Witt kernels of function field extensions in characteristic 2

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Abstract

In characteristic 2 we give a complete characterization of anisotropic symmetric bilinear forms that become metabolic over the function field of a quadratic form. We also study the hyperbolicity of nonsingular quadratic forms over such a field by generalizing some results by Fitzgerald [6]. As an application, we introduce and study the notion of Pfister neighbors for bilinear forms, and classify anisotropic bilinear forms of height 1, i.e. those that become metabolic over their own function fields. Other consequences are also included.

Key words: Bilinear and quadratic forms, hyperbolicity, metabolicity, Witt group and Witt ring of a field, Witt kernels of field extensions.
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1 Introduction

Let F denote a field of characteristic 2, and let $W_q(F)$ (*resp.* $W(F)$) be the Witt group of nonsingular quadratic forms over F (*resp.* the Witt ring of regular symmetric bilinear forms over F). For any field extension K/F, there are homomorphisms $W_q(F) \xrightarrow{i} W_q(K)$ and $W(F) \xrightarrow{j} W(K)$ induced by scalar extension.

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An important problem in the algebraic theory of quadratic and bilinear forms is the study of the kernels $W_q(K/F)$ and $W(K/F)$ of the homomorphisms i and j, respectively.

In this paper we are interested to the case where $K = F(\varphi)$, the function field of a quadratic form φ. For such a field in characteristic different from 2, the study was done by Fitzgerald [6]. Our aim is to treat the case of characteristic 2. More precisely, we will give a complete computation of $W(F(\varphi)/F)$, and for the kernel $W_q(F(\varphi)/F)$ we will extend important results in [6]. Our study includes function fields of possibly singular quadratic forms.

1.1 The kernel $W(F(\varphi)/F)$

Throughout this paper, the expression "bilinear form" means "regular symmetric bilinear form".

For a quadratic form (*resp.* a bilinear form) φ with underlying vector space V, we denote by $D_F(\varphi)$ the set of scalars in F^* represented by φ (*resp.* the set of scalars $\varphi(v,v) \in F^*$ for $v \in V$). A totally singular form of dimension n is a quadratic form isometric to $\sum_{i=1}^{n} a_i x_i^2$ for some $a_i \in F^*$. We denote it by $\langle a_1, \cdots, a_n \rangle$. A bilinear form B is called associated to a totally singular form φ (or φ is associated to B) if $\dim B = \dim \varphi$ and $D_F(B) = D_F(\varphi)$. Note that $D_F(B)$ is nothing but the F^2-vector space inside F generated by the elements in the diagonal of any matrix representing B, and that $\dim_{F^2} D_F(B) \leq \dim B$ with equality if B is anisotropic. Moreover, two totally singular forms associated to the same bilinear form are isometric. We denote by $\widetilde{B}$ the totally singular form associated to a bilinear form B, and by $\mathcal{A}(\varphi)$ the set of bilinear forms associated to a totally singular form φ.

Let $\langle a_1 : b : a_2 \rangle$ denote the 2-dimensional bilinear form whose underlying vector space has a basis $\{e_1, e_2\}$ satisfying: $B(e_i, e_i) = a_i$ and $B(e_1, e_2) = b$. A metabolic plane is a 2-dimensional bilinear form isomorphic to $\langle a : 1 : 0 \rangle$ for some $a \in F$. A metabolic bilinear form is an orthogonal sum of metabolic planes. Recall that a bilinear form is isotropic if and only if it contains a metabolic plane as a subform.

Our first result in this paper is the following proposition which gives the subform theorem version for bilinear forms. It will be very useful to study the metabolicity of bilinear forms over function fields of quadratic forms:

Proposition 1.1 *Let B be an anisotropic bilinear form, and let φ be an anisotropic quadratic form such that B is metabolic over $F(\varphi)$. Then φ is totally singular and for any $\alpha \in D_F(\varphi)D_F(B)$, there exists a subform B' of αB such that $B' \in \mathcal{A}(\varphi)$. In particular,* $\dim \varphi \leq \dim B$.

We also need the notion of norm field introduced in [7]. Recall that the norm field of a nonzero totally singular form φ, denoted by $N_F(\varphi)$, is the field $F^2(ab \mid a, b \in D_F(\varphi))$. The degree of the extension $N_F(\varphi)/F^2$, denoted by $\mathrm{ndeg}_F(\varphi)$, is called the norm degree of φ. If $b_1, \cdots, b_m \in F^*$ are such that $N_F(\varphi) = F^2(b_1, \cdots, b_m)$ and $\mathrm{ndeg}_F(\varphi) = 2^m$, then we denote by φ_{qp} the totally singular form associated to the anisotropic m-fold bilinear Pfister form $\langle 1, b_1\rangle_b \otimes \cdots \otimes \langle 1, b_m\rangle_b$, where $\langle a_1, \cdots, a_n\rangle_b$ denotes the bilinear form $\sum_{i=1}^n a_i x_i y_i$ for $a_i \in F^*$. We prove that the anisotropic part φ_{an} of φ is similar to a subform of φ_{qp}. Moreover, if φ_{an} is similar to a subform of a totally singular form φ' which is associated a bilinear Pfister form, then φ_{qp} is a subform of φ'. For more details on norm fields we refer to [7, Section 8].

Now our criterion concerning the metabolicity of bilinear forms over function fields of quadratic forms is as follows:

Theorem 1.2 *Let φ be an anisotropic quadratic form of dimension ≥ 2 such that $W(F(\varphi)/F) \neq 0$. Then φ is totally singular, and an anisotropic bilinear form B over F becomes metabolic over $F(\varphi)$ if and only if $B \simeq \alpha_1 B_1 \perp \cdots \perp \alpha_r B_r$ for some $\alpha_1, \cdots, \alpha_r \in F^*$ and $B_1, \cdots, B_r$ d-fold bilinear Pfister forms belonging to $\mathcal{A}(\varphi_{\mathrm{qp}})$, where d is such that $\mathrm{ndeg}_F(\varphi) = 2^d$ ($\simeq$ and $\perp$ denote isometry and orthogonal sum, respectively).*

1.2 The kernel $W_q(F(\varphi)/F)$

Contrary to the case of bilinear forms, the hyperbolicity of nonsingular quadratic forms was previously studied over some field extensions, like quadratic and biquadratic extensions. This was done by Baeza [3], [4, Cor. 4.16, Page 128], Mammone and Moresi [17], and Hamza [1, Cor. 2.8], [2]. Recently, the author gave a complete computation of $W_q(K/F)$ for K/F purely inseparable multiquadratic extension [16].

As was investigated in characteristic different from 2 by Elman, Lam, Wadsworth and Fitzgerald [5], [6], an important question related to the computation of $W_q(F(\varphi)/F)$ is to know if these kernels are generated by Pfister forms (up to scalars), and whether the description can be given up to isometry instead of Witt-equivalence. In this sense, we recall the terminology of "Pfister group" and "strong Pfister group" originally due to Elman, Lam and Wadsworth:

Definition 1.3 *For an integer $n \geq 1$, we denote by P_nF the set of quadratic forms isometric to n-fold Pfister forms, and $GP_nF = F^*P_nF$. For a field extension K/F and a subset I of $\mathbb{N}$, we say that:*
(1) $W_q(K/F)$ is an I-Pfister group if it is generated by quadratic forms in $W_q(K/F) \cap GP_nF$ for $n \in I$.

(2) $W_q(K/F)$ is a strong I-Pfister group if any quadratic form in $W_q(K/F)$ is isometric to an orthogonal sum of quadratic forms in $W_q(K/F) \cap GP_nF$ for $n \in I$.

The following theorem describes, up to isometry, nonsingular quadratic forms that become hyperbolic over the function field of a Pfister neighbor or quasi-Pfister neighbor (*cf.* subsection 2.3). In particular, for the quasi-Pfister neighbors case, this generalizes the result [1, Cor. 2.8] by Hamza:

Theorem 1.4 *Let φ be an anisotropic Pfister neighbor or quasi-Pfister neighbor of π. Then we have the following:*
(1) $W_q(F(\varphi)/F) = W_q(F(\pi)/F)$.
(2) If φ is a Pfister neighbor, then any anisotropic form in $W_q(F(\varphi)/F)$ is isometric to $\tau \otimes \pi$ for some bilinear form τ. In particular, $W_q(F(\varphi)/F)$ is a strong n-Pfister group, where n satisfies $\dim \pi = 2^n$.
(3) If φ is a quasi-Pfister neighbor, then any anisotropic form in $W_q(F(\varphi)/F)$ is isometric to $B \otimes \rho$ for some nonsingular quadratic form ρ, where B is a bilinear Pfister form satisfying $\widetilde{B} \simeq \pi$. In particular, $W_q(F(\varphi)/F)$ is a strong $(n+1)$-Pfister group, where n is as in (2).

However, in general, we have no criterion to compute $W_q(F(\varphi)/F)$ for an arbitrary quadratic form φ. The principal idea that we will use to get informations on the kernel $W_q(F(\varphi)/F)$ is to compare it to another one $W_q(F(\varphi')/F)$ provided that φ' satisfies some properties. Our main result in this sense is Theorem 1.5 which treats the case when φ' is dominated by φ, $\dim \varphi = \dim \varphi' + 1$ and $W_q(F(\varphi')/F)$ is a strong n-Pfister group for some $n \geq 1$ (dominated means that φ' is the restriction of φ to a subspace of its underlying vector space; *cf.* subsection 2.2):

Theorem 1.5 *Let φ be an anisotropic quadratic form (possibly singular) of dimension ≥ 3, and let φ' be a quadratic form dominated by φ with codimension 1, i.e. $\dim \varphi = \dim \varphi' + 1$, such that $W_q(F(\varphi')/F)$ is a strong n-Pfister group for some $n \geq 1$. Then $W_q(F(\varphi)/F)$ is an $\{n, n+1\}$-Pfister group. Moreover, if $W_q(F(\varphi)/F) \cap P_nF = \{0\}$, then $W_q(F(\varphi)/F)$ is an $(n+1)$-Pfister group.*

Theorem 1.5 partially extends [6, Prop. 1.2] to characteristic 2, and can be refined in the case of a Pfister neighbor φ' as follows:

Theorem 1.6 *Let φ be an anisotropic quadratic form of dimension ≥ 3, and let φ' be a quadratic form dominated by φ with codimension 1. If φ' is a Pfister neighbor of an n-fold Pfister form, then $W_q(F(\varphi)/F)$ is a strong m-Pfister group where $m = n$ or $n+1$ according as φ is a Pfister neighbor of an n-fold Pfister form or not.*

This paper is organized as follows. In the next section, we recall definitions and notions on bilinear and quadratic forms. Sections 3 is devoted to some preliminaries that we need for the proofs of our results, and then in section 4 we give our proofs. In view of the subform theorem stated in Proposition 1.1, we introduce in section 5 the notion of Pfister neighbors for bilinear forms. We prove preliminary results on such forms, similar to those known for Pfister neighbors in the case of quadratic forms. By using Theorem 1.2, we prove that anisotropic bilinear forms of height 1 are those similar to bilinear Pfister forms (Corollary 5.5). We also give a characterization of Pfister neighbors based on a splitting property over their own function fields, by proving that an anisotropic bilinear form B is a Pfister neighbor if and only if there exists $B' \in \mathcal{A}(\widetilde{B})$ such that the anisotropic part of $B'_{F(B)}$ is defined over F (Corollary 5.6). This is similar to a result by Knebusch for Pfister neighbors in characteristic different from 2 [12], and its generalization to characteristic 2 by the author and Hoffmann [7]. In the last section, we give further results on Witt kernels, and we give a detailed description of $W_q(F(\varphi)/F)$ for φ of small dimension.

2 Backgrounds on bilinear and quadratic forms

Any quadratic form φ of dimension ≥ 1 can be written up to isometry

$$\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \cdots, c_s \rangle .$$

The form $\langle c_1, \cdots, c_s \rangle$ is unique up to isometry, and we call it the quasilinear part of φ. The quadratic form φ is called nonsingular if $s = 0$; singular if $s > 0$; and totally singular if $r = 0$.

Let φ be a nonzero quadratic form of dimension $n \geq 1$. If φ is not isometric to $[0, 0] \perp \langle 0, \cdots, 0 \rangle$ and $\langle a, 0, \cdots, 0 \rangle$ for some $a \in F^*$, then the polynomial $\varphi(x_1, \cdots, x_n)$ given by φ is irreducible. In this case, we define the function field of φ, denoted by $F(\varphi)$, as the quotient field of $\dfrac{F[x_1, \cdots, x_n]}{(\varphi(x_1, \cdots, x_n))}$. In other cases, we set $F(\varphi) = F$.

The function field of a bilinear form B is defined as the field $F(\widetilde{B})$.

For a field extension K/F and a quadratic form (or a bilinear form) φ, the form $\varphi \otimes K$ is denoted by φ_K.

Two quadratic forms (or bilinear forms) φ and φ' are called similar if $\varphi \simeq a\varphi'$ for some scalar $a \in F^*$.

A bilinear form B' is called a subform of another one B, denoted by $B' \subset B$, if $B \simeq B' \perp B''$ for some bilinear form B''.

2.1 Witt decomposition of bilinear and quadratic forms

It was proved in [7, Prop. 2.4] that any nonzero quadratic form φ is uniquely decomposed as follows: $\varphi \simeq i \times \mathbb{H} \perp j \times \langle 0 \rangle \perp \varphi_{\mathrm{an}}$, where φ_{an} is an anisotropic form called the anisotropic part of φ, and $\mathbb{H} = [0, 0]$ is the hyperbolic plane (here $n \times \psi$ denotes the orthogonal sum of n copies of a quadratic form ψ). The integer i (*resp.* j) is called the Witt index of φ and denoted by $i_W(\varphi)$ (*resp.* the defect index of φ and denoted by $i_d(\varphi)$). A nonsingular form φ is called hyperbolic if $\dim \varphi = 2i_W(\varphi)$.

Any bilinear form B is decomposed as follows: $B \simeq M \perp B_{\mathrm{an}}$, where M is a metabolic bilinear form and B_{an} is an anisotropic bilinear form. The form B_{an} is unique [9], [10], and we call it the anisotropic part of B.

Two quadratic forms (or bilinear forms) φ and φ' are called Witt-equivalent, denoted by $\varphi \sim \varphi'$, if $\varphi \perp H \simeq \varphi' \perp H'$ where H and H' are hyperbolic quadratic forms (or metabolic bilinear forms). The condition $\varphi \sim \varphi'$ implies that $\varphi_{\mathrm{an}} \simeq \varphi'_{\mathrm{an}}$.

It is clear that a bilinear form B is isotropic if and only if $\widetilde{B}$ is isotropic.

2.2 Dominated forms and the subform theorem

Let φ and φ' be two quadratic forms with underlying vector spaces V and W, respectively. We say that φ is dominated by φ' (or φ' dominates φ), denoted by $\varphi \prec \varphi'$, if there exists an injective F-linear map $t : (\varphi, V) \longrightarrow (\varphi', W)$ such that $\varphi'(t(v)) = \varphi(v)$ for any $v \in V$. We say that φ is weakly dominated by φ' if $a\varphi \prec \varphi'$ for some $a \in F^*$.

The following proposition gives an equivalent definition of the domination relation:

Proposition 2.1 *([7, Lem. 3.1]) For quadratic forms φ and φ', we have an equivalence between:*
(1) $\varphi \prec \varphi'$.
(2) There exist nonsingular forms Q, R, nonnegative integers $s' \leq s \leq t$, and scalars $c_1, \cdots, c_t, d_1, \cdots, d_{s'} \in F$ such that:
(i) $\varphi \simeq R \perp \langle c_1, \cdots, c_s \rangle$.
(ii) $\varphi' \simeq Q \perp R \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \cdots, c_t \rangle$.

The subform theorem asserts the following:

Theorem 2.2 *([7, Th. 4.2]) If φ and φ' are anisotropic quadratic forms such* *that φ' is nonsingular and becomes hyperbolic over $F(\varphi)$, then $\varphi \prec a\varphi'$ for any scalar $a \in D_F(\varphi)D_F(\varphi')$.*

2.3 On Pfister forms, quasi-Pfister forms and their neighbors

For an n-fold bilinear Pfister form B, the nonsingular form $B \otimes [1, b]$ is called an $(n+1)$-fold Pfister form, where $\otimes$ denotes the action of $W(F)$ on $W_q(F)$ [4]. A quasi-Pfister form is a totally singular form associated to a bilinear Pfister form.

We say that a quadratic form φ is a Pfister neighbor (*resp.* a quasi-Pfister neighbor) if there exists a Pfister form (*resp.* a quasi-Pfister form) π such that $2 \dim \varphi > \dim \pi$ and φ is weakly dominated by π. In this case, the form π is unique (up to isometry), and for any field extension K/F we have that φ_K is isotropic if and only if π_K is isotropic. In particular, $\pi_{F(\varphi)}$ and $\varphi_{F(\pi)}$ are isotropic [7], [13, Prop. 3.1].

3 Preliminaries on bilinear and quadratic forms

We say that a totally singular form φ is quasi-hyperbolic if $\dim \varphi \geq 2 \dim \varphi_{\mathrm{an}}$. This definition of quasi-hyperbolicity is different from that fixed in [15], and presents the advantage that it remains invariant under field extensions. However, it should be noted that the results [15, Prop. 1.4, Th. 1.5] that we will use in our proofs, and which are proved using the first definition of quasi-hyperbolicity, remain true with this new one.

Lemma 3.1 *Let B be a bilinear form and φ a quadratic form. We have:*
(1) If B is metabolic, then $\widetilde{B}$ is quasi-hyperbolic.
(2) If B is anisotropic and $B_{F(\varphi)}$ is isotropic, then φ is totally singular.

Proof. (1) Let V be the underlying vector space of B. If B is metabolic, then V contains a subspace W of half dimension such that $B(W, W) = 0$. In particular, $\widetilde{B}(w) = 0$ for any $w \in W$, and then $\dim \widetilde{B}_{\mathrm{an}} \leq \dim V - \dim W = \dim W = \frac{\dim \widetilde{B}}{2}$.

(2) Since $\widetilde{B}_{F(\varphi)}$ is isotropic and $\widetilde{B}$ is anisotropic, the claim follows from [13, Cor. 3.3]. $\square$

Remark 3.2 *In general, the quasi-hyperbolicity of the totally singular form $\widetilde{B}$ does not imply the metabolicity of B. For example, for x a variable over F*

and $K = F(x)$, the bilinear form $B = \langle 1:1:0\rangle \perp \langle 1:0:x\rangle$ is not metabolic over K as we can see by using the uniqueness of the anisotropic part, but $\widetilde{B} \simeq \langle 1,0,1,x\rangle \simeq \langle 0,0,1,x\rangle$ is quasi-hyperbolic over K.

Proposition 3.3 *An isotropic bilinear Pfister form is metabolic.*

Proof. The proposition is obvious for $F = \mathbb{Z}/2\mathbb{Z}$ or B of dimension 2. So suppose $F \neq \mathbb{Z}/2\mathbb{Z}$ and $\dim B > 2$. Set $B = B' \perp \alpha B'$ for some $\alpha \in F^*$ and B' a bilinear Pfister form. If B' is isotropic, then we conclude by induction on $\dim B$. If not, there exists $\beta \in D_F(B') \cap D_F(\alpha B')$. By [4, Cor. 2.16, Page 101] B' is round, i.e. any scalar $x \in D_F(B')$ satisfies $xB' \simeq B'$. Hence, $B' \simeq \beta B' \simeq \alpha B'$ and thus B is metabolic. $\square$

We need some results concerning the isotropy of bilinear forms over inseparable quadratic extensions:

Lemma 3.4 *Let B be an anisotropic bilinear form and $d \in F^* - F^{*2}$. We have:*
(1) B is isotropic over $F(\sqrt{d}) \iff$ there exist $a, b \in F$ with $a \neq 0$ such that $\langle a:b:ad\rangle \subset B$.
(2) B is metabolic over $F(\sqrt{d}) \iff B \simeq \perp_{i=1}^{r} \langle a_i : b_i : a_i d\rangle$ for some $a_i, b_i \in F$ with $a_i \neq 0$ $(1 \leq i \leq r)$.

Proof. Let V be the underlying vector space of B. In both assertions the implication ($\Longleftarrow$) is clear.

(1) $\Longrightarrow$: Suppose that $B_{F(\sqrt{d})}$ is isotropic. Then there exist vectors $v, v' \in V$, not both zero, such that $B(v,v) = dB(v',v')$. Since B is anisotropic, we have $B(v',v') \neq 0$. Moreover, the vectors v and v' are linearly independent. Otherwise, there would exist $\alpha \in F^*$ such that $v = \alpha v'$. Hence $\alpha^2 B(v',v') = dB(v',v')$, and thus d would belong to F^{*2}. The restriction of B to the space $Fv \oplus Fv'$ is the bilinear form $\langle a:b:ad\rangle$ where $a = B(v',v')$ and $b = B(v,v')$.

(2) $\Longrightarrow$: Suppose that $B_{F(\sqrt{d})}$ is metabolic. Since $B_{F(\sqrt{d})}$ is isotropic, there exist by statement (1) scalars a_1, b_1 with $a_1 \neq 0$ such that $B \simeq \langle a_1 : b_1 : a_1 d\rangle \perp B'$ for some bilinear form B'. If $\dim B = 2$, then we are done. If not, we use the uniqueness of the anisotropic part to get that $B'_{F(\sqrt{d})}$ is metabolic. The claim then follows by induction on $\dim B$. $\square$

We get the following corollary:

Corollary 3.5 *If B is a bilinear form and $d \in F^* - F^{*2}$, then there exists a bilinear form B' defined over F such that $(B_{F(\sqrt{d})})_{\rm an} \simeq B'_{F(\sqrt{d})}$.*

Proof. Without loss of generality, we may suppose that B is anisotropic and becomes isotropic over $F(\sqrt{d})$. By Lemma 3.4, $B \simeq \perp_{i=1}^{r} \langle a_i : b_i : a_i d\rangle \perp B'$ for some anisotropic bilinear B' over $F(\sqrt{d})$, and scalars a_i, b_i with $a_i \neq 0$ $(1 \leq i \leq r)$. By the uniqueness of the anisotropic part we get $(B_{F(\sqrt{d})})_{\rm an} \simeq B'_{F(\sqrt{d})}$. $\square$

The following result will be used in the proof of Theorem 1.2:

Proposition 3.6 *Let φ and φ' be totally singular forms such that φ' is anisotropic of dimension ≥ 2 and dominated by φ. Then we have the inclusion $W(F(\varphi)/F) \subset W(F(\varphi')/F)$.*

To prove this proposition we need some results from the specialization theory of bilinear and quadratic forms. First, we prove the following lemma:

Lemma 3.7 *For φ and φ' as in Proposition 3.6, there exists an F-place from $F(\varphi)$ to $F(\varphi')$.*

Proof. Since $\varphi' \prec \varphi$, there exist integers $2 \leq s \leq t$ and scalars $c_1, \cdots, c_t \in F$ such that $\varphi' \simeq \langle c_1, \cdots, c_s\rangle$ and $\varphi \simeq \varphi' \perp \langle c_{s+1}, \cdots, c_t\rangle$. Without loss of generality, we may suppose $c_1 = 1$. Since $\dim \varphi_{\rm an} \geq \dim \varphi' \geq 2$, the homogeneous polynomials given by φ and φ' are irreducible and we have:

$$F(\varphi) = F(x_2, \cdots, x_t)(\sqrt{\alpha})$$

$$F(\varphi') = F(y_2, \cdots, y_s)(\sqrt{\alpha'}),$$

where $\alpha = \sum_{i=2}^{t} c_i x_i^2$ and $\alpha' = \sum_{i=2}^{s} c_i y_i^2$. By [19, Cor. 6.13, page 162], there exists an F-place λ from $F(x_2, \cdots, x_t)$ to $F(y_2, \cdots, y_s)$ given by:

$$\lambda(x_i) = \begin{cases} y_i & \text{if } 2 \leq i \leq s \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lambda(\alpha) = \alpha'$, the F-place λ restricts to an F^2-place from $F^2(x_2^2, \cdots, x_t^2)(\alpha)$ to $F^2(y_2^2, \cdots, y_s^2)(\alpha')$. Moreover, the squaring map (*resp.* its inverse) yields field isomorphisms $F(\varphi) \longrightarrow F^2(x_2^2, \cdots, x_t^2)(\alpha)$ and $F^2(y_2^2, \cdots, y_s^2)(\alpha') \longrightarrow F(\varphi')$. It is now clear that composing these isomorphisms with λ yields the desired F-place from $F(\varphi)$ to $F(\varphi')$. $\square$

We also need a specialization result by Knebusch:

Proposition 3.8 *([11, Th. 3.1]) Let K and L be fields (of any characteristic), and let $\lambda : K \longrightarrow L \cup \{\infty\}$ be a place. Then there exists a unique additive map $\lambda_* : W(K) \longrightarrow W(L)$ defined as follows: $\lambda_*(\langle a \rangle_b) = \langle \lambda(a) \rangle_b$ for every $a \in K$ such that $\lambda(a) \neq 0, \infty$, and $\lambda_*(\langle a \rangle_b) = 0$ for every $a \in K$ such that $\lambda(ac^2) = 0$ or ∞ for every $c \in K^*$.*

Now we are able to prove Proposition 3.6:

Proof. By Lemma 3.7 there exists an F-place $\lambda : F(\varphi) \longrightarrow F(\varphi') \cup \{\infty\}$. This place induces an additive map $\lambda_* : W(F(\varphi)) \longrightarrow W(F(\varphi'))$ defined as in Proposition 3.8. Since $\lambda_*(\langle a \rangle_b) = \langle \lambda(a) \rangle_b = \langle a \rangle_b$ for every $a \in F^*$, the claim follows. $\square$

We give an analogue of Proposition 3.6 for Witt kernels for quadratic forms:

Proposition 3.9 *Let φ and φ' be quadratic forms such that φ is anisotropic and becomes isotropic over $F(\varphi')$. Then $W_q(F(\varphi)/F) \subset W_q(F(\varphi')/F)$.*

Proof. Suppose $W_q(F(\varphi)/F) \neq 0$, and let $\eta \in W_q(F(\varphi)/F)$ be anisotropic. We proceed by induction on $\dim \eta$. By the subform theorem φ is weakly dominated by η, and thus $\eta_{F(\varphi')}$ is isotropic since $\varphi_{F(\varphi')}$ is also isotropic. Hence $F(\varphi')(\eta)/F(\varphi')$ is purely transcendental.

(1) If η is similar to a Pfister form, then $\eta \in W_q(F(\varphi')/F)$ and we are done.

(2) If η is not similar to a Pfister form, then $\eta_1 := (\eta_{F(\eta)})_{\mathrm{an}} \neq 0$ [14]. The form $\varphi_{F(\eta)}$ is anisotropic, otherwise $F(\eta)(\varphi)/F(\eta)$ would be purely transcendental [13, Cor. 3.4], and thus $\eta_{F(\eta)}$ would be hyperbolic since $\eta_{F(\varphi)} \sim 0$. Since $\varphi_{F(\eta)(\varphi')}$ is isotropic and $(\eta_1)_{F(\eta)(\varphi)} \sim \eta_{F(\eta)(\varphi)} \sim 0$, we deduce by induction that $(\eta_1)_{F(\eta)(\varphi')} \sim 0$. Since $F(\eta)(\varphi') = F(\varphi')(\eta)$ and $F(\varphi')(\eta)/F(\varphi')$ is purely transcendental, the form $\eta_{F(\varphi')}$ is then hyperbolic. Hence the claim. $\square$

Remark 3.10 *Note that in this proof, the use of the subform theorem is necessary when φ' is totally singular. However, for φ' not totally singular, the subform theorem can be avoided since in this case the isotropy of $\varphi_{F(\varphi')}$ implies that $F(\varphi')(\varphi)/F(\varphi')$ is purely transcendental [13, Cor. 3.4], and thus $W_q(F(\varphi)/F) \subset W_q(F(\varphi')/F)$.*

4 Proofs

4.1 Proof of Proposition 1.1

Let B be an anisotropic bilinear form and φ an anisotropic quadratic form such that B is metabolic over $F(\varphi)$. Let $\alpha = uv$ with $u \in D_F(\varphi)$ and $v \in D_F(B) = D_F(\widetilde{B})$. Since $B_{F(\varphi)}$ is metabolic, $\widetilde{B}_{F(\varphi)}$ is quasi-hyperbolic and φ is totally singular (Lemma 3.1), hence it follows from [15, Prop. 1.4] that $u\varphi \subset v\widetilde{B}$, hence $D_F(\varphi) \subset \alpha D_F(\widetilde{B}) = \alpha D_F(B)$. Write $\varphi = \langle a_1, \cdots, a_n \rangle$, and let x_i be elements in the underlying vector space V of B such that $B(x_i, x_i) = \alpha^{-1} a_i$. The x_i are F-linearly independent since the a_i are F^2-linearly independent because of the anisotropy of φ. Let W be the subspace of V generated by the x_i, and $B' \simeq \alpha B \mid_W$. By the anisotropy of B, we conclude that there exists a bilinear form B'' with $\alpha B \simeq B' \perp B''$. Clearly, $D_F(B') = D_F(\varphi)$, $\dim B' = \dim \varphi$, implying the result. $\square$

4.2 Proof of Theorem 1.2

We have to show that if φ is an anisotropic quadratic form of dimension ≥ 2 such that $W(F(\varphi)/F) \neq 0$, then φ is totally singular, and any anisotropic B in $W(F(\varphi)/F) \neq 0$ decomposes into an orthogonal sum of forms similar to bilinear Pfister forms in $\mathcal{A}(\varphi_{\mathrm{qp}})$.

The metabolicity of $B_{F(\varphi)}$ implies that φ is totally singular. We may suppose that φ represents 1, and that $\mathrm{ndeg}_F(\varphi) = 2^d$. We will use induction on d. Suppose $d = 1$. Then necessarily $\varphi \simeq \langle 1, a \rangle$ where $N_F(\varphi) = F^2(a)$. In this situation, $F(\varphi) = F(\sqrt{a})(t)$ for some transcendental element t, and thus B becomes already metabolic over $F(\sqrt{a})$. The theorem now follows immediately from Lemma 3.4(2).

So suppose $d \geq 2$. Then we may write $\varphi = \langle 1, a_1, \cdots, a_d, \cdots \rangle$ such that $N_F(\varphi) = F^2(a_1, \cdots, a_d)$. Let $\varphi' = \langle 1, a_1, \cdots, a_{d-1} \rangle$. Then $N_F(\varphi') = F^2(a_1, \cdots, a_{d-1})$, $\mathrm{ndeg}_F(\varphi') = 2^{d-1}$. By Proposition 3.6, $B_{F(\varphi')}$ is metabolic. By induction, $B \simeq \perp_{i=1}^m x_i C_i$ with $(d-1)$-fold bilinear Pfister forms C_i associated to φ'_{qp}. In particular, $D_F(C_i) \cup \{0\} = D_F(\varphi'_{\mathrm{qp}}) \cup \{0\} = N_F(\varphi') = F^2(a_1, \cdots, a_{d-1})$.

After scaling, we may suppose that $x_1 = 1$. If we write $C' = C_1 = \langle 1, c_1 \rangle_b \otimes \cdots \otimes \langle 1, c_{d-1} \rangle_b$, then $D_F(C') \cup \{0\} = F^2(c_1, \cdots, c_{d-1}) = F^2(a_1, \cdots, a_{d-1}) = N_F(\varphi')$, and we also can write $B \simeq C' \perp B'$ for some bilinear form B'.

Since B is metabolic over $F(\varphi)$, we have that $\widetilde{B}$ is quasi-hyperbolic over $F(\varphi)$.

By [15, Prop. 1.4] and since B represents 1, we have $N_F(\varphi) \subset D_F(B) \cup \{0\} = D_F(\widetilde{B}) \cup \{0\}$, and by [15, Th. 1.5], we know that $\dim B = \dim \widetilde{B}$ is a multiple of 2^d.

So $D_F(C') \cup \{0\} = N_F(\varphi') \subset N_F(\varphi) \subset D_F(C' \perp B') \cup \{0\}$, and since $\mathrm{ndeg}_F(\varphi') = 2^{d-1} < 2^d = \mathrm{ndeg}_F(\varphi)$, there exists $u \in D_F(C') \cup \{0\}$ and $c_d \in D_F(B') \cup \{0\}$ such that $N_F(\varphi')(u+c_d) = F^2(c_1, \cdots, c_{d-1}, u+c_d) = N_F(\varphi)$. Necessarily, $c_d \neq 0$ since $u \in N_F(\varphi')$, and we have $N_F(\varphi')(u + c_d) = N_F(\varphi')(c_d) = N_F(\varphi)$.

Now let $C = C' \otimes \langle 1, c_d \rangle_b$. Note that $D_F(C) \cup \{0\} = N_F(\varphi) = N_F(\varphi_{\mathrm{qp}}) = D_F(\varphi_{\mathrm{qp}}) \cup \{0\}$ and $\dim C = \dim \varphi_{\mathrm{qp}} = 2^d$, implying that C is associated to φ_{qp}.

Consider $C \perp B$. Since $C' \perp \langle c_d \rangle_b \subset B$ and $C' \perp \langle c_d \rangle_b \subset C$, we have that $C \perp B \simeq M \perp B''$ with B'' anisotropic and M metabolic of dimension $\geq 2^d + 2 = 2\dim(C' \perp \langle c_d \rangle_b)$. By dimension count, $\dim B'' < \dim B$. Now $C_{F(\varphi)}$ is clearly isotropic and hence metabolic, so that $B''_{F(\varphi)}$ is metabolic as well, and therefore its dimension is a multiple of 2^d [15, Th. 1.5]. In particular, $\dim B'' \leq \dim B - 2^d$ as $\dim B$ is a multiple of 2^d. In $W(F)$, we therefore have $B \sim C \perp C \perp B \sim C \perp B''$ with B anisotropic and $\dim(C \perp B'') \leq \dim B$. Consequently, $B \simeq C \perp B''$, and the theorem follows by an easy induction on the dimension of B. $\square$

4.3 Proof of Theorem 1.4

Let φ be an anisotropic Pfister neighbor or quasi-Pfister neighbor of π.

(1) Since the forms $\varphi_{F(\pi)}$ and $\pi_{F(\varphi)}$ are isotropic, it follows from Proposition 3.9 that $W_q(F(\varphi)/F) = W_q(F(\pi)/F)$. Hence, for the proofs of statements (2)-(3) we suppose that $\varphi \simeq \pi$.

(2) Suppose that φ is a Pfister form. Let $\psi \in W_q(F(\varphi)/F)$ be anisotropic. By the subform theorem $x_1\varphi \prec \psi$ for some $x_1 \in F^*$. If $\dim \psi = \dim \varphi$, we take $\tau = \langle x_1 \rangle_b$ and we are done. If not, we write $\psi \simeq x_1\varphi \perp \psi'$ for some nonsingular form ψ'. Since $\psi_{F(\varphi)}$ is hyperbolic, the form $\psi'_{F(\varphi)}$ is also hyperbolic. Since $\dim \psi' < \dim \psi$, we get by induction on $\dim \psi$ that $\psi' \simeq x_2\varphi \perp \cdots \perp x_r\varphi$ for some $x_2, \cdots, x_r \in F^*$. Hence, $\psi \simeq \tau \otimes \varphi$ where $\tau = \langle x_1, \cdots, x_r \rangle_b$.

(3) Suppose that φ is a quasi-Pfister form. Let $\psi \in W_q(F(\varphi)/F)$ be anisotropic, and let $B = \langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ be such that $\varphi \simeq \widetilde{B}$. By [8, Rem. 5.2(i)], $\psi \sim B \otimes \rho \in W_q(F)$ for some nonsingular form ρ which we may suppose of minimal dimension among all γ with $\psi \sim B \otimes \gamma$. In particular, ρ is anisotropic. Let $x_i \in F^*$ and $b_i \in F$ such that $\rho \simeq \perp_{i=1}^r x_i[1, b_i]$. Suppose that $B \otimes \rho$ is

isotropic. Then, after reindexing, there exist $2 \leq s \leq r$, $y_i \in D_F(x_i B \otimes [1, b_i])$, $1 \leq i \leq s$, such that $y_1 + \cdots + y_s = 0$. By roundness of the Pfister forms $B \otimes [1, b_i]$, $x_i B \otimes [1, b_i] \simeq y_i B \otimes [1, b_i]$ for $i \leq s$. Let $y_i := x_i$ for $i > s$ and put $\rho' = \perp_{i=1}^r y_i [1, b_i]$. By the above, $\psi \sim B \otimes \rho \simeq B \otimes \rho'$, but ρ' is obviously isotropic, a contradiction. Hence $B \otimes \rho$ is anisotropic and thus $\psi \simeq B \otimes \rho$. $\square$

4.4 Proof of Theorem 1.5

Let φ be an anisotropic quadratic form of dimension ≥ 3, and let φ' be a quadratic form dominated by φ such that $\dim \varphi = \dim \varphi' + 1$ and $W_q(F(\varphi')/F)$ is a strong n-Pfister group for some $n \geq 1$. We have to prove that $W_q(F(\varphi)/F)$ is an $\{n, n+1\}$-Pfister group.

Let $\psi \in W_q(F(\varphi)/F)$. We use induction on $\dim \psi$, the initial step $\dim \psi \leq 2$ being trivial. So we may assume ψ anisotropic of dimension > 2. Since $\psi \in W_q(F(\varphi')/F)$ (Proposition 3.9) and by assumption, there exist forms $\rho_i \in GP_n F \cap W_q(F(\varphi')/F)$ such that $\psi \simeq \perp_{i=1}^r \rho_i$. After scaling, we may assume that φ' and ρ_1 represent 1, so that $\varphi' \prec \rho_1$. Put $\gamma = \perp_{i=2}^r \rho_i$.

Now ρ_1 is hyperbolic over $F(\varphi)$ if and only if φ is weakly dominated by ρ_1 if and only if $\rho_1 \in GP_n F \cap W_q(F(\varphi)/F)$, in which case $\gamma_{F(\varphi)}$ is hyperbolic, and we are done by induction. So we may assume that ρ_1 does not dominate a form similar to φ.

Since ρ_1 dominates φ' but not φ, it follows that $i_t(\varphi \perp \rho_1) = \dim \varphi'$ (*cf.* [7, Cor. 3.13]; here i_t denotes the total index $i_W + i_d$). If σ denotes the anisotropic part of $\varphi \perp \rho_1$, then $\varphi \perp \rho_1 \simeq u \times \langle 0 \rangle \perp v \times \mathbb{H} \perp \sigma$ with $u + v = \dim \varphi'$.

On the other hand, since ψ is hyperbolic over $F(\varphi)$, and since both ψ and φ represent 1, we have that ψ dominates φ, hence $i_t(\varphi \perp \psi) = \dim \varphi = \dim \varphi' + 1$. Therefore,

$$\varphi \perp \psi \simeq u \times \langle 0 \rangle \perp v \times \mathbb{H} \perp \sigma \perp \gamma,$$

and comparing the total index on both sides shows that $\sigma \perp \gamma$ is isotropic, so that there exists $x \in D_F(\sigma) \cap D_F(\gamma)$ since σ and γ are anisotropic.

Now consider $\pi = \rho_1 \perp x\rho_1$ which is anisotropic because it dominates $\rho_1 \perp \langle x \rangle$ which is a Pfister neighbor of π and in turn dominated by the anisotropic form ψ. Then $\varphi \perp \pi \simeq u \times \langle 0 \rangle \perp v \times \mathbb{H} \perp \sigma \perp x\rho_1$ with $\sigma \perp x\rho_1$ isotropic. Hence, $i_t(\varphi \perp \pi) \geq \dim \varphi' + 1 = \dim \varphi$, which by [7, Cor. 3.13] implies equality and that π dominates φ. In particular, $\pi \in P_{n+1} F \cap W_q(F(\varphi)/F)$.

Now let ψ' be the anisotropic part of $\psi \perp \pi$. We have that both ψ and π dominate $\rho_1 \perp \langle x \rangle$, so $i_t(\psi \perp \pi) \geq \dim(\rho_1 \perp \langle x \rangle) = 2^n + 1$, and since

$i_t(\psi \perp \pi) = i_W(\psi \perp \pi)$ by nonsingularity, we have that $\dim \psi' \leq \dim \psi + \dim \pi - 2(2^n + 1) < \dim \psi$. But clearly, $\psi' \in W_q(F(\varphi)/F)$, and the proof readily concludes by induction on the dimension. $\square$

4.5 Proof of Theorem 1.6

Let φ be an anisotropic quadratic form of dimension ≥ 3, and let φ' be a Pfister neighbor of an n-fold Pfister form such that $\varphi' \prec \varphi$ and $\dim \varphi = \dim \varphi' + 1$.

If φ is a Pfister neighbor of an n-fold Pfister form, then $W_q(F(\varphi)/F)$ is a strong n-Pfister group (Theorem 1.4(2)). So suppose that φ is not a Pfister neighbor of an n-fold Pfister form. Our aim is to prove that $W_q(F(\varphi)/F)$ is a strong $(n+1)$-Pfister group. Let ρ be the Pfister form associated to φ' and let $\psi \in W_q(F(\varphi)/F)$ be anisotropic. Then ψ becomes hyperbolic over $F(\varphi')$ and thus over $F(\rho)$. After scaling, we may assume that ψ and φ' represent 1, so that by Theorem 1.4(2), $\psi \simeq \rho \perp \tau \otimes \rho$ for some bilinear form τ. We now repeat the same proof as for Theorem 1.5, with ρ_1 replaced by ρ and γ by $\tau \otimes \rho$, by noting that ρ does not dominate a form similar to φ because by assumption φ is not a Pfister neighbor of some n-fold Pfister form. We also let σ be the anisotropic part of $\varphi \perp \rho$.

As before, it follows that there exists $x \in D_F(\sigma) \cap D_F(\tau \otimes \rho)$. By a standard argument, the roundness of ρ readily implies that $\tau \otimes \rho \simeq x\rho \perp \tau' \otimes \rho$ for some bilinear form τ'.

Now consider $\pi = \rho \perp x\rho \in P_{n+1}F$. As above, we have that φ is dominated by π. Hence, $\pi \in P_{n+1}F \cap W_q(F(\varphi)/F)$. Therefore also $\tau' \otimes \rho \in W_q(F(\varphi)/F)$, and the proof can now readily be completed by induction on $\dim \psi$ as $\psi \simeq \pi \perp \tau' \otimes \rho$. $\square$

5 Pfister neighbors for bilinear forms

In view of the subform theorem given in Proposition 1.1, we suggest the following definition for Pfister neighbors in the case of bilinear forms:

Definition 5.1 *We say that a bilinear form B' is a Pfister neighbor of a bilinear Pfister form B if $2\dim B' > \dim B$ and there exists $B'' \in \mathcal{A}(\widetilde{B'})$ similar to a subform of B.*

As for quadratic forms we prove the following:

Proposition 5.2 *Let B' be a Pfister neighbor of a bilinear Pfister form B.*

Then:
(1) For any field extension K/F, the forms B_K and B'_K are simultaneously isotropic or anisotropic. In particular, $B_{F(B')}$ and $B'_{F(B)}$ are isotropic.
(2) If B' is anisotropic and a Pfister neighbor of another bilinear Pfister form ρ, then $\widetilde{B} \simeq \widetilde{\rho}$.

Proof. (1) Let $B'' \in \mathcal{A}(\widetilde{B'})$ similar to a subform of B. The forms B'_K and B''_K are simultaneously isotropic or anisotropic since $\widetilde{B''} \simeq \widetilde{B'}$. If B_K is isotropic, then it is metabolic and B''_K becomes isotropic as $\dim B'' = \dim B' > \frac{\dim B}{2}$. Obviously, the isotropy of B''_K implies that of B_K.

(2) By statement (1) $B'_{F(\rho)}$ is isotropic and thus $B_{F(\rho)}$ is also isotropic. Since B is anisotropic, $\dim B = \dim \rho$ and $1 \in D_F(B) \cap D_F(\rho)$, it follows from Proposition 1.1 that $B \in \mathcal{A}(\widetilde{\rho})$, i.e. $\widetilde{B} \simeq \widetilde{\rho}$. $\square$

We give an equivalent definition to 5.1:

Proposition 5.3 *Let B and B' be anisotropic bilinear forms such that B is a bilinear Pfister form. Then B' is a Pfister neighbor of B if and only if B becomes isotropic over $F(B')$ and $2\dim B' > \dim B$.*

Proof. If B is isotropic over $F(B')$, then it is metabolic over $F(B')$, and by Proposition 1.1 there exists a bilinear form $B'' \in \mathcal{A}(\widetilde{B'})$ similar to a subform of B. If moreover $2\dim B' > \dim B$, then B' is a Pfister neighbor of B. The converse follows from Proposition 5.2(1). $\square$

Theorem 1.2 allows us important corollaries:

Corollary 5.4 *Let B be an anisotropic bilinear form and φ an anisotropic totally singular form. Suppose that $B_{F(\varphi)}$ is metabolic and $2\dim\varphi > \dim B$. Then B is similar to a bilinear Pfister form ρ, and any bilinear form $B' \in \mathcal{A}(\varphi)$ is a Pfister neighbor of ρ.*

Proof. By Theorem 1.2, $B \simeq \perp_{i=1}^{r} \alpha_i B_i$ for some scalars $\alpha_i \in F^*$ and d-fold bilinear Pfister forms B_i associated to φ_{qp}. Since $\dim\varphi_{\mathrm{qp}} \geq \dim\varphi$ and $2\dim\varphi > \dim B = r\dim\varphi_{\mathrm{qp}}$, it follows that $r = 1$ and thus B is similar to the bilinear Pfister form $\rho = B_1$. Moreover, it follows from Proposition 5.3 that any $B' \in \mathcal{A}(\varphi)$ is a Pfister neighbor of ρ since $2\dim B' = 2\dim\varphi > \dim B = \dim\rho$ and ρ is isotropic over $F(\varphi) = F(B')$. $\square$

This corollary allows us to classify anisotropic bilinear forms of height 1. This makes up the first step for a standard splitting theory of bilinear forms:

Corollary 5.5 *An anisotropic bilinear form B becomes metabolic over $F(B)$ if and only if it is similar to a bilinear Pfister form.*

Proof. We use Proposition 3.3, and Corollary 5.4 applied to the case $\varphi = \widetilde{B}$. $\square$

Corollary 5.6 *Let B be an anisotropic bilinear form. Then the following are equivalent:*
(1) B is a Pfister neighbor of some bilinear Pfister form.
(2) There exist anisotropic bilinear forms B', B'' such that $\dim B'' < \dim B'$, $B' \in \mathcal{A}(\widetilde{B})$ and $B'_{F(B)} \sim B''_{F(B)}$.
(3) There exists a bilinear form $B' \in \mathcal{A}(\widetilde{B})$ such that the anisotropic part of $B'_{F(B)}$ is defined over F.

Proof. (2) $\Longrightarrow$ (1) The assumption $B'_{F(B)} \sim B''_{F(B)}$ implies $(B' \perp B'')_{F(B)} \sim 0$. Since B' is anisotropic and $\dim B'' < \dim B'$, the bilinear form $B' \perp B''$ can not be metabolic. Moreover, $2 \dim \widetilde{B} = 2 \dim B' > \dim(B' \perp B'') \geq \dim(B' \perp B'')_{\mathrm{an}}$. We apply Corollary 5.4 with $\varphi = \widetilde{B}$ to get that $(B' \perp B'')_{\mathrm{an}}$ is similar to a bilinear Pfister form ρ, and B is a Pfister neighbor of ρ since $B \in \mathcal{A}(\widetilde{B})$.

(3) $\Longrightarrow$ (2) If B'' is a bilinear form over F satisfying $\left(B'_{F(B)}\right)_{\mathrm{an}} \simeq B''_{F(B)}$, then clearly $B'_{F(B)} \sim B''_{F(B)}$, and $\dim B'' < \dim B'$ since $B'_{F(B)}$ is isotropic.

(1) $\Longrightarrow$ (3) Let B' and B'' be bilinear forms such that $B' \in \mathcal{A}(\widetilde{B})$, $B' \perp B''$ is similar to a d-fold bilinear Pfister form and $\dim B > 2^{d-1}$. Since $B'_{F(B)}$ is isotropic, we get $B'_{F(B)} \sim B''_{F(B)}$. Moreover, $\dim B'' = 2^d - \dim B' = 2^d - \dim B < 2^{d-1} < \dim B$. By [8, Th. 1.1], $B''_{F(B)}$ is anisotropic and thus $\left(B'_{F(B)}\right)_{\mathrm{an}} \simeq B''_{F(B)}$. $\square$

6 Other results on Witt kernels

We begin this section by a criterion about the metabolicity over inseparable biquadratic extensions. The case of quadratic extensions is covered by Theorem 1.2.

Proposition 6.1 *Let B be an anisotropic bilinear form of dimension ≥ 2 and $a_1, a_2 \in F^*$. Then the following are equivalent:*

(1) B is metabolic over $F(\sqrt{a_1}, \sqrt{a_2})$.
(2) $B \sim (\perp_{i=1}^{r} \alpha_i\lambda_i) \perp (\perp_{i=1}^{s} \beta_i\mu_i)$ for some scalars $\alpha_i, \beta_j \in F^*$, 1-fold bilinear Pfister forms $\lambda_i \in \mathcal{A}(\langle 1, a_1\rangle)$, and $\mu_j = \langle 1, x_j + a_2\rangle_b$ with $x_j \in F^2(a_1)$.

Proof. (1) $\Longrightarrow$ (2) By Theorem 1.2, it suffices to prove the existence of scalars $\beta_j \in F^*$, and bilinear forms $\mu_j = \langle 1, x_j + a_2\rangle_b$ with $x_j \in F^2(a_1)$ such that $\Big(B \perp (\perp_{j=1}^{s} \beta_j\mu_j)\Big)_{F(\sqrt{a_1})} \sim 0$. We proceed by induction on $\dim B$ and we may suppose that $B_{F(\sqrt{a_1})}$ is not metabolic.

By Corollary 3.5 there exists an anisotropic bilinear form B' over F such that $(B_{F(\sqrt{a_1})})_{\mathrm{an}} \simeq B'_{F(\sqrt{a_1})}$. Let $\beta_1 \in D_F(B')$. Since $B'_{F(\sqrt{a_1})}$ is metabolic over $F(\sqrt{a_1}, \sqrt{a_2})$, it follows from Proposition 1.1 that $B'_{F(\sqrt{a_1})} \simeq \beta_1 B'_1 \perp B'_2$ for some $B'_1 \in \mathcal{A}(\langle 1, a_2\rangle_{F(\sqrt{a_1})})$ and a bilinear form B'_2 over $F(\sqrt{a_1})$. Let u, v be vectors in the underlying vector space V of B'_1 such that $B'_1(u,u) = 1$ and $B'_1(v,v) = a_2$. Since $\langle 1, a_2\rangle$ is anisotropic, the vectors u and v are F-linearly independent. Let $\alpha = B'_1(u,v)$. By considering the basis $\{u, \alpha u + v\}$ of V, we get the isometry $B'_1 \simeq \langle 1, x_1 + a_2\rangle_b$, where $x_1 = \alpha^2 \in F^2(a_1)$. In particular, $(B' \perp \beta_1 \langle 1, x_1 + a_2\rangle_b)_{F(\sqrt{a_1})} \sim B'_2$.

If $\dim B = 2$, then $B'_2 = 0$ since we may take $B = B'$. So suppose $\dim B > 2$ and $B'_2 \not\sim 0$. Since $\Big((B' \perp \beta_1 \langle 1, x_1 + a_2\rangle_b)_{F(\sqrt{a_1})}\Big)_{\mathrm{an}} \simeq B'_2$, there exists by Corollary 3.5 an anisotropic bilinear form B'' over F such that $B'_2 \simeq B''_{F(\sqrt{a_1})}$. The form $B''_{F(\sqrt{a_1})}$ is metabolic over $F(\sqrt{a_1}, \sqrt{a_2})$ since B' is also metabolic over $F(\sqrt{a_1}, \sqrt{a_2})$ and $x_1 + a_2$ is a square in $F(\sqrt{a_1}, \sqrt{a_2})$. Since $\dim B'' < \dim B$, we conclude by induction.

(2) $\Longrightarrow$ (1) The forms λ_i are metabolic over $F(\sqrt{a_1})$, and for any $j \in \{1, \cdots, s\}$ the form μ_j is also metabolic over $F(\sqrt{a_1}, \sqrt{a_2})$ since $x_j + a_2$ is a square in this field. $\square$

Theorem 1.6 generalizes [6, Prop. 1.4], and together with Theorem 1.5 it implies the following corollary which is a partial generalization of [6, Th. 2.1]:

Corollary 6.2 *Let φ' be a codimension 1 Pfister neighbor of an n-fold Pfister form, and let φ be a quadratic form which is not a Pfister neighbor and dominates φ' such that $\dim\varphi = \dim\varphi' + 2$. Then $W_q(F(\varphi)/F)$ is an $(n+2)$-Pfister group.*

Proof. We reproduce the same proof like that for [6, Th. 2.1]. We have $W_q(F(\varphi)/F) \cap P_{n+1}F = 0$ since $\dim\varphi = 2^n + 1$ and φ is not a Pfister neighbor. Let φ'' be a quadratic form such that $\varphi' \prec \varphi'' \prec \varphi$ and $\dim\varphi'' = \dim\varphi' + 1$. The form φ'' is not a Pfister neighbor, otherwise

$\varphi'' \in GP_nF$ since $\dim \varphi'' = \dim \varphi' + 1 = 2^n$, and thus φ would be a Pfister neighbor. By Theorem 1.6, $W_q(F(\varphi'')/F)$ is a strong $(n+1)$-Pfister group, and by Theorem 1.5 $W_q(F(\varphi)/F)$ is an $(n+2)$-Pfister group since $W_q(F(\varphi)/F) \cap P_{n+1}F = 0$. $\square$

We combine [1, Cor. 2.8], Theorems 1.4, 1.5, 1.6 and Corollary 6.2 to describe $W_q(F(\varphi)/F)$ for φ of small dimension:

Corollary 6.3 *Let φ be an anisotropic quadratic form of dimension at most 5, and let s denote the dimension of its quasilinear part. Then we have the following description:*

Dimension of φ	**Condition on φ**	**The kernel $W_q(F(\varphi)/F)$**
2	$s=0$	*strong 1-Pfister group*
	$s=2$	*strong 2-Pfister group*
3	$s=1$	*strong 2-Pfister group*
	$s=3$	*strong 3-Pfister group*
4	$s=0$ *and φ is not similar to a 2-fold Pfister form*	*strong 3-Pfister group*
	$s=2$	*strong 3-Pfister group*
	$s=4$ *and φ is similar to a 2-fold quasi-Pfister form*	*strong 3-Pfister group*
	$s=0$ *and φ is similar to a 2-fold Pfister form*	*strong 2-Pfister group*
	$s=4$ *and φ is not similar to a 2-fold quasi-Pfister form*	$\{3,4\}$*-Pfister group*
5	$s=1$ *or* 3 *and φ is not a Pfister neighbor*	4*-Pfister group*
	$s=1$ *or* 3 *and φ is a Pfister neighbor*	*strong 3-Pfister group*
	$s=5$ *and φ is quasi-Pfister neighbor*	*strong 4-Pfister group*
	$s=5$ *and φ is not a quasi-Pfister neighbor*	*?*

Proof. Set $\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \cdots, c_s\rangle$. By [1, Cor. 2.8], we may exclude the case $\dim \varphi = s = 2$, and by Theorem 1.4 we may exclude the case where φ is a Pfister neighbor or a quasi-Pfister neighbor. It remains to treat the following cases:

(1) $\dim \varphi = 4$ and $r > 0$. We apply Theorem 1.6 to $\varphi' = [a_1, b_1] \perp \langle a_2\rangle$ or $[a_1, b_1] \perp \langle c_1\rangle$ according as $s = 0$ or $s = 2$.

(2) $\dim\varphi = 4$ and $r = 0$. We apply Theorem 1.5 to $\varphi' = \langle c_1, c_2, c_3\rangle$ since we know that $W_q(F(\varphi')/F)$ is a strong 3-Pfister group.

(3) $\dim\varphi = 5$ and $r = 1$ or 2. We apply Corollary 6.2 to $\varphi' = [a_1, b_1] \perp \langle c_1\rangle$. $\square$

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