Canonical p-dimensions of algebraic groups and degrees of basic polynomial invariants

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Abstract

In the present notes we provide a new uniform way to compute a canonical p-dimension of a split algebraic group G for a torsion prime p using degrees of basic polynomial invariants described by V. Kac. As an application, we compute the canonical p-dimensions for all split simple algebraic groups.

The notion of a canonical dimension of an algebraic structure was introduced by Berhuy and Reichstein [1]. For a split algebraic group G and its torsion prime p the canonical p-dimension of G was studied by Karpenko and Merkurjev in [3] and [4]. In particular, this invariant was shown to be related with the size of the image of the characteristic map

$$\phi_G : S^*(\hat{T}) \to \mathrm{CH}^*(X), \tag{1}$$

where $\hat{T}$ is the character group of a maximal split torus T, $X = G/B$ is the variety of complete flags and S^* stands for the symmetric algebra. Namely, one has the following formula for the canonical p-dimension of a group G

$$\mathrm{cd}_p(G) = \min\{i \mid \overline{\mathrm{Ch}_i}(X)\}, \tag{2}$$

where $\overline{\mathrm{Ch}_i}(X)$ stands for the image of ϕ_G in the Chow ring $\mathrm{Ch}_i(X) = \mathrm{CH}_i(X; \mathbb{Z}/p\mathbb{Z})$ with $\mathbb{Z}/p\mathbb{Z}$-coefficients. Using this remarkable fact together with the explicit description of the image $R_p = \overline{\mathrm{Ch}}(X)$ Karpenko and Merkurjev computed the canonical p-dimensions for all classical algebraic groups G.

The goal of the present notes is to relate the work by V. Kac [2] devoted to the study of the p-torsion of the Chow ring of an algebraic group G with the canonical p-dimensions of G. In particular we provide a different and uniform approach of computing $cd_p(G)$.

Theorem. *Let G be a split simple algebraic group of rank n and p be a torsion prime. Then*

$$\mathrm{cd}_p(G) = N + n - (d_{1,p} + d_{2,p} + \ldots + d_{n,p}),$$

where N stands for the number of positive roots of G and integers $d_{1,p}, \ldots, d_{1,n}$ are the degrees of basic polynomial invariants modulo p.

Proof. Consider the characteristic map (1) modulo p

$$(\phi_G)_p : S^*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to \mathrm{Ch}^*(X) \tag{3}$$

According to [2, Theorem 1], the kernel of this map I_p is generated by a regular sequence of n homogeneous polynomials of degrees $d_{1,p}, \ldots, d_{n,p}$.

Recall that a Poincare polynomial $P(A,t)$ for a graded module M^* over a field k is defined to be $\sum_i \dim_k M^i \cdot t^i$ (see [5]). Hence, the Poincare polynomial for the $\mathbb{Z}/p\mathbb{Z}$-module R_p is equal to

$$P(R_p, t) = \prod_{i=0}^{n} \frac{1 - t^{d_{i,p}}}{1-t}. \tag{4}$$

Indeed, we identify R_p with the quotient of the polynomial ring in n variables $\mathbb{Z}/p\mathbb{Z}[\omega_1, \ldots, \omega_n]$ modulo the ideal I_p (here ω_i are the fundamental weights). The formula (4) then follows immediately by [5, Cor. 3.3].

According to (2) the canonical p-dimension is equal to the difference $\dim(X) - \deg P(R_p, t)$. □

Corollary 1. *We obtain the following values for the canonical p-dimensions of groups of types F_4, E_6, E_7 and E_8 (in the list below G^{sc} and G^{ad} denote the simply-connected and adjoint forms of G).*

$\mathrm{cd}_2\, F_4 = 3$, $\mathrm{cd}_3\, F_4 = 8$
$\mathrm{cd}_2\, E_6 = 3$, $\mathrm{cd}_3\, E_6^{sc} = 8$, $\mathrm{cd}_3\, E_6^{ad} = 16$
$\mathrm{cd}_2\, E_7^{sc} = 17$, $\mathrm{cd}_2\, E_7^{ad} = 18$ $\mathrm{cd}_3\, E_7 = 8$
$\mathrm{cd}_2\, E_8 = 60$, $\mathrm{cd}_3\, E_8 = 28$ $\mathrm{cd}_5\, E_8 = 24$

Proof. Follows from the list of degrees of basic polynomial invariants provided in [2, Table 2]. □

Corollary 2. *There is the following relation between the canonical p-dimension of G and the p-torsion part of the Chow group of G*

$$\mathrm{cd}_p(G) = \max\{i \mid \mathrm{Ch}^i(G) \neq 0\}$$

Proof. It is known that $\mathrm{Ch}(G) = \mathrm{Ch}(X)/J_p$, where J_p is the ideal generated by the non-constant part of R_p. Since $\mathrm{Ch}(X)$ is a free R_p-module, we obtain $P(\mathrm{Ch}(X)/J_p, t) = P(\mathrm{Ch}(X), t)/P(R_p, t)$. □

Corollary 3. *Let $d_1, d_2, \ldots, d_n$ be the degrees of "usual" basic polynomial invariants of G, i.e., with $\mathbb{Q}$-coefficients. Then the canonical p-dimension of G is equal to*

$$\mathrm{cd}_p(G) = \sum_{i=0}^{n} d'_i \cdot (p^{k_i} - 1),$$

where the integers d'_i and p^{k_i} are the factors of the decompositions $d_i = d'_i \cdot p^{k_i}$, $p \nmid d'_i$. Observe that the degrees d_i for which $k_i > 0$ form the set of p-exceptional degrees introduced in [2].

Proof. Follows by the isomorphism of [2, Theorem 3,(ii)]. □

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References

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