w are distinct, lemma 2.19 implies that $S(\delta) \equiv 1 \bmod K(\delta)^{*2}$ for all irreducible factors $S(x)$ of $R(x)$ or of $P(x)$. It follows that $R(\delta) = \prod_{j=1}^{l} R_j(\delta) \equiv 1 \bmod K(\delta)^{*2}$ and that $P(\delta) = \prod_{k=1}^{l} P_k(\delta) \equiv 1 \bmod K(\delta)^{*2}$. □

2.2. **The construction of the polynomial** h**.** Let A be the Ω-algebra over $K(x)$ as given in theorem 2.13. We now construct the polynomial $h(x) \in K[x]$ in terms of the ramification of A given by the monic irreducible polynomials $Q_i, i = 1, \dots a$, R_j, $j = 1, \dots, e$ and $P_k, k = 1, \dots, l$ of type (1), (2) and (3) respectively.
Consider the field extension $M = K(y_1, \dots, y_e, z_1, \dots, z_l)$, with for $j = 1, \dots, e$, y_j a root of the polynomial Q_j, and for $k = 1, \dots, l$, z_k a root of the polynomials P_k. Then $M = K(i)(\sqrt[d]{\pi})$ for some $d \in \mathbb{N}$, define $n := 4d+1$. We denote $Y = \{y_1, \dots, y_e\}$ and $Z = \{z_1, \dots, z_l\}$.
Using that y_j is a square in $K(y_j)$ and z_k is not a square in $K(z_k)$. And using that $K(y_j, z_k) = K(i)(\sqrt[m]{\pi})$, with $m = 2^{s_k - 1} r$ where r is odd, and s_k is such that $2^{s_k} p_k = [K(z_k) : K]$, with p_k odd. A similar argument as the one used in the proof of lemma 2.20 yields that for the valuation on M all the elements of Y have a value different to the values of all the elements in Z.

In order to define the polynomial h we have to consider four different cases. In what follows we use the following notation. For any finite set W of elements in L, let $m(W)$ denote the element of W with the smallest value and let $M(W)$ denote the element of W with the largest value.

(i) *The element of $Y \cup Z$ with the smallest valuation is an element of Y and the element of $Y \cup Z$ with the largest value is an element of Z.*
In this case we partition Y and Z respectively in subsets $Y_r \subset Y$, $r = 1, \dots, b$ and $Z_t \subset Z$, $t = 1, \dots, b$ such that for all elements $\widetilde{y}_r \in Y_r$ and all elements $\widetilde{z}_t \in Z_t$ the following holds:

$$v(\widetilde{y}_1) < v(\widetilde{z}_1) < \ldots < v(\widetilde{y}_b) < v(\widetilde{z}_b).$$

Put (square brackets means taking integer parts)

$$\begin{aligned}
c_1 &= [\tfrac{4d+1}{2d} v(M(Y_1))] + 1, \\
c_2 &= [\tfrac{4d+1}{2d} v(M(Z_1))] + 1 \\
&\vdots \\
c_{2b-1} &= [\tfrac{4d+1}{2d} v(M(Y_b))] + 1 \\
c_{2b} &= [\tfrac{4d+1}{2d} v(M(Z_b))] + 1
\end{aligned}$$

Then for $i = 1, \dots, b$ we have $v(M(Y_i)) < v(m(Z_i))$ and the following inequalities hold:

$$\frac{4d+1}{2d}(v(m(Z_i)) - v(M(Y_i))) \geq \frac{4d+1}{2d} > 2.$$

This implies

$$\frac{4d+1}{2d} v(M(Y_i)) < c_{2(i-1)+1} < \frac{4d+1}{2d} v(M(Y_i)) + 2 < \frac{4d+1}{2d} v(m(Z_i)),$$

and therefore

$$(4d+1)v(M(Y_i)) < 2dc_{2(i-1)+1} < (4d+1)v(m(Z_i)).$$

We obtain

$$v(M(Y_i)^n) < v(\pi^{2c_{2(i-1)+1}}) < v(m(Z_i)^n).$$

By definition we have for $i = 1,\dots,b-1$, (since the values of the elements in Y are different form the values of the elements in Z), that $v(M(Z_i)) < v(m(Y_{i+1}))$. We get

$$\frac{4d+1}{2d}(v(m(Y_{i+1}) - v(M(Z_i))) \geq \frac{4d+1}{2d} > 2,$$

so

$$(4d+1)v(M(Z_i)) < 2dc_{2i} < (4d+1)v(m(Y_{i+1})),$$

yielding

$$v(M(Z_i)^n) < v(\pi^{2c_i}) < v(m(Y_{i+1})^n).$$

Finally for $j = b$ we get directly from the definition that

$$v(M(Z_b)^n) < v(\pi^{2c_{2b}}).$$

So we verified that

$$\begin{aligned} v((M(Y_1))^n) &< v(\pi^{2c_1}) < v((m(Z_1))^n) \leq v((M(Z_1))^n) < v(\pi^{2c_2}) < \\ & v((m(Y_2))^n) \leq v(M(Y_2))^n) < \dots \leq v((M(Z_b))^n) < v(\pi^{2c_{2b}}). \end{aligned}$$

(ii) *The element of $Y \cup Z$ with the smallest valuation is an element of Y and the element of $Y \cup Z$ with the largest value is an element of Y.*
In this case we can partition Y and Z respectively in subsets $Y_r \subset Y$, $r = 1,\dots,b+1$ and $Z_t \subset Z$, $t = 1,\dots,b$ such that for all elements $\widetilde{y}_r \in Y_r$ and all elements $\widetilde{z}_t \in Z_t$ the following holds:

$$v(\widetilde{y}_1) < v(\widetilde{z}_1) < \dots < v(\widetilde{y}_b) < v(\widetilde{z}_b) < v(\widetilde{y}_{b+1}).$$

As above we define c_j $(j = 1\dots,2b)$,

$$\begin{aligned} c_{2(i-1)+1} &= [\tfrac{4d+1}{2d}M(Y_i)]+1 \\ c_{2i} &= [\tfrac{4d+1}{2d}M(Z_i)]+1, \end{aligned}$$

and verify in a similar way that

$$\begin{aligned} v((M(Y_1))^n) &< v(\pi^{2c_1}) < v((m(Z_1))^n) \leq v((M(Z_1))^n) < v(\pi^{2c_2}) < v((m(Y_2))^n) \leq \\ & \dots \leq v((M(Z_b))^n) < v(\pi^{2c_{2b}}) < v((m(Y_{b+1}))^n). \end{aligned}$$

(iii) *The element of $Y \cup Z$ with smallest value is an element of Z and the element of $Y \cup Z$ with largest value is an element of Y.* We partition $Y \cup Z$ in $2b$ sets $Y_1,\dots,Y_b$ and $Z_1,\dots,Z_b$, such that

$$v(\widetilde{z}_1) < v(\widetilde{y}_1) < \dots < v(\widetilde{z}_b) < v(\widetilde{y}_b).$$

And we define c_j, $(j = 1, \ldots, 2b-1)$ as follows:

$$\begin{aligned} c_{2(i-1)+1} &= [\tfrac{4d+1}{2d} M(Z_i)] + 1 \\ c_{2i} &= [\tfrac{4d+1}{2d} M(Y_i)] + 1, \end{aligned}$$

here $i =, 1 \ldots b-1$. One can verify the following inequalities,

$$\begin{aligned} v((M(Z_1))^n) &< v(\pi^{2c_1}) < v((m(Y_1))^n) \leq v((M(Y_1))^n) < v(\pi^{2c_2}) < v((m(Z_2))^n) \leq \\ &\ldots < v(\pi^{2c_{2b-1}}) < v((m(Y_b))^n). \end{aligned}$$

(iv) *The element of $Y \cup Z$ with smallest value is an element of Z and the element of $Y \cup Z$ with largest value is also an element of Z.*
We partition $Y \cup Z$ in $2b+1$ sets $Y_1, \ldots, Y_b$ and $Z_1, \ldots, Z_{b+1}$. We define the elements c_i $(i = 1, \ldots, 2b+1)$:

$$\begin{aligned} c_{2(i-1)+1} &= [\tfrac{4d+1}{2d} M(Z_i)] + 1 \\ c_{2i} &= [\tfrac{4d+1}{2d} M(Y_i)] + 1, \end{aligned}$$

with $i = 1, \ldots, b$. And one verifies the inequalities,

$$\begin{aligned} & v((M(Z_1))^n) < v(\pi^{2c_1}) < v((m(Y_1))^n) \leq v((M(Y_1))^n) < v(\pi^{2c_2}) < v((m(Z_2))^n) \leq \\ & \qquad \ldots < v(\pi^{2c_{2b}}) < v((m(Z_{b+1}))^n) \leq v((M(Z_{b+1}))^n) < v(\pi^{2c_{2b+1}}). \end{aligned}$$

Definition of h.

Define $m := 2b$ if (i) or (ii) holds, $m := 2b-1$ if (iii) holds and $m := 2b+1$ if (iv) holds.
Let L be the odd degree extension of K defined in remark 2.16. Every polynomial Q_i, $i = 1, \ldots, a$ splits over L in q_i irreducible factors of degree 2, say $Q_{i,t}$, $i = 1, \ldots a$ and $t = 1, \ldots q_i$. Let $x_{i,t}$ be a root of $Q_{i,t}$. Then $x_{i,t} = \pi^{\frac{v(x_{i,t})}{q_i}} w_{i,t}$ where $v(x_{i,t})$ is the value of $x_{i,t}$ in the field $L(x_{i,t})$, $w_{i,t}$ is a unit in $\mathscr{O}_{L(x_{i,t})}$, and q_i is as defined in definition 2.12. Since the residue field of K is infinite we can choose units u_s in K in such a way that $(\overline{u_s})^2 \neq (\overline{w_{i,t}})^n, (\overline{u_s})^2 \neq (\overline{w^{\tau}_{i,t}})^n$ for all $s = 1, \ldots m$, and $j = 1, \ldots, a$, where $\overline{u_s}$ be the residue of u_s in k, $\overline{w_{i,t}}$ be the residue of $w_{i,t}$ in $k(i)$, and $\overline{w^{\tau}_{i,t}}$ is an element conjugated to $\overline{w_{i,t}}$ under τ (the automorphism defined by $i \mapsto -i$).

Definition 2.21. Let m, c_s and u_s for $s = 1, \ldots, m$ be as defined above. Define $a_s = \pi^{c_s} u_s$, $s = 1, \ldots, m$, and define

$$h(x) := \prod_{s=1}^{m} (a_s^2 - x^n).$$

2.3. **Proof of theorem 2.13.** With h as in definition 2.21 the following holds:

Lemma 2.22. *The quaternion algebras $(xQR, \pi P)_{K(x)}$ and $(\pi h(x), xPQR)_{K(x)}$ are isomorphic.*

Proof: First note that since $n-1$ is even, $a_s^2 - x^n = a_s^2 - (x^{\frac{n-1}{2}})^2 x$ is a norm of the quadratic extension $K(x)(\sqrt{x})$. It follows that for each s the quaternion algebra $(a_s^2 - x^n, x)$ is trivial. Hence also (h,x) is trivial. Lemma 2.5 (1) implies that the algebra $(P,\pi) = \otimes_{k=1}^{l}(P_k,\pi)$ is trivial. Expanding $(\pi h, xPQR)$ then yields

$$\begin{aligned}(\pi h, xPOR) &\sim (\pi, xPQR) \otimes (h, xPQR)\\ &\sim (\pi, x) \otimes (QR,\pi) \otimes (h, PQR).\end{aligned}$$

Lemma 2.5 (b) says that (Q,P) and (R,P) are trivial so the expansion of $(xQR,\pi P)$ gives

$$\begin{aligned}(xQR,\pi P) &\sim (x,\pi)\otimes(x,P)\otimes(QR,\pi)\otimes(Q,P)\otimes(R,P)\\ &\sim (x,\pi)\otimes(P,x)\otimes(QR,\pi).\end{aligned}$$

It follows that the isomorphism, $(xQR,\pi P) \cong (\pi h, xPQR)$, which we have to prove, is established if we show that

$$(x,\pi)\otimes(P,x)\otimes(QR,\pi) \cong (\pi,x)\otimes(QR,\pi)\otimes(h,PQR),$$

or equivalently that

$$(P,x) \cong (h,PQR). \tag{1}$$

Note that both are Ω-algebras. The ramification locus of (P,x) consists exactly of the points corresponding to $P_1,\ldots,P_l$. So the isomorphism (1) holds if the ramification locus of the right hand side is also equal to $P_1,\ldots,P_l$ (cf. corollary 2.11). This holds true if

$$\begin{cases} h(x_i) \equiv 1 \bmod K(x_i)^{*2} & \text{for all } i = 1,\ldots,a\\ h(y_j) \equiv 1 \bmod K(y_j)^{*2}, & \text{for all } j = 1,\ldots,e\\ h(z_k) \not\equiv 1 \bmod K(z_k)^{*2}, & \text{for all } k = 1,\ldots,l\\ P(\delta)Q(\delta)R(\delta) \sim 1 \text{in} K(\delta) & \text{for all roots } \delta \text{ of } h.\end{cases} \tag{2}$$

We first verify the last condition in (2). Let δ be a root of h. Note that if $K(\delta)$ is a real field, then we have that $P(\delta)Q(\delta)R(\delta) \equiv 1 \bmod K(\delta)^{*2}$ (since the polynomials P,Q,R are sums of squares in $K[x]$). So we may assume that $K(\delta)$ is a non-real field. According to lemma 2.20 we have $R(\delta) \equiv 1 \bmod K(\delta)^{*2}$, $P(\delta) \equiv 1 \bmod K(\delta)^{*2}$ in $K(\delta)$.
Assume that $Q(\delta) \not\equiv 1 \bmod K(\delta)^{*2}$. Then it follows from lemma 2.1 that $Q(\delta) \not\equiv 1 \bmod L(\delta)^{*2}$ where L is the odd degree extension of K defined in remark 2.16. Then $Q_{i,t}(\delta) \not\equiv 1 \bmod L(\delta)^{*2}$ for some (i,t), where $Q_{i,t}$ are the monic irreducible factors of Q over L (as defined in subsection 2.2). We also fixed a root $x_{i,t}$ of $Q_{i,t}$. According to lemma 2.19, $v(\delta) = v(x_{i,t})$ in $L(x_{i,t},\delta)$. Let $\delta = \pi^{v(\delta)}\varepsilon$, with $h_1 \in \mathbb{Z}$ and ε a unit in $\mathscr{O}_{L(x_{i,t},\delta)}$, since $\delta^n = a_s^2$, for some $s \in \{1,\ldots,m\}$, we have $\varepsilon = \sqrt[n]{u_s^2}$. And by the choice of the units u_s, $s = 1,\ldots,m$ we see that the residues of ε^n and $w_{i,t}^n$ are not equal and not conjugated under the automorphism τ, therefore the same holds for the residues of ε and $w_{i,t}$. Lemma 2.19 then implies $Q_i(\delta) \equiv 1 \bmod L(\delta)^{*2}$, and we have a contradiction. Hence $Q(\delta) \equiv 1 \bmod K(\delta)^{*2}$.
So $P(\delta)Q(\delta)R(\delta) \equiv 1 \bmod K(\delta)^{*2}$, and we proved that the last condition of (2) is satisfied.
We now verify the first condition of (2). To do this we verify for all $s = 1,\ldots,m$ and for all indices i that $a_s^2 - x_i^n \equiv 1 \bmod K(x_i)^{*2}$ in the three possible cases $v(a_s^2) < v(x_i^n)$, $v(a_s^2) > v(x_i^n)$ and $v(a_s^2) = v(x_i^n)$, where v is the valuation of $K(x_i)$.

If $v(a_s^2) < v(x_i^n)$ then $a_s^2 - x_i^n \equiv a_s^2 \equiv 1 \bmod K(x_i)^{*2}$. If $v(a_s^2) > v(x_i^n)$ then $a_s^2 - x_i^n \equiv -x_i^n \equiv 1 \bmod K(x_i)^{*2}$ since $x_i \equiv 1 \bmod K(x_i)^{*2}$ and $K(i) \subset K(x_i)^2$. Finally if $v(a_s^2) = v(x_i^n)$ then $a_s^2 - x_i^n = \pi^{2v(a_s)}(u_s^2 - w_i^n) \equiv 1 \bmod K(x_i)^{*2}$ because $u_s^2 - w_i^n$ is a unit in $\mathcal{O}_{K(x_i)}$ (by construction of the elements u_s).

Finally we verify the second and the third condition of (2), i.e. $h(y) \equiv 1 \bmod K(y)^{*2}$ for all $y \in Y$ and $h(z) \not\equiv 1 \bmod K(z)^{*2}$ for all $z \in Z$. To do this we have to consider the cases (i) - (iv), on which the definition of h depend, seperatly.

We consider case (i). Let $y \in Y_1$ then $h(y) = \prod_{s=1}^{2b}(a_s^2 - y^n) \equiv \prod_{s=1}^{2b}(-y^n) \equiv y^{2bn} \equiv 1 \bmod K(y)^{*2}$. Let $y \in Y_j$ with $j > 1$, then $h(y) = \prod_{s=1}^{2b}(a_s^2 - y^n) \equiv \prod_{s=1}^{2j-2} a_s^2 \prod_{s=2j-1}^{2b}(-y^n) \equiv a_s^{4(j-1)} y^{2(b-j+1)n} \equiv 1 \bmod K(y)^{*2}$. This settles the second condition of (2).
Let $z \in Z_k$ then $h(z) = \prod_{s=1}^{2b}(a_s^2 - z^n) \equiv \prod_{s=1}^{2k-1} a_s^2 \prod_{s=2k}^{2b}(-z^n) \equiv -a_s^{2(2k-1)} z^{(2b-2k+1)n} \equiv z \not\equiv 1 \bmod K(z)^{*2}$, since $(2b-2k+1)n \equiv 1 \bmod 2$ and $z \not\equiv 1 \bmod K(z)^{*2}$. Which proves that the third condition of (2) holds for h.
In a similar way on can verify in each of the three other cases ((ii), (iii) and (iv)) that the polynomial h satisfies the second and the third condition of (2). $\square$

We can now prove our main result.

Proof of theorem 2.13: Let A be an Ω-algebra over $K(x)$ of exponent 2 and with ramification locus as in theorem 2.13. Lemma 2.18 implies that $A \sim (xQR, \pi P) \otimes (\pi, x)$.
According to Lemma 2.22 there exist elements $a_1, \ldots, a_m \in K^*$ and an odd number n, such that $(xQR, \pi P) \sim (\pi \prod_{s=1}^{m}(a_s^2 - x^n), xPQR)$. Since $(a_s^2 - x^n, x) \sim (a_s^2 - (x^{\frac{n-1}{2}})^2 x, x) \sim 1$, we have $(h, x) \sim (\prod_{s=1}^{m}(a_s^2 - x^n), x) \sim 1$. This implies $(\pi, x) \sim (\pi h, x)$. Therefore

$$A \sim (xQR, \pi P) \otimes (\pi, x) \sim (\pi h, xPQR) \otimes (\pi h, x) \sim (\pi h, PQR),$$

as stated in the theorem. $\square$

3. Conic bundle surfaces over Henselian discrete valued fields with real closed residue fields.

As mentioned in part 1.2 of the introduction, the results on Ω-algebras over $K(x)$ with K a Henselian discrete valued field with real closed residue field discussed in section 2, especially theorem 2.8 and theorem 2.13, yield information on the existence of relatively minimal conic bundles $\varphi : X \longrightarrow \mathbb{P}_K^1$ over K with prescribed local data. In this section we give the proper translation of these results.
Let K be a Henselian discrete valued field with real closed residue field k. Let x be the generic point of $\mathbb{P}_K^1$, then $K(x)$ is the function field of $\mathbb{P}_K^1$. A closed point y of $\mathbb{P}_K^1$ such that its residue field $K(y)$ is a finite extension of $K(i)$, corresponds to a monic irreducible polynomial T which is a sum of two squares in $K[x]$. We say that y is of type (1), (2) or (3) if T is of type (1), (2), (3) respectively according to definition 2.12.

Theorem 3.1. *(1) Let S be a finite set of closed points y in $\mathbb{P}^1_K$, such that $K(i) \subset K(y)$. Then there exists a relatively minimal conic bundle $\varphi_S : X \to \mathbb{P}^1_K$ with $S = \{y \in \mathbb{P}^1_K | \varphi \text{ degenerates in } y\}$ such that the fibers X_z in the K-rational points z of $\mathbb{P}^1_K$ are isomorphic to $\mathbb{P}^1_K$.*
Moreover a relatively minimal conic bundle with these properties is unique up to fiber preserving birational isomorphisms, its local invariants are given by $\{L_y/K(y)\}_{y\in S}$, where L_y is the unique totally ramified quadratic extension of $K(y)$.
(2) If S consists only of points of type (1), (2) or (3), then the generic fibre X_x of φ_S is isomorphic to the conic given by an equation of the form $\pi h x_1^2 + PQR x_2^2 = x_3^2$, where PQR is the product of the monic irreducible polynomials associated to the closed points in S and $h \in K[x]$.

Proof: Let $y \in S$. Note that (by the transitivity of the norm) $\mathrm{cor}_{K(y)/K}(K(y)^*) = N_{K(y)/K}(K(y)^*) \subset N_{K(i)/K}(K^*) \subset K^{*2} + K^{*2} = K^{*2}$ (because K is pythagorean). It follows that any element in $\bigoplus_{y\in S} H^1(K(y), \mathbb{Z}/2\mathbb{Z})$ has trivial image under the map $\sum \mathrm{cor}$ in the exact sequence (FES). So Faddeev's exact sequence implies the existence of an algebra A which ramifies exactly in the points of S. Up to multiplying A with an algebra defined over K we may assume A is an Ω-algebra (using proposition 2.9). It follows from theorem 2.8 that A is a quaternion algebra. Proposition 1.4 then yields the existence of a relative minimal conic bundle $\varphi : X \to \mathbb{P}^1_K$ with generic fiber corresponding to the conic associated to A and with local invariants exactly defined in the points of S. Let z be a K-rational point of $\mathbb{P}^1_K$. The fact that A is an Ω-algebra implies (using lemma 2.5,(a)) that $A \otimes_{K(x)} K(x)_z$ is trivial. But A, being unramified in z, defines an element in $\mathrm{Br}(O_z)$. This element must also be trivial since the canonical map $\mathrm{Br}(O_z) \to \mathrm{Br}(K(x)_z)$ is injective. The conic defined by $A \otimes_{O_z} K(z)$ therefore contains a rational point. Proposition 1.4 then implies that $X_z \cong \mathbb{P}^1_K$.
To prove uniqueness up to fiber preserving birational isomorphism, we first note that the set S determines the local invariants. Let $y \in S$ then $K(y)$ is a finite extension of $K(i)$. Therefore (cf. page 6) there is a unique quadratic extension of $K(y)$, given by $K(y)(\sqrt[2n]{\pi})$, with $n = [K(y) : K(i)]$. This means that S defines a unique set of local invariants.
Now let B be an other quaternion algebra over $K(x)$ with ramification locus S. Then Fadeev's exact sequence (FES) implies that $A \otimes_{K(x)} B$ is a constant algebra, i.e. $A \otimes_{K(x)} B$ is Brauer equivalent to an algebra defined over K, or equivalently $B \sim A \otimes_{K(x)} (-1, \varepsilon)_{K(x)}$ with $\varepsilon \in \{1, -1, \pi, -\pi\}$. It follows that either B is isomorphic to A, in the case $\varepsilon = 1$, or $B \otimes_{K(x)} K(x)_\infty \sim (-1, \varepsilon)_{K(x)_\infty}$ with $\varepsilon = -1, \pi$ or $-\pi$. In the later case $B \otimes_{K(x)} K(x)_\infty$ is a division algebra. The fiber at the point at infinity of the conic bundle corresponding to B will be the conic $u^2 + v^2 - \varepsilon w^2$, $(\varepsilon = -1, \pi, -\pi)$, over K, which is a conic without a rational point. So if the relative minimal conic bundle over K with degenerate fibers exactly in the points of S has a fiber at infinity which is isomorphic to $\mathbb{P}^1_K$ it must be a conic bundle corresponding to the quaternion algebra A. It follows from proposition 1.4 that such a relative minimal conic bundle is uniquely determined up to fiber preserving birational isomorphism.
The second part of the theorem follows immediately from proposition 1.4 and theorem 2.13. □

Remark 3.2. (1) In theorem 3.1 the condition on the fibers in the K-rational points can be replaced by the same condition for one fiber in a K-rational point. This follows from the fact that in

the characterization of Ω-algebras (proposition 2.9). The property that $A\otimes K(x)_\infty$ is trivial is equivalent to saying that $A\otimes K(x)_z$, where z is any rational point of $\mathbb{P}^1_K$, is trivial.
(2) One can ask whether theorem 3.1 could be replaced by a stronger version stating that S determines four essentially different relative minimal conic bundles over K, one corresponding to the Ω-algebra A with $\mathrm{Ram}(A)=S$ and three others corresponding to $A\otimes(-1,-1)$, $A\otimes(-1,\pi)$, and $A\otimes(-1,-\pi)$ respectively . This is not possible in general since these three algebras are not always of index 2, but rather could be of index 4 over $K(x)$. We illustrate this with two examples over $K=\mathbb{R}((t))$.
(a) The quaternion algebra $H_1=(1+x^2,t)_{K(x)}$ is a division algebra since it is ramified in the point corresponding to the irreducible polynomial $1+x^2$, (its ramification is equal to t which is not a square in the residue field $K(i)((t))$). It is also an Ω-algebra since one of the entries is a sum of squares. The biquaternion algebra $H_1\otimes(-1,-1)_{K(x)}$ corresponds to the Albert form $\langle 1+x^2,t,-t(1+x^2),1,1,1\rangle$. This quadratic form is anisotropic over the field $\mathbb{R}(x)((t))$; as in the proof of lemma 2.3, this can be seen by considering the residue forms over $\mathbb{R}(x)$. So it is certainly anisotropic over the smaller field $K(x)=\mathbb{R}((t))(x)$, implying that the biquaternion algebra $H_1\otimes(-1,-1)_{K(x)}$ is a division algebra and therefore of index 4 over $K(x)$.
(b) The quaternion algebra $H_2=(1+x^2,-(1+t+x^2))_{K(x)}$ is a division algebra since it is ramified in the point corresponding to the irreducible polynomial $1+x^2\in K[x]$, (the ramification in that point is equal to t, which is not a square in the residue field $K(i)((t))$). H_2 is also an Ω-algebra since one of the entries is a sum of squares. (Note that H_2 is also equal to $(1+x^2,1+t+x^2)_{K(x)}$, a quaternion algebra with both entries equal to a sum of two squares.) In this case the biquaternion algebra $H_2\otimes(-1,-1)_{K(x)}$ is equivalent to $(-(1+x^2),-(1+t+x^2))_{K(x)}$, so it is an algebra of index 2.

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