BAK'S WORK ON LOWER K-THEORY OF RINGS

R. HAZRAT AND N. VAVILOV

Abstract. This paper is a survey of contributions of Anthony Bak to Algebra and (lower) Algebraic K-theory and some of their consequences. We present an overview of his work in these areas, briefly describe the setup and problems as well as the methods introduced by Bak to attack these problems and state some of the crucial theorems. The aim is to analyze in details some of his methods which are quite important and promising for further work in the subject. Among the topics covered are 1) definition of unitary/general quadratic groups over form rings, 2)structure theory and stability for classical groups and their generalisations, 3) quadratic K_2 and the quadratic Steinberg groups 4) nilpotent K-theory and localisation-completion 5) Intermediate subgroups 6) congruence subgroup problem , and 7) dimension theory. On the other hand, we do not touch work of Bak pertaining to or motivated by topology, including global actions, surgery theory, transformation groups and smooth actions.

On the occasion of his sixtieth birthday,
To TONY BAK; with respect and affection

Contents

1. Form rings and their K-theory

Let $GL_n(A)$ be a general linear group, i.e., the group of all invertible elements of a matrix algebra $M_n(A)$ where A is a associative ring with identity. For an ideal I of A, set $GL_n(A,I) = \mathrm{Ker}(GL_n(A) \to GL_n(A/I))$. Let $E_n(A)$ denote the subgroup of $GL_n(A)$ generated by the elementary matrices and let $E_n(A,I)$ denote the normal subgroup of $E_n(A)$ generated by the elementary matrices in $GL_n(A,I)$. There is a standard embedding $GL_n(A) \to GL_{n+1}(A), \sigma \mapsto \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$, called the stabilization map. Define $GL(A) = \varinjlim_n GL_n(A)$ and $E(A) = \varinjlim_n E_n(A)$. The groups $GL(A,I)$ and $E(A,I)$ are defined similarly.

In the mid 60's Bass proves an important result relating the group structure of $GL(A)$ to the ring structure of A. The theorem states that

$$E(A,I) = [E(A), E(A,I)] = [GL(A), GL(A,I)].$$

Date: September 14, 2005.

Moreover, if $H \subset GL(A)$ is a subgroup normalised by $E(A)$, then, for a unique ideal I, $E(A,I) \subset H \subset GL(A,I)$, and H is normal in $GL(A)$. From this it follows that $E(A,I)$ (in particular $E(A)$) is a normal subgroup of $GL(A)$ and the normal subgroups of $GL(A)$ are in one to one correspondence with subgroups of $K_1(A,I) := GL(A,I)/E(A,I)$ for an ideal I in A. Indeed algebraic K-theory was born out of Bass' observation that the group K_1 (and K_0) could be fit in a unified theory with applications in algebra. The group K_2 was suggested by Milnor in the early 1970's as the kernel of the natural map from the Steinberg group $St(A)$ to $E(A)$, which measures how precisely the three relations available for elementary matrices cover all the possible relations among them; $0 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to 0$. Milnor then proved that the homomorphism $St(A) \to E(A)$ is a universal central extension [23].

After this, activity exploded in two directions. One was to prove similar results in the *non-stable* case, that is when n is fixed, and the other was to prove similar results for all classical groups, for example symplectic or orthogonal groups. Both directions have been proved to be much more difficult than the original cases. In the non-stable case, as there is no "room" available for manoeuver as opposed to the stable case (e.g. think of the Whitehead lemma, proving $E(A)$ is normal in $GL(A)$) one forces to put some finiteness assumption on the ring. Indeed there are (counter) examples available that some rings do not follow the pattern of the stable case. For example in [18], it is shown that for any given n, there exist rings A for which $E_n(A)$ are not normal in $GL_n(A)$). A major contribution in this direction is the work of Suslin who showed that if A is a *module finite ring* that is finitely generated as module over its centre, and $n \geq 3$ then $E_n(A)$ is a normal subgroup of $GL_n(A)$. Then van der Kallen and Suslin-Tulenbaev showed that for such ring, the map $St_n(A) \to E_n(A)$ is central when $n \geq 4$ [30]. And finally a series of work by Wilson, Golubchik and Vaserstein settled down the classification of subgroups of $GL_n(A)$ normalised by $E_n(A)$ when $n \geq 3$ and A is module finite. It turned out that in the non-stable case, one should allow the strip a bit looser than the stable case. Let $\widetilde{GL_n}(A,I) = Ker(GL_n(A) \to PGL_n(A/I))$. Obviously $GL_n(A,I) \subseteq \widetilde{GL_n}(A,I)$. The theorem states that if $H \subset GL_n(A)$ is a subgroup normalised by $E_n(A)$, then, for a unique ideal I, $E_n(A,I) \subset H \subset \widetilde{GL_n}(A,I)$. Note that this does not imply that H is normal in $GL_n(A)$ (See §4.2D in [19] and [31],[17]).

Suslin's result makes it possible to define the nonstable $K_{1,n}$ for module finite rings. The study of these non-stable K_1's is known to be very difficult. There are examples due to van der Kallen and Bak [22, 12] which show that non-stable K_1 can be non-abelian and the natural question is how non-abelian it can be?

The breakthrough came with the brilliant work of Bak, who showed that this group is nilpotent by abelian (Theorem 2.1) if $n \geq 3$ and the ring satisfies some dimension condition (e.g. has a centre with finite Krull dimension). His method which consists of some "conjugation calculus" on elementary elements, plus simultaneously applying localization-patching and completion was a source of further work in this and related areas. We analyze this important work in details in Section 2.

In the other direction, in the case of classical groups, as the elementary subgroups have more generators and relations, this makes, among other things, computations in this setting more complicated.

In 1969 Bak introduces, in his Ph.D. thesis, the notion of form ring and the general quadratic group associated to a form ring. These groups include not only most classical groups (symplectic groups, orthogonal groups, classical unitary groups), but give a whole new range of *classical-like groups*. Bak studies systematically the K-theory of form rings and establishes Bass' result (and in his book, Milnor's result) in this setting. Unfortunately, Bak's thesis was not published. It is mentioned in [14] that "many works appearing up till the late 80's, were proving structure theorems for classical groups over rings which had already been covered in [2]".

Although the notion of form ring, form ideals, general quadratic group and its relative groups are now classical and widely used in literature, let us recall the notions for the convenience of the reader (For a general and comprehensive treatment, see [19] and [6]).

Let A be a ring with an involution denoted by $a \mapsto \overline{a}$; thus $\overline{\overline{a}} = a$ and $\overline{ab} = \overline{b}\overline{a}$. Let $\lambda \in \text{Centre}(A)$ such that $\lambda\overline{\lambda} = 1$. Let $\Lambda_{min} = \{a - \lambda\overline{a} | a \in A\}$ and $\Lambda_{max} = \{a \in A | a = -\lambda\overline{a}\}$. Clearly Λ_{min} and Λ_{max} are additive subgroups of A such that $\Lambda_{min} \subseteq \Lambda_{max}$ and satisfy the closure property $a\Lambda_{min}\overline{a} \subseteq \Lambda_{min}$ and $a\Lambda_{max}\overline{a} \subseteq \Lambda_{max}$ for all elements $a \in A$. Let Λ be an additive subgroup of A such that

(1) $\Lambda_{min} \subseteq \Lambda \subseteq \Lambda_{max}$
(2) $a\Lambda\overline{a} \subseteq \Lambda$ for all $a \in A$.

Λ is called a *form parameter* and the pair (A, Λ) is called a *form ring*.

Let (A, Λ) and (A', Λ') be form rings relative, respectively, to λ and λ'. A ring homomorphism $\mu : A \longrightarrow A'$ such that for any $a \in A$, $\mu(\overline{a}) = \overline{\mu(a)}, \mu(\lambda) = \lambda'$ and $\mu(\Lambda) \subseteq \Lambda'$ is called a *morphism of form rings*. A morphism $\mu : (A, \Lambda) \longrightarrow (A', \Lambda')$ of form rings is called surjective if $\mu : A \longrightarrow A'$ is a surjective ring homomorphism and $\mu(\Lambda) = \Lambda'$.

In order to construct later relative groups for the general quadratic group, we introduce now the notion of form ideal in a form ring, due to Bak. Let I be an ideal of A which is invariant under the involution of A, i.e., $\overline{I} = I$. Let $\Gamma_{max} = I \cap \Lambda$ and $\Gamma_{min} = \{x - \lambda\overline{x} | x \in I\} + \{x\alpha\overline{x} | x \in I, \alpha \in \Lambda\}$. Clearly Γ_{min} and Γ_{max} depend only on the form parameter Λ and the ideal I and satisfy the closure property $a\Gamma_{min}\overline{a} \subseteq \Gamma_{min}$ and $a\Gamma_{max}\overline{a} \subseteq \Gamma_{max}$ for all elements $a \in A$. A *relative form parameter of* I is an additive subgroup of Γ of I such that

(1) $\Gamma_{min} \subseteq \Gamma \subseteq \Gamma_{max}$
(2) $a\Gamma\overline{a} \subseteq \Gamma$ for all $a \in A$.

The pair (I, Γ) is called a *form ideal* in (A, Λ).

Let V be a right A-module and f a *sesquilinear* form on V, i.e., a biadditive map $f : V \times V \longrightarrow A$ such that $f(ua, vb) = \overline{a}f(u, v)b$ for all $u, v \in V$ and $a, b \in A$. Define the maps $h : V \times V \longrightarrow A$ and $q : V \longrightarrow A/\Lambda$ by $h(u, v) = f(u, v) + \lambda\overline{f(v, u)}$ and $q(v) = f(v, v) + \Lambda$. The function q is called a Λ-*quadratic form* on V and h its associated λ-*Hermitian form*. The triple (V, h, q) is called a *quadratic module over* (A, Λ). It is called *nonsingular*, if V is finitely generated and projective over A and the map $V \to Hom_A(V, A), v \mapsto h(v, -)$ is bijective, i.e. the Hermitian form h is nonsingular. A morphism $(V, h, q) \to (V', h', q')$ of quadratic modules over (A, Λ) is an A-linear map $V \to V'$ which preserves the Hermitian and Λ-quadratic forms.

Define the *general quadratic group* $G(V, h, q)$ to be the group of all automorphisms of (V, h, q). Thus

$$G(V, h, q) = \{\sigma \in GL(V) | h(\sigma u, \sigma v) = h(u, v), q(\sigma v) = q(v) \text{ for all } u, v \in V\}$$

where $GL(V)$ denotes as usual the group of all A-linear automorphisms of V. Suppose h and q are defined by the sesquilinear form f. If (I, Γ) is a form ideal in (A, Λ), define the *relative general quadratic group*

$$G(V, h, q, (I, \Gamma)) = \{\sigma \in G(V, h, q) | \sigma \equiv 1 \, mod I, f(\sigma v, \sigma v) - f(v, v) \in \Gamma \text{ for all } v \in V\}$$

In his thesis Bak proved that, if (V, h, q) is nonsingular then the group $G(V, h, q, (I, \Gamma))$ is well defined, i.e. does not depend on the choice of f, and is normal in $G(V, h, q)$. Published proofs for the special case $G_{2n}(A, \Lambda)$ which is defined below and is all we need here, are found in section 5.2 of the book of Hahn-O'Meara [19] or in the recent paper of Bak and Vavilov [14].

We recall now the group $G_{2n}(A, \Lambda)$. Let V denote a free right A-module with ordered basis $e_1, e_2, \cdots, e_n, e_{-n}, \cdots, e_{-1}$. If $u \in V$, let $u_1, \cdots, u_n, u_{-n}, \cdots, u_{-1} \in A$ such that $u = \sum_{i=-n}^{n} e_i u_i$. Let $f : V \times V \to A$ denote the sesquilinear map defined by $f(u, v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n}$. It is easy to see that if h and q are the Hermitian and Λ-quadratic forms defined by f then

$$h(u, v) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n} + \lambda \overline{u}_{-n} v_n + \cdots + \lambda \overline{u}_{-1} v_1$$

and

$$q(u) = \overline{u}_1 u_{-1} + \cdots + \overline{u}_n u_{-n} + \Lambda.$$

The ordered basis we chose here was used in [14]. As it is mentioned in [14], in all previously published works, where general quadratic groups over form rings were considered, either the ordered basis $e_1, e_{-1}, \cdots, e_n, e_{-n}$, or the ordered basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$ is used. For example this latter one is used in the book of Bak [6] (See also [21]).

Using the basis above, we can identify $G(V, h, q)$ with a subgroup of the general linear group $GL_{2n}(A)$ of rank $2n$. This subgroup will be denoted by $G_{2n}(A, \Lambda)$ and is called the *general quadratic group over* (A, Λ) *of rank* n. Using the basis, we can identify the relative subgroup $G(V, h, q, (I, \Gamma)) \subseteq G(V, h, q)$ with a subgroup denoted by $G_{2n}(I, \Gamma)$ of $G_{2n}(A, \Lambda)$. Bak describes the matrices in $G_{2n}(A, \Lambda)$ and $G_{2n}(I, \Gamma)$ in his thesis (See also [14], [19]).

Next we recall the definition of the elementary quadratic subgroup. For $i \in \Delta_n = \{1, \cdots, n, -n, \cdots, -1\}$, let $\varepsilon(i)$ denote the sign of i, i.e., $\varepsilon(i) = 1$ if $i \geq 0$ and $\varepsilon(i) = -1$ if $i < 0$. Let $i, j \in \Delta_n$ such that $i \neq j$. The *elementary transvection* $T_{ij}(a)$ is defined as follows:

$$T_{ij}(a) = \begin{cases} e + a e_{ij} - \lambda^{(\varepsilon(j) - \varepsilon(i))/2} \overline{a} e_{-j,-i} & \text{where } a \in A, \text{ if } i \neq -j \\ e + a e_{i,-i} & \text{where } a \in \lambda^{-(\varepsilon(i)+1)/2} \Lambda, \text{ if } i = -j. \end{cases}$$

Once one knows the form of matrices in $G_{2n}(A, \Lambda)$, it is easy to check that $T_{ij}(a) \in G_{2n}(A, \Lambda)$. The symbol T_{ij} where $i \neq -j$ is called a *short root* whereas $T_{i,-i}$ is called a *long root*.

The subgroup generated by all elementary transvections is called the *elementary quadratic group* and is denoted by $E_{2n}(A, \Lambda)$. This group is the quadratic version of the elementary group in the theory of general linear group. Note that elementary transvections corresponding to long roots are elementary matrices in $E_{2n}(A)$ and elementary transvections corresponding to short roots are a product of two elementary matrices in $E_{2n}(A)$. Let (I, Γ) be a form ideal of (A, Λ). The subgroup which is generated by all (I, Γ)-elementary transvections is denoted by $F_{2n}(I, \Gamma)$, i.e.,

$$F_{2n}(I, \Gamma) = \langle T_{ij}(x), T_{i,-i}(y) | x \in I, y \in \lambda^{-(\varepsilon(i)+1)/2} \Gamma \rangle.$$

The normal closure ${}^{E_{2n}(A,\Lambda)}F_{2n}(I,\Gamma)$ of $F_{2n}(I,\Gamma)$ in $E_{2n}(A,\Lambda)$ is denoted by $E_{2n}(I,\Gamma)$ and is called the *relative (or principal) elementary quadratic subgroup of* $G_{2n}(A,\Lambda)$ *of level* (I,Γ). In this note we sometimes do not distinguish between short and long roots and simply write $T_{ij}(x)$, assuming that $x \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$ whenever $i=-j$.

There are standard relations among the elementary transvections, which are analogous to those for the elementary matrices in the general linear group. Here we follow [19] and particularly [14]. These are much simpler than the original generators and relations (up to 28 relations) in the book of Bak, which in return makes computations with them less painful (See e.g. [21],[25]). We also need these relations to define the quadratic Steinberg group.

(R1) $T_{ij}(a) = T_{-j,-i}(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{a})$
(R2) $T_{ij}(a)T_{ij}(b) = T_{ij}(a+b)$
(R3) $[T_{ij}(a), T_{hk}(b)] = 1$ where $h \neq j, -i$ and $k \neq i, -j$
(R4) $[T_{ij}(a), T_{jh}(b)] = T_{ih}(ab)$ where $i, h \neq \pm j$ and $i \neq \pm h$
(R5) $[T_{ij}(a), T_{j,-i}(b)] = T_{i,-i}(ab - \lambda^{-\varepsilon(i)}\overline{b}\overline{a})$ where $i \neq \pm j$
(R6) $[T_{i,-i}(a), T_{-i,j}(b)] = T_{ij}(ab)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{b}ab)$ where $i \neq \pm j$

There is a standard embedding

$$G_{2n}(A,\Lambda) \to G_{2(n+1)}(A,\Lambda), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix}$$

called the stabilization map (Exactly how this embedding looks like, depends on the ordered basis we choose). Define $G(A,\Lambda) = \varinjlim_n G_{2n}(A,\Lambda)$ and $E(A,\Lambda) = \varinjlim_n E_{2n}(A,\Lambda)$. The groups $G(I,\Gamma)$ and $E(I,\Gamma)$ are defined similarly. As one expects the quadratic version of the Bass' result holds here too. Namely

$$E(I,\Gamma) = [E(A,\Lambda), E(I,\Gamma)] = [G(A,\Lambda), G(I,\Gamma)].$$

Moreover if $H \subseteq G(A,\Lambda)$ is a subgroup normalised by $E(A,\Lambda)$, then, for a unique form ideal $(I,\Gamma), E(I,\Gamma) \subseteq H \subseteq G(I,\Gamma)$ and H is normal in $G(A,\Lambda)$. The proof of this has been written by Bass [16] (See also §5.4D in [19]) which uses the linear result. The much more difficult and demanding task was done by Bak, by imposing some finiteness on the ring to prove the nonstable version of this theorem in the quadratic case parallel to the linear case [2, 3]. Here again, not only we have to loose the strip a bit but also to slightly revise the upper bound group. Let $\widetilde{G_{2n}}(I,\Gamma) = \{\sigma \in G_{2n}(A,\Lambda) | [\sigma, E_{2n}(A,\Lambda)] \subseteq E_{2n}(I,\Gamma)\}$. Then Bak's theorem is as follows,

Theorem 1.1 **(Bak).** *Let A be a module finite ring. Furthermore assume that the centre of A is Noetherian with (Krull) dimension d. Let $n \geq max\{3, d+2\}$. Then*

(1) *For any form ideal (I,Γ), $E_{2n}(I,\Gamma)$ is a normal subgroup of $G_{2n}(A,\Lambda)$.*
(2) *If $H \subseteq G_{2n}(A,\Lambda)$ is a subgroup normalised by $E_{2n}(A,\Lambda)$, then, for a unique form ideal (I,Γ), $E_{2n}(I,\Gamma), \subseteq H \subseteq \widetilde{G_{2n}}(I,\Gamma)$.*

In fact Bak's original theorem imposes a weaker assumption on the centre of A. Plus one can observe that in some cases (e.g. $n >$ s-rank A) the group $\widetilde{G_{2n}}(I,\Gamma)$ is the analogous of $\widetilde{GL_n}(A,I)$.

Recently Bak and Vavilov in [14] could remove the assumption on the dimension of the center of A and prove that if A is finite module over its centre as above then

$E_{2n}(A,\Lambda)$ is a normal subgroup of $G_{2n}(A,\Lambda)$ when $n \geq 3$ (See also [13]). This has been done by adopting Suslin's method for the linear case to the quadratic case [28, 30]. Removing this assumption for the second part of Bak's Theorem 1.1 is remained to be done.

Analog to the linear case one can define the quadratic Steinberg group and the quadratic K_2 group. The *quadratic Steinberg group* $St_{2n}(A,\Lambda)$ is a group generated by $X_{ij}(a)$ where $i,j \in \Delta_n$ and $a \in A$, subjected to the relations R(1) to R(6), with X_{ij} instead of T_{ij}. Here $n \geq 3$ as for $n = 1$ or 2 some of R(1) to R(6) are not valid. The $K_{2,2n}(A,\Lambda)$ is defined as the kernel of the natural epimorphism $St_{2n}(A,\Lambda) \to E_{2n}(A,\Lambda)$. For the *stable* version of these groups, consider the map $St_{2n}(A,\Lambda) \to St_{2(n+1)}(A,\Lambda)$, and define $St(A,\Lambda) = \varinjlim_n St_{2n}(A,\Lambda)$ and $K_2(A,\Lambda)$ as the kernel of the natural epimorphism $St(A,\Lambda) \to E(A,\Lambda)$. Similar to the linear case one can proceed to show that the map $St(A,\Lambda) \to E(A,\Lambda)$ is a central extension. In fact Bak proves

Theorem 1.2. *The homomorphism $St(A,\Lambda) \to E(A,\Lambda)$ is a universal central extension of $E(A,\Lambda)$.*

The proof of this theorem can be found in the book of Bak [6] or §5.5 of [19]. Bak and his student Tang has recently announced that by adopting Suslin-Tulenbaev's proof of centrality of nonstable K_2 in the linear case to the quadratic case, they could show the centrality of quadratic K_2 for a ring module finite over its centre and n bigger than 7 (See also P. 543 [19]). The presence of both short and long roots in the elementary quadratic subgroup, among other things, make the proof quite involved.

2. Nonstable K-theory of rings

In 1991, Bak introduced in his paper [12] his *localisation-completion* method. Using this method he was able to prove that nonstable $K_{1,n}(A) = GL_n(A)/E_n(A)$ is nilpotent by abelian group providing A is module finite over its centre, with finite Bass-Serre dimension and $n \geq 3$. Recall that a group G is nilpotent by abelian, if there is a normal subgroup H, such that G/H is abelian and H is nilpotent. This clearly implies that G is solvable. In fact he proved more:

Theorem 2.1 (**Bak**). *Let A be a quasi-finite R-algebra, i.e. a direct limit of module finite R-subalgebras and $n \geq 3$. Then there is a filtration*

$$GL_n(A) \supseteq SL_n^0(A) \supseteq SL_n^1(A) \supseteq \cdots \supseteq E_n(A)$$

where $GL_n(A)/SL_n^0(A)$ is abelian and $SL_n^0(A) \supseteq SL_n^1(A) \cdots$ is a descending central series. Moreover if $i \geq \delta(A)$ where $\delta(A)$ is the Bass-Serre dimension of A, then $SL_n^i(A) = E_n(A)$.

As the methodology employed to prove this theorem was the main motivation for developing later on *dimension theory* in arbitrary categories and for subsequent works by others, we describe here Bak's localisation-completion method. For making the method as transparent as possible, we suppose the rings involved are commutative Noetherian [1]. Let us start by recalling Bass-Serre dimension for a

[1] Since a ring is a direct limit of its Noetherian subrings and the functors GL_n and E_n preserve the direct limit, the proof smoothly reduces to Noetherian rings. Bak introduced his *finite completion* method, to take care of this when working with general rings

commutative ring R. Consider the topological space $Spec(R)$ of all prime ideals of R under the Zariski topology and let $Max(R)$ denote the subspace consisting of all maximal ideals of R. The Bass-Serre dimension of R, denoted by $\delta(R)$, is defined to be the smallest nonnegative integer d such that the space $Max(R) = X_1 \cup \cdots \cup X_r$ is a finite union of irreducible Noetherian subspaces X_i, with topological dimension not greater than d. It is easy to see that $\delta(R) = 0$ if and only if R is a semi-local ring. As a first step for invoking induction on $\delta(R)$ later on, Bak proves an *induction lemma*: If $s \in R$ such that for each $X_k (1 \leq k \leq r)$, s does not lie in some member of X_k, then $\delta(\hat{R}_s) < \delta(R)$ where $\hat{R}_s = \varprojlim_{p \geq 0} R/s^pR$ is the completion of R at s. With this lemma, Bak sets the stage for simultaneously employing *localisation-patching* to make computations in zero dimensional rings and *completion* for applying induction on $\delta(R)$. This is done as follows. For any multiplicative set S in R, let $S^{-1}R$ denote the ring of S-fractions of R. If $s \in R$, let $\langle s \rangle = \{s^i \mid i \geq 0\}$ denote the multiplicative set generated by s. For each k $(1 \leq k \leq r)$ above, pick a maximal ideal $\mathfrak{m}_k \in X_k$ and let S denote the multiplicative set $S = R \setminus (\mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r)$. For each $s \in S$, consider a diagram

$$\hat{R}_s \longleftarrow R \longrightarrow \langle s \rangle^{-1} R$$

and the direct limit $\varinjlim_s (\langle s \rangle^{-1} R) = S^{-1}R$. The ring $S^{-1}R$ is a semilocal and any completion over $S^{-1}R$ involving only finite number of elements actually takes place in some $\langle s \rangle^{-1}R$. Furthermore $\delta(\hat{R}_s) < \delta(R)$ for any $s \in S$ by the induction lemma. Bak's strategy is to use the rings $\langle s \rangle^{-1}R$ for making computations and the rings $\hat{R}_s$ for applying induction on the dimension of R. To carry out this strategy, he introduces his dimension filtration,

$$SL_n^d(R) = \bigcap_{\substack{R \to A \\ \delta(A) \leq d}} Ker\Big(GL_n(R) \to GL_n(A)/E_n(A)\Big).$$

It is clear that the following homomorphism is injective,

$$GL_n(R)/SL_n^0(R) \to \prod_{\delta(A)=0} GL_n(A)/E_n(A).$$

Since $\delta(A) = 0$, A is semi-local, and thus $GL_n(A)/E_n(A)$ is abelian. It follows that $GL_n(R)/SL_n^0(R)$ is an abelian group. To show that the second half of the filtration in the Theorem 2.1 is a descending central series, it suffices to show that for any $x \in SL_n^0(R)$ and $y \in SL_n^{d-1}(R)$, the commutator $[x, y] \in SL_n^d(R)$. Since the filtration is functorial, we can assume $d = \delta(R)$.

Here is where Bak uses his localisation-completion technique. Consider the diagram

$$GL_n(\hat{R}_s) \xleftarrow{\hat{F}_s} GL_n(R) \xrightarrow{F_s} GL_n(\varinjlim_s \langle s \rangle^{-1} R)$$

Since $\delta(S^{-1}R) = 0$, the image of x in $GL_n(S^{-1}R)$ lies in $E_n(S^{-1}R)$. Since $S^{-1}R = \varinjlim R_s$, there is an $s \in S$ such that $F_s(x) \in E_n(\langle s \rangle^{-1}R)$. On the other hand by induction on the dimension of R, $\hat{F}_s(y) \in E_n(\hat{R}_s)$.

It remains only to show that

$$[F_s{}^{-1}(E_n(\langle s \rangle^{-1}R)), \hat{F}_s^{-1}(E_n(\hat{R}_s))] \subseteq E_n(R). \tag{1}$$

Let $x \in {F_s}^{-1}(E_n(\langle s\rangle^{-1}R))$ and $y \in \hat{F}_s^{-1}(E_n(\hat{R}_s))$. It is clear that after replacing s by a large enough power s^i, we can assume that $x \in F^{-1}(E^K(\frac{1}{s}R))$ for some K, where for any subset $A \subset \langle s\rangle^{-1}R$, $E^K(A)$ denotes the subset of $E_n(\langle s\rangle^{-1}R)$ consisting of all products of K or fewer A-elementary matrices. Since $E_n(R)$ is dense in $E_n(\hat{R}_s)$ in the s-adic topology, it is clear that $y \in E_n(R)GL_n(s^kR)$ for any $k \geq 0$. Thus in order to prove Inclusion 1, it is enough to prove that given any $0 \neq s \in S$, and any natural number K, there exist a natural number k such that

$$\big[F_s^{-1}\big(E^K\big(\frac{1}{s}R\big)\big), GL_n(s^kR)\big] \subseteq E_n(R). \tag{2}$$ [2]

Since R is Noetherian, there is an m, such that the map $F_s : s^mR \to R_s$ is injective. This is a key point, because $F_s|_{GL_n(s^mR)}$ will be injective. Bak then shows that there is a natural number $k_m \geq m$ such that

$$\big[E^K\big(\frac{1}{s}R\big), F_s(GL_n(s^{k_m}R))\big] \subseteq F_s((E_n(s^mR)) \tag{3}$$

Inclusion 2 is a trivial consequence of 3 and the injectivity of F_s on $GL_n(s^mR)$.

Note that once this is proved, it follows by taking $s = 1$ that $E_n(R)$ is normal in $GL_n(R)$ proving a well known result of Suslin.

We shall present the main idea of the proof of 3. It is enough to work with $K = 1$. The general case follows by induction on K. Let $\epsilon(a/s) \in E^1\big(\frac{1}{s}R\big)$ and $\sigma' \in F_s(GL_n(s^kR))$. If we show that for any maximal ideal $\mathfrak{m}$ of R, there is an element $t_\mathfrak{m} \in R \setminus \mathfrak{m}$, and an integer $l_\mathfrak{m}$ such that $[\epsilon(t_\mathfrak{m}^{l_\mathfrak{m}}a/s), \sigma'] \in F_s(E_n(s^qR))$ for suitable q, then since a finite number of $t_\mathfrak{m}$ generate R, it can be seen that $[\epsilon(a/s), \sigma'] \in F_s(E_n(s^pR))$. But to show this we have to use two localisations at the same time (see the diagram below). Suppose $\sigma \in GL_n(s^kR)$, such that $F_s(\sigma) = \sigma'$. For any ideal $\mathfrak{m}$, since $R_\mathfrak{m}$ is semi-local, the image of σ in $R_\mathfrak{m}$ can be written as $\epsilon'\delta'$ where δ' is a diagonal and $\epsilon' \in E(s^{k/2}R_\mathfrak{m})$[3]. By a direct limit argument, there is a $t \in R \setminus \mathfrak{m}$, such that the image of σ in R_t is $\epsilon''\delta''$ where $\epsilon'' \in E(s^{k/2}/tA)$. On the other hand since R is a Noetherian, one can see that for a suitable q the map $t^q\langle s\rangle^{-1}R \to \langle st\rangle^{-1}R$ is injective and thus $GL_n(t^q\langle s\rangle^{-1}R) \to GL_n(\langle st\rangle^{-1}R)$ is injective. Since $GL_n(t^q\langle s\rangle^{-1}R)$ is a normal subgroup of $GL_n(\langle s\rangle^{-1}R)$, for an integer $l \geq q$, $[\epsilon(t^la/s), \sigma'] \in GL_n(t^q\langle s\rangle^{-1}R)$.

$$\begin{tikzcd}
GL_n(s^kR) \arrow[rr, "\epsilon''\delta''"] \arrow[dr, dotted, "\epsilon'\delta'"] \arrow[dd, "\sigma'"'] & & GL_n(\langle t\rangle^{-1}R) \arrow[dl, dotted] \arrow[dd, "\bar{\epsilon}\bar{\delta}"] \\
& GL_n(s^kR_\mathfrak{m}) \subseteq E_n(R_\mathfrak{m}, s^kR_\mathfrak{m})\Delta(s^kA) & \\
GL_n(\langle s\rangle^{-1}R) \trianglerighteq GL_n(t^q\langle s\rangle^{-1}R) \arrow[rr, hook] & & GL_n(\langle st\rangle^{-1}R)
\end{tikzcd}$$

[2]There are some small details here; One can see that although $y = uz$ where $u \in E_n(R)$ and $z \in GL_n(s^kR)$ but u is not harmful and it is enough to prove Equation 2.

[3]It is not hard to see that $E_n(R, q^2) \subseteq E_n(q)$ where q is an ideal of R, and $E_n(R, q)$ is the normal closure of $E_n(q)$

Now pushing σ' from one hand and $\epsilon''\delta''$ from the other hand to the ring $\langle st\rangle^{-1}R$, since these are images of the element σ, they coincide:

$$[F_t(\epsilon(\frac{t^l a}{s})), F_t(\sigma')] = [F_t(\epsilon(\frac{t^l a}{s})), F_s(E(\frac{s^{k/2}}{t}a'))\bar{\delta}].$$

It is easy to see that $\bar{\delta}$ disappear and we are left to show that for any p and q there are suitable integers l and k such that

$$[E(\frac{t^l}{s}A), E(\frac{s^k}{t}A)] \subseteq E(s^p t^q A).$$

Bak achieves this by careful study of the "conjugation calculus" of elementary matrices (Lemma 4.1 to 4.3 in [12]).

Localization-completion method; Application to nonstable K-theory of classical like groups. Bak's method is the main source for further work. In [21] the first author adopts the same method to study nonstable K_1 of quadratic modules, establishing Theorem 2.1 for the general quadratic group $G_{2n}(A,\Lambda)$. In [20] the authors apply the localisation-completion method in the setting of Chevalley groups to prove that K_1 of Chevalley groups are nilpotent by abelian. More precisely, let Φ be a reduced irreducible root system of rank at least two and let R be a commutative ring of finite Bass-Serre dimension or finite Krull dimension. Let $G(\Phi,R)$ be a Chevalley group of type Φ over R and let $E(\Phi,R)$ be its elementary subgroup. Then $K_1(\Phi,R) = G(\Phi,R)/E(\Phi,R)$ is nilpotent-by-abelian. In particular, $E(\Phi,R)$ is a characteristic subgroup of $G(\Phi,R)$. This fact plays a crucial role in A. Stepanov's recent spectacular results, partly positive partly negative, solving the oversubgroup problem for $E(\Phi,R)$ embedded in $G(\Phi,R')$ where R' is an overring of R [27]. Also the recent classification of subgroups of the general linear group $GL_{2n}(A)$ which contain the elementary quadratic subgroup $E_{2n}(A,\Lambda)$ made possible by using Bak's view of using "conjugation calculus" along the localization-patching [25, 32].

3. Bak's dimension theory

In 1995 Bak gave a lecture series in Buenos Aires, sketching a general theory of group-valued functors on arbitrary categories with structure and dimension [11].

An arbitrary category $\mathcal{C}$ is structured by fixing a class of commutative diagrams in $\mathcal{C}$, called structure diagrams and a class of functors taking values in $\mathcal{C}$, called infrastructure functors. A function $d : Obj(\mathcal{C}) \to (ordinal\ numbers)$ is called a dimension function if it satisfies a certain property called reduction relating it to the structure on $\mathcal{C}$. The structure diagram and infrastructure functors one takes depend on the landscape being modelled and the results being sought. We shall describe fundamental concepts and results of the theory in term of structure diagrams of the kind $\bullet \leftarrow \bullet \rightarrow \bullet$ (which are automatically commutative) and dimension function taking values in $\mathbb{Z}^{\geq 0} \cup \{\infty\}$. A short description of the theory in terms of structure diagram of the kind

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

is found in [10] as well as applications. We begin by defining the notion of a category with structure, using structure diagrams of the kind $\bullet \leftarrow \bullet \rightarrow \bullet$.

Definition 3.1. A *category with structure* is a category $\mathcal{C}$ together with a class $\mathcal{S}(\mathcal{C})$ of diagrams $C \leftarrow A \rightarrow B$ in $\mathcal{C}$ called *structure diagrams* and a class $\mathcal{I}(\mathcal{C})$ of functors $F : I \rightarrow \mathcal{C}$ from directed quasi-ordered sets I to $\mathcal{C}$ called *infrastructure functors*, satisfying the following conditions.

(1) $\mathcal{S}(\mathcal{C})$ is closed under isomorphism of diagrams.
(2) For each object A of $\mathcal{C}$, the *trivial* diagram i.e., $A \leftarrow A \rightarrow A$ is in $\mathcal{S}(\mathcal{C})$.
(3) $\mathcal{I}(\mathcal{C})$ is closed under isomorphism of functors.
(4) For each object A of $\mathcal{C}$, the *trivial* functor $F_A : \{*\} \longrightarrow \mathcal{C}, * \mapsto A$, is in $\mathcal{I}(\mathcal{C})$, where $\{*\}$ denotes the directed quasi-ordered set with precisely one element $*$.
(5) For each $F : I \longrightarrow \mathcal{C}$ in $\mathcal{I}(\mathcal{C})$, the direct limit $\varinjlim_I F$ exists in $\mathcal{C}$.

Next a category with dimension is defined. To do this, we need first the notion of reduction.

Definition 3.2. Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure. Let $d : \mathcal{O}bj(\mathcal{C}) \longrightarrow \mathbb{Z}^{\geq 0} \cup \infty$ be a function which is constant on isomorphism classes of objects. Let $A \in \mathcal{O}bj(\mathcal{C})$ such that $0 < d(A) < \infty$. A *d-reduction* of A is a set

$$C_i \leftarrow A \rightarrow B_i \ (i \in I)$$

of structure diagrams where I is a directed quasi-ordered set and $B : I \longrightarrow \mathcal{C}, i \mapsto B_i$, is an infrastructure functor such that the following holds.

(1) If $i \leq j \in I$ then the triangle

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ B_i \arrow[r] & B_j \end{tikzcd}$$

commutes.
(2) $d(\varinjlim_I B) = 0$.
(3) $d(C_i) < d(A)$ for all $i \in I$.

A function d is called a *dimension function* on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ if any object A of $\mathcal{C}$, such that $0 < d(A) < \infty$, has a d-reduction. In this case, the quadruple $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ is called a *category with dimension*.

Remark 3.3. Bak has a more comprehensive theory where he also introduces *virtual isomorphisms* in the category and the notion of *type* and an object can have *d*-reductions of a specific type [11].

Even in this early stage of the theory, one can prove that for any dimension function, there exist a universal one, as the following theorem shows

Theorem 3.4. *Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure and $\mathcal{C}^0$ a nonempty class of objects of $\mathcal{C}$, closed under isomorphism. Then there is a dimension function δ on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ called the universal dimension function for $\mathcal{C}^0$ such that $\mathcal{C}^0$ is the class of* 0*-dimensional objects of δ and such that if d is any other dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$, whose* 0*-dimensional objects are contained in $\mathcal{C}^0$, then $\delta \leq d$.*

For the rest of this section $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ will denote a category with dimension and

$$\mathcal{G}, \mathcal{E} : \mathcal{C} \longrightarrow \mathcal{G}roup$$

a pair of group valued functors on $\mathcal{C}$ such that $\mathcal{E} \subseteq \mathcal{G}$.

Bak then defines the dimension filtration

$$\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots \supseteq \mathcal{E}$$

of $\mathcal{G}$ with respect to $\mathcal{E}$ by

$$\mathcal{G}^n(A) = \bigcap_{\substack{A \to B \\ d(B) \le n}} Ker\Big(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B)\Big).$$

This filtration generalises that in Section 2 for GL_n.

Definition 3.5. A pair $\mathcal{G}, \mathcal{E}$ of group valued functors on $\mathcal{C}$ is called *good* if the following holds.

(1) $\mathcal{E}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) For any A of $\mathcal{C}$, $\mathcal{E}(A)$ is a perfect group.
(3) For any zero dimensional object A, $K_1(A) := \mathcal{G}(A)/\mathcal{E}(A)$ is an abelian group.
(4) For any diagram
$$C \leftarrow A \rightarrow B$$
let $H = Ker(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B))$ and $L = Ker(\mathcal{G}(A) \to \mathcal{G}(C)/\mathcal{E}(C))$. Then the mixed commutator $[H, L] \subseteq \mathcal{E}(A)$.

The following theorem is a central result in Bak's theory of group valued functors on categories with dimension.

Theorem 3.6. *Let $\mathcal{C} = (\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ be a category with dimension and $(\mathcal{G}, \mathcal{E})$ be a good pair of group valued functors on $\mathcal{C}$. Then the dimension filtration*

$$\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$$

of $\mathcal{G}$ with respect to $\mathcal{E}$ is a normal filtration of $\mathcal{G}$ such that the quotient functor $\mathcal{G}/\mathcal{G}^0$ takes its values in abelian groups and the filtration $\mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$ is a descending central series such that if $d(A)$ is finite then $\mathcal{G}^n(A) = \mathcal{E}(A)$ whenever $n \ge d(A)$. In particular, if $d(A)$ is finite then $\mathcal{E}(A)$ is a characteristic subgroup of $\mathcal{G}(A)$.

Remark 3.7. Bak has also an alternative version of the theorem above in which structure diagrams are commutative squares

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

instead of diagrams $\bullet \leftarrow \bullet \rightarrow \bullet$ and a good pair $(\mathcal{G}, \mathcal{E})$ is replaced by a natural transformation $\mathcal{S} \to \mathcal{G}$ of group valued functors such that the following holds

(1) $\mathcal{S}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) $\mathcal{S}(A)$ is perfect for any A.
(3) $\mathcal{G}(A)/\text{image}(\mathcal{S}(A) \to \mathcal{G}(A))$ is abelian for any zero dimensional object A.
(4) $Ker(\mathcal{S}(A) \to \mathcal{G}(A)) \subseteq$ Center$(\mathcal{S}(A))$ for any finite dimensional object A.
(5) The extension $\mathcal{S} \to \mathcal{G}$ satisfies excision on any structure square.

The conclusion of the alternative version is the same as that above (See [10]).

In the light of Section 2.1 one can see that the localisation-completion method can serve as a way structuring the category of rings (with possibly some supplementary algebraic structure like form parameters) so that the resulting category with structure forms together with Bass-Serre dimension a category with dimension. One can then readily see that the functors GL_n and E_n satisfy conditions 1,2 and 3 of Definition 3.5 and the functors GL_n and St_n, where St_n denotes the Steinberg group, satisfy conditions 1, 2 and 3 of Remark 3.7. Thus in order to obtain Theorem 3.6 one needs only to check that the remaining conditions hold for the functors in question. In [24] Bak's student Mundkur uses the functors GL_n and St_n and the alternative version of 'good' functors described in Remark 3.7 to recover Bak's theorem by the machinery of dimension theory. But he had to pay a price and assume $n \geq 4$, as it is known only that St_n of commutative rings is a central extensions of E_n if $n \geq 4$.

On the other hand in the case of quadratic modules it has not yet been established that quadratic St_n is a central extension of quadratic E_n (Some progress has been made for a big n, see Section 1). Thus in [21], the first author had to adopt the notion of good pair in Definition 3.5 and check condition 4, which because of the presence of both short and long roots in the elementary quadratic subgroup, needs extra effort.

Bak and Stepanov use in [10] the alternative version of good functors in Remark 3.7 to study the nonstable K-theory of *net* general linear groups introduced by Borewicz and Vavilov.

4. Congruence subgroup problem

The development of (lower) algebraic K-theory is closely related to the classification of congruence subgroups in general linear groups (see also Section 1).

Let A be a division algebra with centre K and O a subring of A. Consider the restriction of the reduced norm map $Nrd : GL_n(O) \to K^*$ and let $SL_n(O)$ be the kernel of this homomorphism. For any two sided ideal $\mathfrak{a}$ of O, recall the group $GL_n(O, \mathfrak{a})$ from Section 1 and set $SL_n(O, \mathfrak{a}) = SL_n(O) \cap GL_n(O, \mathfrak{a})$. The groups $GL_n(O, \mathfrak{a})$ and $SL_n(O, \mathfrak{a})$ are called the *congruence subgroups of level* $\mathfrak{a}$ of $GL_n(O)$ and $SL_n(O)$, respectively.

For the moment assume $A = \mathbb{Q}$ is the field of rational numbers and $O = \mathbb{Z}$ is the ring of integers. For any ideal $\mathfrak{a}$ of $\mathbb{Z}$, the quotient $\mathbb{Z}/\mathfrak{a}$ is finite. It follows that $GL_n(O)/GL_n(O, \mathfrak{a})$ is finite and this in return implies that $SL_n(O)/SL_n(O, \mathfrak{a})$ is finite. Thus $SL_n(O, \mathfrak{a})$ has finite index in $SL_n(O)$. In 1964, Bass, Lazard and Serre (and independently Mennicke) proved that if $n \geq 3$ then the converse is also valid, namely any finite index subgroup of $SL_n(O)$ contains a congruence subgroup (For almost a hundred years before this theorem was proved, it was known to Klein that $SL_2(\mathbb{Z})$ does not follow this pattern!).

Generally let D denote a finite, central division algebra over a global field K. Let Σ denote a nonempty finite set of nonequivalent valuations of K which contains all archimedean valuations of K. The ring $R = \cap_{v \notin \Sigma} R_v$ where R_v is the valuation ring of v is called the *ring of* Σ*-integers* of K. Let O be a maximal R-order on D. If $\mathfrak{a}$ is a two sided ideal of O then $O/\mathfrak{a}$ is finite and thus as above the congruence subgroup $SL_n(O, \mathfrak{a})$ has finite index in $SL_n(O)$. The *congruence subgroup problem* asks whether the converse is true as in the case $D = \mathbb{Q}$ and $O = \mathbb{Z}$ (i.e., does any finite index subgroup of $SL_n(O)$ contain a congruence subgroup?).

Serre formulates the congruence subgroup problem in terms of computing a certain group defined as follows. Let $\overline{SL_n(O)}$ (resp $\widehat{SL_n(O)}$) denote the completion of $SL_n(O)$ with respect to the topology defined by the family of congruence subgroups $SL_n(O, \mathfrak{a})$ (resp. family of subgroups of finite index). There is a canonical surjective homomorphism $\widehat{SL_n(O)} \to \overline{SL_n(O)}$. Let $C(\Sigma, SL_n(O))$ be the kernel of this map. It is called the *congruence kernel*. One can check easily that $C(\Sigma, SL_n(O))$ is trivial if and only if the congruence subgroup problem has a positive answer. Serre's formulation of the problem is as follow.

Congruence subgroup problem according to Serre. If $n \geq 3$ or if $n = 2$ and Σ has at least two elements, then is $C(\Sigma, SL_n(O))$ finite? When is $C(\Sigma, SL_n(O))$ trivial?

In 1967 Bass, Milnor and Serre answered the congruence problem for SL_n where $n \geq 3$ and Sp_{2n} where $n \geq 2$ in the case $D = K$ is a global field and computed the groups $C(\Sigma, SL_n(O))$ and $C(\Sigma, Sp_{2n}(O))$. In 1970 Serre considered the group SL_2 when Σ has at least two elements and computed the group $C(\Sigma, SL_2(O))$. In all the cases above the group $C(\Sigma, G)$ where G is one of SL_n or Sp_{2n} can be described as follows:

$$C(\Sigma, G) = \begin{cases} 1 & \text{if } \Sigma \text{ contains a noncomplex valuation} \\ \mu(K) & \text{if } \Sigma \text{ is totally complex} \end{cases}$$

where $\mu(K)$ is the group of roots of unity of K.

Serre also showed that if $G = SL_2$ and S has only one element, then $C(S, G)$ is an infinite group (covering the counter-example of Klein).

In 1981, Bak and Rehmann studied the congruence subgroup problem in the general situation of a finite, central division algebra D over a global field K [8] and discovered that the computation of the congruence kernel $C(\Sigma, SL_n(O))$ depends on $\Sigma \setminus Pl_{D/K}(\Sigma)$ rather than Σ where

$$Pl_{D/K}(\Sigma) = \{v \in \Sigma | \frac{K_v^*}{Nrd(D_v^*)} \neq 1\}.$$

Their Theorem is as follows

Theorem 4.1 (Bak-Rehmann). *Let $n \geq 2$. If $n = 2$ then suppose that $|\Sigma| > 2$. In addition if $D \neq K$ then suppose that $SL_1(O)$ is infinite. Then*

$$C(\Sigma, SL_n(O)) = \begin{cases} 1 & \text{if } \Sigma \setminus Pl_{D/K}(\Sigma) \text{ contains a noncomplex valuation} \\ \mu(K) & \text{if } \Sigma \setminus Pl_{D/K}(\Sigma) \text{ is empty or totally complex, unless} \end{cases}$$

$2|[D : K]$ *and for every* 2*-power root of unity* $\xi \neq \pm 1$*, one has* $\xi - \xi^{-1} \notin K$*. In this case* $C(\Sigma, SL_n(O))$ *is either* $\mu(K)$ *or* $\mu(K)/\pm 1$*.*

It is still a conjecture that the case $\mu(K)/\pm 1$ never happens and $C(\Sigma, SL_n(O))$ is either trivial or $\mu(K)$.

Sketch of the Proof. One shows first that

$$C(\Sigma, SL_n(O)) \cong \varprojlim_{\mathfrak{a} \neq 0} SL_n(O, \mathfrak{a})/E_n(O, \mathfrak{a}).$$

By Bass stability for the functor K_1, we know that

$$SL_n(O, \mathfrak{a})/E_n(O, \mathfrak{a}) \cong SK_1(O, \mathfrak{a}) := SL(O, \mathfrak{a})/E(O, \mathfrak{a}).$$

Thus

$$C(\Sigma, SL_n(O)) \cong \varprojlim_{\mathfrak{a}\neq 0} SK_1(O,\mathfrak{a}).$$

To compute $SK_1(O,\mathfrak{a})$ one uses the following exact sequence of Bak [[6],7.36]

$$K_2(D) \to \coprod_{v\notin\Sigma} \operatorname{coker}\Big(K_2(O_v,\mathfrak{a}_v) \to K_2(D_v)\Big) \to SK_1(O,\mathfrak{a})$$

$$\to \prod_{v\notin\Sigma} SK_1(O_v,\mathfrak{a}_v) \oplus SK_1(D) \to \amalg_{v\notin\Sigma}(SK_1(D_v), SK_1(O_v)),$$

where $\amalg$ denotes the restricted direct product [[6],§7E] [4]. Bak and Rehmann then show that for almost all $v's$, the group $SK_1(O_v)$ is trivial. On the other hand since the group SK_1 is trivial for division algebras over local and global fields (Theorems of Nakayama-Matsushima and Wang respectively) $\amalg_{v\notin\Sigma}(SK_1(D_v), SK_1(O_v)) = 1$ and the above exact sequence reduces to

$$K_2(D) \xrightarrow{\phi} \coprod_{v\notin\Sigma} \frac{K_2(D_v)}{K_2(O_v,\mathfrak{a}_v)} \to SK_1(O,\mathfrak{a}) \to \prod_{v\notin\Sigma} SK_1(O_v,\mathfrak{a}_v) \to 1.$$

Since we are interested in the inverse limit of $SK_1(O,\mathfrak{a})$ over all nonzero $\mathfrak{a}$, we can restrict to the case of "small" two sided ideals $\mathfrak{a}$. Bak and Rehmann then conclude that $SK_1(O_v,\mathfrak{a}_v) = 1$ ([8], Lemma 6.1). Here one can also note that for small ideals, say $\mathfrak{a}' \subseteq \mathfrak{a}''$, the canonical map $K_2(O,\mathfrak{a}') \to K_2(O,\mathfrak{a}'')$ is an isomorphism. For any small ideal $\mathfrak{a}$, set $K_2(O_v) = K_2(O_v,\mathfrak{a}_v)$. Then the exact sequence above takes the form

$$K_2(D) \xrightarrow{\phi} \coprod_{v\notin\Sigma} \frac{K_2(D_v)}{K_2(O_v)} \to SK_1(O,\mathfrak{a}) \to 1.$$

Thus it remains to determine the cokernel of the map ϕ above. This is made possible by using results of Rehmann describing K_2 of division rings in terms of symbols. (These results are similiar to those of Matsumoto describing K_2 of fields). Making use of these descriptions, Bak and Rehmann define maps $\psi : K^* \otimes Nrd(D^*) \to K_2(D)$ and $\psi_v : \mu(K_v) \to K_2(D_v)/k_2(O_v)$ and obtain a commutative diagram

$$\begin{tikzcd}
K_2(D) \arrow[r, "\phi"] & \coprod_{v\notin\Sigma} \frac{K_2(D_v)}{K_2(O_v,\mathfrak{a}_v)} \\
K^* \otimes Nrd(D^*) \arrow[u, "\psi"] \arrow[r, "\phi'"] & \coprod_{v\notin\Sigma} \frac{\mu(K_v)}{\mu(a_v)} \arrow[u, "\amalg_{v\notin\Sigma}\psi_v"']
\end{tikzcd}$$

In the above diagram ϕ is the canonical map. The important map ψ is defined as follows; $\psi(a,b) = (a,\beta)$ where $b = Nrd(\beta)$. (Note that if $Nrd(D^*) = K^*$ and ψ happens to be an isomorphism, then one can show that $\psi^{-1} : K_2(D) \to K_2(K)$ is

[4]This exact sequence holds not only for general linear groups, but for all classical groups GQ (see §1), and not only for rings of integers in division rings, but for a much larger class of rings. It contains the basic strategy for computing the answer to the congruence subgroup and metaplectic problems for all classical groups [7] and also for computing surgery groups in differential topology. Examples of surgery computations are provided in [4] and [5]. The paper [4] is of special importance, because it shows that odd dimensional surgery groups of finite odd order groups vanish. This means that there is no obstruction to performing surgery on an odd dimensional smooth compact manifold whose fundamental group is finite of odd order [1].

the reduced norm map for the functor K_2. The reduced norm for K_2 is defined in the general case by Suslin in 1987 using geometric tools). The maps ϕ' and ψ_v come from the residue norm homomorphim and its combination with the map defined above (see [8], §3 and §4 for implicit constructions). One can prove that $\coprod_{v\notin\Sigma}\psi_v$ is surjective. In the case that there is a noncomplex $v\in\Sigma$, such that $Nrd(D_v)=K_v$, one can observe that ϕ' is also surjective and thus from above commutative diagram it follows that the cokernel of ϕ is trivial and thus $SK_1(O,\mathfrak{a})=1$. This readily implies the first case of the Theorem 4.1. The rest of the work is to show that in other cases the map $\coprod_{v\notin\Sigma}\psi_v$ is an isomorphism and $\mathrm{Im}(\phi\psi)=\mathrm{Im}\phi$ and thus $\mathrm{coker}\phi\cong\mathrm{coker}\phi'$. This is achieved by embedding the above diagram to a larger one using a suitable global splitting field for D and careful analysis of this new diagram ([8] §5).

We will finish this section by mentioning another result of Bak and Rehmann on K_2 of global fields. Recall that, since the norm map of finite field extensions coincides with the transfer map on K_1, the *Hasse norm theorem* can be stated in terms of K_1 functors as follows: *If L is a cyclic extension of a global field K, then an element of $K_1(K)$ lies in the image of transfer map $N_{L/K}:K_1(L)\to K_1(K)$ if and only if its image in each $K_1(K_v)$ lies in the image of the transfer map $N_{L_w/K_v}:K_1(L_w)\to K_1(K_v)$.* Note that here K_v and L_w are the completions of K and L with respect to v and w and w is an extension of v to L. A natural question is whether the result is true for higher K-groups.

In [9] Bak and Rehmann succeeded in proving the Hasse norm theorem for K_2. In fact, they prove it is true not only for cyclic extensions of global fields, but for all finite extensions.

Theorem 4.2 (Bak-Rehmann). *If L is a finite field extension of a global field K, then an element of $K_2(K)$ lies in the image of the transfer map $N_{L/K}:K_2(L)\to K_2(K)$ if and only if its image in each $K_2(K_v)$ lies in the image of the transfer map $\prod_{w/v}N_{L_w/K_v}:\prod_{w/v}K_2(L_w)\to K_2(K_v)$.*

Note that here one considers all extensions w of v to L. This does not change Hasse's original theorem, since if L is Galois over K then

$$\mathrm{image}(N_{L_w/K_v})=\mathrm{image}(\prod_{w/v}N_{L_w/K_v}).$$

To prove their theorem, Bak and Rehmann observe that it is equivalent to the exactness of the sequence

$$K_2(L)\xrightarrow{N_{L/K}}K_2(K)\xrightarrow{\coprod\lambda_v}\coprod_{v\in\Sigma_{L/K}}\mu(K_v)\longrightarrow 1$$

where λ is the composition of the canonical map $K_2(K)\to K_2(K_v)$ with the norm residue map $K_2(K_v)\to\mu(K_v)$ and $\Sigma_{L/K}$ is the set of all real v such that any extension of v to L is complex. They then prove the exactness of the sequence.

It is worth mentioning that Bak and Rehmann prove their theorem without using any Galois cohomology machinery and Tate's result connecting K_2 with Galois cohomology. Similarly Bak and Rehmann show that the Hasse-Schilling norm theorem for K_2 is equivalent to the exactness of the sequence

$$K_2(D)\xrightarrow{Nrd}K_2(K)\xrightarrow{\coprod\lambda_v}\coprod_{v\in\Sigma_{D/K}}\mu(K_v)$$

and prove the result in this form. The reader is encouraged to compare their Theorem 3 in [9] with Suslin's theorem 26.7 in [29].

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Department of Pure Mathematics, Queen's University Belfast, Belfast BT7 1NN, Northern Ireland, U.K.

E-mail address: r.hazrat@qub.ac.uk

Current address: Max Planck Institute For Mathematics, P.O Box 7280, D-53072 Bonn, Germany

E-mail address: hazrat@mpim-bonn.mpg.de

Department of Mathematics and Mechanics, Saint-Petersburg State University, Saint-Petersburg 198904, Russia

E-mail address: vavilov@pdmi.ras.ru