On classification of projective homogeneous varieties up to motivic isomorphism

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Abstract

We give a complete classification of anisotropic projective homogeneous varieties of dimension less than 6 up to motivic isomorphism. We give several criteria for anisotropic flag varieties of type A_n to have isomorphic motives.

1 Introduction

The present paper can be viewed as an application of the methods and results obtained by the authors in [CPSZ].

Let k be a field of characteristic not 2 and k_s denotes its separable closure. For a variety X over k we denote by X_s the base change $X \times_k k_s$. By $\mathcal{M}(X)$ we denote the Chow motive of X. Recall (see [MPW, § 1]) that X is a twisted flag variety of inner type over k if $X = {}_\xi(G/P)$ is a twisted form of the projective homogeneous variety G/P, where G is an adjoint simple split algebraic group over k, P its parabolic subgroup and the twisting is given by a 1-cocycle $\xi \in Z^1(k, G(k_s))$.

The present paper is devoted to the following

Problem. *Describe all pairs (X, Y) of non-isomorphic twisted flag varieties X and Y of inner type over k which have isomorphic Chow motives.*

This problem can be subdivided into two subproblems:

(i) Describe all such pairs (X, Y) with $X_s \simeq Y_s$;

(ii) Describe all such pairs (X, Y) with $X_s \not\simeq Y_s$.

Let us briefly remind what is known so far. The complete solution of the problem (i) is known for quadrics and Severi-Brauer varieties due to Izhboldin, Karpenko, Merkurjev, Rost, Vishik and others (see [Izh98], [Ka96], [Ka00], [Ro98], [Vi03]). Concerning (ii), the example (of dimension 5) was provided by Bonnet in [Bo03]. It deals with twisted flag varieties of type G_2. For exceptional varieties of type F_4 a similar example was provided in [NSZ].

In the present paper we provide a complete solution of the mentioned above problem for projective homogeneous varieties of dimension less than 6. Namely, we prove the following (using the notation of 2.1)

1.1 Theorem. *Let X and Y be non-isomorphic twisted flag varieties of dimension ≤ 5 of inner type over k which have isomorphic Chow motives.*

(i) *If $X_s \simeq Y_s$, then either*
$X = \mathrm{SB}(A)$ *and* $Y = \mathrm{SB}(A^{\mathrm{op}})$ *are Severi-Brauer varieties corresponding to a central simple algebra A and its opposite A^{op}, where* $\deg(A) = 3, 4, 5, 6$ *and* $\exp(A) > 2$
or
$X = \mathrm{SB}_{1,2}(A)$ *and* $Y = \mathrm{SB}_{1,2}(A^{\mathrm{op}})$ *are twisted forms of the flag varieties corresponding to central simple algebras A such that* $\deg(A) = \exp(A) = 4$.

(ii) *If $X_s \not\simeq Y_s$, then either*
$X = \mathrm{SB}_{1,3}(A)$ *and* $Y = \mathrm{SB}_{1,2}(A')$ *are twisted forms of the flag varieties corresponding to central simple algebras A and A' such that* $\deg(A) = \deg(A') = 4$ *and $A \simeq A'$ or A'^{op},*
or
$X = {}_\xi(\mathrm{G}_2/P_1)$ *and* $Y = {}_\xi(\mathrm{G}_2/P_2)$ *are twisted forms of the variety G/P_i, $i = 1, 2$, where G is a split exceptional group of type* G_2 *and P_i is one of its maximal parabolic subgroups,*
or
$X = \mathbb{P}^n$ *and $Y = Q^n$ is the projective space and the split quadric respectively, where $n = 3, 5$.*
or
$X = \mathbb{P}^5$ *and* $Y = \mathrm{G}_2/P_2$ *is the projective space and the split Fano variety of type* G_2.

1.2 Remark. Observe that the case $X = {}_\xi(\mathrm{G}_2/P_1)$ and $Y = {}_\xi(\mathrm{G}_2/P_2)$ of the theorem is the example of Bonnet mentioned above and, hence, is the minimal one in the sense of dimension.

1.3 Remark. The case $X = \mathrm{SB}_{1,3}(A)$ and $Y = \mathrm{SB}_{2,3}(A')$ with $A \simeq A', A'^{\mathrm{op}}$ provides another minimal example of two non-isomorphic varieties that have isomorphic Chow motives.

Apart from Theorem 1.1, we prove the following

1.4 Theorem. *Let $X = \mathrm{SB}_{n_1,\ldots,n_r}(A)$ and $Y = \mathrm{SB}_{m_1,\ldots,m_r}(A')$ be twisted flag varieties of inner type A_n, $n \geq 2$, over k, where the central simple algebras A and A' have exponents 1, 2, 3, 4, or 6. Assume that*

(i) $\mathcal{M}(X_s) \simeq \mathcal{M}(Y_s)$;

(ii) $n_1 = 1$ *or* $n_r = n$

(iii) $m_1 = 1$ *or* $m_r = n$.

Then $\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow A \simeq A'$ *or* A'^{op}.

As an immediate consequence we obtain

1.5 Corollary. *Let $X = \mathrm{SB}_{1,n}(A)$ and $Y = \mathrm{SB}_{n-1,n}(A')$, where A and A' are central simple algebras of degree $n+1$, $n \geq 3$, and exponent 1,2,3,4 or 6. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow A \simeq A' \text{ or } A'^{\mathrm{op}}.$$

1.6 Remark. The varieties X and Y of 1.5 provide examples of twisted flag varieties that satisfy **(b)**. In fact, $X_s \not\simeq Y_s$, since they have different automorphism groups by [De77].

The paper is organized as follows. In section 2 we consider the case of a split group and state several facts which will be extensively used in the proofs. Section 3 is devoted to the case by case proof of Theorem 1.1. In the section 4 we prove Theorem 1.4 and provide several results that we need for the proof of 1.1.

2 Preliminaries

In the paper we use the following notation.

2.1. Let G be a split simple algebraic group defined over a field k. We fix a maximal split torus T of G and a Borel subgroup B of G containing T and defined over k. Denote by Φ the root system of G, by $\Pi = \{\alpha_1, \ldots, \alpha_{\operatorname{rk} G}\}$ the set of simple roots of Φ corresponding to B, by W the Weyl group, and by $S = \{s_1, \ldots, s_{\operatorname{rk} G}\}$ the corresponding set of fundamental reflections. Let P_Θ be the standard parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P_\Theta = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta\rangle$. Denote by P_i the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$. By Φ/P_Θ we denote the flag variety G/P_Θ. The root enumeration follows Bourbaki.

The notation $\mathrm{SB}_{n_1,\ldots,n_r}(A)$, $1 \le n_1 < \ldots < n_r \le n$, is used for the twisted form of the variety A_n/P_Θ, where $\Theta = \Pi \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_r}\}$ and A is a central simple algebra of degree $n+1$ corresponding to the twisting. Observe that $\mathrm{SB}_{n_1,\ldots,n_r}(A) = X(A; n_1, \ldots, n_r)$ in the notation of [MPW, Appendix] and $\mathrm{SB}(A) = \mathrm{SB}_1(A)$ is the usual Severi-Brauer variety defined by A. By $\operatorname{ind}(A)$ we denote the index of A and by $\exp(A)$ its exponent. A split projective quadric of dimension n is denoted by Q^n.

2.2. According to [Ko91] the Chow motive of the flag variety $X = G/P_\Theta$, when G is a split group, is isomorphic to

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{\dim X} \mathbb{Z}(i)^{\oplus a_i(X)},$$

where $\mathbb{Z}(i)$ are the twists of the Lefschetz motive and the positive integers (ranks) $a_i(X)$ are the coefficients of the generating polynomial $p_X(z) = \sum_{i=0}^{\dim X} a_i(X) z^i$. The latter is defined by the following explicit formula:

$$p_X(z) = \Big(\prod_{i=1}^{\operatorname{rk} G} \frac{z^{d_i(W)} - 1}{z - 1}\Big) / \Big(\prod_{j=1}^{m} \prod_i \frac{z^{d_i(W_j)} - 1}{z - 1}\Big).$$

Here $W_1 \times \ldots \times W_m$ is the decomposition of W_Θ into the product of Weyl groups corresponding to the irreducible root systems and $d_i(W_j)$ are the degrees of the respective fundamental polynomial invariants (see [Ca72, 9.4 A]).

The following observation follows from the above isomorphism.

2.3. *The motives of flag varieties X and Y of dimension n over a separably closed field are isomorphic iff the corresponding sequences of ranks $(a_0(X), \dots, a_n(X))$ and $(a_0(Y), \dots, a_n(Y))$ are equal.*

We shall need the following two facts:

2.4. (See [Ka00, Criterion 7.1]) Let A, A' be central simple algebras over k and $\mathrm{SB}(A)$, $\mathrm{SB}(A')$ be the respective Severi-Brauer varieties. Then

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \mathcal{M}(\mathrm{SB}(A')) \Leftrightarrow A \simeq A', A'^{\mathrm{op}}.$$

2.5. (see [Izh98, Cor. 2.9 and Prop. 3.1]) Let q, q' be regular quadratic forms of rank n and X_q, $X_{q'}$ be the respective projective quadrics. If n is odd or $n < 7$, then

$$\mathcal{M}(X_q) \simeq \mathcal{M}(X_{q'}) \Leftrightarrow X_q \simeq X_{q'}.$$

Finally, we shall need the following observation:

2.6. (See [Ka00, Proof of Lemma 2.3]) Let X and Y be smooth projective varieties over k with isomorphic Chow motives. Then there is an isomorphism of abelian groups

$$\mathrm{Coker}(\mathrm{CH}_0(X) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(X_s)) \simeq \mathrm{Coker}(\mathrm{CH}_0(Y) \xrightarrow{\mathrm{res}} \mathrm{CH}_0(Y_s)).$$

3 Small dimensions

In this section we classify all pairs (X, Y) of non-isomorphic twisted flag varieties of inner type over k of dimension ≤ 5 which have isomorphic Chow motives and hence prove Theorem 1.1.

Dimension 1. Twisted flag varieties of dimension 1 are the twisted forms of the projective line $\mathbb{P}^1$. The twisted forms of $\mathbb{P}^1$ are Severi-Brauer varieties $\mathrm{SB}(H)$, where H is a quaternion algebra. By 2.4

$$\mathcal{M}(\mathrm{SB}(H)) \simeq \mathcal{M}(\mathrm{SB}(H')) \Leftrightarrow H \simeq H', H'^{\mathrm{op}}$$

Since $H \simeq H^{\mathrm{op}}$, we conclude that the motives are isomorphic iff the varieties are isomorphic.

Dimension 2. All twisted flags of dimension 2 are the twisted forms of the projective space $\mathbb{P}^2$ or the split quadric surface $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Observe that Q^2 is a projective homogeneous variety for a group of type D_2 which is not simple, but semisimple. Nevertheless, we shall consider this case too.

The motives of $\mathbb{P}^2$ and Q^2 are not isomorphic, since the respective sequences of ranks $(1,1,1)$ and $(1,2,1)$ are different.

The twisted forms of Q^2 of inner type over k are 2-dimensional quadrics (see [Inv, Cor. (15.12)]). By 2.5 the motives of two quadrics of dimension 2 are isomorphic iff the quadrics are isomorphic.

The twisted forms of $\mathbb{P}^2$ are Severi-Brauer varieties $\mathrm{SB}(A)$, where A is a central simple algebra of degree 3. Again by 2.4 we have

$$\mathcal{M}(\mathrm{SB}(A)) \simeq \mathcal{M}(\mathrm{SB}(A')) \Leftrightarrow A \simeq A', A'^{\mathrm{op}}.$$

Since the varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A^{\mathrm{op}})$ are isomorphic iff A is split, we conclude that all pairs of non-isomorphic varieties which have isomorphic motives are of the kind $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 3.

Dimension 3. Computing generating functions (see 2.2) we conclude that there are only three projective homogeneous varieties of dimension 3 over k_s. Namely, the projective space $\mathbb{P}^3$, the quadric Q^3 and the variety of complete flags A_2/B (B denotes a Borel subgroup). The respective sequences of ranks look as follows:

$$\begin{array}{rcl} \mathbb{P}^3 \simeq \mathrm{A}_3/P_1 & : & (1,1,1,1) \\ Q^3 \simeq \mathrm{B}_2/P_1 & : & (1,1,1,1) \\ \mathrm{A}_2/B & : & (1,2,2,1) \end{array}$$

In particular, we see that the motives of $\mathbb{P}^3$ and Q^3 are isomorphic but the motives of Q^3 and A_2/B are not.

By 2.4 all non-isomorphic twisted forms of $\mathbb{P}^3$ which have isomorphic motives form pairs $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 4 and exponent 4. Observe that all non-isomorphic twisted forms of Q^3 are quadrics as well and by 2.5 the motive of a quadric determines this quadric uniquely. Therefore it remains to describe all possible motivic isomorphisms between the twisted forms ${}_\xi\mathbb{P}^3$ and ${}_\zeta Q^3$ and the twisted forms ${}_\xi(\mathrm{A}_2/B)$ and ${}_\zeta(\mathrm{A}_2/B)$ of the variety of complete flags A_2/B.

According to Corollary 4.4 there are no non-isomorphic twisted forms of A_2/B which have isomorphic Chow motives. And the next lemma shows that there are no such (non-trivial) twisted forms of $\mathbb{P}^3$ and Q^3.

3.1 Lemma. *Let ξ, ζ be 1-cocycles. Then $\mathcal{M}({}_\xi\mathbb{P}^3) \simeq \mathcal{M}({}_\zeta Q^3)$ iff ξ and ζ are trivial.*

Proof. This is a particular case of a more general result (see Lemma 4.2) proven using Index Reduction Formula. Here we give an elementary proof. It uses only well-known facts about quadrics and Severi-Brauer varieties.

Observe that any twisted form of $\mathbb{P}^3$ is a Severi-Brauer variety $\mathrm{SB}(A)$ for some central simple algebra A of degree 4 and any twisted form of Q^3 is a non-singular quadric of dimension 3.

As in 2.6 for a variety X consider the abelian group $\mathrm{Coker}(\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_s))$. If $X = \mathrm{SB}(A)$ is a Severi-Brauer variety of a central simple algebra A, then this cokernel is equal to $\mathbb{Z}/\operatorname{ind}(A)\mathbb{Z}$ (see [Ka00]), where $\operatorname{ind}(A)$ is the index of A. In particular, this cokernel is trivial iff A is split. If X is a quadric then this cokernel is trivial iff X is isotropic (see [Sw89]). In the case X is an anisotropic quadric this cokernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In our case we have two varieties $X = \mathrm{SB}(A)$ and $Y = {}_\zeta Q^3$ which have isomorphic motives. Hence, by 2.6 the respective cokernels must be isomorphic.

Hence, if the quadric Y is isotropic, then the algebra A is split. The latter implies that the motive $\mathcal{M}(\mathrm{SB}(A))$ splits into the direct sum of Lefschetz motives and so is $\mathcal{M}(Y)$, i.e., Y is split as well by 2.5.

Assume q is anisotropic, then there exists a quadratic field extension l/k such that the Witt index of $Y_l = Y \times_k l$ is one (see [Vi03, §7.2]). Since the motives of X and Y are still isomorphic over l, we conclude that A is split over l. Then Y_l is split as well. This leads to a contradiction. $\square$

3.2 Remark. Observe that the pair of twisted forms $({}_\xi(\mathrm{B}_2/P_1), {}_\xi(\mathrm{B}_2/P_2))$ can be viewed as a low-dimensional analog of the pair $({}_\xi(\mathrm{G}_2/P_1), {}_\xi(\mathrm{G}_2/P_2))$ considered by Bonnet. The lemma says that contrary to G_2-case the motives of ${}_\xi(\mathrm{B}_2/P_1)$ and ${}_\xi(\mathrm{B}_2/P_2)$ are not isomorphic (if ξ is non-trivial).

Dimension 4. There are three non-isomorphic projective homogeneous varieties of dimension 4 over k_s. Namely, the projective space $\mathbb{P}^4$, the 4-dimensional quadric $Q^4 \simeq \mathrm{Gr}(2,4)$ and the variety of complete flags B_2/B. The respective sequences of ranks in these cases are all different and look as follows:

$$\begin{array}{rcl} \mathbb{P}^4 \simeq \mathrm{A}_4/P_1 & : & (1,1,1,1,1) \\ Q^4 \simeq \mathrm{A}_3/P_2 & : & (1,1,2,1,1) \\ \mathrm{B}_2/B & : & (1,2,2,2,1) \end{array}$$

Hence, the motives of $\mathbb{P}^4$, Q^4 and B_2/B are non-isomorphic to each other.

By 2.4 all non-isomorphic twisted forms of $\mathbb{P}^4$ which have isomorphic motives form pairs $(\mathrm{SB}(A), \mathrm{SB}(A^{\mathrm{op}}))$, where A is a division algebra of degree 5. By Corollary 4.5 there are no non-isomorphic twisted forms of B_2/B which have isomorphic Chow motives. Therefore the only case left is the case of inner twisted forms of Q^4.

The inner forms of Q^4 are the generalized Severi-Brauer varieties $\mathrm{SB}_2(A)$, where A is a central simple algebra of degree 4. The next lemma shows that there are no non-isomorphic forms of $\mathrm{SB}_2(A)$ which have isomorphic motives.

3.3 Lemma. *Let A, A' be central simple algebras of degree* 4. *Then*

$$\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}_2(A')) \Leftrightarrow \mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A')$$

Proof. Let $\mathcal{M}(\mathrm{SB}_2(A)) \simeq \mathcal{M}(\mathrm{SB}_2(A'))$. It suffices to prove that for all field extensions l/k one has $\mathrm{ind}(A_l) = \mathrm{ind}(A'_l)$. Indeed, by [Ka00, Lemma 7.13] $\langle A\rangle = \langle A'\rangle$ in $\mathrm{Br}(k)$, hence, $A \simeq A'$ or A'^{op}. But $\mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A^{\mathrm{op}})$ for any central simple algebra A of degree 4.

Assume that there exists a field extension l/k such that $\mathrm{ind}(A_l) \neq \mathrm{ind}(A'_l)$. Depending on the indices of A and A' we distinguish the following cases:

Case 1. $\mathrm{ind}(A) = 4$ and $\mathrm{ind}(A') = 1$ or 2.

In this case $\mathrm{SB}_2(A')$ has a rational point. By [Inv, Case $\mathrm{A}_3 = \mathrm{D}_3$], the variety $\mathrm{SB}_2(A')$ is isotropic, hence, the group

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}_2(A')) \to \mathrm{CH}_0(\mathrm{SB}_2(A'_{k_s}))$$

is trivial. By 2.6 the cokernel

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}_2(A)) \to \mathrm{CH}_0(\mathrm{SB}_2(A_{k_s}))$$

must be trivial as well. If $\exp(A) = 2$, then A is a biquaternion algebra and by [Inv, Cor. (15.33)] $\mathrm{SB}_2(A)$ is an anisotropic quadric. Then the cokernel above must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction. If $\exp(A) = 4$, then by [Inv, Cor. (15.33)] $A \simeq C^{\pm}(B, \sigma, f)$, where $(B, \sigma, f) \in {}^1\mathrm{D}_3$ and B is a central simple algebra of degree 6 and index 2. By Merkurjev's theorem (see [Me95]) the cokernel above must be again isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction.

Case 2. $\operatorname{ind}(A) = 2$ and $\operatorname{ind}(A') = 1$.

In this case A' is split, hence, the corresponding variety is a split quadric. From the other hand, $\mathrm{SB}_2(A) \simeq X_q$, where q is some 6-dimensional quadratic form and X_q is the corresponding projective quadric. Using 2.5, we conclude that $\mathrm{SB}_2(A) \simeq \mathrm{SB}_2(A')$, a contradiction. □

Dimension 5. There are five non-isomorphic projective homogeneous varieties over k_s of dimension 5. Namely, the projective space $\mathbb{P}^5$, the quadric Q^5, the exceptional Fano variety G_2/P_2, the flag varieties $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$. The respective sequences of ranks look as follows:

$$\begin{array}{ccc} \mathbb{P}^5 \simeq \mathrm{A}_5/P_1 & : & (1,1,1,1,1,1) \\ Q^5 \simeq \mathrm{B}_3/P_1 & : & (1,1,1,1,1,1) \\ \mathrm{G}_2/P_2 & : & (1,1,1,1,1,1) \\ \mathrm{A}_3/P_{\{\alpha_1\}} \simeq \mathrm{A}_3/P_{\{\alpha_3\}} & : & (1,2,3,3,2,1) \\ \mathrm{A}_3/P_{\{\alpha_2\}} & : & (1,2,3,3,2,1) \end{array}$$

Therefore, the motives of $\mathbb{P}^5$, Q^5 and G_2/P_2 are isomorphic and the motives of $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$ are isomorphic.

As was mentioned before, the twisted forms of $\mathbb{P}^5$ and Q^5 were completely classified up to motivic isomorphisms by Karpenko and Izhboldin (see 2.4 and 2.5). Moreover, by Lemma 4.2 there there is only one pair $({}_\xi\mathbb{P}^5, {}_\zeta Q^5)$ of twisted forms that have isomorphic motives.

By the result of Bonnet [Bo03] the motive of the twisted form ${}_\xi(\mathrm{G}_2/P_2)$ is isomorphic to the motive of ${}_\xi(\mathrm{G}_2/P_1)$ which is a 5-dimensional quadric.

By Corollary 1.5 the motives of the twisted forms of $\mathrm{A}_3/P_{\{\alpha_1\}}$ and $\mathrm{A}_3/P_{\{\alpha_2\}}$ are isomorphic iff the respective central simple algebras of degree 4 are isomorphic or opposite. This provides the last example (see 1.1) of a pair of non-isomorphic varieties of dimension 5 that have isomorphic motives.

4 Arbitrary dimensions

In the present section we prove several classification results. We start with the following

4.1 Lemma. *Let X and Y be twisted flag varieties of inner type over k which have isomorphic Chow motives. Assume X is not of type E_8 and splits over its function field $k(X)$, i.e., the group corresponding to X splits over $k(X)$. Then X splits over the function field of Y.*

Proof. Since the motives are isomorphic, there is an isomorphism of cokernels (see 2.6) and, hence, an isomorphism of cokernels over $k(Y)$

$$\mathrm{Coker}(\mathrm{CH}_0(X_{k(Y)}) \to \mathrm{CH}_0(X_{k(Y)_s})) \simeq \mathrm{Coker}(\mathrm{CH}_0(Y_{k(Y)}) \to \mathrm{CH}_0(Y_{k(Y)_s}))$$

Since $Y_{k(Y)}$ is isotropic, the right cokernel is trivial and so is the left one. The fact that the map $\mathrm{res} : \mathrm{CH}_0(X_{k(Y)}) \to \mathrm{CH}_0(X_{k(Y)_s})$ is surjective and the group $\mathrm{CH}_0(X_{k(Y)_s})$ is a free abelian group of rank one generated by the class of a rational point $[pt]$ implies that the preimage $\mathrm{res}^{-1}([pt])$ is a 0-cycle of degree 1 in $\mathrm{CH}_0(X_{k(Y)})$. Then, the variety $X_{k(Y)}$ is isotropic (see [To04, Q. 0.2]).

By [KR94, 3.16.(iii)] the function field $k(X)$ is a generic splitting field for the respective parabolic subgroup P. Since $X_{k(Y)}$ is isotropic, the field $k(Y)$ is a k-specialization of $k(X)$ (see [KR94, Def. 1.2]). Since X splits over $k(X)$, $k(X)$ is a splitting field for the respective group G. Then, by [KR94, 3.9.(iii)], $k(Y)$ is a splitting field of G as well, i.e., $X_{k(Y)}$ splits. □

4.2 Proposition. *Let γ, δ be 1-cocycles and $X = {}_\gamma\mathbb{P}^n$, $Y = {}_\delta Q^n$ be the respective twisted forms for $n > 1$ odd. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow \gamma \text{ and } \delta \text{ are trivial.}$$

Proof. Observe that X is a Severi-Brauer variety corresponding to a central simple algebra A and Y is a n-dimensional quadric.

Assume that $\mathcal{M}(X) \simeq \mathcal{M}(Y)$ and γ is not trivial. By Lemma 4.1 applied to X and Y, the algebra $A_{k(Y)}$ splits, i.e., $\mathrm{ind}(A_{k(Y)}) = 1$. From the other hand by Index Reduction Formula (see [MPW]) we obtain

$$\mathrm{ind}(A_{k(Y)}) = \min\{\mathrm{ind}(A), 2^{(n-1)/2}\,\mathrm{ind}(A \otimes_k C_0(q))\} > 1,$$

where $C_0(q)$ is the even part of the Clifford algebra of the quadric corresponding to Y. This leads to a contradiction. □

Note that the same proof works for twisted forms of types B_n and C_n. Namely,

4.3 Proposition. *Let γ, δ be 1-cocycles and $X = {}_\gamma(\mathrm{C}_n/P_l)$, $Y = {}_\delta(\mathrm{B}_n/P_l)$ be the respective twisted forms for an odd $1 \le l < n$. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow \gamma \text{ and } \delta \text{ are trivial.}$$

Proof. For any simple algebraic group G as above consider a twisted flag variety $W = {}_\xi(G/P_\Theta)$ over k. On the Tits diagram (see [Ti66]) of G over $k(W)$ all vertices corresponding to simple roots from $\Pi \setminus \Theta$ are circled.

In our case since l is odd, this implies that $X_{k(X)}$ is split (see [Ti66] for a complete list of Tits diagrams). The rest of the proof repeats the proof of 4.2. $\square$

The rest of this section is devoted to the twisted forms of flag varieties. In particular, we obtain the description of motivic isomorphisms for twisted forms of the flag varieties A_2/B, B_2/B and $\mathrm{A}_3/P_{\{\alpha_i\}}$, $i = 1, 2, 3$. We start with the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume $\mathcal{M}(X) \simeq \mathcal{M}(Y)$. Since X and Y are twisted forms of flags containing the subspace of dimension 1 (we may assume $n_1 = m_1 = 1$), the motives of X and Y can be decomposed into a direct sum of twisted motives of Severi-Brauer varieties (see [CPSZ, Thm. 2.1]). Namely,

$$\mathcal{M}(X) \simeq \bigoplus_i \mathcal{M}(\mathrm{SB}(A))(i), \quad \mathcal{M}(Y) \simeq \bigoplus_j \mathcal{M}(\mathrm{SB}(A'))(j). \qquad (*)$$

This together with 2.6 implies the isomorphism of abelian groups

$$\mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}(A)) \to \mathrm{CH}_0(\mathbb{P}^n)) \simeq \mathrm{Coker}(\mathrm{CH}_0(\mathrm{SB}(A')) \to \mathrm{CH}_0(\mathbb{P}^n))$$

and, hence, the isomorphism $\mathbb{Z}/\operatorname{ind}(A)\mathbb{Z} \simeq \mathbb{Z}/\operatorname{ind}(A')\mathbb{Z}$, i.e., $\operatorname{ind}(A) = \operatorname{ind}(A')$. Since the motivic isomorphism is preserved under the base extensions, we obtain that $\operatorname{ind}(A_l) = \operatorname{ind}(A'_l)$ for any finite field extension l/k. In fact, by [Ka00, Lemma 7.13] the latter is equivalent to $\langle A \rangle = \langle A' \rangle$ in $\mathrm{Br}(k)$. In particular, if $\exp(A) = \exp(A')$ is $2, 3, 4, 6$, we obtain $A \simeq A'$ or A'^{op}.

In the opposite direction, let $A \simeq A'$ or A'^{op}. By conditions (i) and (iii) one has two motivic decompositions (*) with the same sets of indices $\{i\}$ and $\{j\}$. Now according to 2.4 the motives of $\mathrm{SB}(A)$ and $\mathrm{SB}(A')$ are isomorphic and, hence, so are $\mathcal{M}(X)$ and $\mathcal{M}(Y)$. $\square$

The following obvious consequences of Theorem 1.4 are used in the proof of Theorem 1.1.

4.4 Corollary. *Let $X = \mathrm{SB}_{1,\dots,n}(A)$ and $Y = \mathrm{SB}_{1,\dots,n}(A')$ be twisted forms of the variety of complete flags of type A_n. Assume the respective central simple algebras A and A' have exponents 1,2,3,4 or 6. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow X \simeq Y.$$

4.5 Corollary. *Let X and Y be twisted forms of the variety of complete flags B_2/B. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \Leftrightarrow X \simeq Y.$$

Proof. The proof repeats the proof of 1.4 observing that the motivic decompositions (*) is provided by [CPSZ, Cor. 2.9]. □

4.6. Consider the pseudo-abelian completion $\mathcal{M}(G, R)$ of the category of motives of projective G-homogeneous varieties with R-coefficients, where G is a group of inner type A_n and R is a ring of coefficients. Such categories were defined and extensively studied in [CM04]. In particular, it was proven that any object of $\mathcal{M}(G, R)$, where R is a discrete valuation ring, has a unique direct sum decomposition into indecomposable objects. Modulo this result the proof of 1.4 immediately implies the following

4.7 Corollary. *Let G be an adjoint semi-simple group of inner type A_n. Let R be a ring such that any object of $\mathcal{M}(G, R)$ has a unique direct sum decomposition into indecomposable objects. Let $X = \mathrm{SB}_{n_1,\dots,n_r}(A)$ and $Y = \mathrm{SB}_{m_1\dots m_r}(A')$ be two twisted flag varieties of type A_n given by central simple algebras A and A' of prime degree. Assume that $\mathcal{M}(X_s) \simeq \mathcal{M}(Y_s)$. Then*

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \text{ in } \mathcal{M}(G, R) \Longleftrightarrow \langle A \rangle = \langle A' \rangle \text{ in } \mathrm{Br}(k).$$

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