On the Chow groups of Quadratic Grassmannians

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1 Introduction

The current article is devoted to the computation of certain invariants of smooth projective quadrics. Among the invariants of quadrics one can distinguish those which could be called *discrete*. These are invariants whose values are (roughly speaking) collections of integers. For a quadric of given dimension such an invariant takes only finitely many values. The first example is the usual dimension of anisotropic part of q. More sophisticated example is given by the *splitting pattern* of Q, or the collection of *higher Witt indices* - see [4] and [6]. The question of describing the set of possible values of this invariant is still open. Some progress in this direction was achieved by considering the interplay of the splitting pattern invariant with another discrete invariant, called, *motivic decomposition type* - see [9]. The latter invariant measures in what pieces the Chow-motive of a quadric Q could be decomposed. The splitting pattern invariant can be interpreted in terms of the existence of certain cycles on various flag varieties associated to Q, and the motivic decomposition type can be interpreted in terms of the existence of certain cycles on $Q \times Q$. So, both these invariants are faces of the following invariant $GDI(Q)$, which we will call *(quite) generic discrete invariant.* Let Q be a quadric of dimension d, and, for any $1 \leqslant m \leqslant [d/2]+1$, let $G(m,Q)$ be the grassmanian of projective subspaces of dimension $(m-1)$ on Q. Then $GDI(Q)$ is the collection of the subalgebras

$$C^*(G(m,Q)) := image(\mathrm{CH}^*(G(m,Q))/2 \to \mathrm{CH}^*(G(m,Q)|_{\overline{k}})/2).$$

It should be noticed that this invariant has a "noncompact form", where one uses powers of quadrics $Q^{\times r}$ instead of $G(m,Q)$. The equivalence of both forms follows from the fact that the Chow-motive of $Q^{\times r}$ can be decomposed into the direct sum of the Tate-shifts of the Chow-motives of $G(m,Q)$. The varieties $G(m,Q)|_{\overline{k}}$ have natural cellular structure, so Chow-ring for them is a finite-dimensional $\mathbb{Z}$-algebra with the fixed basis parametrized by the Young diagrams of some kind. This way, $GDI(Q)$ appears as a rather combinatorial object.

The idea is to try to describe the possible values of $GDI(Q)$, rather than that of the certain faces of it. In the present article we will address the computation of $GDI(m,Q)$ for the biggest possible $m=[d/2]+1$. This case corresponds to the Grassmannian of middle-dimensional planes on Q. It should be noticed, that it is sufficient to consider the case of odd-dimensional quadrics. This follows from the fact that for the quadric P of even dimension $2n$ and arbitrary codimension 1 subquadric Q in it, $G(n+1,P)=G(n,Q)\times_{\mathrm{Spec}(k)}\mathrm{Spec}(k\sqrt{det_{\pm}(P)})$.

Below we will show that, for $m=[d/2]+1$, the $GDI(m,Q)$ can be described in a rather simple terms - see Main Theorem 5.8 and Definition 5.11. The restriction on the possible values here is given by the Steenrod operations - see Proposition 5.12. And at the moment there is no other restrictions known - see Question 5.13 (the author would expect that there is none). Finally, in the last section we show that in the case of a *generic* quadric, the Grassmannian of middle-dimensional planes is 2-incompressible, which gives a new proof of the conjecture of G.Berhuy and Z.Reihstein (see [1, Conjecture 12.4]). Also, we formulate a conjecture describing the canonical dimension of arbitrary quadric - see Conjecture 6.6.

Most of these results were announced at the conference on "Quadratic forms" in Oberwolfach in May 2002. This text was written while I was a member at the Institute for Advanced Study at Princeton, and I would like to express my gratitude to this institution for the support, excellent working conditions and very stimulating atmosphere. The support of the the Weyl Fund is deeply appreciated. Finally, I'm very gratefull to G.Berhuy for the numerous discussions concerning canonical dimension, which made it possible for the final section of this article to appear.

2 The Chow ring of the last grassmannian

Let k be a field of characteristic different from 2, and q be a nondegenerate quadratic form on a $(2n+1)$-dimensional k-vector space W_q. Denote as $G(n, Q)$ the grassmannian of n-dimensional totally isotropic subspaces in W_q. If q is completely split, then the corresponding grassmanian will be denoted as $G(n)$, and the underlying space of the form q will be denoted as W_n. For small n, examples are: $G(1) \cong \mathbb{P}^1$, $G(2) \cong \mathbb{P}^3$, and $G(3) \cong Q_6$ - the 6-dimensional hyperbolic quadric.

The Chow ring $\mathrm{CH}^*(G(n))$ has $\mathbf{Z}$-basis, consisting of the elements of the type z_I, where I runs over all subsets of $\{1, \ldots, n\}$ (including the empty one). In particular, $\mathrm{rank}(\mathrm{CH}^*(G(n))) = 2^n$. The degree (codimension) of z_I is $|I| = \sum_{i \in I} i$, and this cycle can be defined as the collection of such n-dimensional totally isotropic subspaces $A \subset W_q$, that

$$\dim(A \cap \pi_{n+1-j}) \geqslant \#(i \in I, i \geqslant j), \quad \text{for all } 1 \leqslant j \leqslant n,$$

where $\pi_1 \subset \ldots \subset \pi_n$ is the fixed flag of totally isotropic subspaces in W_q. The element $z_\emptyset$ is the ring unit $1 = [G(n)]$.

Other parts of the landscape are: the tautological n-dimensional bundle V_n on $G(n)$, and the embedding $G(n-1) \stackrel{j_{n-1}}{\to} G(n)$ given by the choice of a rational point $x \in Q$.

Fixing such a point x, let $M_n \subset G(n) \times G(n)$ be the closed subvariety of pairs (A,B), satisfying the conditions:

$$x \in B, \quad \text{and} \quad \operatorname{codim}(A \cap B \subset A) \leqslant 1.$$

The projection on the first factor $(A,B) \mapsto A$ defines a birational map $g_n : M_n \to G(n)$. In particular, by the projection formula, the map $g_n^* : \mathrm{CH}^*(G(n)) \to \mathrm{CH}^*(M_n)$ is injective. On the other hand, the rule $(A,B) \mapsto (B/x)$ defines the map $\pi : M_n \to G(n-1)$. Tautological bundle V_n is naturally a subbundle in the trivial $2n+1$-dimensional bundle $pr^*(W_n)$, which we will denote still by W_n. The variety M_n can be also described as the variety of pairs $B \subset C \subset W_n$, where B is totally isotropic, $dim(B) = n$, $dim(C) = n+1$, and $x \in B$. In other words, $M_n \cong \mathbb{P}_{G(n-1)}(j_{n-1}^*(W_n/V_n))$, where the identification is given by the rule:

$$(A,B) \mapsto (A+B)/B \quad \text{(respectively,} \quad (B,B) \mapsto B^{\perp}/B \ \text{)}.$$

Clearly, $j_{n-1}^*(W_n/V_n) = (W_{n-1}/V_{n-1}) \oplus \mathcal{O}$, and $W_{n-1}/V_{n-1} \cong (V_{n-1}^{\perp})^{\vee}$. V_{n-1} is a subbundle of $V_{n-1}^{\perp}$, and $V_{n-1}^{\perp}/V_{n-1} = \Lambda^{2n-1}W_{n-1} \cong \mathcal{O}$. Thus, $M_n \cong \mathbb{P}_{G(n-1)}(Y_{n-1})$, where $[Y_{n-1}] = [V_{n-1}^{\vee}] + 2[\mathcal{O}] \in \mathrm{K}_0(G(n-1))$. We get a diagram

$$G(n) \overset{g_n}{\leftarrow} \mathbb{P}_{G(n-1)}(Y_{n-1}) \overset{\pi_{n-1}}{\rightarrow} G(n-1).$$

Using the exact sequences $0 \to A \to (A+B) \to (A+B)/A \to 0$ and $0 \to B \to (A+B) \to (A+B)/B \to 0$, and the fact that q defines a nondegenerate pairing between the spaces $(A+B)/B \cong A/(A \cap B)$ and $(A+B)/A \cong B/(A \cap B)$ for all pairs (A,B) aside from the codimension > 1 subvariety $(\Delta(G(n)) \cap M_n) \subset M_n$, we get the exact sequences:

$$0 \to g_n^*(V_n) \to X_{n-1} \to \mathcal{O}(1) \to 0, \quad \text{and}$$

$$0 \to \pi_{n-1}^*(V_{n-1}) \oplus \mathcal{O} \to X_{n-1} \to \mathcal{O}(-1) \to 0,$$

where X_{n-1} is the bundle with the fiber C. In particular,

$$[g_n^*(V_n)] = [\pi_{n-1}^*(V_{n-1})] + [\mathcal{O}] + [\mathcal{O}(-1)] - [\mathcal{O}(1)]. \quad \text{Also,}$$

$$\mathrm{CH}^*(\mathbb{P}_{G(n-1)}(Y_{n-1})) = \mathrm{CH}^*(G(n-1))[\rho]/(\rho^2 \cdot c(V_{n-1}^{\vee})(\rho)),$$

where $\rho = c_1(\mathcal{O}(1))$.

Consider the open subvariety $\tilde{M}_n := g_n^{-1}(G(n) \backslash j_{n-1}(G(n-1))) \subset M_n$ The map $\tilde{g}_n : \tilde{M}_n \to G(n) \backslash j_{n-1}(G(n-1))$ is an isomorphism, and $\tilde{\pi}_{n-1} : \tilde{M}_n \to G(n-1)$ is an n-dimensional affine bundle over $G(n-1)$.

Proposition 2.1 *There is split exact sequence*

$$0 \to \mathrm{CH}^{*-n}(G(n-1)) \stackrel{j_{n-1*}}{\to} \mathrm{CH}^*(G(n)) \stackrel{j_{n-1}{}^*}{\to} \mathrm{CH}^*(G(n-1)) \to 0.$$

Proof: Consider commutative diagram:

$$\begin{tikzcd}
G(n) & M_n \arrow[l, "g_n"'] \arrow[r, "\pi_{n-1}"] & G(n-1) \\
G(n)\backslash j_{n-1}(G(n-1)) \arrow[u, "\varphi"] & \tilde{M}_n \arrow[l, "\tilde{g}_n"] \arrow[r, "\tilde{\pi}_{n-1}"'] \arrow[u, "\psi"] & G(n-1). \arrow[u, equal]
\end{tikzcd}$$

Notice that the choice of a point $y \in Q\backslash T_{x,Q}$ gives a section $s : G(n-1) \to \tilde{M}_n$ of the affine bundle $\tilde{\pi}_{n-1} : \tilde{M}_n \to G(n-1)$. And the composition $\varphi \circ \tilde{g}_n \circ s$ is equal to j'_{n-1}, where j'_{n-1} is constructed from the point $y \in Q$ in the same way as j_{n-1} was constructed from the point x. Thus, the isomorphism $\mathrm{CH}^*(G(n)\backslash j_{n-1}(G(n-1))) \xrightarrow{(\tilde{\pi}^*_{n-1})^{-1}\tilde{g}^*_n} \mathrm{CH}^*(G(n-1))$ together with the localization at $G(n-1) \stackrel{j_{n-1}}{\to} G(n) \stackrel{\varphi}{\leftarrow} (G(n)\backslash G(n-1))$ gives us exact sequence

$$\mathrm{CH}^{*-n}(G(n-1)) \stackrel{j_{n-1*}}{\to} \mathrm{CH}^*(G(n)) \stackrel{j'_{n-1}{}^*}{\to} \mathrm{CH}^*(G(n-1)) \to 0.$$

Thus, $ker(j'_{n-1}{}^*) = im(j_{n-1*})$. Since it is true for arbitrary pair of points $x, y \in Q$ satisfying the condition that the line passing through them does not belong to a quadric, we get: $ker(j_{n-1}{}^*) = im(j_{n-1*})$. On the other hand, the map $j'_{n-1*} : \mathrm{CH}^{*-n}(G(n-1)) \to \mathrm{CH}^*(G(n))$ is split injective, since $(\tilde{\pi}_{n-1})_* \circ \tilde{g}^*_n \circ \varphi^* \circ j'_{n-1*} = id$. Then the same is true for j_{n-1*}. And we get the desired split exact sequence. $\square$

Lemma 2.2 *The ring* $\mathrm{CH}^*(G(n))$ *is generated by the elements of degree* $\leqslant n$.

Proof: It easily follows by induction with the help of Proposition 2.1, and projection formula. $\square$

Proposition 2.3 *Let q be $2n+1$-dimensional split quadratic form. Then*

(1) *The group $O(q)$ acts trivially on* $\mathrm{CH}^*(G(n))$.

(2) *The maps j^*_{n-1} and j_{n-1*} do not depend on the choice of a point $x \in Q$.*

Proof: Use induction on n. For $n = 1$ the statement is trivial. Suppose it is true for $(n-1)$. Let $j_{n-1,x} : G(n-1)_x \to G(n)$ be the map corresponding to the point $x \in Q$. For any $\varphi \in O(q)$ such that $\varphi(x) = y$, we have the map $\varphi_{x,y} : G(n-1)_x \to G(n-1)_y$ such that $j_{n-1,y} \circ \varphi_{x,y} = \varphi \circ j_{n-1,x}$. By the inductive assumption, the maps $\varphi^*_{x,y}$ and $(\varphi_{x,y})_* = ((\varphi_{x,y})^*)^{-1}$ define canonical identification of $CH^*(G(n-1)_x)$ and $CH^*(G(n-1)_y)$ which does not depend on the choice of φ. And under this identification,

$$j^*_{n-1,x} \circ \varphi^* = j^*_{n-1,y} \text{ and } \varphi_* \circ (j_{n-1,x})_* = (j_{n-1,y})_*.$$

Let $\varphi \in O(q)$ be arbitrary element, and $x, y \in Q$ be a such (rational) points that $\varphi(x) = y$. Let z be arbitrary point on Q such that neither of lines $l(x, z)$, $l(y, z)$ lives on Q. Consider reflections $\tau_{x,z}$ and $\tau_{y,z}$. They are rationally connected in $O(q)$. Consequently, for $\psi := \tau_{y,z} \circ \tau_{x,y}$, $\psi^* = id = \psi_*$. Thus, $(j_{n-1,x})^* = (j_{n-1,y})^*$ and $(j_{n-1,x})_* = (j_{n-1,y})_*$.

From Proposition 2.1 we get the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^{*-n}(G(n-1)_y) \arrow[r, "(j_{n-1,y})_*"] \arrow[d, equal] & \mathrm{CH}^*(G(n)) \arrow[r, "j^*_{n-1,x}"] & \mathrm{CH}^*(G(n-1)_x) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{CH}^{*-n}(G(n-1)_x) \arrow[r, "(j_{n-1,x})_*"] & \mathrm{CH}^*(G(n)) \arrow[r, "j^*_{n-1,y}"] \arrow[u, "\varphi^*=(\varphi_*)^{-1}"'] & \mathrm{CH}^*(G(n-1)_y) \arrow[r] & 0.
\end{tikzcd}$$

It implies that φ^* is identity on elements of degree $\leqslant n$. Since, by the Lemma 2.2, such elements generate $\mathrm{CH}^*(G(n))$ as a ring, $\varphi^* = id = \varphi_*$, and j^*_{n-1}, j_{n-1*} are well-defined. $\square$

Proposition 2.4 *There is unique set of elements $z_i \in \mathrm{CH}^i(G(n))$ defined for all $n \geqslant 1$ and satisfying the properties:*

(0) *For $G(1) \cong \mathbb{P}^1$, z_1 is the class of a point.*

(1) *As a $\mathbb{Z}$-module, $\mathrm{CH}^*(G(n)) = \oplus_{I \subset \{1,\dots,n\}} \mathbb{Z} \cdot \prod_{i \in I} z_i$.*

(2) $j_{n-1*}(1) = z_n$.

(3) $j^*_{n-1} : \mathrm{CH}^*(G(n)) \to \mathrm{CH}^*(G(n-1))$ *is given by the following rule on the additive generators above:*

$$\prod_{i \in I} z_i \mapsto \begin{cases} 0, & \text{if } n \in I; \\ \prod_{i \in I} z_i, & \text{if } n \notin I. \end{cases}$$

Proof: Let us introduce the elementary cycles $z_i \in \mathrm{CH}^i(G(n))$ inductively as follows: For $n = 1$, $G(1) \cong \mathbb{P}^1$, and z_1 is just the class of a point. Let $z_i \in \mathrm{CH}^i(G(n-1))$, for $1 \leqslant i \leqslant n-1$ are defined and satisfy the condition $(1) - (3)$. Let us define similar cycles on $G(n)$.

From the Proposition 2.1 we get: ${j_{n-1}}^*$ is an isomorphism on CH^i, for $i < n$. Now, for $1 \leqslant i \leqslant n-1$, we define $z_i \in \mathrm{CH}^i(G(n))$ as unique element corresponding under this isomorphism to $z_i \in \mathrm{CH}^i(G(n-1))$. And put: $z_n := {j_{n-1}}_*(1)$. We automatically get (2) satisfied.

Let $J \subset \{1, \ldots, n-1\}$. From the projection formula we get:

$${j_{n-1}}_*(\prod_{j \in J} z_j) = z_n \cdot \prod_{j \in J} z_j.$$

Applying once more Proposition 2.1, we get condition (1) and (3). $\square$

Remark: The cycle z_i we constructed is given by the set of n-dimensional totally isotropic subspaces $A \subset W_n$ satisfying the condition: $A \cap \pi_{n+1-i} \neq 0$ for fixed totally isotropic subspace π_{n+1-i} of dimension $(n+1-i)$.

Consider the commutative diagram:

$$\begin{array}{ccccc}
G(n) & \xleftarrow[g_n]{} & \mathbb{P}_{G(n-1)}(Y_{n-1}) & \xrightarrow[\pi_{n-1}]{} & G(n-1) \\
{\scriptstyle j_{n-1}}\uparrow & & {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j_{n-2}} \\
G(n-1) & \xleftarrow[g_{n-1}]{} & \mathbb{P}_{G(n-2)}(Y_{n-2}) & \xrightarrow[\pi_{n-2}]{} & G(n-2).
\end{array}$$

By Proposition 2.4, the ring homomorphism

$$j^* : \mathrm{CH}^*(\mathbb{P}_{G(n-1)}(Y_{n-1})) \to \mathrm{CH}^*(\mathbb{P}_{G(n-2)}(Y_{n-2}))$$

is a surjection, and it's kernel is generated as an ideal by the elements $\pi_{n-1}^*(z_{n-1})$ and $\rho^2 \cdot c(V_{n-2}^\vee)(\rho)$. In particular, $(j^*)^k$ is an isomorphism for all $k < n-1$, and the kernel of $(j^*)^{n-1}$ is additively generated by $\pi_{n-1}^*(z_{n-1})$.

Theorem 2.5 *Let V_n be tautological bundle on $G(n)$, z_i be elements defined in Proposition* 2.4*, and $\rho = c_1(\mathcal{O}(1))$. Then:*

(1) $g_n^*(z_k) = \rho^k + 2\sum_{0<i<k} \rho^{k-i}\pi_{n-1}^*(z_i) + \pi_{n-1}^*(z_k)$*, for all* $0 < k < n$.

(2) $g_n^*(z_n) = \rho^n + 2\sum_{0<i<n} \rho^{n-i}\pi_{n-1}^*(z_i)$.

(3) $c(V_n)(t) = t^n + 2\sum_{1\leqslant i\leqslant n}(-1)^i z_i t^{n-i}$.

Proof: $G(1)$ is a conic and $V_1 \cong \mathcal{O}(-1)$. Hence, $c(V_1)(t) = t - 2z_1$. Let now $(1)_{m,k}$ is proven for all $m < n$ and all $0 < k < m$, and $(2)_m$ and $(3)_m$ are proven for all $m < n$.

Since $(j^*)^k : \mathrm{CH}^k(\mathbb{P}_{G(n-1)}(Y_{n-1})) \to \mathrm{CH}^k(\mathbb{P}_{G(n-2)}(Y_{n-2}))$ is an isomorphism for $k < n-1$, the condition $(1)_{n,k}$ follows from $(1)_{n-1,k}$ for all such k in view of $j^*(\rho) = \rho$ and $j^*(\pi_{n-1}^*(z_k)) = \pi_{n-2}^*(z_k)$ (Proposition 2.4(3)). Analogously, since the kernel $(j^*)^{n-1}$ is additively generated by $\pi_{n-1}^*(z_{n-1})$, the condition $(2)_{n-1}$ implies that $g_n^*(z_{n-1}) = \rho^{n-1} + 2\sum_{1\leqslant i<n-1} \rho^{n-1-i}\pi_{n-1}^*(z_i) + \lambda \cdot \pi_{n-1}^*(z_{n-1})$, where $\lambda \in \mathbf{Z}$. Since $Y_{n-1} = \mathcal{O} \oplus (V_{n-1}^{\perp})^{\vee}$, the projection $\pi_{n-1} : \mathbb{P}_{G(n-1)}(Y_{n-1}) \to G(n-1)$ has the section s (given by the rule: $(B/x) \mapsto (B, B)$). It satisfies: $g_n \circ s = j_{n-1}$. Since $s^*(\pi_{n-1}^*(z_{n-1})) = z_{n-1}$, $s^*(\rho) = 0$ and $j_{n-1}^*(z_{n-1}) = z_{n-1}$, we get $\lambda = 1$, which implies $(1)_{n,n-1}$.

Choose some rational point $y \in Q\backslash T_{x,Q}$. By Propositions 2.3(2) and 2.4(2), the cycle z_n is defined as the set of such planes A, that $y \in A$. Then the cycle $g_n^*(z_n)$ is the set of such pairs (A, B), that $y \in A$, $x \in B$ and $\dim(A + B/A) \leqslant 1$. Thus $A + B = y + B$, and $g_n^*(z_n)$ is given by the section $\mathbb{P}_{G(n-1)}(\mathcal{O}) \subset \mathbb{P}_{G(n-1)}(Y_{n-1})$. Since $c(Y_{n-1})(t) = t^2 \cdot c(\pi_{n-1}^*(V_{n-1}^{\vee}))(t)$, this class can be expressed as $\rho \cdot c(\pi_{n-1}^*(V_{n-1}^{\vee}))(\rho)$. The last expression is equal to $\rho^n + 2\rho^{n-1}\pi_{n-1}^*(z_1) + \ldots + 2\rho\pi_{n-1}^*(z_{n-1})$ because of $(3)_{n-1}$. The statement $(2)_n$ is proven.

Finally, since $[g_n^*(V_n)] = [\pi_{n-1}^*(V_{n-1})] + [\mathcal{O}] + [\mathcal{O}(-1)] - [\mathcal{O}(1)]$, $g_n^*(c(V_n)(t)) = \pi_{n-1}^*(c(V_{n-1})(t)) \cdot \frac{t\cdot(t-\rho)}{t+\rho}$. In the light of $(3)_{n-1}$, this is equal to

$$\left(t^{n-1} + 2\sum_{1\leqslant i\leqslant n-1}(-1)^i\pi_{n-1}^*(z_i)t^{n-1-i}\right) \cdot \frac{t\cdot(t-\rho)}{t+\rho}.$$

Using the equality $\rho^2(\rho^{n-1} + 2\rho^{n-2}\pi_{n-1}^*(z_1) + \ldots + 2\pi_{n-1}^*(z_{n-1})) = 0$, as well as the conditions $(1)_{n,k}$ and $(2)_n$, we can rewrite the last expression as:

$$t^n + 2\sum_{1\leqslant i\leqslant n}(-1)^i g_n^*(z_i)t^{n-i}.$$

Since g_n^* is injective (the map g_n is birational), we get:

$$c(V_n)(t) = t^n + 2\sum_{1\leqslant i\leqslant n}(-1)^i z_i t^{n-i}.$$

The statement $(3)_n$ is proven. $\square$

3 Multiplicative structure

The multiplicative structure of $\mathrm{CH}^*(G(n))$ was studied extensively by H.Hiller, B.Boe, J.Stembridge, P.Pragacz and J.Ratajski - see [3], [8].

We can compute this ring structure from Theorem 2.5. Although, we restrict our consideration only to $(\,mod\,2)$ case, it should be pointed out that the integral case can be obtained in exactly the same way.

Let us denote as $\overline{u}$ the image of u under the map $\mathrm{CH}^* \to \mathrm{CH}^*/2$.

Proposition 3.1

$$\mathrm{CH}^*(G(n))/2 = \bigotimes_{1\leqslant d\leqslant n;\, d-odd} (\mathbb{Z}/2[\overline{z}_d]/(\overline{z}_d^{2^{m_d}})),$$

where $m_d = [log_2(n/d)] + 1$.

Proof: Consider the diagram:

$$G(n) \xleftarrow{g_n} \mathbb{P}_{G(n-1)}(Y_{n-1}) \xrightarrow{\pi_{n-1}} G(n-1).$$

From Theorem 2.5, $g_n^*(\overline{z}_k) = \overline{\rho}^k + \pi_{n-1}^*(\overline{z}_k)$, for $k < n$, and $g_n^*(\overline{z}_n) = \overline{\rho}^n$. Then it easily follows by the induction on n, that $\overline{z}_k^2 = \overline{z}_{2k}$ (where we assume $\overline{z}_r = 0$ if $r > n$).

Thus, we have surjective ring homomorphism

$$\bigotimes_{1\leqslant d\leqslant n;\, d-\mathrm{odd}} (\mathbb{Z}/2[\overline{z}_d]/(\overline{z}_d^{2^{m_d}})) \to \mathrm{CH}^*(G(n))/2.$$

Since the dimensions of both rings are equal to 2^n, it is an isomorphism. $\square$

Let J be a set. Let us call a *multisubset* the collection $\Lambda = \coprod_{\beta\in B} \Lambda_\beta$ of disjoint subsets of J. For a subset I of J, we will denote by the same symbol I the multisubset $\coprod_{i\in I}\{i\}$. Let $B = \coprod_{\gamma\in C} B_\gamma$, and $\Lambda'_\gamma = \coprod_{\beta\in B_\gamma} \Lambda_\beta$. Then the multisubset $\Lambda' := \coprod_{\gamma\in C} \Lambda'_\gamma$ is called the *specialization* of Λ. We call the specialization *simple* if $\#(B_\gamma) \leqslant 2$, for all $\gamma \in C$.

Let J now be some set of natural numbers (it may contain multiple entries). Then to any finite multisubset $\Lambda = \coprod_{\beta\in B} \Lambda_\beta$ of J we can assign the set of natural numbers $\overline{\Lambda} := \{\sum_{i\in\Lambda_\beta} i\}_{\beta\in B}$. We call the specialization Λ *good* if $\overline{\Lambda} \subset \{1,\dots,n\}$.

Suppose I be some finite set of natural numbers. Let us define the element $\overline{z}_I \in \mathrm{CH}^*(G(n))/2$ by the formula:

$$\overline{z}_I = \sum_{\Lambda} \prod_{j\in\overline{\Lambda}} \overline{z}_j,$$

where we assume $\overline{z}_r = 0$, if $r > n$, and the sum is taken over all simple specializations Λ of the multisubset $I = \coprod_{i\in I}\{i\}$.

Lemma 3.2 *If $I \not\subset \{1,\dots,n\}$, then $\overline{z}_I = 0$.*

Proof: If I contains an element $r > n$, then $\overline{z}_I$ is clearly zero. Suppose now that I contains some element i twice, say as i_1 and i_2. Consider the subgroup $Z_2 \subset S_s$ interchanging i_1 and i_2 and keeping all other elements in place. We get $\mathbb{Z}/2$-action on our specializations. The terms which are not stable under this action will appear with multiplicity 2, so, we can restrict our attention to the stable terms. But such specializations have the property that $\{i_1, i_2\}$ is disjoint from the rest of i's, and the corresponding sum looks as: $\sum_M \prod_{j\in\overline{M}} \overline{z}_j \cdot (\overline{z}_i^2 + \overline{z}_{2i})$, where the sum is taken over all simple specializations of the multisubset $I\backslash\{i_1, i_2\}$. Since $\overline{z}_i^2 = \overline{z}_{2i}$, this expression is zero. $\square$

We immediately get the (modulo 2) version of the Pieri formula proved by H.Hiller and B.Boe:

Proposition 3.3 ([8])

$$\overline{z}_I \cdot \overline{z}_j = \overline{z}_{I\cup j} + \sum_{i\in I} \overline{z}_{(I\backslash i)\cup(i+j)},$$

where we omit terms $\overline{z}_J$ with $J \not\subset \{1,\dots,n\}$ (in particular, if J contains some element with multiplicity > 1).

Proof: $\overline{z}_{I\cup j} = \sum_\Lambda \prod_{l\in\overline{\Lambda}} \overline{z}_l$, where the sum is taken over all simple specializations of the multisubset $I \cup j$. We can distinguish two types of specializations: 1) j is separated from I; 2) j is not separated from I, that is, there is β such that $\Lambda_\beta = \{i, j\}$, for some $i \in I$. Let us call the latter specializations to be of type $(2, i)$. Clearly, the sum over specializations of the first kind is equal to $\overline{z}_I \cdot \overline{z}_j$, and the sum over the specializations of the type $(2, i)$ is equal to $\overline{z}_{(I\backslash i)\cup(i+j)}$. Finally, the terms with $J \not\subset \{1, \dots, n\}$ could be ommited by Lemma 3.2. $\square$

We also get the expression of monomials on z_i's in terms of z_I's.

Proposition 3.4 (1) *The set* $\{\overline{z}_I\}_{I\subset\{1,\dots,n\}}$ *is a basis of* $\mathrm{CH}^*(G(n))/2$.

(2) $\prod_{i\in I}\overline{z}_i = \sum_\Lambda \overline{z}_{\overline{\Lambda}}$, *where sum is taken over all good specializations of* I.

Proof:

(1) On the $\mathbb{Z}/2$-vector space $\mathrm{CH}^*(G(n))/2 = \oplus_{I\subset\{1,\dots,n\}}\mathbb{Z}/2 \cdot \prod_{i\in I}\overline{z}_i$ we have lexicographical filtration. Consider the linear map $\varepsilon : \mathrm{CH}^*(G(n))/2 \to \mathrm{CH}^*(G(n))/2$ sending $\prod_{i\in I}\overline{z}_i$ to $\overline{z}_I$. Then the associated graded map: $gr(\varepsilon)$ is the identity. Thus, ε is invertible, and the set $\{\overline{z}_I\}_{I\subset\{1,\dots,n\}}$ form a basis.

(2) Consider the $\mathbb{Z}/2$-vector spaces $W_1 := \oplus_\Lambda \mathbb{Z}/2 \cdot x_\Lambda$, and $W_2 := \oplus_\Lambda \mathbb{Z}/2 \cdot y_\Lambda$, where Λ runs over all finite multisubsets of $\mathbb{N}$.

Consider the linear maps $\psi : W_2 \to W_1$ which sends y_Λ to the $\sum_{\Lambda'} x_{\Lambda'}$, where the sum is taken over all specializations of Λ, and $\varphi : W_1 \to W_2$ which sends x_Λ to the $\sum_{\Lambda'} y_{\Lambda'}$, where the sum is taken over all simple specializations Λ' of Λ. It is an easy exercise to show that φ and ψ are mutually inverse.

Consider the linear surjective maps: $w_1 : W_1 \to \mathrm{CH}^*(G(n))/2$ and $w_2 : W_2 \to \mathrm{CH}^*(G(n))/2$ given by the rule: $w_1(x_\Lambda) := \overline{z}_{\overline{\Lambda}}$, and $w_2(y_\Lambda) := \prod_{j\in\overline{\Lambda}}\overline{z}_j$.

Then, by the definition of $\overline{z}_I$, $w_1 = w_2 \circ \varphi$. Then $w_2 = w_1 \circ \psi$, which implies that $\prod_{i\in I}\overline{z}_i = \sum_\Lambda \overline{z}_{\overline{\Lambda}}$, where the sum is taken over all specializations of I. It remains to notice, that nongood specializations do not contribute to the sum (by Lemma 3.2). $\square$

Examples: 1) $\overline{z}_i \cdot \overline{z}_j = \overline{z}_{i,j} + \overline{z}_{i+j}$, where the first term is ommited if $i = j$ and the second if $i + j > n$. 2) $\overline{z}_{i,j,k} = \overline{z}_i \cdot \overline{z}_j \cdot \overline{z}_k + \overline{z}_{i+j} \cdot \overline{z}_k + \overline{z}_{j+k} \cdot \overline{z}_i + \overline{z}_{i+k} \cdot \overline{z}_j$.

4 Action of the Steenrod algebra

On the Chow-groups modulo prime l there is the action of the Steenrod algebra. Such action was constructed by V.Voevodsky in the context of arbitrary motivic cohomology - see [10], and then a simpler construction was given by P.Brosnan for the case of usual Chow groups - see [2]. For quadratic grassmannians we will be interested only in the case $l = 2$.

We can compute the action of the Steenrod squares on the cycles $\overline{z}_i$. Let S be the total Steenrod opeartion $id + \mathrm{S}^1 + \mathrm{S}^2 + \ldots$.

Theorem 4.1

$$\mathrm{S}(\overline{z}_i) = \sum_{m=i}^{\min(n,2i)} \binom{i}{m-i} \cdot \overline{z}_m$$

Proof: Use induction on n. The base is trivial. Suppose the statement is true for $(n-1)$. By Theorem 2.5, $g_n^*(\overline{z}_i) = \overline{\rho}^i + \pi_{n-1}^*(\overline{z}_i)$, where we assume $\overline{z}_n \in \mathrm{CH}^*(G(n-1))/2$ to be zero. Also, $\overline{c}(V_n)(t) = t^n$, which implies that $\overline{\rho}^{n+1} = 0$. Using the fact that S commutes with the pull-back morphisms (see [2]), the latter equality, and inductive assumption, we get:

$$g_n^*(\mathrm{S}(\overline{z}_i)) = \mathrm{S}(g_n^*(\overline{z}_i)) = \mathrm{S}(\overline{\rho}^i) + \pi_{n-1}^*\left(\sum_{m=i}^{\min(n-1,2i)} \binom{i}{m-i} \cdot \overline{z}_m\right) =$$

$$\sum_{m=i}^{2i} \binom{i}{m-i}\overline{\rho}^m + \sum_{m=i}^{\min(n-1,2i)} \binom{i}{m-i} \cdot (\overline{\rho}^m + g_n^*(\overline{z}_m)) =$$

$$\sum_{m=i}^{\min(n,2i)} \binom{i}{m-i} g_n^*(\overline{z}_m).$$

Now, the statement follows from the injectivity of g_n^*. $\square$

5 Main theorem

Let X be some variety over the field k. We will denote:

$$C^*(X) := image(\mathrm{CH}^*(X)/2 \to \mathrm{CH}^*(X|_{\overline{k}})/2).$$

Let now Q be a smooth projective quadric of dimension $2n-1$, and $X = G(n, Q)$ be the grassmanian of middle-dimensional projective planes on it. Then $X|_{\overline{k}} = G(n)$. In this section we will show that, as an algebra, $C^*(G(n, Q))$ is generated by the elementary cycles $\overline{z}_i$ contained in it.

Let $F(n,Q)$ be the variety of complete flags $(l_0 \subset l_1 \subset \ldots \subset l_{n-1})$ of projective subspaces on Q. Then $F(n,Q)$ is naturally isomorphic to the complete flag variety $F_{G(n,Q)}(V_n)$ of the tautological n-dimensional bundle on $G(n,Q)$. On the variety $F(n,Q)$ there are natural (subquotient) line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$. The first chern classes $c_1(\mathcal{L}_i)$, $1 \leqslant i \leqslant n$ generate the ring $\mathrm{CH}^*(F(n,Q))$ as an algebra over $\mathrm{CH}^*(G(n,Q))$, and the relations among them depend only on the chern classes $c_j(V_n)$. Let F_n be the variety of complete flags of subspaces of the n-dimensional vector space V. It also has natural line bundles $\mathcal{L}'_1, \ldots, \mathcal{L}'_n$. Again, the first chern classes $c_1(\mathcal{L}'_i)$ generate the ring $\mathrm{CH}^*(F_n)$. By Theorem 2.5 (3), modulo 2, all chern classes $c_j(V_n)$ are the same as the chern classes of the trivial n-dimensional bundle $\oplus_{i=1}^n \mathcal{O}$. Thus, modulo 2, the Chow ring of $F_{G(n,Q)}(V_n)$ is isomorphic to the Chow ring of $F_{G(n,Q)}(\oplus_{i=1}^n \mathcal{O})$. We get:

Theorem 5.1 *There is a ring isomorphism*

$$\mathrm{CH}^*(F(n,Q))/2 \cong \mathrm{CH}^*(G(n,Q))/2 \otimes_{\mathbb{Z}/2} \mathrm{CH}^*(F_n)/2,$$

where the map $\mathrm{CH}^*(G(n,Q)) \to \mathrm{CH}^*(F(n,Q))$ *is induced by the natural projection* $F(n,Q) \to G(n,Q)$*, and the map* $\mathrm{CH}^*(F_n)/2 \to \mathrm{CH}^*(F(n,Q))/2$ *is given on the generators by the rule:* $c_1(\mathcal{L}'_i) \mapsto c_1(\mathcal{L}_i)$.

□

Notice, that the map $ac : \mathrm{CH}^*(F_n) \to \mathrm{CH}^*(F_n|_{\overline{k}})$ is an isomorphism, and, $C^*(F(n,Q)) = C^*(G(n,Q)) \otimes_{\mathbb{Z}/2} \mathrm{CH}^*(F_n|_{\overline{k}})/2$. Thus, we have:

Statement 5.2 *Let* $v_1, \ldots, v_s$ *be linearly independent elements of* $\mathrm{CH}^*(F_n|_{\overline{k}})/2$*, and* $x_i \in \mathrm{CH}^*(G(n,Q)|_{\overline{k}})/2$*, then* $x = \sum_{i=1}^s x_i \cdot v_i$ *belongs to* $C^*(F(n,Q))$ *if and only if all* $x_i \in C^*(G(n,Q))$.

□

The ring $\mathrm{CH}^*(F_n)$ can be described as follows. Let us denote $c_1(\mathcal{L}_j)$ as h_j, and the set $\{h_j, \ldots, h_n\}$ as $\underline{h}(j)$ (and $\underline{h}(1)$ as $\underline{h}$). For arbitrary set of variables $\underline{u} = \{u_1, \ldots, u_r\}$ let us define the degree i polynomials $\sigma_i(\underline{u})$ and $\sigma_{-i}(\underline{u})$ from the equation:

$$\prod_l (1 + u_l) = \sum_i \sigma_i(\underline{u}) = \left(\sum_i \sigma_{-i}(\underline{u}) \right)^{-1}.$$

It is well-known that:

Statement 5.3

$$\mathrm{CH}^*(F_n) = \mathbb{Z}[\underline{h}]/(\sigma_i(\underline{h}),\ 1 \leqslant i \leqslant n) = \mathbb{Z}[\underline{h}]/(\sigma_{-i}(\underline{h}(i)),\ 1 \leqslant i \leqslant n).$$

Since $\sigma_{-i}(\underline{h}(i))$ is the $\pm$-monic polynomial in h_i with coefficients in the subring, generated by $\underline{h}(i+1)$, we get: $\mathrm{CH}^*(F_n)$ is a free module over the subring $\mathbb{Z}[\underline{h}(n)]/(\sigma_{-n}(\underline{h}(n))) = \mathbb{Z}[h_n]/(h_n^n)$.

Let $\pi : F(n, Q) \to F(n-1, Q)$ be the natural projection between full flag varieties. We will denote by the same symbol $\overline{z}_I$ the images of $\overline{z}_I$ in $\mathrm{CH}^*(F(n, Q))/2$.

The following statement is the key for the Main Theorem.

Proposition 5.4

$$\pi^*\pi_*(\overline{z}_I) = \sum_{i \in I} \overline{z}_{(I \backslash i)} \cdot \pi^*\pi_*(\overline{z}_i).$$

Proof: $F(n, Q)$ is a conic bundle over $F(n-1, Q)$ inside the projective bundle $\mathbb{P}_{F(n-1,Q)}(V)$, where, in $K_0(F(n, Q))$, $\pi^*[V] = [\mathcal{L}_n] + [\mathcal{L}_n^{-1}] + [\mathcal{O}]$. Sheaf $\mathcal{L}_n$ is nothing else but the restriction of the sheaf $\mathcal{O}(-1)$ from $\mathbb{P}_{F(n-1,Q)}(V)$ to $F(n, Q)$.

Lemma 5.5 *Let V be a 3-dimensional bundle over some variety X equipped with the nondegenerate quadratic form p. Let $\pi : Y \to X$ be conic bundle of p-isotropic lines in U. Then there is a $CH^*(X)$-algebra automorphism $\phi : CH^*(Y) \to CH^*(Y)$ of exponent 2 such that*

(1) $\phi(c_1(\mathcal{O}(-1)|_Y)) = c_1(\mathcal{O}(1)|_Y)$.

(2) $\pi^*\pi_*(x) \cdot c_1(\mathcal{O}(1)) = x - \phi(x)$

Proof: Consider variety $Y \times_X Y$ with the natural projections π_1 and π_2 on the first and second factor, respectively. Then divisor $\Delta(Y) \subset Y \times_X Y$ defines an invertible sheaf $\mathcal{L}$ on $Y \times_X Y$ such that $\mathcal{L}^2 \cong \pi_1^*(\mathcal{O}(1)) \otimes \pi_2^*(\mathcal{O}(1))$ and $\Delta^*(\mathcal{L}) = \mathcal{O}(1)$. Consider the map $f := \Delta \circ \pi_2 : Y \times_X Y \to Y \times_X Y$. Define $\phi : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y)$ as $id - \Delta^* \circ f_* \circ \pi_1^*$.

The described maps fit into the diagram:

$$\begin{tikzcd}
Y \arrow[d, "\pi"'] & Y \times_X Y \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] & \\
X & Y \arrow[l, "\pi"] \arrow[r, "\Delta"'] & Y \times_X Y
\end{tikzcd}$$

with the transversal cartesian square ($\pi_1^*(T_\pi) = T_{\pi_2}$). Consequently, $\pi^* \circ \pi_* = \pi_{2*} \circ {\pi_1}^*$. Since $\mathcal{O}(\Delta(Y))|_Y = \mathcal{O}(1)$, we have:

$$\pi^*\pi_*(x) \cdot c_1(\mathcal{O}(1)) = \Delta^*\Delta_*\pi_{2*}\pi_1^*(x) = x - \phi(x).$$

Consider the map $\psi := id - f_* : \mathrm{CH}^*(Y \times_X Y) \to \mathrm{CH}^*(Y \times_X Y)$. We claim that ψ is a ring homomorphism. Really, $Y \times_X Y \cong \mathbb{P}_Y(U)$, where the projection $\mathbb{P}_Y(U) \to Y$ is given by π_2 and $c(U)(t) = t(t - c_1(\mathcal{O}(1)))$. Thus $\mathrm{CH}^*(Y \times_X Y) = \mathrm{CH}^*(Y)[\rho]/(\rho(\rho - c_1(\mathcal{O}(1))))$, where the map $\mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y \times_X Y)$ is π_2^*. Notice, that $f_*\pi_2^* = \Delta_*\pi_{2*}\pi_2^* = 0$, that is, $\psi|_{\mathrm{CH}^*(Y)}$ is the identity. At the same time, $\psi(\rho) = \rho - \rho = 0$. Since $\mathrm{CH}^*(Y \times_X Y)$ is free $\mathrm{CH}^*(Y)$-module of rank 2 with the basis 1, ρ, by the projection formula, we get that ψ is an endomorphism of $\mathrm{CH}^*(Y \times_X Y)$ considered as an $\mathrm{CH}^*(Y)$-algebra.

Since, $\phi = \Delta^* \circ \psi \circ \pi_1^*$, it is a homomorphism of $\mathrm{CH}^*(X)$-algebras. Also, $\phi(c_1(\mathcal{O}(1))) = -c_1(\mathcal{O}(1))$. Finally, since the composition $\pi_{2*}\pi_1^*\Delta^*\Delta_* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y)$ is equal $2 \cdot id$, we get

$$(\Delta^* \circ f_* \circ \pi_1^*)^{\circ 2} = 2(\Delta^* \circ f_* \circ \pi_1^*),$$

which is equivalent to: $\phi^{\circ 2} = id$. Thus, ϕ is an automorphism of exponent 2. $\square$

Let us compute the action of ϕ on basis elements $\overline{z}_I$. Let σ_i be elementary symmetric functions in h_i's. Since $h_i \in CH^*(F(n-1, Q))$, for $i < n$, we have equality $\phi(h_i) = h_i$ for them, and $\phi(h_n) = -h_n$. We know that $\sigma_i = (-1)^i 2z_i$. We immediately conclude:

Lemma 5.6 $\phi(z_i) = z_i + \sum_{0<l<i} 2z_{i-l}h_n^l + h_n^i$.

Lemma 5.7 $\phi(\overline{z}_I) = \overline{z}_I + \sum_{i \in I} \overline{z}_{(I \setminus i)}\overline{h}_n^i$.

Proof: Let us define the *size* $s(I)$ of I as the number of it's elements. Use induction on the size of I. The case of size $= 1$ is OK by the previous lemma. Suppose the statement is known for sizes $< s(I)$.

Let i be some element of I. We know from Proposition 3.3 that $\overline{z}_I = \overline{z}_{(I\backslash i)} \cdot \overline{z}_i + \sum_{j\in I, j\neq i} \overline{z}_{(I\backslash\{i,j\})\cup(i+j)}$. Since ϕ is a ring homomorphism, we get:

$$\phi(\overline{z}_I) = \phi(\overline{z}_{(I\backslash i)}) \cdot \phi(\overline{z}_i) + \sum_{j\in I\backslash i} \phi(\overline{z}_{(I\backslash\{i,j\})\cup(i+j)}) =$$

$$\left(\overline{z}_{(I\backslash i)} + \sum_{l\in I\backslash i} \overline{z}_{(I\backslash\{i,l\})} h_n^l\right) \cdot (\overline{z}_i + \overline{h}_n^i) +$$

$$\sum_{j\in I\backslash i} \left(\overline{z}_{(I\backslash\{i,j\})\cup(i+j)} + \overline{z}_{(I\backslash\{i,j\})} \overline{h}_n^{i+j} + \sum_{m\in I\backslash\{i,j\}} \overline{z}_{(I\backslash\{i,j,m\})\cup(i+j)} \overline{h}_n^m\right) =$$

$$\overline{z}_I + \overline{z}_{(I\backslash i)} \overline{h}_n^i + \sum_{l\in I\backslash\{i\}} (\overline{z}_{(I\backslash\{i,l\})} \cdot \overline{z}_i) \overline{h}_n^l + \sum_{m\neq j\in I\backslash\{i\}} \overline{z}_{(I\backslash\{i,j,m\})\cup(i+j)} \overline{h}_n^m =$$

$$\overline{z}_I + \overline{z}_{(I\backslash i)} \overline{h}_n^i + \sum_{j\in I\backslash i} \overline{z}_{(I\backslash j)} \overline{h}_n^j + 2 \cdot \sum_{m\neq j\in I\backslash\{i\}} \overline{z}_{(I\backslash\{i,j,m\})\cup(i+j)} \overline{h}_n^m =$$

$$\overline{z}_I + \sum_{j\in I} \overline{z}_{(I\backslash j)} \overline{h}_n^j$$

(as usually, one should omit $\overline{z}_J$ with $J \not\subset \{1, \ldots, n\}$). $\square$

Let $p = q \perp \mathbb{H}$. Then Q can be identified with the quadric of projective lines on P passing through fixed rational point y. This identifies the complete flag variety $F(r, Q)$ with the subvariety of $F(r+1, P)$ consisting of flags containing our point y. We get an embedding $i_r : F(r, Q) \to F(r+1, P)$. It is easy to see that the diagram

$$\begin{array}{ccc} F(n,Q) & \xrightarrow{i_n} & F(n+1,P) \\ \pi \downarrow & & \downarrow \pi' \\ F(n-1,Q) & \xrightarrow[i_{n-1}]{} & F(n,P) \end{array}$$

is cartesian, and since π' is smooth, we have an equality: $\pi^* \circ \pi_* \circ i_n^* = i_n^* \circ \pi'^* \circ \pi'_*$.

It follows from Lemmas 5.5 and 5.7 that

$$\pi'^*\pi'_*(\overline{z}_I) \cdot \overline{h}_{n+1} = \sum_{i\in I} \overline{z}_{(I\backslash i)} \overline{h}^{i}_{n+1}.$$

Thus, modulo the kernel of multiplication by $\overline{h}_{n+1}$, $\pi'^*\pi'_*(\overline{z}_I) \equiv \sum_{i\in I} \overline{z}_{(I\backslash i)} \overline{h}^{i-1}_{n+1}$. But, by the Statement 5.3 and Theorem 5.1, such kernel is generated by $\overline{h}^{n}_{n+1}$.

Since $i_n^*(\overline{z}_I) = \overline{z}_I$, $i_n^*(\overline{h}_{n+1}) = \overline{h}_n$ and $\overline{h}^{n}_{n} = 0$ on $F(n, Q)$, we get:

$$\pi^*\pi_*(\overline{z}_I) = i_n^*\pi'^*\pi'_*(\overline{z}_I) = \sum_{i\in I} \overline{z}_{(I\backslash i)} \overline{h}^{i-1}_{n} = \sum_{i\in I} \overline{z}_{(I\backslash i)} \pi^*\pi_*(\overline{z}_i).$$

□

Notice, that the elements $\pi^*\pi_*(z_i)$ belong to $\mathrm{CH}^*(F_n)/2$, and they are linearly independent (being nonzero and having different degrees).

As a corollary, we get:

Main Theorem 5.8 *As an algebra, $C^*(G(n, Q))$ is generated by the elementary classes $\overline{z}_i$ contained in it.*

Proof: Let $\overline{z}$ be an element of $C^*(G(n, Q))$. It can be expressed as a linear combination of the basis elements $\overline{z}_I$'s. Let us define the *size* $s(\overline{z})$ of the element $\overline{z} = \sum \overline{z}_{I_a}$ as the maximum of sizes of I_a involved. Let $m(\overline{z})$ be the main term of $\overline{z}$, that is, $\sum_{a:s(I_a)=s(\overline{z})} \overline{z}_{I_a}$.

Lemma 5.9 *Let $\overline{z} = \sum_a \overline{z}_{I_a} \in C^*(G(n, Q))$. Let $s(I_a) = s(\overline{z})$, and $i \in I_a$. Then the elementary cycle $\overline{z}_i$ belongs to $C^*(G(n, Q))$.*

Proof: Let $i \in I_a$, and $I_a\backslash i = \{j_2, \ldots, j_s\}$. Denote the operation $\pi^*\pi_*$ as D. Then $D(\overline{z}) = \sum_{1\leqslant j\leqslant n} d_j(\overline{z}) \cdot D(\overline{z}_j)$, where $d_j(\overline{z}) \in \mathrm{CH}^*(G(n, Q)|_{\overline{k}})/2$, and the elements $D(\overline{z}_j) \in \mathrm{CH}^*(F_n)/2$ are linearly independent. Since D is defined over the base field, $D(\overline{z}) \in C^*(F(n, Q))$, and, by the Statement 5.2, $d_j(\overline{z}) \in C^*(G(n, Q))$. Clearly, $m(d_j(\overline{z})) = d_j(m(\overline{z}))$. It is easy to see that $d_{j_s} \ldots d_{j_2}(\overline{z}) = \overline{z}_i$, since for arbitrary I_b with $s(I_b) < s = s(\overline{z})$ we have: $d_{j_s} \ldots d_{j_2}(\overline{z}_{I_b}) = 0$, or 1, and for $I_c \neq I_a$ with $s(I_c) = s$, $d_{j_s} \ldots d_{j_2}(\overline{z}_{I_c})$ is either 0, or has degree different from i. Thus, $\overline{z}_i \in C^*(G(n, Q))$. □

Let us prove by induction on the size of $\overline{z}$, that $\overline{z}$ belongs to the subring of $C^*(G(n,Q))$ generated by $\overline{z}_j$'s. The base of induction, $s=1$ is trivial. Notice that $m(\prod_{i\in I}\overline{z}_i)=\overline{z}_I$. Thus, the size of $\overline{z}'=\overline{z}-\sum_{a:s(I_a)=s(\overline{z})}\prod_{i\in I_a}\overline{z}_i$ is smaller than that of $\overline{z}$. But by the Lemma 5.9, all the $\overline{z}_i$'s appearing in this expression belong to $C^*(G(n,Q))$. By the inductive assumption, $\overline{z}'$ belongs to the subring of $C^*(G(n,Q))$ generated by $\overline{z}_j$'s. Then so is $\overline{z}$. $\square$

Remark. Actually, for the proof of the Main Theorem one just needs the statement of the Lemma 5.5(1).

Corollary 5.10 *For arbitrary smooth projective quadric Q,*

$$C^*(G(n,Q))=\bigotimes_{1\leqslant d\leqslant n;\, d-odd}(\mathbb{Z}/2[\overline{z}_{d2^{l_d}}]/(\overline{z}_{d2^{l_d}}^{2^{(m_d-l_d)}})),$$

for certain $0\leqslant l_d\leqslant m_d=[log_2(n/d)]+1$.

Proof: It immediately follows from the Main Theorem 5.8, Proposition 3.1, and the fact that $\overline{z}_s^2=\overline{z}_{2s}$ (or 0, if $2s>n$). $\square$

Now we can introduce:

Definition 5.11 (1) *Let Q be a quadric of dimension $2n-1$. Denote as $J(Q)$ the subset of $\{1,\ldots,n\}$ consisting of those i, for which $\overline{z}_i\in C^*(G(n,Q))$.*

(2) *Let P be a quadric of dimension $2n$. Let Q be arbitrary subquadric of codimension 1 in P. Then $J(P)$ is a subset of $\{0,1,\ldots,n\}$, where $0\in J(P)$ iff $det_{\pm}(P)=1$ and, for $i>0$, $i\in J(P)$ iff $i\in J(Q|_{k\sqrt{det_{\pm}(P)}})$.*

Remark: The definition (2) above is motivated by the fact that $G(n+1,P)$ is isomorphic to $G(n,Q)\times_{\mathrm{Spec}(k)}\mathrm{Spec}(k\sqrt{det_{\pm}(P)})$.

It follows from the Main Theorem 5.8 that $C^*(G(n,Q))$ is exactly the subring of $\mathrm{CH}^*(G(n,Q)|_{\overline{k}})/2$ generated by $\overline{z}_i$, $i\in J(Q)$. In particular, $J(Q)$ carries all the information about $C^*(G(n,Q))$. Notice, that the same information is contained in the sequence $\{l_d\}_{d-\mathrm{odd};1\leqslant d\leqslant n}$.

What restrictions do we have on the possible values of $J(Q)$? Because of the action of the Steenrod operations, we get:

Proposition 5.12 *Let $i \in J(Q)$, and $r \in \mathbb{N}$ is such that $\binom{i}{r} \equiv 1\,(\,mod\,2)$, and $i + r \leqslant n$. Then $(i + r) \in J(Q)$.*

Proof: j belongs to $J(Q)$ if and only if the cycle $\overline{z}_j \in \mathrm{CH}^j(G(n, Q)|_{\overline{k}})/2$ is defined over the base field. Since $\overline{z}_i$ has such a property, and the Steenrod operation S^r is defined over the base field too, we get: $\overline{z}_{(i+r)} = \mathrm{S}^r(\overline{z}_i)$ is also defined over the base field. □

The natural question arises:

Question 5.13 *Do we have other restrictions on $J(Q)$? In other words, let $J \subset \{1, \dots, n\}$ be a subset satisfying the conditions of Proposition* 5.12*. Does there exist a quadric such that $J(Q) = J$?*

It is not difficult to check that, at least, for $n \leqslant 4$, there is no other restrictions.

6 On the canonical dimension of quadrics

In this section we will show that in the case of a *generic* quadric the variety $G(n, Q)$ is 2-incompressible, and also will formulate the conjecture describing the *canonical dimension* of arbitrary quadric. I would like to point out that the current section would not appear without the numerous discussions with G.Berhuy, who brought this problem to my attention.

We start by computing the characteristic classes of the variety $G(n, Q)$.

Let W be $(2n + 1)$-dimensional vector space over k equipped with the nondegenerate quadratic form q. Let $F(r) = F(r, Q)$ be the variety of complete flags $(\pi_1 \subset \ldots \subset \pi_r)$ of totally isotropic subspaces in W. Thus, $F(0) = \mathrm{Spec}(k)$, $F(1) = Q$, etc. ... . We get natural smooth projective maps: $\varepsilon_r : F(r + 1) \to F(r)$ with fibers - quadrics of dimension $2n - 2r + 1$.

Let $\mathcal{L}_i$ be the standard subquotient linear bundles $\mathcal{L}_i := \pi_i/\pi_{i-1}$, and $h_i = c_1(\mathcal{L}_i)$. The bundle $\mathcal{L}_i$ is defined on $F(r)$, for $r \geqslant i$. These divisors h_i are the roots of the tautological vector bundle V_n studied above.

Proposition 6.1 *The Chern polynomial of the tangent bundle $T_{F(r)}$ is equal to:*

$$c(T_{F(r)}) = \frac{\prod_{1 \leqslant i \leqslant r}(1 - h_i)^{2n+1}}{\prod_{1 \leqslant i \leqslant r}(1 - 2h_i) \cdot \prod_{1 \leqslant i < j \leqslant r}\left((1 - h_j + h_i) \cdot (1 - h_j - h_i)\right)}.$$

Proof: Let V_r be a tautological vector bundle on $F(r)$. Then V_r is an isotropic subbundle of $\eta^*(W)$ ($\eta : F(r) \to \mathrm{Spec}(k)$ is the projection), and on the subquotient $W_r := V_r^{\perp}/V_r$ we have a nondegenerate quadratic form $q_{\{r\}}$. Then the variety $F(r+1)$ is defined as zeroes of this quadratic form. Thus, $F(r+1)$ is the divisor of the sheaf $\mathcal{O}(2)$ on the projective bundle $\mathbb{P}_{F(r)}(W_r)$, and we have exact sequence:

$$0 \to \mathcal{O}(2)|_{F(r+1)} \to T_{\mathbb{P}_{F(r)}(W_r)}|_{F(r+1)} \to T_{F(r+1)} \to 0.$$

On the other hand, we have sequences:

$$0 \to T_{\varepsilon_r} \to T_{\mathbb{P}_{F(r)}(W_r)} \to \varepsilon^*(T_{F(r)}) \to 0 \ \text{ and}$$

$$0 \to \mathcal{O} \to W_r \otimes \mathcal{L}_{r+1}^{-1} \to T_{\varepsilon_r} \to 0.$$

It remains to notice, that in K_0, $[W_r] = (2n+1)[\mathcal{O}] - \sum_{i=1}^{r}([\mathcal{L}_i] + [\mathcal{L}_i^{-1}])$, to get the equality:

$$c(T_{F(r+1)}) = \varepsilon_r^*(c(T_{F(r)})) \cdot \frac{(1-h_{r+1})^{2n+1}}{(1-2h_{r+1}) \cdot \prod_{i=1}^{r}(1-h_{r+1}+h_i)(1-h_{r+1}-h_i)}$$

The statement now easily follows by induction on r. $\square$

Now, it is easy to compute the characteristic classes of the quadratic grassmanians.

Proposition 6.2

$$c(T_{G(r)}) = \frac{\prod_{1\leqslant i\leqslant r}(1-h_i)^{2n+1}}{\prod_{1\leqslant i\leqslant r}(1-2h_i) \cdot \prod_{1\leqslant i<j\leqslant r}\left((1-(h_j-h_i)^2)\cdot(1-h_j-h_i)\right)},$$

where h_j are the roots of the tautological vector bundle V_r on $G(r)$.

Proof: Consider the (forgetting) projection $\delta_r : F(r) \to G(r) = G(r, Q)$. We have natural identification of $F(r)$ with the variety of complete flags corresponding to the tautological bundle V_r on $G(r)$ (we will permit ourselves to use the same notation for the tautological bundles on $G(r)$ and $F(r)$ - this is justified by the fact that they are related by the map δ_r^*). One readily deduces:

$$c(T_{\delta_r}) = \prod_{1\leqslant i<j\leqslant r}(1+h_j-h_i),$$

and the statement follows. $\square$

Now we can prove the following Conjecture of G.Berhuy (proven by him for $n \leqslant 4$):

Theorem 6.3 $degree(c_{dim(G(n))}(-T_{G(n)})) \equiv 2^n \ (mod\, 2^{n+1})$.

Proof: By Proposition 6.2, Chern classes of $(-T_{G(n)})$ can be expressed as polynomials in the Chern classes of the tautological vector bundle V_n. From Theorem 2.5 we know that $c_j(V_n) = \sigma_j = (-1)^j 2z_j$, where z_j are elementary cycles defined in Proposition 2.4.

Since, in K_0, $[V_n] + [V_n^{\vee}] = 2n[\mathcal{O}]$, we get the relations on σ_j:

Lemma 6.4 $\sigma_i^2 = 2(-1)^i(\sigma_{2i} + \sum_{1\leqslant j<i}(-1)^j\sigma_j \cdot \sigma_{2i-j})$.

Proof: It is just the component of degree $2i$ of the relation

$$\left(1+\sum_i \sigma_i\right)\cdot\left(1+\sum_i(-1)^i\sigma_i\right) = 1.$$

$\square$

Let $A := \mathbb{Z}[\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n]$. We have ring homomorphism $\psi : A \to \mathrm{CH}^*(G(n))$ sending $\tilde{\sigma}_i$ to σ_i. It follows from the Lemma 6.4, that for arbitrary $f \in A$ there exists some $g \in A$ such that g does not contain squares, and $\psi(f-g) \in 2^{n+1}\,\mathrm{CH}^*(G(n))$. If f has degree $= dim(G(n))$, then g got to be monomial $\lambda \cdot \prod_{1\leqslant i\leqslant n}\sigma_i$. Moreover, if f was a monomial divisible by 2, or containing square, then λ will be divisible by 2. Consider ideal $L \subset A$ generated by 2 and squares of elements of positive degree. Let R be a quotient ring, and $\varphi : A \to R$ be the projection.

Since $\prod_{1\leqslant i\leqslant n}\sigma_i = (-1)^{n(n+1)/2}2^n\prod_{1\leqslant i\leqslant n} z_i$, and $\prod_{1\leqslant i\leqslant n} z_i$ is the class of a rational point (by Proposition 2.4), we get that for arbitrary $f \in A$, the $degree(\psi(f))$ is divisible by 2^n, and for $f \in L$ the degree is divisible by 2^{n+1}. Thus, modulo 2^{n+1}, the degree of $\psi(f)$ depends only on $\varphi(f)$.

In R we have the following equalities:

$$\varphi(\prod_{1\leqslant i<j\leqslant n}(1-(h_i-h_j)^2)) = 1, \text{ and}$$

$$\varphi(\prod_{1\leqslant i<j\leqslant n}(1-h_i-h_j)) = \varphi(\prod_{j=1}^{n-1}(1+\sum_{1\leqslant s\leqslant j}\sigma_s)).$$

Also, clearly, $\varphi((1+f)^2)=1$ for any f of positive degree. Thus,

$$\varphi(c(-T_{G(n)})) = \varphi(\prod_{j=1}^{n}(1+\sum_{1\leqslant s\leqslant j}\sigma_s)).$$

And the component of this polynomial of dimension $= dim(G(n))$ is just $\varphi(\prod_{1\leqslant i\leqslant n}\sigma_i)$. Consequently, $degree(c_{dim(G(n))}(-T_{G(n)})) \equiv 2^n \ (\,mod\, 2^{n+1})$.

□

We recall from [7] that a variety X is p-compressible if there is a rational map $X \dashrightarrow Y$ to some variety Y such that $dim(Y) < dim(X)$ and $v_p(n_X) \leqslant v_p(n_Y)$, where n_Z is the image of the degree map $deg : \mathrm{CH}_0(Z) \to \mathbb{Z}$.

From the Rost degree formula ([7, Theorem 6.4]) for the characteristic number $c_{dim(G(n))}$ modulo 2 (see [7, Corollary 7.3, Proposition 7.1]), we get:

Proposition 6.5 *Let Q be a smooth $2n+1$-dimensional quadric, all splitting fields of which have degree divisible by 2^n (we call such Q - generic). Then the variety $G(n, Q)$ is 2-incompressible.*

□

Call two smooth varieties X and Y equivalent if there are rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. Then let $d(X)$ be the minimal dimension of a variety equivalent to X. Recall from [1] that a *canonical dimension $cd(q)$* of a quadratic form q is defined as $d(G(n, Q))$, where $n = [dim(q)/2] + 1$.

Proposition 6.5 gives another proof of the fact that the canonical dimension of a generic $(2n+1)$-dimensional form is $n(n+1)/2$, which computes the canonical dimension of the groups SO_{2n+1} and SO_{2n+2} (cf. [5, Theorem 1.1, Remark 1.3]).

Our computations of the generic discrete invariant $GDI(m, Q)$ permit to conjecture the answer in the case of arbitrary smooth quadric Q:

Conjecture 6.6 *Let Q be smooth projective quadric of dimension d. Then*

$$cd(Q) = \sum_{j\in\{1,\dots,[d+1/2]\}\backslash J(Q)} j,$$

where $J(Q)$ is the invariant from the Definition 5.11.

If Q is generic, then $J(Q)$ is empty, and $cd(Q)$ is indeed equal $\sum_{1\leqslant i\leqslant n} i = n(n+1)/2$.

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