Hermitian forms and the u-invariant

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Abstract

We study the notion of hermitian u-invariant. We give some estimates of the u-invariant of a division algebra with involution in terms of the u-invariant of some subalgebras stable under the involution. We also find some finiteness results for comparing the u-invariant of a division algebra with involution and that of its centre. Some results about the values of this invariant are also given. A description of the Tits index of some algebraic groups of classical type over $\mathbb{Q}_p(t)$, $p \neq 2$ is given as an application.

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1 Introduction

The u-invariant is one of the most interesting invariants in the algebraic theory of quadratic forms. This invariant was introduced by Kaplansky over non formally real fields and by Elman and Lam [2] over formally real fields. (In this work we always assume that characteristic $\neq 2$).

Determination of this invariant for a given field, has been of great importance in the literature. For a wide class of fields, the values of this invariant are not known and sometimes it is not clear if the u-invariant of such fields is finite. Obtaining lower and upper bounds for this invariant and comparing the u-invariant of a given field K with the u-invariant of its subfields or the fields containing K are good approaches to the problem.

Systems of quadratic forms and exact sequences of Witt groups are two powerful tools used in the literature to obtain good lower and upper bounds for the u-invariant. For example, systems of quadratic forms have been used in [8] by Leep to obtain the bound $\mathrm{u}(L) \leqslant \frac{n+1}{2}\,\mathrm{u}(K)$ for a field extension L/K of degree n.

Using exact sequences of Witt groups is however less common than using systems of quadratic forms, but one can mention an exact triangle of Witt groups used by Elman and Lam (cf. [2], [3]) to compare the u-invariant of a field with the u-invariant of a quadratic extension of this field.

In this work, we try to adapt these tools in the context of hermitian forms over a division algebra with involution. To our best knowledge, the notion of the u-invariant of a division algebra with involution appears, for the first time, in Pfister's paper [13] in connection with systems of quadratic forms over a formally real field and with exception of [13], this notion has not been further studied in the literature.

Our initial motivation for studying this notion was to describe the Tits index of some classical groups over $\mathbb{Q}_p(t)$, $p \neq 2$ for which we would have need to obtain some information about the maximal dimension of anisotropic hermitian forms over some division algebras with involution over $\mathbb{Q}_p(t)$, $p \neq 2$. The results concerning Tits indices are given in the last section of this paper as applications of our main results. We later come back to this subject.

Let D be a division algebra with an involution σ and let ε be an element of the centre of D with $\sigma(\varepsilon)\varepsilon = 1$. The u-invariant with respect to ε, is by definition the supremum over the dimension of anisotropic ε-hermitian forms over (D, σ). This number is denoted by $\mathrm{u}(D, \sigma, \varepsilon)$ (value ∞ is admitted also). Let E be a subdivision algebra of D stable under σ. We are interested in the following questions:

Question 1.1. (Going up)
Under which conditions does the finiteness of $\mathrm{u}(E, \sigma|_E, \varepsilon)$ imply that of $\mathrm{u}(D, \sigma, \varepsilon)$ and can one obtain an upper bound for $\mathrm{u}(D, \sigma, \varepsilon)$ in terms of $\mathrm{u}(E, \sigma|_E, \varepsilon)$?

Question 1.2. (Going down)
Under which conditions does the finiteness of $\mathrm{u}(D, \sigma, \varepsilon)$ imply that of $\mathrm{u}(E, \sigma|_E, \varepsilon)$ and can one obtain a lower bound for $\mathrm{u}(D, \sigma, \varepsilon)$ in terms of $\mathrm{u}(E, \sigma|_E, \varepsilon)$?

A substantial part of this work is devoted to these questions. In certain situations, we are able to give precise answers. We will especially deal with the following cases:

- D/E is a quadratic extension of fields (cf. Remark 3.5, Proposition 5.1, Proposition 5.10).

- D is a quaternion algebra and E is a maximal subfield of D (cf. Corollary 3.4, Remark 3.5, Corollary 5.8).

- more generally if there exist two invertible elements λ and μ in D such that $\lambda\mu = -\mu\lambda$, $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$ and $K(\lambda)$ is a quadratic extension of K where K is the centre of D. Let E be the centralizer of $K(\lambda)$ in D (cf. Proposition 3.1, Remark 3.2, Proposition 5.7).

- E is the fixed field of the restriction of σ to the centre of D (cf. Proposition 3.6).

- D is the tensor product $E \otimes_k L$ where k is the fixed field of $\sigma|_K$, K is the centre of D and L/k is a field extension (cf. Proposition 3.8, Proposition 3.10, Remark 3.11).

In particular the Question 1.1 has an affirmative answer when E is the fixed field of $\sigma|_K$ where K is the centre of D. By contrast, the finiteness of $\mathrm{u}(D,\sigma,\varepsilon)$ does not imply, in general, that of $\mathrm{u}(k)$, for example if $D = (-1,-1)_{\mathbb{R}}$ is the algebra of Hamiltonian quaternions and σ is its canonical involution we have $\mathrm{u}(D,\sigma,-1) = 1$ and $\mathrm{u}(\mathbb{R}) = \infty$. In fact we prove that if $\dim_K D$ is a power of 2, and if both $\mathrm{u}(D,\sigma,\varepsilon)$ and $\mathrm{u}(D,\sigma,-\varepsilon)$ are finite then $\mathrm{u}(K,\sigma|_K) < \infty$ (cf. Theorem 6.2). In particular if σ is of the first kind, then $\mathrm{u}(K)$ is finite if and only if $\mathrm{u}(D,\sigma,1)$ and $\mathrm{u}(D,\sigma,-1)$ are finite.

The principal ideas to prove these results are to use some exact sequences of Witt groups, mainly the exact sequence of Milnor-Husemoller [19, Ch.10, 1.2], Lewis [9] and Parimala-Sridharan-Suresh [1, Appendix 2] and to use the results of Leep [8] and Pfister [13] on systems of quadratic and hermitian forms. We also use a variation of the exact sequence of Parimala-Sridharan-Suresh discussed in [4].

On one occasion, we need to compare the hermitian u-invariant of a field with that of some of its subfields. We do this by using an exact sequence of Witt groups of biquadratic extensions. We present this exact sequence in §4, (see Theorem 4.3). This exact sequence may be regarded as a particular case of a general exact sequence of L-groups and projective Witt groups due to Ranicki, see [16] and [17, p.242].

Another natural question is to ask about possible values for the u-invariant for some particular type of division algebra with involution. We consider the case of a quadratic extension L/K together with the nontrivial automorphism $\bar{}$ and we prove that $\mathrm{u}(L,\bar{}) \neq 3,5,7$ (cf. Proposition 7.3).

This result is a hermitian analogue of a classical theorem in the theory of quadratic forms which states that the u-invariant cannot be equal to 3, 5 and 7 (cf. [19, Ch.2, 16.2]). Our proof is of course similar to the proof of this theorem.

One could also define the notion of u-invariant for central simple algebras with involution. But thanks to Morita theory, the problem can be translated to division algebras with involution, so we are only interested in this situation.

As David Lewis pointed out to me, one can also define the notion of the u-invariant for G-equivariant forms, i.e., the forms stable under the action of a finite group G and with the same ideas one can obtain some results that are similar to those stated here.

Patrick Morandi pointed out to me that by using a version of a theorem of Springer for hermitian forms obtained by Larmour [7], one can calculate the u-invariant of a valued division algebra over a henselian valued field, in terms of the u-invariant of its residue division algebra.

Finally in section §8, we give some applications of these results to describe the Tits index of some classical groups over $\mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$. We recall that according to a generalized Witt decomposition theorem ([21]), semisimple algebraic groups defined over an arbitrary field K are determined up to isomorphism by their K^{sep} class, their anisotropic kernel and their Tits index, see Tits [21] or Springer [20, Ch. 17] for these notions. Describing the Tits index is therefore important. For the groups of classical type, these indices can be described in terms of algebras with involution. For finite fields, the field of real numbers, number fields and p-adic fields, all possible indices are known (see [21]). We try to describe these indices over $\mathbb{Q}_{\mathrm{p}}(t)$, $p \neq 2$ (cf. §8).

In consequence of the results due to Saltman [18] on the structure of the group $\mathrm{Br}_2(\mathbb{Q}_{\mathrm{p}}(t))$, the only division algebras over $\mathbb{Q}_{\mathrm{p}}(t)$ with an involution of the first kind are split ones, quaternions and biquaternions. For split case, by a result due to Parimala and Suresh [12] we have $8 \leqslant \mathrm{u}(\mathbb{Q}_{\mathrm{p}}(t)) \leqslant 10$ (an earlier result due to Hoffmann and Van Geel [5] states that $8 \leqslant \mathrm{u}(\mathbb{Q}_{\mathrm{p}}(t)) \leqslant 22$). By using these results and by examining anisotropic hermitian and skew hermitian forms over quaternion algebras and biquaternions algebras over $\mathbb{Q}_{\mathrm{p}}(t)$ we obtain some information about possible indices.

2 The u-invariant of a division algebra with involution

Let K be a field of characteristic different from 2. For a division algebra D over K, a K/k-involution σ on D is an involution where k is the fixed field of $\sigma|_K$.

Let us denote by $S^{\varepsilon}(D,\sigma)$ the semigroup of isometry classes of ε-hermitian forms over (D,σ) and by $W^{\varepsilon}(D,\sigma)$ the Witt group of ε-hermitian forms over (D,σ).

We refer to [19] and [6] for basic notions about quadratic and hermitian forms and algebras with involution.

Every division algebra over a field K considered in this paper is implicitly assumed to be K-central.

Let D be a division algebra over a field K with an involution σ. Let $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$. A system $h = (h_1, \cdots, h_r) : V \times V \to D^r$ of r, ε-hermitian forms over a right D-vector space V is called *anisotropic* if $x \in V$, $h(x,x) = 0$ implies that $x = 0$.

We consider the u-invariant in the sense of Kaplansky:

Definition 2.1. ([14, Ch. 9, Definition 2.4])

$$\mathrm{u}_r(D,\sigma,\varepsilon) = \sup\{\dim_D V;\ \text{there exists an anisotropic } \varepsilon\text{-hermitian map } h : V \times V \to D^r\}$$

Let us simplify the notation by writing $\mathrm{u}(D,\sigma,\varepsilon)$ instead of $\mathrm{u}_1(D,\sigma,\varepsilon)$ and by $\mathrm{u}(D,\sigma)$ instead of $u(D,\sigma,1)$.

We recall that according to a result due to Leep [8], the system u-invariant $\mathrm{u}_r = \mathrm{u}(k)$ satisfies $\mathrm{u}_r \leqslant r\,\mathrm{u}_1 + \mathrm{u}_{r-1}$ and $\mathrm{u}_r \leqslant \frac{r(r+1)}{2}\,\mathrm{u}_1$. This result has been generalized by Pfister [13] to the system u-invariant of ε-hermitian forms over a division algebra with involution and also over skew fields with involution not necessarily finite-dimensional over their centres.

As a first observation, we have:

Proposition 2.2. *Let D be a division algebra over a field K. Let σ and τ be two involutions on D with the same restriction to K: $\sigma|_K = \tau|_K$. Let $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$.*

(1) *If σ and τ are of the second kind and $\varepsilon' \in K$ with $\varepsilon'\tau(\varepsilon') = 1$ then* $\mathrm{u}(D,\sigma,\varepsilon)$ *and* $\mathrm{u}(D,\tau,\varepsilon')$ *do not depend on the choice of ε and ε'; moreover* $\mathrm{u}(D,\sigma,\varepsilon) = \mathrm{u}(D,\tau,\varepsilon')$.

(2) *If σ and τ are of the first kind then we have:* $\mathrm{u}(D,\sigma,\varepsilon) = \mathrm{u}(D,\tau,\varepsilon)$ *if σ and τ have the same type, otherwise we have* $\mathrm{u}(D,\sigma,\varepsilon) = \mathrm{u}(D,\tau,-\varepsilon)$.

Proof. For (1), there exists an element $b \in D^*$ such that $\sigma = \mathrm{Int}(b) \circ \tau$. We have $b\tau(b)^{-1} = \lambda \in K^*$. We obtain then $\lambda\tau(\lambda) = 1$. Let $\lambda' = \lambda^{-1}\varepsilon'\varepsilon^{-1}$. We have $\lambda'\tau(\lambda') = 1$. It follows from Hilbert 90 that there exists $\mu \in K^*$ such that $\mu\tau(\mu^{-1}) = \lambda'$.

If we take $c = \mu b$, then $\sigma = \mathrm{Int}(b) \circ \tau = \mathrm{Int}(c) \circ \tau$. Now the correspondence $\varphi \mapsto c^{-1}\varphi$ gives a bijection between the semigroups $S^{\varepsilon}(D,\sigma)$ and $S^{\varepsilon'}(D,\tau)$. This bijection preserves isometry, orthogonal sum and dimension. So we conclude that $\mathrm{u}(D,\sigma,\varepsilon) = \mathrm{u}(D,\tau,\varepsilon')$.

For (2), we use the same argument. There exists $b \in D^*$ such that $\sigma = \mathrm{Int}\,(b) \circ \tau$. We have $\lambda := b\tau(b)^{-1} \in K^*$. Moreover $\lambda = 1$ if σ and τ are of the same type and $\lambda = -1$ if σ and τ are of different type. The correspondence $\varphi \mapsto b^{-1}\varphi$ from $S^{\varepsilon}(D,\sigma)$ to $S^{\lambda\varepsilon}(D,\tau)$ gives $\mathrm{u}(D,\sigma,\varepsilon) = \mathrm{u}(D,\tau,\lambda\varepsilon)$. □

Remark 2.3. As pointed out to me by the Referee, the preceding proposition actually says that for a given division algebra D, there are three possible u-invariants, let us say a unitary, an orthogonal and a symplectic one; namely for any σ and ε, the u-invariant of (D,σ,ε) coincides with one of them, depending on the type of σ and the value of ε. Note that for unitary case, the preceding proposition states that for a given D, $\mathrm{u}(D,\sigma,\varepsilon)$ depends only to the restriction of σ to the centre of D. This leads us to introduce the notation $\mathrm{u}^+(D)$ for the orthogonal u-invariant, $\mathrm{u}^-(D)$ for the symplectic u-invariant (this was also suggested to me by Karim Becher). In this way, if τ_1 is a symplectic involution and τ_2 is an orthogonal involution on D we have $\mathrm{u}^+(D) = \mathrm{u}(D,\tau_1,-1) = \mathrm{u}(D,\tau_2,1)$ and $\mathrm{u}^-(D) = \mathrm{u}(D,\tau_1,1) = \mathrm{u}(D,\tau_2,-1)$. This point of view might lead to some simplification in the presentation of some parts of this paper, cf. for instance Corollary 3.4 and Proposition 3.6 or Theorem 6.2.

3 Going up results

Let D be a division algebra over a field K with an involution σ. We suppose that there exist two invertible elements λ and μ in D such that $\lambda\mu = -\mu\lambda$, $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$ and $K(\lambda)$ is a quadratic extension of K.

Let $\widetilde{D}$ be the centralizer of $L = K(\lambda)$ in D. According to [1, Appendix 2], we have $\mu\widetilde{D}\mu^{-1} = \widetilde{D}$, $\mu^2 \in \widetilde{D}$, $\mu^2 \in \widetilde{D}$ and $D = \widetilde{D} \oplus \mu\widetilde{D}$. On $\widetilde{D}$ we have two natural involutions $\sigma_1 = \sigma|_D$ and $\sigma_2 = \mathrm{Int}(\mu^{-1}) \circ \sigma_1$. We have $\deg \widetilde{D} = \frac{1}{2}\deg D$. The involution σ_1 is always of the second kind. The involution σ_2 is of the same kind as σ but of different type if σ is of the first kind. See [1, §3.1] for more details.

Let $\pi_i : D \to \widetilde{D}$ be the L-linear projections $\pi_1(\alpha+\mu\beta) = \alpha$ and $\pi_2(\alpha+\mu\beta) = \beta$. If $h : V \times V \to D$ is a ε-hermitian space over (D, σ), then $h_i : V \times V \to \widetilde{D}$ is defined by $h_i(x, y) = \pi_i(h(x, y))$. It is easily verified that h_1 is an ε-hermitian space over $(\widetilde{D}, \sigma_1)$ and h_2 is an $-\varepsilon$-hermitian space over $(\widetilde{D}, \sigma_2)$. See [1, Appendix 2] for more details.

We prove the following proposition which plays an important role in this paper.

Proposition 3.1. *Let D be a division algebra over a field K with a K/k-involution σ. Suppose that there exist λ, $\mu \in D^*$ such that $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$, $\lambda\mu = -\mu\lambda$ and $L = K(\lambda)$ is a quadratic extension of K. Let $\widetilde{D}$ be the centralizer of L in D, $\sigma_1 = \sigma|_{\widetilde{D}}$ and $\sigma_2 = \mathrm{Int}(\mu^{-1}) \circ \sigma_1$ and $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$. Then we have:*

$$\mathrm{u}(D, \sigma, \varepsilon) \leqslant \frac{1}{2}\,\mathrm{u}(\widetilde{D}, \sigma_2, -\varepsilon) + \mathrm{u}(\widetilde{D}, \sigma_1, \varepsilon).$$

Proof. Let π_1 and π_2 be the projections from D to $\widetilde{D}$ induced by the decomposition $D = \widetilde{D} \oplus \mu\widetilde{D}$, i.e, $\pi_i(d_1 + \mu d_2) = d_i$, for $i = 1, 2$. Let (V, h) be a nondegenerate ε-hermitian space over (D, σ) and $h_1 = \pi_1 h$, $h_2 = \pi_2 h$. We have $\dim_{\widetilde{D}}(h_1) = \dim_{\widetilde{D}}(h_2) = 2\dim_D(h)$. If $\dim_{\widetilde{D}}(h_2) \geqslant \mathrm{u}(\widetilde{D}, \sigma_2, -\varepsilon) + 2m - 1$ for some positive integer $m \geqslant 1$, then h_2 contains an orthogonal sum of m hyperbolic planes. Consequently h_2 is totally isotropic over a $\widetilde{D}$-vector subspace W of V of dimension m. If moreover $m \geqslant \mathrm{u}(\widetilde{D}, \sigma_1, \varepsilon) + 1$ then h_1 is isotropic over W. In this way, in order that h be isotropic, it is sufficient to have $2n = \dim_{\widetilde{D}}(h_2) \geqslant \mathrm{u}(\widetilde{D}, \sigma_2, -\varepsilon) + 2\,\mathrm{u}(\widetilde{D}, \sigma_1, \varepsilon) + 1$. This is equivalent to

$$\mathrm{u}(D, \sigma, \varepsilon) \leqslant \tfrac{1}{2}\,\mathrm{u}(\widetilde{D}, \sigma_2, -\varepsilon) + \mathrm{u}(\widetilde{D}, \sigma_1, \varepsilon). \qquad \square$$

Remark 3.2. In the proof of the previous proposition, one may interchange the role of h_1 et h_2, in this way we obtain:

$$\mathrm{u}(D, \sigma, \varepsilon) \leqslant \mathrm{u}(\widetilde{D}, \sigma_2, -\varepsilon) + \frac{1}{2}\,\mathrm{u}(\widetilde{D}, \sigma_1, \varepsilon).$$

Remark 3.3. One may also give an alternative proof of the previous result by using an exact sequence of Parimala-Sridharan-Suresh. See the proof of Proposition 5.7 which uses this idea.

Corollary 3.4. *Let $Q = (a, b)_K$ be a quaternion division algebra over a field K. Let $^-$ be the canonical involution of Q and $\hat{}$ an orthogonal involution of Q and let $L = K(\sqrt{a}) \subset Q$ which is stable under $^-$, then we have:*

$$\begin{aligned}
&\mathrm{u}(Q, \hat{}\,) = \mathrm{u}(Q, {}^-, -1) \leqslant \min\{\tfrac{1}{2}\,\mathrm{u}(L) + \mathrm{u}(L, {}^-), \mathrm{u}(L) + \tfrac{1}{2}\,\mathrm{u}(L, {}^-)\},\\
&\mathrm{u}(Q, {}^-) = \mathrm{u}(Q, \hat{}, -1) \leqslant \tfrac{1}{2}\,\mathrm{u}(L, {}^-).
\end{aligned}$$

Remark 3.5. Let L/K be a quadratic extension and let $^-$ be its nontrivial automorphism. We have the bound $\mathrm{u}(L, {}^-) \leqslant \frac{1}{2}\mathrm{u}(K)$ because to every anisotropic hermitian form over $(L, {}^-)$ of dimension n, one can associate an anisotropic quadratic form over K of dimension $2n$. In the same way, if Q is a quaternion algebra over a field K with the canonical involution $^-$, we have $\mathrm{u}(Q,{}^-) \leqslant \frac{1}{4}\,\mathrm{u}(K)$. See also Proposition 3.6 which states a more general result.

Proposition 3.6. *Let D be a division algebra of degree m over its centre K with a K/k-involution σ and let $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$. Then :*

$$\mathrm{u}(D,\sigma,\varepsilon) \leqslant \frac{r(r+1)}{2m^2[K:k]}\,\mathrm{u}(k) \tag{1}$$

where r is the dimension of k-vector space of ε-hermitian elements of D. In particular, if $\mathrm{u}(k)$ is finite then so is $\mathrm{u}(D,\sigma,\varepsilon)$.

Proof. Let (V,h) be an anisotropic ε-hermitian space over (D,σ) of dimension n. Take D^ε the k-vector space of ε-hermitian elements of D, in other words:

$$D^\varepsilon = \{x \in D : \sigma(x) = \varepsilon x\}.$$

It is well known that $D = D^{+1} \oplus D^{-1}$ and $\dim_k(D^{+1}) = \dim_k(D^{-1}) = m^2$ when σ is of the second kind and $\dim_k(D^{+1}) = \frac{1}{2}m(m+1)$ or $\frac{1}{2}m(m-1)$ when σ is of the first kind. Let $\{e_1,\cdots,e_r\}$ be a k-basis of D^ε and $\{f_1,\cdots,f_s\}$ a k-basis of $D^{-\varepsilon}$. One can write h in the form:

$$\begin{array}{rcl} h(x,y) & = & \varphi_1(x,y)e_1 + \cdots + \varphi_r(x,y)e_r \\ & + & \psi_1(x,y)f_1 + \cdots + \psi_s(x,y)f_s \end{array}$$

where $\varphi_1,\cdots,\varphi_r$ are symmetric bilinear forms and $\psi_1,\cdots,\psi_s$ are skew symmetric bilinear forms over k (the forms φ_i $(1 \leqslant i \leqslant r)$ and ψ_i $(1 \leqslant i \leqslant s)$ are possibly degenerate). So $\psi_i(x,x) = 0$ for all $x \in V$. We deduce that

$$h(x,x) = \varphi_1(x,x)e_1 + \cdots + \varphi_r(x,x)e_r.$$

As h is anisotropic, $\varphi_1,\cdots,\varphi_r$ have no common isotropic vector. Now by using a result due to Leep (cf. [14, Ch 9, 2.1] or [19, Ch.2, 16.5]) we obtain:

$$m^2[K:k]n = \dim_k(V) \leqslant \mathrm{u}_r(k) \leqslant \frac{r(r+1)}{2}\,\mathrm{u}(k).$$

Therefore $n \leqslant \frac{r(r+1)}{2m^2[K:k]}\,\mathrm{u}(k)$ which implies the claimed inequality (1). □

Remark 3.7. The previous result is a finiteness statement. For many situations, one may have better estimates for $\mathrm{u}(D,\sigma,\varepsilon)$. For example for quadratic extensions and quaternion algebras, see Proposition 3.1, Remark 3.2, Corollary 3.4, Proposition 5.1, Proposition 3.8, Proposition 3.10, Remark 3.11. Nevertheless this bound is optimal for $m = 1$ or ($m = 2$ and σ symplectic).

Proposition 3.8. *Let D be a division algebra over a field K with a K/k-involution σ. Let L/k be an extension of degree n and suppose that $D \otimes_k L$ is also a division algebra. Then*

$$\mathrm{u}(D \otimes_k L, \sigma \otimes \mathrm{id}, \varepsilon) \leqslant \frac{n+1}{2}\,\mathrm{u}(D,\sigma,\varepsilon). \tag{2}$$

In particular the finiteness of $\mathrm{u}(D,\sigma,\varepsilon)$ implies that of $\mathrm{u}(D \otimes_k L, \sigma \otimes \mathrm{id}, \varepsilon)$.

Proof. Let (V, φ) be an ε-hermitian space over $(D \otimes_k L, \sigma \otimes \mathrm{id})$. We choose a k-basis $\{e_1, \cdots, e_n\}$ of L. We can write $\varphi(x, y) = (\varphi_1(x, y) \otimes e_1)) + \cdots + (\varphi_n(x, y) \otimes e_n)$ where $\varphi_1, \cdots, \varphi_n$ are ε-hermitian forms (possibly degenerate) over (D, σ). If φ is anisotropic and $\dim(\varphi) = m$, then $\varphi_1, \cdots, \varphi_n$ have no common isotropic vector, therefore $mn = \dim(\varphi_i) \leqslant \mathrm{u}_n(D, \sigma, \varepsilon)$ so:

$$\mathrm{u}(D \otimes_k L, \sigma \otimes \mathrm{id}, \varepsilon) \leqslant \frac{1}{n} \mathrm{u}_n(D, \sigma, \varepsilon).$$

According to a result due to Pfister (cf. [14, Ch.9, 2.5] or [13]) we have: $\mathrm{u}_n(D, \sigma, \varepsilon) \leqslant \frac{n(n+1)}{2} u(D, \sigma, \varepsilon)$. This implies (2). □

Remark 3.9. If in the previous statement we take $D = k$, then we retrieve Leep's estimate $\mathrm{u}(L) \leqslant \frac{n+1}{2}\, \mathrm{u}(k)$.

Proposition 3.10. *Let D be a division algebra over a field K with a K/k-involution σ. Let L/k be a quadratic extension and let $^- : L \longrightarrow L$ be the nontrivial k-automorphism of L. Suppose that $D \otimes_k L$ is a division algebra. Then we have:*

$$\mathrm{u}(D \otimes_k L, \sigma \otimes {}^-, \varepsilon) \leqslant \frac{1}{2}\, \mathrm{u}(D, \sigma, -\varepsilon) + \mathrm{u}(D, \sigma, \varepsilon).$$

Proof. Let $L = k(\xi)$ with $\xi^2 \in k$ and $\bar{\xi} = -\xi$. Let (V, φ) be an ε-hermitian space over $(D \otimes_k L, \sigma \otimes {}^-)$. We can write φ in the form

$$\varphi(x, y) = \varphi_1(x, y) \otimes 1 + \varphi_2(x, y) \otimes \xi,$$

where φ_1 is an ε-hermitian form and φ_2 is a $-\varepsilon$-hermitian form over (D, σ). By repeating the argument given in the proof of Proposition 3.1 we conclude the result. □

Remark 3.11. In the proof of Proposition 3.10, one can interchange the role of φ_1 and φ_2, in this way we obtain:

$$\mathrm{u}(D \otimes_k L, \sigma \otimes {}^-, \varepsilon) \leqslant \mathrm{u}(D, \sigma, -\varepsilon) + \frac{1}{2}\, \mathrm{u}(D, \sigma, \varepsilon).$$

4 An exact sequence of Witt groups for biquadratic extensions

Let $L = K_1 \otimes_k K_2/k$ be a field extension of degree 4 where K_1/k and K_2/k are two quadratic extensions with nontrivial automorphisms τ_1 et τ_2 (resp.). Suppose that $K_2 = k(\lambda)$ with $\lambda \in K_2$, $\lambda^2 \in k$ and $\tau_2(\lambda) = -\lambda$. Every element α of L can be uniquely written in the form $\alpha_1 \otimes 1 + \alpha_2 \otimes \lambda$ where $\alpha_1,\ \alpha_2 \in K_1$. We consider two projections:

$$\begin{array}{ll} \pi_1 : L \longrightarrow K_1 & \pi_2 : L \longrightarrow K_1 \\ \alpha \mapsto \alpha_1 & \alpha \mapsto \alpha_2 \end{array}$$

For every nondegenerate hermitian space (V, h) over $(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$ (resp. over $(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$), we associate the hermitian space $(V, \pi_1 h)$ (resp. $(V, \pi_2 h)$) over (K_1, τ_1) defined by

$$\begin{array}{l} (\pi_1 h)(x, y) = \pi_1(h(x, y)),\ x,\ y \in V \\ (\pi_2 h)(x, y) = \pi_2(h(x, y)),\ x,\ y \in V \end{array}$$

It is easy to check that $\pi_1 h$ and $\pi_2 h$ are nondegenerate hermitian forms over (K_1, τ_1).

For every nondegenerate hermitian space (W, f) over (K_1, τ_1), we associate the hermitian spaces $(W \otimes_k K_2, \rho_1 f)$ over $(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$ and $(W \otimes_k K_2, \rho_2 f)$ over $(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$ by

$$\begin{array}{ll} (\rho_1 f)(x \otimes \alpha, y \otimes \beta) = f(x, y) \otimes \alpha\beta, & x,\ y \in W;\ \ \alpha,\ \beta \in K_2 \\ (\rho_2 f)(x \otimes \alpha, y \otimes \beta) = f(x, y) \otimes \tau_2(\alpha)\beta, & x,\ y \in W;\ \ \alpha,\ \beta \in K_2 \end{array}$$

Proposition 4.1. (1) *Let (V, h) be a nondegenerate hermitian form of dimension 1 over $(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$ with $h \simeq \langle d \rangle$ where $d \in K_1 \otimes_k K_2$ and $(\tau_1 \otimes \tau_2)(d) = d$. Then $\pi_1 h$ is isometric to*

$$\begin{pmatrix} d_1 & d_2\lambda^2 \\ -d_2\lambda^2 & -d_1\lambda^2 \end{pmatrix}$$

where $d_1 = \pi_1(d)$ and $d_2 = \pi_2(d)$.
(2) *Let (V, h) be a nondegenerate hermitian space of dimension 1 over $(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$ with $h \simeq \langle d \rangle$ where $d \in K_1 \otimes_k K_2$ and $(\tau_1 \otimes \mathrm{id})(d) = d$. Then $\pi_2 h$ is isometric to*

$$\begin{pmatrix} d_2 & d_1 \\ d_1 & d_2\lambda^2 \end{pmatrix}$$

where $d_1 = \pi_1(d)$ and $d_2 = \pi_2(d)$.
(3) *For the one dimensional form $f = \langle a \rangle$ over (K_1, τ_1) we have*

$$\rho_1(f) \simeq \langle a \otimes 1 \rangle, \quad \rho_2(f) \simeq \langle a \otimes 1 \rangle.$$

Proof. (1) Let $0 \neq x \in V$ with $h(x, x) = d$. We have a basis $\{x, x\lambda\}$ for the K_1-vector space V. In this basis we have:

$$\begin{array}{ll} (\pi_1 h)(x, x) = d_1, & (\pi_1 h)(x, x\lambda) = d_2\lambda^2 \\ (\pi_1 h)(x\lambda, x) = -d_2\lambda^2 & (\pi_1 h)(x\lambda, x\lambda) = -d_1\lambda^2 \end{array}$$

These relations imply the isometry we are looking for. The proof of (2) is similar and (3) is obvious. □

Proposition 4.2. (1) *Let f be an anisotropic hermitian form over (K_1, τ_1) such that $\rho_1 f$ is isotropic. Then f contains a subform isometric to $\pi_1(\langle d \rangle)$ where $\langle d \rangle$ is a one dimensional form over $(K_1 \otimes K_2, \tau_1 \otimes \tau_2)$.*
(2) *Let h be an anisotropic hermitian form over $(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$ such that $\pi_2 h$ is isotropic , then there exists a one dimensional form $\langle a \rangle$ over (K_1, τ_1)*

such that f contains a subform isometric to $\rho_1(\langle a\rangle)$.
(3) Let f be an isotropic hermitian form over (K_1,τ_1) such that $\rho_2 f$ is isotropic. Then f contains a subform isometric to $\pi_2(\langle d\rangle)$ where $\langle d\rangle$ is a one dimensional hermitian form over $(K_1\otimes_k K_2,\tau_1\otimes \mathrm{id})$.

Proof. (1) Let $v=x_1\otimes 1+y_1\otimes\lambda\neq 0$ be an isotropic vector for $\rho_1(f)$, i.e., $\rho_1(f)(v,v)=0$. This relation implies that

$$\begin{cases} f(x_1,x_1)+f(y_1,y_1)\lambda^2=0 \\ f(x_1,y_1)+f(y_1,x_1)=0 \end{cases} \tag{3}$$

The vectors x_1 and y_1 are linearly independent over K_1. In fact if $x_1=\alpha y_1$ for some $\alpha\in K_1$, the previous system gives

$$\begin{cases} \alpha\tau_1(\alpha)+\lambda^2=0 \\ \tau_1(\alpha)+\alpha=0 \end{cases}$$

which implies that $\alpha^2=\lambda^2$ so $K_1\simeq K_2$, contradiction because $K_1\otimes_k K_2$ is a field. Now consider the K_1-vector space W generated by x_1 and y_1. For $d_1=f(y_1,y_1)$ and $d_2=f(y_1,x_1)\lambda^{-2}$, the representing matrix of $f|_W$ in the basis $\{y_1,x_1\}$ is

$$\begin{pmatrix} d_1 & d_2\lambda^2 \\ -d_2\lambda^2 & -d_1\lambda^2 \end{pmatrix}$$

According to Proposition 4.1, for a hermitian element $d=d_1\otimes 1+d_2\otimes\lambda$ with respect to $(\tau_1\otimes\tau_2)$ we have

$$\pi_1(\langle d\rangle)\simeq \begin{pmatrix} d_1 & d_2\lambda^2 \\ -d_2\lambda^2 & -d_1\lambda^2 \end{pmatrix}$$

so f contains a subform isometric to $\pi_1(\langle d\rangle)$.
(2) Let $x\neq 0$ be an isotropic vector for $\pi_2 h$, i.e., $(\pi_2 h)(x,x)=0$. This relation implies that $h(x,x)=a\otimes 1\in K_1\otimes_k K_2$ for some $a\in K_1$. Since h is anisotropic, we have $a\neq 0$. We deduce then that h contains a subform isometric to $\rho_1(\langle a\rangle)$.
(3) The argument is similar to that of (1). Let $v=x_1\otimes 1+y_1\otimes\lambda\neq 0$ be an anisotropic vector for $\rho_2 h$, i.e., $(\rho_2 h)(v,v)=0$. This relation implies that

$$\begin{cases} f(x_1,x_1)-f(y_1,y_1)\lambda^2=0 \\ f(x_1,y_1)-f(y_1,x_1)=0 \end{cases}$$

The vectors x_1 and y_1 are linearly independent over K_1. Let W be the K_1-vector space generated by x_1 and y_1. The matrix of the form $f|_W$ with respect to the basis $\{y_1,x_1\}$ is

$$\begin{pmatrix} d_2 & d_1 \\ d_1 & d_2\lambda^2 \end{pmatrix}$$

According to Proposition 4.1, for the one dimensional hermitian form $\langle d\rangle$ over $(K_1\otimes_k K_2,\tau_1\otimes\mathrm{id})$ where $d=d_1\otimes 1+d_2\otimes\lambda$ we have:

$$\pi_2(\langle d\rangle)\simeq \begin{pmatrix} d_2 & d_1 \\ d_1 & d_2\lambda^2 \end{pmatrix}$$

which completes the proof. $\square$

Theorem 4.3. *We have the following exact sequence of Witt groups:*

$$W(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2) \stackrel{\pi_1}{\to} W(K_1, \tau_1) \stackrel{\rho_1}{\to} W(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$$

$$\stackrel{\pi_2}{\to} W(K_1, \tau_1) \stackrel{\rho_2}{\to} W(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$$

Proof. Thanks to the previous proposition, it is enough to verify that this sequence is a complex. Let $\langle d \rangle$ be a one dimensional hermitian form over $(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$ where $d = d_1 \otimes 1 + d_2 \otimes \lambda$ where $d_1,\ d_2 \in K_1$. According to Proposition 4.1 we have

$$\pi_1(\langle d \rangle) \simeq \begin{pmatrix} d_1 & d_2\lambda^2 \\ -d_2\lambda^2 & -d_1\lambda^2 \end{pmatrix}.$$

We obtain then

$$\rho_1\pi_1(\langle d \rangle) \simeq \begin{pmatrix} d_1 \otimes 1 & d_2\lambda^2 \otimes 1 \\ -d_2\lambda^2 \otimes 1 & -d_1\lambda^2 \otimes 1 \end{pmatrix}$$

A simple calculation shows that

$$v = \begin{bmatrix} 1 \otimes 1 \\ 1 \otimes \lambda^{-1} \end{bmatrix}$$

is an isotropic vector for $\rho_1\pi_1(\langle d \rangle)$. Consequently we have $\rho_1\pi_1 = 0$.

In order to show that $\pi_2\rho_1 = 0$, we consider a one dimensional form $\langle a \rangle$ over (K_1, τ_1). We have $\rho_1(\langle a \rangle) \simeq \langle a \otimes 1 \rangle$. So

$$\pi_2\rho_1(\langle a \rangle) \simeq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

which is a hyperbolic form.

In order to show that $\rho_2\pi_2 = 0$, we consider a one dimensional form $\langle d \rangle$ over $(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id})$ where $d = d_1 \otimes 1 + d_1 \otimes \lambda$ is a hermitian element with respect to $(\tau_1 \otimes \mathrm{id})$ with $d_1,\ d_2 \in K_1$. We have:

$$\rho_2\pi_2(\langle d \rangle) \simeq \begin{pmatrix} d_2 \otimes 1 & d_1 \otimes 1 \\ d_1 \otimes 1 & d_2\lambda^2 \otimes 1 \end{pmatrix}.$$

This form is hyperbolic because

$$v = \begin{bmatrix} 1 \otimes 1 \\ 1 \otimes \lambda^{-1} \end{bmatrix}$$

is an isotropic vector. □

5 Going down results

Proposition 5.1. *Let L/K be a quadratic extension and let $^-$ be its nontrivial automorphism. Then* $\mathrm{u}(L)$ *and* $\mathrm{u}(L,^-)$ *are finite if and only if* $\mathrm{u}(K)$ *is finite, moreover we have* $\mathrm{u}(K) \leqslant 2\,\mathrm{u}(L,^-) + \mathrm{u}(L)$ *and* $\mathrm{u}(K) \leqslant \mathrm{u}(L,^-) + 2\,\mathrm{u}(L)$.

Proof. We write $L = K(\sqrt{a})$ where $a \in K^*$. Let q be an anisotropic form of dimension n over K. Consider the following exact sequence of Witt groups due to Milnor-Husemoller [19, Ch.10, 1.2]:

$$0 \to W(L,^-) \xrightarrow{\pi} W(K) \xrightarrow{r^*} W(L)$$

In this exact sequence, π is the transfer map induced by the projection $\pi : L \to K,\ x + y\sqrt{a} \mapsto x$ and r^* is the restriction map. If $r^*(q)$ is anisotropic, then q contains a subform isometric to $\langle b, -ab\rangle$ for some $b \in K$. We can then write (by induction)

$$q \simeq (\langle 1, -a\rangle \otimes q_1) \oplus q_2$$

where q_1, q_2 are two nondegenerate quadratic forms over K so that $r^*(q_2)$ is anisotropic. Let φ be the hermitian form over $(L,^-)$ induced by q_1. We have $\pi(\varphi) \simeq \langle 1, -a\rangle \otimes q_1$. Therefore we have a Witt r^*-decomposition, i.e., there exists an orthogonal decomposition $q \simeq q' \perp q''$ where $q' \simeq \pi(\varphi)$ for some nondegenerate hermitian form φ over $(L,^-)$ (in particular $r^*(q')$ is hyperbolic) and $r^*(q'')$ is anisotropic. We may suppose that φ is anisotropic, therefore: $\dim(q) = \dim(\pi(\varphi)) + \dim(q'') = 2\dim(\varphi) + \dim(q'') \leqslant 2\,\mathrm{u}(L,^-) + \mathrm{u}(L)$ which implies the result. If we use the same argument with the following exact sequence of Witt groups due to Lewis [9]:

$$W(L) \xrightarrow{s_*} W(K) \xrightarrow{r^*} W(L,^-),$$

we obtain $\mathrm{u}(K) \leqslant \mathrm{u}(L,^-) + 2\,\mathrm{u}(L)$. In this exact sequence, s_* is the Scharlau transfer map and r^* is the restriction map.

Now suppose that we have $\mathrm{u}(K) < \infty$. According to a result due to Elman and Lam (cf. [2, theorem 4.3]) we have $\mathrm{u}(L) \leqslant \frac{3}{2}\,\mathrm{u}(K) < \infty$. The finiteness of $\mathrm{u}(L,^-)$ is easy to check. In fact if $\varphi = \langle a_1, \cdots, a_n\rangle$ is a form over $(L,^-)$ where $a_i \in K$, its trace form is isometric to $\psi = \langle a_1, \cdots, a_n, -aa_1, \cdots, -aa_n\rangle$. Isotropy of φ and ψ are equivalent. The finiteness of $\mathrm{u}(K)$ implies therefore that of $\mathrm{u}(L,^-)$. $\square$

Remark 5.2. One can regard the estimates of Proposition 5.1 as an improvement, in some situations, of the estimate $\mathrm{u}(K) \leqslant 4\,\mathrm{u}(L)$ for a non formally real field K due to Elman [3, Theorem 3.1(iii)], because one can find many examples where

$$\min\{2\,\mathrm{u}(L,^-) + \mathrm{u}(L)\ ,\ \mathrm{u}(L,^-) + 2\,\mathrm{u}(L)\} < 4\,\mathrm{u}(L).$$

For example if K is a p-adic field then $\mathrm{u}(K) = \mathrm{u}(L) = 4$ and $\mathrm{u}(L,^-) \leqslant 2$. Our bound is independent of the formally real nature of L or K. To our best knowledge, the going down result of Elman, is the best one for arbitrary fields.

In the sequel, we need the following lemma which is an immediate consequence of a theorem of Springer.

Lemma 5.3. *Let L/K be a quadratic extension and let τ be its nontrivial automorphism. Let M/K be an extension of odd degree and let φ be an anisotropic hermitian form over (L,τ). Then φ remains anisotropic over $(L\otimes_K M, \tau\otimes \mathrm{id})$.*

Proof. For a hermitian space (V, φ) over (L, τ), we denote its trace form by $tr(\varphi)$, defined by $tr(\varphi)(x, y) = tr_{L/K}(\varphi(x, y))$ for every $x,\ y \in V$. We have $tr(\varphi|_{(L\otimes_K M)}) \simeq tr(\varphi)|_M$. If $\varphi|_{(L\otimes_K M)}$ is isotropic, then $tr(\varphi)|_M$ is also isotropic. The strong version of Springer's theorem implies that $tr(\varphi)$ is isotropic. Consequently φ is isotropic. $\square$

Corollary 5.4. *With the notation of the previous lemma we have:*

$$\mathrm{u}(L, \tau) \leqslant \mathrm{u}(L \otimes_K M, \tau \otimes \mathrm{id}).$$

Let D be a division algebra over a field K with an involution σ. Suppose that there exist $\lambda,\ \mu \in D^*$ such that $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$, $\lambda\mu = -\mu\lambda$ and $L = K(\lambda)$ is a quadratic extension of K. In this situation, we have the following exact sequence of Witt groups due to Parimala, Sridharan and Suresh [1, Appendix 2]:

$$W^{\varepsilon}(D, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\widetilde{D}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(D, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{D}, \sigma_2) \qquad (4)$$

In this sequence, the map π_1^{ε} and $\pi_2^{-\varepsilon}$ are transfers induced by the projections π_1 and π_2 defined in §3. The map ρ_1^{ε} is a restriction map defined by the multiplication by λ and the usual restriction map r^*:

$$\begin{tikzcd} W^{\varepsilon}(\widetilde{D}, \sigma_1) \arrow[rr, "\rho_1"] \arrow[dr, "\times\lambda"'] & & W^{-\varepsilon}(D, \sigma) \\ & W^{-\varepsilon}(\widetilde{D}, \sigma_1) \arrow[ur, "r^*"'] & \end{tikzcd}$$

See [1, Appendix 2] for more details. These maps also induce homomorphisms between semigroups of isometry classes of hermitian forms:

$$S^{\varepsilon}(D, \sigma) \xrightarrow{\pi_1^{\varepsilon}} S^{\varepsilon}(\widetilde{D}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} S^{-\varepsilon}(D, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} S^{\varepsilon}(\widetilde{D}, \sigma_2) \qquad (5)$$

Now we can reformulate the exact sequence of (4) in the following way:

Proposition 5.5. *Let $\varphi \in S^{\varepsilon}(\widetilde{D}, \sigma_1)$ be an anisotropic form. Then φ has a Witt ρ_1^{ε}-decomposition, i.e., there exists an orthogonal decomposition $\varphi \simeq \varphi_1 \oplus \varphi_2$ such that $\rho_1^{\varepsilon}(\varphi_1)$ is hyperbolic and $\rho_1^{\varepsilon}(\varphi_2)$ is anisotropic. Moreover there exists $\psi \in S^{\varepsilon}(D, \sigma)$ such that $\varphi_1 \simeq \pi_1^{\varepsilon}(\psi)$.*

Proof. If $\rho_1^{\varepsilon}(\varphi)$ is anisotropic, we take $\varphi_2 = \varphi$. If $\rho_1^{\varepsilon}(\varphi)$ is isotropic then φ contains a subform φ_0 which comes from $S^{\varepsilon}(D, \sigma)$, i.e., $\varphi_0 \simeq \pi_1^{\varepsilon}(\psi_0)$ for some $\psi_0 \in S^{\varepsilon}(D, \sigma)$; see the proof of (4) in [1, Appendix 2] where this has been implicitly proved, see also [4, 4.4]. We have then an orthogonal decomposition $\varphi \simeq \varphi_0 \oplus \varphi'$ for some $\varphi' \in S^{\varepsilon}(\widetilde{D}, \sigma_1)$. As $\rho_1^{\varepsilon}(\varphi_0)$ is hyperbolic and $\dim \varphi' < \dim \varphi$ we can use induction on $\dim \varphi$ to finish the proof. $\square$

We have also the following exact sequence of Witt groups:

$$W^{-\varepsilon}(D, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{D}, \sigma_2) \xrightarrow{\rho_2^{\varepsilon}} W^{-\varepsilon}(D, \sigma) \qquad (6)$$

which is a variation of the exact sequence of Parimala, Sridharan and Suresh, see [4]. In this sequence, ρ_2^ε is the composition of the multiplication by $-\lambda$ and the usual restriction map r^* and the multiplication by $-\mu$:

$$\begin{tikzcd} W^\varepsilon(\widetilde{D},\sigma_2) \arrow[r, "\rho_2"] \arrow[d, "-\lambda"] & W^{-\varepsilon}(D,\sigma) \\ W^\varepsilon(\widetilde{D},\sigma_2) \arrow[r, "r^*"'] & W^\varepsilon(D,\operatorname{Int}(\mu^{-1})\circ\sigma) \arrow[u, "\times\mu"'] \end{tikzcd}$$

We state a Witt-decomposition-like result for the exact sequence of (6):

Proposition 5.6. *Let $\varphi \in S^\varepsilon(\widetilde{D},\sigma_2)$ be an anisotropic form. Then φ has a Witt ρ_2^ε-decomposition, i.e., there exists an orthogonal decomposition $\varphi \simeq \varphi_1 \oplus \varphi_2$ such that $\rho_2^\varepsilon(\varphi_1)$ is hyperbolic and $\rho_2^\varepsilon(\varphi_2)$ is anisotropic. Moreover there exists $\psi \in S^{-\varepsilon}(D,\sigma)$ such that $\varphi_1 \simeq \pi_2^{-\varepsilon}(\psi)$.*

Proof. We use the same argument as in Proposition 5.5. If $\rho_2^\varepsilon(\varphi)$ is isotropic then φ contains a subform φ_0 which comes from $S^{-\varepsilon}(D,\sigma)$, i.e., $\varphi_0 \simeq \pi_2^\varepsilon(\psi_0)$ for some $\psi_0 \in S^{-\varepsilon}(D,\sigma)$; see [4, 4.4]. □

Proposition 5.7. *Let D, $\widetilde{D}$, σ_1, σ and σ_2 be as in Proposition 3.1, then we have:*
(1) $\mathrm{u}(\widetilde{D},\sigma_1,\varepsilon) \leqslant \mathrm{u}(D,\sigma,-\varepsilon) + 2\,\mathrm{u}(D,\sigma,\varepsilon)$.
(2) $\mathrm{u}(\widetilde{D},\sigma_2,\varepsilon) \leqslant 3\,\mathrm{u}(D,\sigma,-\varepsilon)$.

Proof. (1) Let φ be an anisotropic ε-hermitian form over $(\widetilde{D},\sigma_1)$. According to Proposition 5.5, there exists an ε-hermitian form ψ over (D,σ) such that $\varphi \simeq \pi_1(\psi) \oplus \varphi'$ for some form φ' over $(\widetilde{D},\sigma_1)$ such that $\rho(\varphi')$ is anisotropic. As φ is anisotropic, so is ψ, therefore $\dim(\psi) \leqslant \mathrm{u}(D,\sigma,\varepsilon)$. As $\rho(\varphi')$ is anisotropic, $\dim(\varphi') = \dim\rho(\varphi') \leqslant \mathrm{u}(D,\sigma,-\varepsilon)$, consequently $\dim(\varphi) = \dim(\pi_1(\psi)) + \dim(\varphi') = 2\dim(\psi) + \dim(\varphi') \leqslant 2\,\mathrm{u}(D,\sigma,\varepsilon) + \mathrm{u}(D,\sigma,-\varepsilon)$.

For (2) we apply the same argument by using Proposition 5.6. □

Corollary 5.8. *With the notation of Corollary 3.4 we have:*
(1) $\mathrm{u}(L,{}^-) \leqslant 2\,\mathrm{u}(Q,{}^-) + \mathrm{u}(Q,\hat{\ }) = 2\,\mathrm{u}^-(Q) + \mathrm{u}^+(Q)$
(2) $\mathrm{u}(L) \leqslant 3\,\mathrm{u}(Q,\hat{\ }) = 3\,\mathrm{u}^+(Q)$

Proposition 5.9. *With the notation of §4, let $\varphi \in S(K_1,\tau_1)$ be an anisotropic form. Then φ has a Witt ρ_1-decomposition, i.e., there exists an orthogonal decomposition $\varphi \simeq \varphi_1 \oplus \varphi_2$ such that $\rho_1(\varphi_1)$ is hyperbolic and $\rho_1(\varphi_2)$ is anisotropic. Moreover there exists $\psi \in S(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$ such that $\varphi_1 \simeq \pi_1(\psi)$.*

Proof. If $\rho_1(\varphi)$ is isotropic then according to Proposition 4.2, φ contains a subform φ_0 which comes from $S(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$, i.e., $\varphi_0 \simeq \pi_1(\psi_0)$ for some $\psi_0 \in S(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2)$, we can then use an induction argument as in Proposition 5.5 □

Proposition 5.10. *With the notation of Theorem 4.3, we have*

$$\mathrm{u}(K_1, \tau_1) \leqslant 2\,\mathrm{u}(K_1 \otimes_k K_2, \tau_1 \otimes \mathrm{id}) + \mathrm{u}(K_1 \otimes_k K_2, \tau_1 \otimes \tau_2).$$

Proof. The proof is similar to that of Proposition 5.7; we use Proposition 5.9. □

Remark 5.11. Using [4, 4.4], and Proposition 4.2, one can state similar Witt-decomposition-like results for other maps involved in these exact sequences. For example by a Witt π_1^{ε}-decomposition result one can give an alternative proof of Proposition 3.1.

6 A Finiteness result

In the article [1], one can find several useful results about extensions of odd degree and their connections with substructures of codimension 2 of central simple algebras with involution, see ([1], Lemma 3.1.1, Lemma 3.3.1, Lemma 3.3.2, Lemma 3.3.3). From these results we can derive the following proposition which has been proved in [11].

Proposition 6.1. *Let D be a noncommutative K-division algebra and σ a K/k-involution on D. Suppose that the degree of D is a 2-power. Then there exists an extension M/k of odd degree such that $D_M = D \otimes_k M$ contains the elements λ, μ such that $\tau(\lambda) = -\lambda$ and $\tau(\mu) = -\mu$ and $\lambda\mu = -\mu\lambda$ and $[F(\lambda) : F] = 2$ where $F = KM = K \otimes_k M$ and τ is the involution $\sigma \otimes \mathrm{id}$ when σ is of second kind or σ is symplectic and D is a quaternion algebra otherwise $\tau = \mathrm{Int}(\mu) \circ (\sigma \otimes \mathrm{id})$.*

Theorem 6.2. *Let D be a division algebra of dimension a power of 2 over its centre K with a K/k-involution σ. Suppose that $\mathrm{u}(D, \sigma, \varepsilon) < \infty$ for $\varepsilon = 1$ and $\varepsilon = -1$, then $\mathrm{u}(K, \sigma|_K) < \infty$. In particular if both $\mathrm{u}^+(D)$ and $\mathrm{u}^-(D)$ are finite then $\mathrm{u}(K)$ is finite too.*

Proof. We prove this result by induction on $\dim_K(D)$. For $\dim_K(D) = 1$ the conclusion is evident. Suppose that $\dim_K(D) > 1$.

First suppose that σ is of the second kind. According to Proposition 6.1, there exists an extension M/k of odd degree such that $D_M = D \otimes_k M$ contains the elements μ and λ such that $\tau(\lambda) = -\lambda$, $\tau(\mu) = -\mu$ and $\lambda\mu = -\mu\lambda$ for $\tau = \sigma \otimes \mathrm{id}$ and $F(\lambda)/F$ is a quadratic extension where $F = K \otimes_k M$. As M/k is an extension of odd degree, $E = D_M$ is a division algebra. According to Proposition 3.8 we have $\mathrm{u}(E, \tau, \varepsilon) < \infty$ and $\mathrm{u}(E, \tau, -\varepsilon) < \infty$. By applying Proposition 5.7 we obtain: $\mathrm{u}(\widetilde{E}, \tau_1, \pm\varepsilon) < \infty$ and $\mathrm{u}(\widetilde{E}, \tau_2, \pm\varepsilon) < \infty$ where $\widetilde{E} = C_E(F(\lambda))$. As τ is of the second kind, so are τ_1 and τ_2. By the induction hypothesis, we have $\mathrm{u}(L, \tau_1|_L) < \infty$ and $\mathrm{u}(L, \tau_2|_L) < \infty$ where $L = F(\lambda)$ is the centre of $\widetilde{E}$. Let F' be the fixed field of $\tau_2|_L$. We have

$$(L, \tau_1|_L) \simeq (F \otimes_M F', \tau|_F \otimes \tau')$$
$$(L, \tau_2|_L) \simeq (F \otimes_M F', \tau|_F \otimes \mathrm{id})$$

where τ' is the nontrivial automorphism of F'/M. Proposition 5.10 implies that $\mathrm{u}(F, \tau|_F) < \infty$ and from Corollary 5.4 we deduce that $\mathrm{u}(K, \sigma|_K) \leqslant \mathrm{u}(F, \tau|_F) < \infty$.

Now consider the case where σ is of the first kind. If D is a quaternion algebra and σ is its canonical involution, then there exist $\lambda,\ \mu \in D$ with $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$, $\lambda\mu = -\mu\lambda$ et $[K(\lambda) : K] = 2$.

Otherwise there exists an extension M/K of odd degree such that $E = D \otimes_K M$ contains λ and μ with $\tau(\lambda) = -\lambda$, $\tau(\mu) = -\mu$, $\lambda\mu = -\mu\lambda$ and $[M(\lambda) : M] = 2$ where $\tau = \mathrm{Int}(\mu) \circ (\sigma \otimes \mathrm{id})$ (cf. Proposition 6.1). Them involution τ is of the first kind but of a different type from that of σ. In any case take:

$$\tau = \begin{cases} \sigma & \dim_K(D) = 4,\ \sigma \text{ symplectic} \\ \mathrm{Int}(\mu) \circ (\sigma \otimes \mathrm{id}) & \text{otherwise} \end{cases}$$

As M/K is an extension of odd degree, E is a division algebra. According to Proposition 3.8 we have $\mathrm{u}(E, \tau, \varepsilon) < \infty$ and $\mathrm{u}(E, \tau, -\varepsilon) < \infty$ (note that according to Proposition 2.2, the finiteness hypothesis (which is for σ) is still valid for τ). By applying Proposition 5.7 we obtain: $\mathrm{u}(\widetilde{E}, \tau_1, \pm\varepsilon) < \infty$ and $\mathrm{u}(\widetilde{E}, \tau_2, \pm\varepsilon) < \infty$ where $\widetilde{E} = C_E(M(\lambda))$. The involution τ_1 is unitary and by the first part of the proof the condition $\mathrm{u}(\widetilde{E}, \tau_1, \pm\varepsilon) < \infty$ implies that $\mathrm{u}(L, \tau_1|_L) < \infty$ where $L = M(\lambda)$. The involution τ_2 is of the first kind. The condition $\mathrm{u}(\widetilde{E}, \tau_2, \pm\varepsilon) < \infty$ states that both $\mathrm{u}^-(\widetilde{E})$ and $\mathrm{u}^+(\widetilde{E})$ are finite. So we conclude by induction that $\mathrm{u}(L) = \mathrm{u}(L, \tau_2|_L) < \infty$. Here $\tau_1|_L$ is the nontrivial automorphism of L/M. Now Proposition 5.1 implies that $\mathrm{u}(M) < \infty$ and from the strong version of Springer's theorem we deduce that $\mathrm{u}(K) \leqslant \mathrm{u}(M) < \infty$. $\square$

Remark 6.3. Note that in preceding theorem, according to Proposition 2.2 one of the two hypotheses $\mathrm{u}(D, \sigma, \varepsilon) < \infty$ and $\mathrm{u}(D, \sigma, -\varepsilon) < \infty$ is actually enough in the unitary case.

7 Values of the u-invariant, a particular case

Let L/K be a quadratic extension and let $^-$ be its nontrivial automorphism. The signed discriminant $\mathrm{d}_\pm$ defines a map from $W(L, ^-)$ to $K^*/N(L^*)$. Unfortunately this map is not a homomorphism. But for forms φ and ψ of even dimension we have $\mathrm{d}_\pm(\varphi \oplus \psi) = \mathrm{d}_\pm(\varphi).\,\mathrm{d}_\pm(\psi)$. Let $I(L, ^-) \subset W(L, ^-)$ be the classes of all nondegenerate hermitian forms over $(L, ^-)$. The group $W(L, ^-)$ has a natural ring structure.

Proposition 7.1. (1) *The map* $e_1 : I(L, ^-) \to K^*/N(L^*)$ *defined by* $e_1(\varphi) = \mathrm{d}_\pm(\varphi)$ *is a surjective homomorphism.*
(2) $\ker e_1 = I^2(L, ^-)$.
(3) *Via* e_1*, the group* I/I^2 *is isomorphic to* $K^*/N(L^*)$.

Proof. (1) The map e_1 is surjective because $\mathrm{d}_\pm(\langle 1, -a\rangle) = aN(L^*)$.
(2) The group $I = I(L, ^-)$ is generated by hermitian forms $\langle a, b\rangle$ where $a,\ b \in K^*$. Thus I^2 is generated by the hermitian forms:

$$\varphi = \langle a, b\rangle \otimes \langle c, d\rangle = \langle ac, ad, bc, bd\rangle$$

We have $\mathrm{d}_\pm(\varphi) = 1$. So $I^2 \subset \ker e_1$. Conversely suppose that $\varphi \in I$ with $e_1(\varphi) = 1$. The form φ is represented by $\langle a_1, \cdots, a_{2n}\rangle$ with $n \geqslant 1$.

For $n = 1$, we have $\varphi = \langle a_1, a_2\rangle$ with $-a_1a_2 \in N(L^*)$. So $\varphi \simeq \langle a_1, -a_1\rangle$, therefore φ is hyperbolic and $\varphi = 0$ in $W(L, ^-)$.

Now suppose that $n \geq 2$. We can write: $\varphi = \langle a_1, a_2, a_3\rangle \oplus \langle a_4, \cdots, a_{2n}\rangle$. So $\varphi \sim \langle a_1, a_2, a_3, a_1a_2a_3\rangle \oplus \langle -a_1a_2a_3, a_4, \cdots, a_{2n}\rangle$. We have

$$\langle a_1, a_2, a_3, a_1a_2a_3\rangle \simeq \langle a_1, a_2\rangle \otimes \langle 1, a_1a_3\rangle \in I^2.$$

The dimension of $\varphi' = \langle -a_1a_2a_3, a_4, \cdots, a_{2n}\rangle$ is $2(n-1)$ and $\mathrm{d}_\pm\,\varphi' = 1$. By induction we obtain $\varphi' \in I^2$.
(3) is deduced from (1) and (2). □

Let D be a division algebra over a field K with a K/k-involution τ. Here we call a hermitian form φ over (D, τ) an n-fold Pfister form if φ is the restriction of an n-fold Pfister form q over k to D. This notion appears in [10] for quaternion algebras. A hermitian form φ induced by the n-fold Pfister form $q = \langle\langle a_1, \cdots, a_n\rangle\rangle$, is still denoted by $\varphi = \langle\langle a_1, \cdots, a_n\rangle\rangle$.

Definition 7.2. Let D be a division algebra over a field K with a K/k-involution τ and let ε be an element of K with $\varepsilon\tau(\varepsilon) = 1$. An ε-hermitian form φ over (D, τ) is called *universal* if φ represents all nonzero ε-hermitian elements of D.

Proposition 7.3. *Let L/K be a quadratic extension and let $^-$ be the nontrivial automorphism of L/K. Then we have* $\mathrm{u}(L, ^-) \neq 3, 5, 7$.

Proof. Suppose that $\mathrm{u}(L, ^-) < 4$. Every 2-fold hermitian Pfister form $\langle\langle a, b\rangle\rangle$ is hyperbolic. The hermitian form $\langle 1, a, b\rangle$ is a hermitian neighbor of $\langle\langle a, b\rangle\rangle$ and therefore it is isotropic. We deduce that every hermitian form $\langle a, b, c\rangle$ of dimension 3 over $(L, ^-)$ is isotropic, so we have $\mathrm{u}(L, ^-) \leqslant 2$.

Suppose that $\mathrm{u}(L, ^-) < 8$. We conclude that every 3-fold hermitian Pfister form $\langle\langle a, b, -c\rangle\rangle$ is hyperbolic. Thus for every $a,\ b,\ c \in K^*$ we have:

$$\langle\langle a, b\rangle\rangle \simeq c\langle\langle a, b\rangle\rangle$$

Every form in $I^2 = I^2(L, ^-)$ is an orthogonal sum of the forms $\langle\langle a_i, b_i\rangle\rangle$. As for every hyperbolic plane $\mathbb{H}$ and $c \in K^*$ we have $\mathbb{H} \simeq c\mathbb{H}$, the Witt cancellation theorem implies that $\varphi \simeq c\varphi$ for every $\varphi \in I^2$. In particular φ is universal over $(L, ^-)$ (in the sense of Definition 7.2).

Now suppose that $u = \mathrm{u}(L, ^-) = 5$ or 7. Let φ be an anisotropic hermitian form of dimension u. In particular φ represents its discriminant $d = \mathrm{d}_\pm(\varphi)$. We have then $\varphi = \psi \oplus \langle d\rangle$ where ψ is a form of dimension 4 or 6 and $\mathrm{d}_\pm(\psi) = 1$.

As $\psi \in I$, Proposition 7.1 implies that $\psi \in I^2$. We have already shown that ψ is universal, consequently φ is isotropic which is a contradiction to the choice of φ. $\square$

Remark 7.4. In a similar way, if $D = (a, b)_K$ is a quaternion division algebra with the canonical involution $^-$, then $\mathrm{u}(D, ^-) \neq 3, 5, 7$. However the value 3 for the u-invariant $\mathrm{u}(D, ^-, -1)$ is possible, for example if K is a p-adic field then $\mathrm{u}(D, ^-, -1) = 3$.

Proposition 7.5. *Let L_0 be a field with $\mathrm{u}(L_0) = n$. Let $L = L_0((x))$ be the field of Laurent series over L_0 and let σ be the L_0-automorphism of L induced by $x \mapsto -x$. Then $\mathrm{u}(L, \sigma) = n$.*

Proof. Let K be the fixed field of σ. We have $K = L_0((x^2))$ so $\mathrm{u}(K) = 2n$. Consequently $\mathrm{u}(L, \sigma) \leqslant n$ (cf. Remark 3.5). Let $q = \langle a_1, \cdots, a_n \rangle$ be an anisotropic quadratic form of dimension n over L_0. The restriction of q to L is anisotropic. In fact the isotropy of $q|_{(L,\sigma)}$ is equivalent to that of the quadratic form $q \oplus x^2 q = \langle a_1, \cdots, a_n, -x^2 a_1, \cdots, -x^2 a_n \rangle$ over K. The anisotropy of this form is equivalent to that of q over L_0. $\square$

Remark 7.6. Proposition 7.5 state in particular that the possible values for the hermitian u-invariant of commutative fields contain the possible values of the usual u-invariant.

8 Classical groups over $\mathbb{Q}_p(t)$, $p \neq 2$

We refer to [21] and [20] for basic notions about Tits's indices. In the symbol ${}^g X^t_{n,r}$, where $X = A,\ B,\ C,\ D$, the integers n and r are respectively the absolute and relative rank of the considered classical group G, g denotes the order of the quotient of the Galois group $\Gamma = Gal(k^{sep}/k)$ which operates effectively on the Dynkin diagram. In case the diagram has no nontrivial automorphism, g is necessarily 1. If $g = 1$, G is called of *inner* type, otherwise G is called of *outer* type. The integer t is the degree of a certain division algebra which occurs in the definition of the considered group. If t or g are omitted in the symbol, they are necessarily 1.

Type A_n

Lemma 8.1. *Let k be a function field of a p-adic field with $p \neq 2$. Let L/k be a quadratic extension and let $^-$ be the nontrivial automorphism of L/k. Then $\mathrm{u}(L, ^-) \leqslant 4$.*

Proof. Let $\varphi \simeq \langle a_1, \cdots, a_5 \rangle$ be a hermitian form of dimension 5 over $(L, ^-)$, where $a_i \in k$, $i = 1, \cdots, 5$. Then, φ is isotropic if and only if the quadratic form $q = \langle a_1, \cdots, a_5, -aa_1, \cdots, -aa_5 \rangle$ is isotropic over k where $L = k(\sqrt{a})$. But this form is isotropic over k, because its Hasse invariant is $c(q) = (-a, d)$ where $d = \det(\varphi)$, which has index $\leqslant 2$ in the Brauer group of k and according to the Theorem 4.6 of [12], q is isotropic. We have in particular $\mathrm{u}(L, ^-) \leqslant 4$. $\square$

Proposition 8.2. *Let $k = \mathbb{Q}_p(t)$ with $p \neq 2$. The index ${}^1A_{n,r}^{(d)}$ occurs over k for every positive integers d, r and n satisfying $rd = n + 1$.*

Proof. According to [21] or [20, 17.1.3], ${}^1A_{n,r}^{(d)}$ occurs over k if and only if there exists a division algebra D over k of degree d with $rd = n+1$. The existence of such algebras comes from the fact that there exist division algebras of arbitrary degree over $\mathbb{Q}_p$. □

Proposition 8.3. *Let $k = \mathbb{Q}_p(t)$ with $p \neq 2$.*

(1) *If the index ${}^2A_{n,r}^{(1)}$ occurs over k, then $n + 1 - 2r \in \{0, 1, 2, 3, 4\}$. All the anisotropic indices ${}^2A_{1,0}^{(1)}$, ${}^2A_{2,0}^{(1)}$ ${}^2A_{3,0}^{(1)}$ and ${}^2A_{4,0}^{(1)}$ occur over k.*

(2) *If the index ${}^2A_{n,r}^{(2)}$ occurs over k then $n + 1 - 4r \in \{0, 2, 4, 6\}$.*

Proof. According to [21] or [20, 17.1.6], ${}^2A_{n,r}^{(d)}$ occurs over k if and only if there exist a quadratic extension E/k, a division algebra D over E and an involution σ on D of the second kind such that k is the fixed field of $\sigma|_E$ and a nondegenerate hermitian form h over (D, σ) of dimension $d^{-1}(n + 1)$ and of Witt index r.

(1) For $d = 1$, $D = E$. According to Lemma 8.1 we have $\mathrm{u}(E, \sigma) \leqslant 4$. We conclude that $0 \leqslant n + 1 - 2r \leqslant 4$. As for $E = \mathbb{Q}_p(t)(\sqrt{t})$, we have $\mathrm{u}(E, \sigma) = 4$, the indices ${}^2A_{1,0}^{(1)}$, ${}^2A_{2,0}^{(1)}$ ${}^2A_{3,0}^{(1)}$ and ${}^2A_{4,0}^{(1)}$ occur over k.

(2) For $d = 2$, D is a quaternion algebra over E. Thank to Proposition 3.6 we have $\mathrm{u}(D, \sigma) \leqslant 3$. We have then $0 \leqslant \frac{n+1}{2} - 2r \leqslant 3$. □

Type B_n

Proposition 8.4. *Let $k = \mathbb{Q}_p(t)$ with $p \neq 2$. If the index $B_{n,r}$ occurs over k then we have: $n - 4 \leqslant r \leqslant n$.*

Proof. According to [21] or [20, 17.2.3], $B_{n,r}$ can occur over k if and only if there exists a nondegenerate quadratic form q of dimension $2n + 1$ and of Witt index r. Let q_a be the unique anisotropic part of q up to isometry. We have:

$$\dim q_a = 2n + 1 - 2r \tag{7}$$

According to [12], the dimension of q_a cannot exceed 10. We obtain then $0 \leqslant 2n - 2r + 1 \leqslant 10$ and so $n - 4 \leqslant r \leqslant n$. □

Proposition 8.5. *The anisotropic indices $B_{1,0}$, $B_{2,0}$ and $B_{3,0}$ can occur over $k = \mathbb{Q}_p(t)$ with $p \neq 2$.*

Proof. According to (7), it is enough to find anisotropic quadratic forms of dimension 3, 5 and 7 (resp.), which is possible because $\mathrm{u}(\mathbb{Q}_p(t)) \geqslant 8$. □

Remark 8.6. According to (7), the existence of $B_{4,0}$ over $\mathbb{Q}_p(t)$ is equivalent to $\mathrm{u}(\mathbb{Q}_p(t)) \geqslant 9$. According to a conjecture, one believes that $\mathrm{u}(\mathbb{Q}_p(t)) = 8$ (cf. [14, Chapter 5, 2.5], [5] and [12]).

Type C_n

Proposition 8.7. *Let $k = \mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$. If the index $C_{n,r}^{(d)}$ occurs over k then we have $d \in \{1, 2, 4\}$. For $d = 1$ we have $n = r$. For $d = 2$ we have $n - 2r \in \{0, 1, 2\}$. For $d = 4$, we have $n - 4r \in \{0, 2, 4, 6, 8, 10\}$.*

Proof. According to [21] or [20, 17.2.10], $C_{n,r}^{(d)}$ can occur over k if and only if there exists a division algebra D over k of degree d with an orthogonal involution σ and a skew hermitian form h over (D, σ) of dimension $2d^{-1}n$ and of Witt index r. As D has an involution of the first kind, D lies in the 2-torsion of the Brauer group $\mathrm{Br}_2(k)$ (cf. [19, Ch. 8, 8.4]). According to a result due to Saltman [18], we know that $d \in \{1, 2, 4\}$. Let h_a be the anisotropic part of h. We have

$$\dim h_a = 2d^{-1}n - 2r \tag{8}$$

If $d = 1$, h is alternating and in this case we have $r = n$.

If $d = 2$, D is a quaternion algebra over k. According to Corollary 3.4 and Lemma 8.1 we have $\mathrm{u}(D, \sigma, -1) = \mathrm{u}(D, ^-) \leqslant \frac{1}{2}\mathrm{u}(L, ^-) \leqslant 2$ where $^-$ is the canonical involution of D and L is a maximal subfield of D stable under $^-$. We deduce that $0 \leqslant \dim h_a \leqslant 2$. Now (8) implies that $n - 2r \in \{0, 1, 2\}$.

If $d = 4$, according to a result due to Albert (cf. [6, 16.1]), D is isomorphic to a biquaternion algebra. We write $D = D_1 \otimes D_2$ where D_1 and D_2 are quaternion algebras. According to Proposition 2.2 we have $\mathrm{u}(D, \sigma, -1) = \mathrm{u}(D_1 \otimes D_2, ^- \otimes ^-, -1)$ where $^-$ (resp. $^-$) is the canonical involution of D_1 (resp. D_2). Let L be a maximal subfield of D_2 stable under $^-$. Thanks to Proposition 3.1 we have:

$$\mathrm{u}(D_1 \otimes D_2, ^- \otimes ^- - 1) \quad \leqslant \mathrm{u}(D_1 \otimes L, ^- \otimes \mathrm{id}) + \tfrac{1}{2}\,\mathrm{u}(D_1 \otimes L, ^- \otimes ^-, -1)$$

Thanks to Proposition 3.6 we have $\mathrm{u}(D_1 \otimes L, ^- \otimes \mathrm{id}) \leqslant \frac{1}{4}\,\mathrm{u}(L)$. According to [12], $\mathrm{u}(L) \leqslant 10$. We conclude then $\mathrm{u}(D_1 \otimes L, ^- \otimes \mathrm{id}) \leqslant 2$. Now by using Proposition 3.10 and Proposition 3.1 we obtain

$$\mathrm{u}(D_1 \otimes L, ^- \otimes ^-, -1) \leqslant 6.$$

We obtain then $\mathrm{u}(D_1 \otimes D_2, ^- \otimes ^-, -1) \leqslant 5$. We conclude that $0 \leqslant \dim h_a \leqslant 5$ and (8) implies that $n/2 - 2r \in \{0, 1, \cdots, 5\}$. $\square$

Proposition 8.8. *The anisotropic indices $C_{1,0}^{(2)}$, $C_{2,0}^{(2)}$ and $C_{2,0}^{(4)}$ occur over $k = \mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$.*

Proof. According to (8), it is enough to find a skew hermitian form of dimension 2 over (D, σ) where D is a suitable quaternion division algebra over $\mathbb{Q}_{\mathrm{p}}(t)$ and σ is an orthogonal involution of D. This choice is possible because $\mathrm{u}(D, \sigma, -1) = 2$ for $D = (-p, u)_{\mathbb{Q}_{\mathrm{p}}(t)}$ where $u \in \mathbb{Z}_p^* \backslash \mathbb{Z}_p^{*2}$. In fact by Proposition 3.6, $\mathrm{u}(D, \sigma, -1) \leqslant \frac{1}{4}\,\mathrm{u}(\mathbb{Q}_{\mathrm{p}}(t))$. We obtain then $\mathrm{u}(D, \sigma, -1) \leqslant 2$, because $\mathrm{u}(\mathbb{Q}_{\mathrm{p}}(t)) \leqslant 10$ according to [12]. So it is enough to construct an anisotropic skew hermitian form of dimension 2 over (D, σ) or equivalently an anisotropic hermitian form of dimension 2 over $(D, ^-)$ where $^-$ is the canonical involution of D. We may take for example the hermitian form $\langle 1, t \rangle$. $\square$

Lemma 8.9. *Let D be a division algebra over a field K with a K/k-involution σ. Consider the division algebra $D(t) = k(t) \otimes_k D$ with the involution $\widehat{\sigma} = \mathrm{id} \otimes \sigma$. Let (V, h_V) and (W, h_W) be two anisotropic ε-hermitian spaces over (D, σ). Let $V(t) = k(t) \otimes_k V$, $W(t) = k(t) \otimes_k W$, $\widehat{h}_V$ and $\widehat{h}_W$ the restrictions of h_V and h_W (resp.) to $D(t)$. Then the hermitian form $\widehat{h}_V \oplus t\widehat{h}_W$ is anisotropic over $(D(t), \widehat{\sigma})$. In particular $\mathrm{u}(D(t), \widehat{\sigma}, \varepsilon) \geqslant 2\,\mathrm{u}(D, \sigma, \varepsilon)$ where $\varepsilon \in K$ satisfies $\varepsilon\sigma(\varepsilon) = 1$.*

Proof. Let $x_1 \oplus x_2 \in V(t) \oplus W(t)$ be a nonzero anisotropic vector for $\widehat{h}_V \oplus t\widehat{h}_W$, i.e., $\widehat{h}_V \oplus t\widehat{h}_W(x_1 \oplus x_2, x_1 \oplus x_2) = 0$. This relation implies that:

$$\widehat{h}_V(x_1, x_1) + t\widehat{h}_W(x_2, x_2) = 0 \tag{9}$$

By using the embedding $V(t) \subset V((t))$ we may suppose that $x_1 = \Sigma_{i=N}^{\infty} v_i t^i$ and $x_2 = \Sigma_{i=M}^{\infty} w_i t^i$ where $v_i \in V$, $w_i \in W$, $v_N \neq 0$ and $w_M \neq 0$. We consider two cases: $N \leqslant M$ and $N > M$. If $N \leqslant M$, (9) implies that $h_V(v_M, v_M) = 0$ which is a contradiction because h_V is anisotropic. If $N > M$, (9) implies that $h_W(w_N, w_N) = 0$ which is a contradiction because h_W is anisotropic. □

Proposition 8.10. *Let $k = \mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$. Then if the index ${}^1D_{n,r}^{(d)}$ occurs over k then $d \in \{1, 2, 4\}$. Moreover:*

1. *For $d = 1$ we have $n - r \in \{0, 2, 3, 4, 5\}$*
2. *For $d = 2$ we have $n - 2r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9\}$*
3. *For $d = 4$ we have $n - 4r \in \{0, 2, 4, \cdots, 20\}$.*

Proof. According to [21] or [20, 17.3.13], ${}^1D_{n,r}^{(d)}$ occurs over k if and only if there exist a division algebra D over k of degree d with an orthogonal involution σ and a nondegenerate hermitian form h of dimension $2nd^{-1}$ with trivial discriminant and of Witt index r, and moreover $d \geqslant 1$, $rd \leqslant n$ and $n \neq rd + 1$. As D has an involution of the first kind, according to a result of Albert (cf. [19, Ch. 8, 8.4]), D lies in the 2-torsion of $\mathrm{Br}(k)$. By using a result of Saltman [18], we obtain $d \in \{1, 2, 4\}$. Let h_a be the anisotropic part of h. We have:

$$\dim h_a = 2d^{-1}n - 2r, \quad \mathrm{d}_{\pm}\, h_a = 1 \tag{10}$$

(1) If $d = 1$, we have $D = k$ and h_a is a symmetric bilinear form over k and $\dim h_a$ is even. By using a result due to Parimala-Suresh [12] the dimension of h_a cannot exceed 10, we obtain then $2n - 2r = \dim h_a \in \{0, 2, 4, 6, 8, 10\}$.

The case $\dim h_a = 2$ is impossible because in this case $\mathrm{d}_{\pm}\, h_a = 1$ implies the isotropy of h_a. Consequently $2n - 2r = 2$ is impossible.

The case $\dim h_a = 4$ is possible, we can take the anisotropic form $h_a = \langle 1, p, -u, -pu \rangle$ which has trivial discriminant where $u \in {\mathbb{Z}_p}^* \backslash {\mathbb{Z}_p}^{*2}$. This form is anisotropic over $\mathbb{Q}_{\mathrm{p}}(t)$. The case $2n - 2r = 4$ is then possible.

The case $\dim h_a = 6$ is possible. Consider the biquaternion division algebra $(u,t) \otimes_k (t+1,p)$. The fact that this algebra is a division algebra can be found in an appendix of the Saltman's paper [18] due to W. Jacob and J.-P. Tignol. Let $h_a = \langle u, t, -ut, -(t+1), -p, (t+1)p\rangle$ be the Albert form of this algebra. This form is anisotropic and has trivial discriminant. Consequently $2n - 2r = 6$ is possible.

The case $\dim h_a = 8$ is also possible. We can take the anisotropic form $h_a = \langle 1, t\rangle \otimes \langle 1, p, -u, -pu\rangle$. The anisotropy of h_a can be deduced for example from Lemma 8.9.

(2) If $d = 2$, D is a quaternion division algebra over k. Now (10) implies that $\mathrm{d}_\pm h = \mathrm{d}_\pm h_a = 1$. We write $D = (a,b)_k$, the quaternion division algebra over k generated by i, j with $i^2 = a \in k^*$, $j^2 = b \in k^*$, $ij = -ji$. Let $L = k(i) = k(\sqrt{a}) \subset D$. According to Proposition 2.2, $\mathrm{u}(D,\sigma) = \mathrm{u}(D, {}^-, -1)$ where ${}^-$ is the canonical involution of D.

As L is the function field of a p-adic field ($p \neq 2$), we have $\mathrm{u}(L) \leqslant 10$ (cf. [12, 4.5]). According to Lemma 8.1, $\mathrm{u}(L, {}^-|_L) \leqslant 4$. Now by applying Corollary 3.4 we obtain:

$$\mathrm{u}(D, {}^-, -1) \leqslant \frac{10+8}{2} = 9.$$

Consequently $\dim h_a \leqslant 9$. We have then $n - 2r \in \{0, 1, 2, \cdots, 9\}$.

The case $\dim h_a = 1$ is impossible because the discriminant of every skew hermitian form over D of dimension 1 is different from 1. Consequently the case $n - 2r = 1$ is impossible.

The case $\dim h_a = 2$ is possible. To construct, consider an arbitrary nondegenerate skew hermitian form h_0 of dimension 1 over (D_0, σ_0) where D_0 is the unique quaternion division algebra over $\mathbb{Q}_\mathrm{p}$ and σ_0 is its canonical involution. Consider the skew hermitian form

$$h = h_0 \oplus t h_0 \tag{11}$$

According to Lemma 8.9, h is anisotropic, moreover its discriminant is trivial. We conclude that the case $n - 2r = 2$ is possible.

It is well known that, there exists an anisotropic skew hermitian form of dimension 3 over D_0 (cf. [19, Ch. 10, 3.6], [15] or [22]). The restriction of this form to (D, σ) is anisotropic. Consequently the case $\dim h_a = 3$ or $n - 2r = 3$ is possible.

The cases $\dim h_a = 4$ or 6 are similar; we take an anisotropic skew hermitian form h_0 over (D_0, σ_0) of dimension 2 or 3 and we consider the anisotropic form h defined in (11).

For the case $\dim h_a = 5$, we first consider a subform of dimension 5 of the anisotropic skew hermitian form of dimension 6 over (D, σ) that we constructed above. By multiplying this form by its discriminant, we obtain an anisotropic form of dimension 5 with trivial discriminant. Therefore $n - 2r = 5$ is possible.

(3) If $d = 4$, D is a biquaternion algebra. We write $D = D_1 \otimes D_2$ where D_1 and D_2 are two quaternion algebras. We have $\mathrm{u}(D, \sigma) = \mathrm{u}(D_1 \otimes D_2, {}^- \otimes {}^-)$ where ${}^-$ (resp. ${}^-$) is the canonical involution of D_1 (resp. D_2). Let L be a maximal

subfield of D_2 stable under $^-$. According to Proposition 3.1 and Proposition 3.10 we have

$$\begin{aligned}\mathrm{u}(D_1\otimes D_2, {}^-\otimes{}^-) &\leqslant \tfrac{1}{2}\,\mathrm{u}(D_1\otimes L, {}^-\otimes \mathrm{id}, -1) + \mathrm{u}(D_1\otimes L, {}^-\otimes{}^-)\\ &\leqslant \tfrac{9}{2} + 6 = \tfrac{21}{2},\end{aligned}$$

We obtain then $\mathrm{u}(D,\sigma) \leqslant 10$. Now (10) implies that $0 \leqslant n/2 - 2r \leqslant 10$. $\square$

Corollary 8.11. *The anisotropic indices ${}^1D^{(1)}_{n,0}$ for $2 \leqslant n \leqslant 4$ and ${}^1D^{(2)}_{n,0}$ for $2 \leqslant n \leqslant 6$ occur over $k = \mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$.*

Type D_n outer

Proposition 8.12. *Let $k = \mathbb{Q}_{\mathrm{p}}(t)$ with $p \neq 2$. If the index ${}^2D^{(d)}_{n,r}$ occurs over k then $d \in \{1, 2, 4\}$. Moreover:*

(1) *For $d = 1$ we have $n - r \in \{0, 1, 2, 3, 4, 5\}$*

(2) *For $d = 2$ we have $n - 2r \in \{0, 1, 2, 3, 5, 6, 7, 8, 9\}$*

(3) *For $d = 4$ we have $n - 4r \in \{0, 2, \cdots, 20\}$.*

Proof. We use the notation of the proof of Proposition 8.10. The criterion for the existence of ${}^2D^{(d)}_{n,r}$ is the same as for ${}^1D^{(d)}_{n,r}$ except $\mathrm{d}_{\pm}\, h \neq 1$ (cf. [21]). Therefore we have $d \in \{1, 2, 4\}$.

(1) For $d = 1$, $\dim h_a = 2$ is possible; it is enough to choose the anisotropic form $\langle 1, p\rangle$. Consequently $n - r = 1$ is possible.

The case $\dim h_a = 4$ is possible. In fact it is enough to choose a subform of dimension 4 of the anisotropic form $\langle 1, t\rangle \otimes \langle 1, p, -u, -pu\rangle$ with nontrivial discriminant. We can choose for example $\langle p, -u, -pu, t\rangle$. Consequently $n-r = 2$ is possible.

For $\dim h_a = 6$ we can choose $\langle p, -u, -pu, tp, -tu, -tpu\rangle$. Consequently $n - r = 3$ is possible.

(2) For $d = 2$, D is a quaternion algebra over k. We have that $\dim h_a \leqslant 9$ as in the proof of Proposition 8.10.

For the case $\dim h_a = 1$, it is enough to consider a skew hermitian form of dimension 1 which has necessarily nontrivial discriminant. Consequently the case $n - 2r = 1$ is possible.

For the case $\dim h_a = 2$, we consider two arbitrary skew symmetric elements of $D_0 = (-p, u)_{\mathbb{Q}_{\mathrm{p}}}$ with different reduced norm modulo ${\mathbb{Q}_{\mathrm{p}}}^{*2}$. We denote these elements by a and b. Now consider the skew hermitian form $h = \langle a, tb\rangle$ over $D = D_0 \otimes_{\mathbb{Q}_{\mathrm{p}}} \mathbb{Q}_{\mathrm{p}}(t)$. The form h is anisotropic form with nontrivial discriminant. Consequently $n - 2r = 2$ is possible.

The case $\dim h_a = 3$ is possible. In fact, there exists a skew hermitian form h_0 of dimension 2 with discriminant equal to $c \neq 1 \in {\mathbb{Q}_{\mathrm{p}}}^*/{\mathbb{Q}_{\mathrm{p}}}^{*2}$ (cf. [19, Ch. 10, 3.6], [15] or [22]). We also consider a skew hermitian form h'_0 over (D_0, σ_0) of dimension 1 and with the discriminant equal to $c' \neq 1 \in {\mathbb{Q}_{\mathrm{p}}}^*/{\mathbb{Q}_{\mathrm{p}}}^{*2}$. We may suppose that $c' \neq c$ (this choice is possible because $\mathrm{Card}(\mathbb{Q}_p{}^*/\mathbb{Q}_p{}^{*2}) = 4$). Thanks to Lemma 8.9, the form $h_a = h_0 \oplus th'_0$ is an anisotropic skew hermitian form over (D, σ) of dimension 3 and with nontrivial discriminant. Consequently $n - 2r = 3$ is also possible.

For $\dim h_a = 5$, we consider an anisotropic skew hermitian form h_0 over (D_0, σ_0) of dimension 3 and with trivial discriminant and an anisotropic skew hermitian form h'_0 over (D_0, σ_0) of dimension 2 and with nontrivial discriminant (According to [19, Ch. 10, 3.6], [15] or [22] these choices are possible). Now consider the form $h = h_0 \oplus th'_0$ over (D, σ) which is of dimension 5 and with nontrivial discriminant Consequently $n - 2r = 5$ is possible.

Suppose that $h \simeq \langle a_1, a_2, \cdots, a_5 \rangle$ is an anisotropic skew hermitian form of dimension 5 and with nontrivial discriminant constructed as in the preceding paragraph. At least one of the forms $\langle a_{i_1}, \cdots, a_{i4} \rangle$ has nontrivial discriminant where $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 5$. Consequently $\dim h_a = n - 2r = 4$ is possible.

(3) For $d = 4$, D is a biquaternion algebra. As in the proof of Proposition 8.10 (3), we obtain $n/2 - 2r \leqslant 10$. □

Corollary 8.13. *The anisotropic indices* ${}^2D_{n,0}^{(1)}$ *for* $2 \leqslant n \leqslant 3$ *and* ${}^2D_{n,0}^{(2)}$ *for* $1 \leqslant n \leqslant 5$ *occur over* $k = \mathbb{Q}_p(t)$ *with* $p \neq 2$.

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