RATIONAL SURFACES AND CANONICAL DIMENSION OF $\mathbf{PGL}_6$

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Summary : The "canonical dimension" of an algebraic group over a field by definition is the maximum of the canonical dimensions of principal homogenous spaces under that group. Over a field of characteristic zero, we prove that the canonical dimension of the projective linear group $\mathbf{PGL}_6$ is 3. We give two distinct proofs, both of which rely on the birational classification of rational surfaces over a nonclosed field. One of the proofs involves taking a novel look at del Pezzo surfaces of degree 6.

1. Introduction

Let F be a field and let $\mathcal{C}$ be a class of field extensions of F. A field $E \in \mathcal{C}$ is called *generic* if for any $L \in \mathcal{C}$ there is an F-place of E with values in L.

Example 1.1. Let X be a variety over F and let $\mathcal{C}_X$ be the class of field extensions L of F such that $X(L) \neq \emptyset$. If X is a smooth irreducible variety, the field $F(X)$ is generic in $\mathcal{C}$ by [7, Lemma 4.1].

The *canonical dimension* $\operatorname{cdim}(\mathcal{C})$ of the class $\mathcal{C}$ is the minimum of $\operatorname{tr.deg}_F E$ over all generic fields $E \in \mathcal{C}$. If X is a variety over F, we write $\operatorname{cdim}(X)$ for $\operatorname{cdim}(\mathcal{C}_X)$ and call it the *canonical dimension of* X. If X is smooth irreducible then by Example 1.1,

$$\operatorname{cdim}(X) \leq \dim X. \tag{1}$$

If X is smooth, proper and irreducible, the canonical dimension of X is the least dimension of a closed irreducible subvariety $Y \subset X$ such that there exists a rational dominant map $X \dashrightarrow Y$ [7, Cor .4.6].

Example 1.2. Let A be a central simple F-algebra of degree n. Consider the class $\mathcal{C}_A$ of all splitting fields of A. Let X be the Severi-Brauer variety $\mathrm{SB}(A)$ of right ideals in A of dimension n. We have $\dim X = n - 1$. Since A is split over a field extension E/F if and only if $X(E) \neq \emptyset$, we have $\mathcal{C}_A = \mathcal{C}_X$ and therefore $\operatorname{cdim}(\mathcal{C}_A) = \operatorname{cdim}(X)$.

Let A be a central simple F-algebra of degree $n = q_1 q_2 \dots q_r$ where the q_i are powers of distinct primes. Write A as a tensor product $A_1 \otimes A_2 \otimes \dots \otimes A_r$, where A_i is a central simple F-algebra of degree q_i. A field extension E/F splits A if and only if E splits A_i for all i. By Example 1.2, the varieties

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$\mathrm{SB}(A)$ and $Y := \mathrm{SB}(A_1) \times \mathrm{SB}(A_2) \times \cdots \times \mathrm{SB}(A_r)$ have the same classes of splitting fields and hence

$$\operatorname{cdim} \mathrm{SB}(A) = \operatorname{cdim}(Y) \leq \dim(Y) = \sum_{i=1}^{r}(q_i - 1) \tag{2}$$

by inequality (1).

It looks plausible that the inequality in (2) is actually an equality. This is proven in [2, Th. 11.4] in the case when $r = 1$, i.e., when $\deg(A)$ is power of a prime.

In the present paper we prove the equality in the case $n = 6$.

Theorem 1.3. *Let A be a division central algebra of degree 6 over a field of characteristic zero. Then* $\operatorname{cdim} \mathrm{SB}(A) = 3$.

The proof builds upon the classification of geometrically rational surfaces. Starting from this classification, we give two independent proofs of the theorem, each of which seems to have its own interest. The first proof uses a novel approach to del Pezzo surfaces of degree 6 (Section 4). The second proof involves a systematic study of the kernel of the map from the Brauer group of a field F to the Brauer group of the function field of a geometrically rational surface over F (Section 5).

Let G be an algebraic group over F. The *canonical dimension of G* is the maximum of $\operatorname{cdim}(X)$ over all G-torsors X over all field extensions of F.

Corollary 1.4. *The canonical dimension of $\mathbf{PGL}_6$ over a field of characteristic zero is equal to* 3.

Proof. Isomorphism classes of $\mathbf{PGL}_6$-torsors over a field extension E/F are in 1-1 correspondence with isomorphism classes of central simple E-algebras of degree 6. Moreover, if a torsor X corresponds to an algebra A then the classes of splitting fields of X and A coincide. Therefore $\operatorname{cdim}(X) = \operatorname{cdim} \mathrm{SB}(A) \leq 3$. There is a field extension E/F possessing a division E-algebra A of degree 6. By Theorem 1.3, $\operatorname{cdim} \mathrm{SB}(A) = 3$ and therefore, $\operatorname{cdim}(\mathbf{PGL}_6) = 3$. □

Remark 1.5. In view of results of Berhuy and Reichstein [2, Rem. 13.2] and Zainoulline [18], Corollary 1.4 completes classification of simple groups of canonical dimension 2 in characteristic zero. Those are $\mathbf{SL}_{3m}/\boldsymbol{\mu}_3$ with m prime to 3.

Let F be a field, $\overline{F}$ an algebraic closure of F. An F-variety, or a variety over F, is a separated F-scheme of finite type. Let X be an F-variety. We let $\overline{X} = X \times_F \overline{F}$.

We shall use the following notation. For a variety X over a field F we write n_X for the *index of X* defined as the greatest common divisor of the degrees $[F(x) : F]$ over all closed points $x \in X$. If there exists an F-morphism $X \to Y$ of F-varieties then n_Y divides n_X. If X is a nonempty open set of a smooth integral quasi-projective F-variety Y then $n_X = n_Y$ (this may be proved by reduction to the case of a curve). Thus if X and Y are two smooth, projective, integral F-varieties which are F-birational, then $n_X = n_Y$ (see also [13, Rem. 6.6]).

2. Rational curves and surfaces

We shall need the following

Theorem 2.1. *Let X be an integral projective variety of dimension at most 2 over a perfect field F. Then there is a smooth integral projective variety X' over F together with a birational morphism $X' \to X$.*

This special case of Hironaka's theorem has been known for a long time. In dimension 1, it is enough to normalize. Modern proofs in the two-dimensional case ([10] [11] [1]) handle arbitrary excellent, noetherian two-dimensional integral schemes: given such a scheme X they produce a birational morphism $X' \to X$ with X' regular. A regular scheme of finite type over a perfect field F is smooth over F.

In this paper an integral variety X over F is called *rational*, resp. *unirational* if there exists a birational, resp. dominant F-rational map from projective space $\mathbb{P}^n_F$ to X, for some integer n. A geometrically integral F-variety X is called *geometrically rational*, resp. *geometrically unirational*, if there exists a birational, resp. dominant $\overline{F}$-rational map from projective space $\mathbb{P}^n_{\overline{F}}$ to $\overline{X}$, for some integer n. Rational integral varieties are unirational. For varieties of small dimension the converse holds under mild assumptions as the following two well known statements show.

Theorem 2.2 (Lüroth)**.** *A unirational integral curve X over F is rational, i.e., X is birationally isomorphic to $\mathbb{P}^1_F$.*

Theorem 2.3 (Castelnuovo)**.** *A unirational integral surface X over an algebraically closed field field F of characteristic zero is rational, i.e., X is birationally isomorphic to $\mathbb{P}^2_F$.*

Proof. See [9, III.2, Theorem 2.4 p. 170] , or [3]. The assumption on $\operatorname{char} F$ is necessary (cf. [9, p. 171]). Surfaces given by an equation $z^p = f(x, y)$ in characteristic p are unirational but in general not rational. □

The following theorem has its origin in a paper of F. Enriques ([5], 1897). The theorem as it stands was proved by V. A. Iskovskikh (1980) after work by Yu. I. Manin (1966, 1967). A proof of the theorem along the lines of modern classification theory (the cone theorem) was given by S. Mori (1982).

For a smooth F-variety X one lets $K = K_X \in \operatorname{Pic} X$ denote the class of the canonical bundle.

A smooth proper F-variety X if called *F-minimal* if any birational F-morphism from X to a smooth proper F-variety is an isomorphism. By Castelnuovo's criterion, a smooth projective surface over a perfect field F is not F-minimal if and only if X contains an exceptional curve of the first kind.

Theorem 2.4 (Iskovskikh, Mori)**.** *Let X be a smooth, projective, geometrically integral surface over a field F. Assume that X is geometrically rational. The group* $\operatorname{Pic} X$ *is free of finite type. Let ρ denote its rank. One of the following statements holds:*

(i) The surface X is not F-minimal.

(ii) We have $\rho = 2$ and X is a conic bundle over a smooth conic.

(iii) We have $\rho = 1$ and the anticanonical bundle $-K_X$ is ample.

Proof. See [6], [15, Thm. 2.7] and [9, Chapter III, Section 2]. See also the notes [3] (where characteristic zero is assumed). □

Smooth projective surfaces whose anticanonical bundle is ample are known as *del Pezzo surfaces.* They automatically are geometrically rational. Let X/F be a del Pezzo surface and $d = \deg(K_X^2)$. In particular, n_X divides d. We have $1 \leq d \leq 9$. For all this, see [12], [9, Chap III.3], [3].

Over a separably closed field F, a del Pezzo surface is either isomorphic to $\mathbb{P}^1 \times_F \mathbb{P}^1$ or it is obtained from $\mathbb{P}^2$ by blowing up a finite set of points (at most 8, in general position). The Picard group of $\mathbb{P}^2$ is $\mathbb{Z}h$, where h is the class of a line, and $K = -3h$. The Picard group of $\mathbb{P}^1 \times_F \mathbb{P}^1$ is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, where e_1 and e_2 are the classes of the two rulings, and $K = -2e_1 - 2e_2$. Given the behaviour of the canonical class under blow-up [12, Chap. III, Prop. 20.10] we therefore have:

Lemma 2.5. *Let X/F be a del Pezzo surface over a separably closed field F. Then one of the following mutually exclusive possibilities holds:*

(i) X is isomorphic to $\mathbb{P}^2$.

(ii) X is isomorphic to $\mathbb{P}^1 \times_F \mathbb{P}^1$.

(iii) The canonical class K_X is not a proper mutliple of another element in $\operatorname{Pic} X$*.*

3. Reduction to a problem on rational surfaces

Lemma 3.1. *Let W be a regular, proper, geometrically unirational variety over a field F of characteristic* 0*. Assume that the canonical dimension* $\operatorname{cdim}(W) = d \leq 2$*. Then there exists a geometrically rational closed F-subvariety $X \subset W$ of dimension d and a dominant rational map $W \dashrightarrow X$.*

Proof. By a property of canonical dimension recalled at the very beginning of this paper, there exist a closed irreducible F-subvariety $X \subset W$ of dimension d and a dominant rational map $W \dashrightarrow X$. By assumption W and therefore X are geometrically unirational. By Theorems 2.2 and 2.3, X is a geometrically rational variety. □

Proposition 3.2. *Let A be a division central algebra of degree 6 over a field F of characteristic zero. Write $A = C \otimes D$, where C and D are central simple F-algebras of degree* 2 *and* 3 *respectively. Consider the Severi-Brauer* *varieties* $Y = \mathrm{SB}(C)$ *and* $Z = \mathrm{SB}(D)$ *of dimension* 1 *and* 2 *respectively. Assume* $\mathrm{cdim}(\mathrm{SB}(\mathrm{A})) \leq 2$*. Then there exists a geometrically irreducible smooth projective* F*-surface* X *such that*

(*i*) X *is* F*-minimal.*

(*ii*) n_X *is divisible by* 6*.*

(*iii*) X *has a point over* $F(Y \times_F Z)$*.*

(*iv*) $Y \times_F Z$ *has a point over* $F(X)$*.*

Proof. Since $\mathrm{cdim}(Y \times_F Z) = \mathrm{cdim}(\mathrm{SB}(\mathrm{A})) \leq 2$, then by Lemma 3.1, there exist a geometrically rational closed F-subvariety $X_1 \subset Y \times Z$ of dimension at most 2 and a dominant rational map $Y \times_F Z \dashrightarrow X_1$. By Theorem 2.1, there is a birational F-morphism $X_2 \to X_1$ with X_2 smooth and projective. Note that since A is a division algebra, we have $n_{Y \times Z} = 6$. Since we have F-morphisms $X_2 \to X_1 \to Y \times_F Z$, the numbers n_{X_1} and n_{X_2} are divisible by 6. There is a dominant rational map $Y \times_F Z \dashrightarrow X_2$.

Suppose that $\dim X_2 = 1$, i.e., X_2 is a geometrically rational curve. Then X_2 is a conic curve (twisted form of the projective line) and n_{X_2} divides 2, a contradiction. It follows that X_2 is a surface. Let $X_2 \to X$ be a birational F-morphism with X an F-minimal smooth projective surface. Since both X_2 and X are smooth projective we have $n_X = n_{X_2}$. □

4. Del Pezzo surfaces of degree 6

In this Section, F is an arbitrary field.

Let us first recall a few facts about del Pezzo surfaces of degree 6. We refer to [12] for background and proofs.

Let us first assume that F is algebraically closed. A del Pezzo surface of degree 6 is the blow-up of $\mathbb{P}^2$ in 3 points not on a line. Because $\mathbf{PGL}_3$ acts transitively on the set of 3 noncolinear points in $\mathbb{P}^2$, all del Pezzo surfaces of degree 6 are isomorphic. A concrete model is provided by the surface S in $\mathbb{P}^2 \times_F \mathbb{P}^2$ with bihomogeneous coordinates $[x_0 : x_1 : x_2;\ y_0 : y_1 : y_2]$ defined by the system of bihomogeneous equations $x_0y_0 = x_1y_1 = x_2y_2$. Projection of S onto either $\mathbb{P}^2$ identifies S with the blow-up of $\mathbb{P}^2$ in the 3 points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. There are 6 "lines" (exceptional curves of the first kind) on S, the inverse images E_1, E_2, E_3 of the 3 points on the first $\mathbb{P}^2$ and the inverse images F_1, F_2, F_3 of the 3 points on the second $\mathbb{P}^2$. The configuration of these lines is that of a (regular) hexagon : two curves E_i do not meet, two curves F_i do not meet, and $(E_i.F_j) = 1$ if $i \neq j$, while $(E_i.F_i) = 0$.

The torus $T = (\mathbb{G}_m)^3/\mathbb{G}_m$ over F where $\mathbb{G}_m$ is diagonally embedded in $(\mathbb{G}_m)^3$ acts on $\mathbb{P}^2 \times_F \mathbb{P}^2$ in the following manner: (t_0, t_1, t_2) sends $[x_0 : x_1 : x_2;\ y_0 : y_1 : y_2]$ to $[t_0x_0 : t_1x_1 : t_2x_2;\ t_0^{-1}y_0 : t_1^{-1}y_1 : t_2^{-1}y_2]$. This action induces an action on $S \subset \mathbb{P}^2 \times \mathbb{P}^2$. The torus T sends each line into itself. The action of T on the complement U of the 6 lines in S is faithful and transitive. If one identifies U with T by the choice of a rational point in U, the variety S with its open set $U = T$ has the structure of a toric variety. The symmetric group $S_2 = \mathbb{Z}/2$ acts on $S \subset \mathbb{P}^2 \times_F \mathbb{P}^2$ by permuting the factors. This globally preserves the lines, the generator of S_2 induces on the hexagon of lines the permutation of each E_i with each F_i, i.e. opposite sides of the hexagon are exchanged. The group S_3 acts on $S \subset \mathbb{P}^2 \times_F \mathbb{P}^2$ by simultaneous permutation on each factor. This globally preserves the lines. The actions of S_2 and S_3 commute. The induced action of the group $H := S_2 \times S_3$ on the hexagon of lines realizes the automorphism group of the hexagon.

Let the group H act on $T = (\mathbb{G}_m)^3/\mathbb{G}_m$ in such a way that the generator of S_2 sends $t \in T$ to t^{-1} and S_3 acts by permutation of the factors. Let T' be the semidirect product of T and H with respect to this action. The above construction yields an isomorphism from T' to the algebraic group $\mathbf{Aut}(S)$ of automorphisms of the surface S. Indeed any σ in $\mathbf{Aut}(S)$ may be multiplied by an element of H so that the action on the hexagon becomes trivial. By general properties of blow-ups, this implies that any of the projections $S \to \mathbb{P}^2$ factorizes through the contraction $S \to \mathbb{P}^2$, i.e. comes from an automorphism of $\mathbb{P}^2$ which respects each of the points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. Any such automorphism is given by an element of T.

Let now F be an arbitrary field and S a del Pezzo surface of degree 6 over F. Over a separable closure $\overline{F}$ of F the del Pezzo surface $\overline{S} = S \times_F \overline{F}$ is split, i.e. isomorphic to the model given above by [4]. Since the 6 lines are globally stable under the action of the Galois group, there exists a Zariski open set $U \subset X$ whose complement over $\overline{F}$ consists of the 6 lines. The Galois action on the 6 lines induces an automorphism of the hexagon of lines, hence a homomorphism $\mathrm{Gal}(\overline{F}/F) \to H = S_2 \times S_3$. There is thus an associated étale quadratic extension K/F and an étale cubic extension L/F. Let T be the connected component of identity in the F-group $\mathbf{Aut}(S)$. Then T is an algebraic torus and U is a principal homogeneous space under T as they are so over $\overline{F}$. The group of connected components of $\mathbf{Aut}(S)$ is a twisted form of $S_2 \times S_3$: it is the F-group of automorphisms of the finite F-scheme associated to the configuration of the 6 lines on $\overline{F}$. The F-torus T will be identified below (Remark 4.4).

Let K be an étale quadratic F-algebra and B an Azumaya K-algebra of rank 9 over K with unitary involution τ trivial on F [8, §2.B]. Thus B is a central simple K-algebra of dimension 9 if K is a field, or B is isomorphic to the product $A \times A^{op}$, where A is a central simple F-algebra of dimension 9 and A^{op} is the opposite algebra if $K \simeq F \times F$.

We consider the F-subspace of τ-symmetric elements

$$\mathrm{Sym}(B,\tau) = \{b \in B \quad \text{such that} \quad \tau(b) = b\}$$

of dimension 9.

Lemma 4.1. *For a right ideal $I \subset B$ of rank 3 over K, the F-subspace $\big(I \cdot \tau I\big) \cap \mathrm{Sym}(B,\tau)$ of $\mathrm{Sym}(B,\tau)$ is 1-dimensional. The correspondence $I \mapsto \big(I \cdot \tau I\big) \cap \mathrm{Sym}(B,\tau)$ gives rise to a closed embedding of varieties*

$$s_{B,\tau} : R_{K/F}\,\mathrm{SB}(B) \to \mathbb{P}\big(\mathrm{Sym}(B,\tau)\big).$$

Proof. It is sufficient to prove the lemma in the split case, i.e., when

$$B = \mathrm{End}(V) \times \mathrm{End}(V^*),$$

where V is a 3-dimensional vector space over F, and τ is the exchange involution $\tau(f, g^*) = (g, f^*)$ for all $f, g \in \mathrm{End}(V)$ (cf. [8, Prop. 2.14]). We identify $\mathrm{Sym}(B, \tau)$ with $\mathrm{End}(V)$ via the embedding $f \mapsto (f, f^*)$ into B.

A right ideal I of B of rank 3 over $K = F \times F$ is of the form

$$I = \mathrm{Hom}(V, U) \times \mathrm{Hom}(V^*, W^*),$$

where U and W are a subspace and a factor space of V of dimension 1 respectively. We have

$$\tau I = \mathrm{Hom}(W, V) \times \mathrm{Hom}(U^*, V^*),$$

and

$$I \cdot \tau I = \mathrm{Hom}(W, U) \times \mathrm{Hom}(U^*, W^*).$$

The F-space

$$\big(I \cdot \tau I\big) \cap \mathrm{Sym}(B, \tau) = \mathrm{Hom}(W, U)$$

therefore is 1-dimensional. Under the identification of $R_{K/F}\,\mathrm{SB}(B)$ with $\mathbb{P}(V) \times \mathbb{P}(V^*)$, the morphism $s_{B,\tau}$ takes a pair of lines (U, W^*) to $\mathrm{Hom}(W, U) = U \otimes W^*$, i.e., it coincides with the Segre closed embedding

$$\mathbb{P}(V) \times \mathbb{P}(V^*) \to \mathbb{P}(V \otimes V^*) = \mathbb{P}(\mathrm{End}(V)). \qquad \square$$

Let $\mathrm{Trd} : B \to K$ be the reduced trace linear form. For any $x, y \in \mathrm{Sym}(B, \tau)$ we have

$$\tau\,\mathrm{Trd}(xy) = \mathrm{Trd}(\tau(xy)) = \mathrm{Trd}(\tau(y)\tau(x)) = \mathrm{Trd}(yx) = \mathrm{Trd}(xy),$$

hence $\mathrm{Trd}(xy) \in F$. We therefore have an F-bilinear form $\mathfrak{b}(x, y) = \mathrm{Trd}(xy)$ on $\mathrm{Sym}(B, \tau)$. The form $\mathfrak{b}$ is non-degenerate as it is so over $\overline{F}$.

Let L be a cubic étale F-subalgebra of B that is contained in $\mathrm{Sym}(B, \tau)$. Write $L^\perp$ for the orthogonal complement of L in $\mathrm{Sym}(B, \tau)$ with respect to the form $\mathfrak{b}$. As L is étale, we have $L \cap L^\perp = 0$. Consider the 7-dimensional F-subspace $F \oplus L^\perp$ of $\mathrm{Sym}(B, \tau)$ and set

$$S(B, \tau, L) = s_{B,\tau}^{-1}(\mathbb{P}(F \oplus L^\perp)).$$

Thus $S(B, \tau, L)$ is a closed subvariety of $R_{K/F}\,\mathrm{SB}(B)$ (cf. [16]).

An *isomorphism* between two triples (B, τ, L) and (B', τ', L') is an F-algebra isomorphism $f : B \xrightarrow{\sim} B'$ such that $f \circ \tau = \tau' \circ f$ and $f(L) = L'$. The automorphism group of (B, τ, L) is a subgroup of the algebraic group $\mathbf{Aut}(B, \tau)$.

The construction of the scheme $S(B, \tau, L)$ being natural, an automorphism of a triple (B, τ, L) induces an automorphism of $S(B, \tau, L)$, i.e., we have an algebraic group homomorphism

$$\mu : \mathbf{Aut}(B, \tau, L) \to \mathbf{Aut}\big(S(B, \tau, L)\big). \tag{3}$$

Theorem 4.2. *Let F be an arbitrary field. Let B be a rank 9 Azumaya algebra with unitary involution τ over a quadratic étale algebra K over F and a cubic étale F-subalgebra L of B contained in* $\mathrm{Sym}(B,\tau)$.

(i) The variety $S(B,\tau,L)$ is a del Pezzo surface of degree 6.

(ii) Any del Pezzo surface of degree 6 over F is isomorphic to $S(B,\tau,L)$ for some B, τ and L.

(iii) Two surfaces $S(B,\tau,L)$ and $S(B',\tau',L')$ are isomorphic if and only if the triples (B,τ,L) and (B',τ',L') are isomorphic.

(iv) The homomorphism μ is an isomorphism.

(v) The étale quadratic algebra K/F and the étale cubic algebra L/F are naturally isomorphic to the ones associated to the Galois action on the lines of the del Pezzo surface $S(B,\tau,L)$ over $\overline{F}$.

Proof. (i): We may assume that F is separably closed. We claim that any triple (B,τ,L) is isomorphic to the *split triple* $\big(M_3(F)\times M_3(F),\varepsilon,F^3\big)$, where:

(1) $\varepsilon(a,b)=(b^t,a^t)$, ($t$ denotes the transpose matrix), in particular $\mathrm{Sym}\big(M_3(F)\times M_3(F),\varepsilon\big)$ consists of matrices of the shape (a,a^t);

(2) F^3 is identified with the subalgebra of diagonal matrices in $\mathrm{Sym}\big(M_3(F)\times M_3(F),\varepsilon\big)$, i.e. those of the shape (a,a) with a diagonal;

(3) $K=F\times F\subset M_3(F)\times M_3(F)$ is the obvious map from $F\times F$ to the center of $M_3(F)\times M_3(F)$.

Indeed, as K and B are split, (B,τ) is isomorphic to $\big(M_3(F)\times M_3(F),\varepsilon\big)$ by [8, Prop. 2.14]. Let $L'\subset M_3(F)\times M_3(F)$ be the image of the (split) étale cubic subalgebra L under this isomorphism. In particular, $(B,\tau,L)\simeq\big(M_3(F)\times M_3(F),\varepsilon,L'\big)$. Any of the two projections to $M_3(F)$ identifies L' with a split étale cubic subalgebra of $M_3(F)$. Any two split étale cubic subalgebras of $M_3(F)$ are conjugate, i.e., there is an $a\in M_3(F)^\times$ such that $aL'a^{-1}=F^3$. Then the conjugation by $(a,(a^{-1})^t)$ yields an isomorphism between $\big(M_3(F)\times M_3(F),\varepsilon,L'\big)$ and $\big(M_3(F)\times M_3(F),\varepsilon,F^3\big)$. The claim is proved.

So we may assume that (B,τ,L) is the split triple. Then $F\oplus L^\perp$ is the space of all pairs (b,b^t) with a matrix b all diagonal elements of which are equal. Let $[x_0:x_1:x_2;\ y_0:y_1:y_2]$ be the projective coordinates in $\mathbb{P}^2\times_F\mathbb{P}^2$. The Segre embedding $s_{B,\tau}$ takes $[x_0:x_1:x_2;\ y_0:y_1:y_2]$ to the point of $\mathbb{P}(M_3(F))$ given by the matrix $(x_iy_j)_{i,j=1,2,3}$ (we here identify an element $(a,a^t)\in\mathrm{Sym}\big(M_3(F)\times M_3(F),\varepsilon\big)$ with $a\in M_3(F)$).

Therefore $S(B,\tau,L)$ is a closed subvariety of $\mathbb{P}^2\times_F\mathbb{P}^2$ given by the equations $x_0y_0=x_1y_1=x_2y_2$, that is a split del Pezzo surfaces of degree 6.

(iv): We may assume that F is separably closed and hence we are in the split situation of the proof of (i). Let the torus T, the semidirect product T' and the split del Pezzo surface S be as in the initial discussion of del Pezzo surfaces of degree 6. We let T' act on B by F-algebra automorphisms as follows. The groups T, respectively S_3, act on B by the formula $x(a,b)=(xax^{-1},(x^{-1})^tbx^t)$, where x is in T, respectively is the monomial matrix corresponding to an element of S_3. The generator of S_2 takes a pair (a,b) to (b,a). The action of T' defined this way commutes with τ and preserves L elementwise and therefore induces an algebraic group homomorphism $\varphi : T' \to \mathbf{Aut}(B, \tau, L)$. The composite map of φ with the homomorphism

$$\mathbf{Aut}(B, \tau, L) \to \mathbf{Aut}(K) \times \mathbf{Aut}(L) = S_2 \times S_3.$$

is a surjective homomorphism which coincides with the one described at the beginning of this section. We claim that φ is an isomorphism. Let G be the kernel of the above homomorphism. It suffices to show that the restriction $\psi : T \to G$ of φ is an isomorphism. We view G as a subgroup of the connected component $\mathbf{Aut}(B, \tau)^+$ of identity in $\mathbf{Aut}(B, \tau)$. We have an isomorphism between $\mathbf{PGL}_3$ and $\mathbf{Aut}(B, \tau)^+$ taking an a to the conjugation by $(a, (a^{-1})^t)$ (cf. [8, §23]). The composite map

$$T \xrightarrow{\psi} G \hookrightarrow \mathbf{Aut}(B, \tau)^+ \xrightarrow{\sim} \mathbf{PGL}_3$$

identifies T with the maximal torus $\widetilde{T}$ of the classes of diagonal matrices in $\mathbf{PGL}_3$. The image of G in $\mathbf{PGL}_3$ coincides with the centralizer of $\widetilde{T}$ in $\mathbf{PGL}_3$, hence it is equal to $\widetilde{T}$. Thus ψ is an isomorphism. The claim is proved.

The composite map

$$T' \xrightarrow{\varphi} \mathbf{Aut}(B, \tau, L) \xrightarrow{\mu} \mathbf{Aut}\big(S(B, \tau, L)\big)$$

coincides with the isomorphism in the initial discussion of del Pezzo surfaces of degree 6. Therefore, μ is an isomorphism.

(ii) and (iii): By the proof of (i), any triple (B, τ, L) over $\overline{F}$ is isomorphic to the split triple. Moreover, any del Pezzo surface of degree 6 splits over $\overline{F}$. The homomorphism μ in (3) is an isomorphism by (iv), therefore the statements follow by the standard technique in [8, §26].

(v): The étale algebras K and L are associated to the Galois action on the set of 6 minimal diagonal idempotents e_i and f_i of the algebra $F^3 \times F^3$ $(i = 1, 2, 3)$ where the e_i (resp. the f_i) are diagonal idempotents in $F^3 \times 0$ (resp. $0 \times F^3$). The statement follows from the fact that the correspondence $e_i \mapsto E_i$, $f_i \mapsto F_i$ establishes an isomorphism of the $(S_2 \times S_3)$-sets of minimal idempotents $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ and exceptional lines $\{E_1, E_2, E_3, F_1, F_2, F_3\}$. □

Remark 4.3. With notation as in the beginning of this section, the natural exact sequence of Galois modules

$$0 \to \overline{F}[U]^\times/\overline{F}^\times \to \mathrm{Div}_{\overline{S} \setminus \overline{U}}(\overline{S}) \to \mathrm{Pic}\,\overline{S} \to \mathrm{Pic}\,\overline{U},$$

where $\mathrm{Div}_{\overline{S} \setminus \overline{U}}(\overline{S})$ denotes the group of divisors of $\overline{S}$ with support on the complement of $\overline{U}$ and the first map is the divisor map, yields the exact sequence of Galois lattices:

$$0 \to \hat{T} \to \mathbb{Z}[KL/F] \to \mathrm{Pic}\overline{S} \to 0,$$

which defines the 2-dimensional F-torus T with character group $\hat{T} = \overline{F}[U]^\times/\overline{F}^\times$. The F-variety U is a principal homogeneous space under T. The 6-dimensional Galois module $\mathbb{Z}[KL/F]$ is the permutation module on the 6 lines.

Direct computation over $\overline{F}$ shows that there is an exact sequence of Galois lattices

$$0 \to \hat{T} \to \mathbb{Z}[KL/F] \to \mathbb{Z}[L/F] \oplus \mathbb{Z}[K/F] \to \mathbb{Z} \to 0.$$

Here $\mathbb{Z}[L/F]$ is the 3-dimensional permutation lattice on the set of opposite pairs of lines in the hexagon and $\mathbb{Z}[K/F]$ is the 2-dimensional permutation lattice on the set of triangles of triples of skew lines in the hexagon. The map $\mathbb{Z}[KL/F] \to \mathbb{Z}[L/F]$ sends a line to the pair it belongs to, and the map $\mathbb{Z}[KL/F] \to \mathbb{Z}[K/F]$ sends a line to the triangle it belongs to. The map $\mathbb{Z}[L/F] \oplus \mathbb{Z}[K/F] \to \mathbb{Z}$ is the difference of the augmentation maps. Note that this Galois homomorphism has an obvious Galois equivariant section.

¿From this we conclude that there exist an isomorphism of Galois lattices

$$\operatorname{Pic} \overline{S} \oplus \mathbb{Z} \simeq \mathbb{Z}[L/F] \oplus \mathbb{Z}[K/F]$$

and an exact sequence of F-tori

$$1 \to \mathbb{G}_{m,F} \to R_{L/F}\mathbb{G}_m \times R_{K/F}\mathbb{G}_m \to R_{KL/F}\mathbb{G}_m \to T \to 1.$$

Taking F-points and using Hilbert's theorem 90, we conclude that $T(F)$ is the quotient of $(KL)^\times$ by the subgroup spanned by $K^\times$ and $L^\times$. We also see that the F-torus is stably rational. More precisely $T \times_F R_{K/F}\mathbb{G}_m \times_F R_{L/F}\mathbb{G}_m$ is F-birational to $\mathbb{G}_{m,F} \times_F R_{KL/F}\mathbb{G}_m$. The F-torus T actually is rational (Voskresenskiĭ proved that all 2-dimensional tori are rational).

Remark 4.4. It follows from the proof of Theorem 4.2 that T is a maximal F-torus of the connected component of the identity $\mathbf{Aut}(B,\tau)^+$ of the automorphism group of the pair (B,τ). By [8, §23], the group of F-points of $\mathbf{Aut}(B,\tau)^+$ coincides with

$$\{b \in B^\times \mid b \cdot \tau(b) \in F^\times\}/K^\times.$$

It follows that

$$T(F) = \{x \in (KL)^\times \mid N_{KL/L}(x) \in F^\times\}/K^\times.$$

We leave it to the reader to compare this description with the one produced in the previous remark.

Remark 4.5. If the quadratic algebra K is split, i.e., $K = F \times F$ and $B = A \times A^{op}$ with the switch involution τ, where A is a central simple F-algebra of dimension 9, the surface $S(B,\tau,L)$ is a closed subvariety of $\mathrm{SB}(A) \times_F \mathrm{SB}(A^{op})$ and the projection $S(B,\tau,L) \to \mathrm{SB}(A)$ is a blow-up with center a closed subvariety of $\mathrm{SB}(A)$ isomorphic to $\operatorname{Spec} L$. In particular, the surface $S(B,\tau,L)$ is not minimal.

Lemma 4.6. *Let $S = S(B,\tau,L)$ be a del Pezzo surfaces of degree 6. Then*
(i) If $n_S = 6$, then K and the K-algebra B are not split.
(ii) If $S(F) \neq \emptyset$, then the K-algebra B is split.

Proof. If K is split then $n_S \leq 3$ by Remark 4.5. By the same remark, if B is split then S has a rational point over K, hence $n_S \leq 2$. Finally, if S has a rational point, then so does $R_{K/F}\operatorname{SB}(B)$ as S is a closed subvariety of $R_{K/F}\operatorname{SB}(B)$, and therefore B is split. □

We may now give our *first proof of Theorem* 1.3. Let notation be as in Proposition 3.2.

By Theorem 2.4, X is either a conic bundle over a smooth conic or a del Pezzo surface of degree $d = 1, \dots, 9$. In the first case, X has a rational point over a field extension of degree dividing 4, therefore, n_X divides 4, a contradiction. In the latter case, we have $6 \mid n_X \mid d$, i.e., $d = 6$ and X is a del Pezzo surface of degree 6.

By Theorem 4.2, we have $X = S(B, \tau, L)$ for a rank 9 Azumaya algebra B with unitary involution τ over a quadratic étale algebra K over F and a cubic étale F-subalgebra L of B contained in $\operatorname{Sym}(B, \tau)$. It follows from Lemma 4.6 (i) that K and B are not split. By (iii) of Proposition 3.2 and by Lemma 4.6 (ii), the $K(Y \times_F Z)$-algebra $B \otimes_K K(Y \times_F Z)$ is split. The field extension $K(Y \times_F Z)/K(Z)$ is the function field of a conic over $K(Z)$ and $B \otimes_K K(Z)$ is an algebra of degree 3 over $K(Z)$, hence $B \otimes_K K(Z)$ is also split. By a theorem of Châtelet (recalled below), the K-algebra B, which is not split, is similar to D_K or to $D_K^{\otimes 2}$. Since B carries an involution of the second kind we have $\operatorname{cor}_{K/F}([B]) = 0$ by [8, Th. 3.1]. From $2[D] = \operatorname{cor}_{K/F}([B])$ we conclude that D and therefore B is split, a contradiction.

5. Splitting properties of geometrically rational varieties of canonical dimension at most 2

In this section we study the kernel of the natural homomorphism of Brauer groups $\operatorname{Br} F \to \operatorname{Br} F(X)$ for a geometrically unirational smooth variety X of canonical dimension at most 2.

Let us recall the well known:

Proposition 5.1. *Let F be a field, $\overline{F}$ a separable closure, $\mathfrak{g} = \operatorname{Gal}(\overline{F}/F)$ the absolute Galois group. Let X/F be a proper, geometrically integral variety. We then have a natural exact sequence*

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathfrak{g}} \to \operatorname{Br} F \to \operatorname{Br} X,$$

where $\operatorname{Br} X = H^2_{\acute{e}t}(X, \mathbb{G}_m)$. *If moreover X/F is smooth, then the map* $\operatorname{Br} X \to \operatorname{Br} F(X)$ *is injective, and we have the exact sequence*

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathfrak{g}} \to \operatorname{Br} F \to \operatorname{Br} F(X).$$

We write $\operatorname{Br}(F(X)/F)$ for the kernel of $\operatorname{Br} F \to \operatorname{Br} F(X)$.

The following well known result is due to F. Châtelet. In dimension 1, i.e. for A a quaternion algebra and X a conic, it goes back to Witt.

Proposition 5.2. *Let $X = \mathrm{SB}(A)$ be the Severi-Brauer variety of A. Then $\mathrm{Br}(F(X)/F)$ is the subgroup of $\mathrm{Br}\, F$ generated by the class of A.*

Proposition 5.3. *Let F be a perfect field and X a geometrically rational surface over F. Then we have one of the following possibilities:*

(i) X is F-birational to a Severi-Brauer surface, i.e. a twisted form of $\mathbb{P}^2$. Then $\mathrm{Br}(F(X)/F)$ is 0 or $\mathbb{Z}/3$, and is spanned by the class of a central simple algebra of degree 3.

(ii) X is F-birational to a twisted form of $\mathbb{P}^1 \times_F \mathbb{P}^1$. Then $\mathrm{Br}(F(X)/F)$ is 0 or $\mathbb{Z}/2$ (spanned by the class of a quaternion or biquaternion algebra) or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (spanned by the classes of two quaternion algebras).

(iii) X is F-birational to a conic bundle over a smooth projective conic. Then $\mathrm{Br}(F(X)/F)$ is 0 or $\mathbb{Z}/2$, or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and is spanned by the classes of two quaternion algebras.

(iv) $\mathrm{Br}(F(X)/F) = 0$, i.e., the natural map $\mathrm{Br}\, F \to \mathrm{Br}\, F(X)$ is injective.

Proof. By resolution of singularities (Theorem 2.1) we may assume that X is smooth, projective and F-minimal.

Assume X is the Severi-Brauer surface $\mathrm{SB}(A)$ associated to a central simple F-algebra A of index 3. That $\mathrm{Br}(F(X)/F)$ is 0 or $\mathbb{Z}/3$, and is spanned by the class of a central simple algebra of degree 3, follows from Proposition 5.2.

Assume X is a twisted form of $\mathbb{P}^1 \times_F \mathbb{P}^1$. As the automorphism group of $\mathbb{P}^1 \times_F \mathbb{P}^1$ is the semidirect product of $\mathbf{PGL}_2 \times \mathbf{PGL}_2$ with the cyclic group of order 2 permuting the components, we have $X = R_{K/F}(C)$ where K/F is an étale quadratic F-algebra and C is a conic curve over K. If K is a field then by [14, Cor. 2.12], we have

$$\mathrm{Br}\big(F(C)/F\big) = \mathrm{cor}_{K/F}\big(\mathrm{Br}\big(K(C)/K\big)\big).$$

By Proposition 5.2, $\mathrm{Br}\big(K(C)/K\big)$ is generated by the class of a quaternion algebra over K and therefore, $\mathrm{Br}\big(F(C)/F\big)$ is generated by the corestriction of a quaternion algebra that is either 0 or a quaternion algebra, or a biquaternion algebra.

If $K = F \times F$ then $X = C \times_F C'$ where C and C' are conics over F. In this case $\mathrm{Br}\big(F(C)/F\big)$ is a quotient of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, spanned by the classes of the quaternion F-algebras associated with C_1 and C_2.

Let X/F be a conic bundle over a conic Y. Then $\mathrm{Br}\big(F(Y)/F\big)$ is 0 or $\mathbb{Z}/2$, spanned by the class of the quaternion algebra Q associated to Y and the kernel of $\mathrm{Br}\, F(Y) \to \mathrm{Br}\, F(X)$ is 0 or $\mathbb{Z}/2$. Thus the order of the kernel of $\mathrm{Br}\, F \to \mathrm{Br}\, F(X)$ divides 4. Let A/F be a nontrivial division algebra in $\mathrm{Br}\big(F(X)/F\big)$ different from Q. It suffices to show that $\mathrm{Br}\big(F(X)/F\big)$ contains a division quaternion algebra different from Q. The index of A over the function field $F(Y)$ is at most 2. By the index reduction formula [17, Th. 1.3], the index of one of the F-algebras A and $A \otimes_F Q$ is at most 2, i.e., one of these two algebras is similar to a division quaternion algebra different from Q.

If X is not F-isomorphic to a twisted form of $\mathbb{P}^2$, to $\mathbb{P}^1 \times_F \mathbb{P}^1$ or to a conic bundle over a conic then according to Theorem 2.4 and Lemma 2.5, X is a del

Pezzo surface with $\operatorname{Pic} X$ of rank 1 such that the canonical class, which is in $\operatorname{Pic} X$, is not divisible in $\operatorname{Pic} \overline{X}$. As the cokernel of the natural map $\operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathfrak{g}}$ is torsion, the group $(\operatorname{Pic} \overline{X})^{\mathfrak{g}}$ is free of rank 1. Therefore $(\operatorname{Pic} \overline{X})^{\mathfrak{g}}$ is generated by the canonical class and hence the map $\operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathfrak{g}}$ is an isomorphism. By Proposition 5.1, this implies that $\operatorname{Br}(F(X)/F) = 0$. □

Remark 5.4. The proof of this proposition uses Theorem 2.4 and Lemma 2.5 in a critical fashion but it requires no discussion of del Pezzo surfaces other than Severi-Brauer surfaces and twisted forms of $\mathbb{P}^1 \times_F \mathbb{P}^1$. The same comment applies to Theorem 5.5 and Corollary 5.7 hereafter, hence to the *second proof of Theorem* 1.3 given at the end of this section.

Theorem 5.5. *Let W be a smooth, proper, geometrically unirational variety over a field F of characteristic zero.*

(i) If $\operatorname{cdim}(W) = 1$ *then* $\operatorname{Br}\big(F(W)/F\big)$ *is* 0 *or* $\mathbb{Z}/2$.

(ii) If $\operatorname{cdim}(W) = 2$ *then* $\operatorname{Br}\big(F(W)/F\big)$ *is one of* 0, $\mathbb{Z}/2$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ *or* $\mathbb{Z}/3$. *The kernel is spanned either by a quaternion algebra, or two quaternion algebras, or one biquaternion algebra, or a cubic algebra.*

Proof. By Lemma 3.1, there exists a closed geometrically rational F-subvariety $X \subset W$ of dimension 1 in case (i) and of dimension 2 in case (ii). As $\operatorname{Br} W$ injects into $\operatorname{Br} F(W)$, we have

$$\operatorname{Br}\big(F(W)/F\big) \subset \operatorname{Br}\big(F(X)/F\big).$$

If the dimension of X is 1, then X is a smooth conic. The kernel of $\operatorname{Br} F \to \operatorname{Br} F(X)$ is 0 or $\mathbb{Z}/2$.

If the dimension of X is 2, then the possibilities for $\operatorname{Br}\big(F(X)/F\big)$ were listed in Proposition 5.3. □

Remark 5.6. The same theorem holds if the hypothesis that W is geometrically unirational is replaced by the hypothesis that W is a geometrically rationally connected variety. These hypotheses indeed imply that the variety X is geometrically rationally connected. Since X is of dimension at most 2 and $\operatorname{char}(F) = 0$ this forces X to be geometrically rational.

Corollary 5.7. *Let W/F be a smooth, proper, geometrically unirational variety over a field F of characteristic zero. Assume* $\operatorname{cdim}(W) \leq 2$. *Let A and A' be central division F-algebras. If there is an F-rational map from W to the product* $\operatorname{SB}(A) \times_F \operatorname{SB}(A')$ *then one of the following occurs:*

(1) *A and A' are cubic algebras.*
(2) *A and A' are quaternion or biquaternion algebras.*

Proof. If there is such a rational map, then the classes of $A \in \operatorname{Ker}[\operatorname{Br} F \to \operatorname{Br} \operatorname{SB}(A)]$ and $A' \in \operatorname{Ker}[\operatorname{Br} F \to \operatorname{Br} \operatorname{SB}(A')]$ belong to $\operatorname{Br}\big(F(W)/F\big)$. The result then follows from Theorem 5.5. □

Remark 5.8. Corollary 5.7 holds if the hypothesis that W is geometrically unirational is replaced by the hypothesis that W is a geometrically rationally connected variety.

We now give our *second proof of Theorem* 1.3. This proof does not use Section 4. Let notation be as in Proposition 3.2, so that Y, resp. Z, is the Severi-Brauer variety attached to a quaternion algebra, resp. to an algebra of degree 3. Assume $\mathrm{cdim}(\mathrm{SB}(A)) \leq 2$. Then $\mathrm{cdim}(Y \times_F Z) \leq 2$. If we apply the above corollary 5.7 to the identity map of $Y \times_F Z$ then we get a contradiction. We could also combine Proposition 3.2 and Proposition 5.3. It is then clear that we here use statement (iv) of Proposition 3.2, as opposed to our use of statement (iii) of that same proposition in our *first proof* (end of Section 4) of Theorem 1.3.

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