Cohomology of buildings and S–arithmetic groups over function fields

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Dedicated to Jean-Pierre Serre
on the occasion of his eightieth birthday

1 Introduction

This paper is a supplement to the article "Cohomologie d' immeubles et de groupes S–arithméthiques" by A. Borel and J.P. Serre from 1976 (see [BS]). They deal with S–arithmetic subgroups Γ of reductive algebraic groups G in the case of number fields and discrete cocompact groups, so the general case for function fields F ($[F : \mathbb{F}_q(t)] < \infty$, $q = p^r$) is missing. For a torsion–free group Γ they show that Γ is of type FL, which implies, that Γ is also of type FP_∞. For this purpose they compute the cohomology of the spherical Tits–building Y_v of the group $G(F_v)$, where F_v ist the completion of F with respect to $v \in S$, providing Y_v with the F_v–analytic topology. Using Y_v as a boundary at infinity of the Bruhat–Tits–building X_v of $G(F_v)$, thereby compactifying X_v, they can also compute the cohomology with compact supports of $X = \prod_{v \in S} X_v$ and prove, that it vanishes in all dimensions, except the top dimension d, which is the sum $\sum_{v \in S} \dim X_v$ (in the number field case S contains also archimedean places and X_v is then the corresponding symmetric space with corners); in dimension d the cohomology with coefficients in $\mathbb{Z}$ is free.

On the other hand, U. Stuhler considers in his paper "Homological properties of certain arithmetic groups in the function field case" (see [St]) the groups $\Gamma = \mathrm{PGL}_2(\mathcal{O}_S)$ for $\mathcal{O}_S \subset F$ (S finite) and shows in the second part, that the "finiteness length" of Γ is bounded, more precisely, that Γ is not of type $FP_{|S|}$. Following a remark of J.P. Serre he uses the spectral sequence for the cohomology groups of stabilizers Γ_σ (σ a simplex in X/Γ) with coefficients in the finite field $\mathbb{F}_p$ and deduces that the vector spaces $H^r(X/\Gamma; H^s(\Gamma_\sigma; \mathbb{F}_p))$ are finite–dimensional for $0 \leq r \leq |S|$, $0 \leq s < |S|$, but definitively not for $r = 0$, $s = |S|$. As a consequence $H^{|S|}(\Gamma; \mathbb{F}_p)$ has also infinite dimension, thus Γ cannot be of type $FP_{|S|}$. For this result Stuhler employs a filtration of X/Γ and computes step by step.

We shall now combine the methods of these two papers: We observe first, that Borel–Serre's computation also works for certain subcomplexes of Y_v and X_v which are finite modulo a stabilizer–group Γ_σ $(\sigma \in X)$, if we assume that Γ_σ has only p–torsion and moreover consists only of elements contained in the unipotent radical U of a minimal F–parabolic group of G. This can be established by changing from Γ to a (congruence) subgroup Γ_0 of finite index: the existence of Γ_0 follows from reduction theory in the formulation of Harder. In this way we obtain that the cohomology groups of Γ_σ vanish with exception of the top dimension, where we have a description as locally constant functions on $\Gamma_\sigma \subset U(F)$ — a restriction of the results of Borel–Serre. Consequently Stuhler's spectral sequence degenerates to the isomorphism $H^d(\Gamma;\mathbb{F}_p) \simeq H^0(X/\Gamma; H^d(\Gamma_\sigma;\mathbb{F}_p))$ and it remains to prove, that this space is infinite–dimensional. This may again be done by a filtration of X/Γ, but it suffices to find an infinite sequence of vertices σ, where Γ_σ becomes bigger and defines new elements of $H^0(X/\Gamma, H^d(\Gamma_\sigma;\mathbb{F}_p))$. Thus we can at first generalize Borel–Serre's theorem on the cohomology of S–arithmetic groups Γ with coefficients in $\mathbb{Z}[\Gamma]$ to the function field case, when Γ has only p–torsion and secondly prove that all S–arithmetic subgroups Γ of almost simple groups G, defined over F cannot be of type FP_d, where d is the sum of the local ranks of G over the fields F_v for $v \in S$. (The problem for reductive groups can be reduced to this special case: See [B3], 2.6 c)

Another proof of the last theorem was recently given by K.U. Bux and K. Wortman (see [BW]).

It is conjectured, that these groups are of type FP_{d-1}, which has been shown in special cases or with additional assumptions on the growth of F with respect to the rank: see the precise statements at the end of this paper.

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2 Cohomology of spherical buildings

In order to fix the notations and to make clear that Borel–Serre's computations also work for certain subcomplexes of spherical buildings we have to report [BS], §1 and §2.

Let k be a non–archimedean local field, G a connected semi–simple k-group of k–rank l and Y the Tits–building of $G(k)$. It is well known by the Solomon–Tits–theorem, that Y has the homotopy–type of a bouquet of (l-1)–spheres — with respect to its simplicial topology. The first step in [BS] is to provide Y with the analytic topology, induced by the valuation on k and to prove an analogue, by computing the Alexander–Spanier–cohomology.

2.1 Denote by P a minimal k–parabolic subgroup of G, by T a maximal k–split torus of P, by Δ the basis of simple roots on T, which defines P and by S the associated set of reflections in the Weyl group W of P, $W = N(T)/Z(T)$. We may identify both Δ and S with the index set $\mathbb{N}_l = \{1, \ldots, l\}$, then P_I is the parabolic subgroup generated by P and $Z(T_I)$, T_I the kernel of all $a_i \in \Delta_I$ for $I \subseteq \mathbb{N}_l$, such that $\Delta \setminus \Delta_I$ is the set of simple roots of the semi–simple factor of P_I; we have $P_\emptyset = P$, $P_{\mathbb{N}_l} = G$ and $J \subset I \Leftrightarrow P_J \subset P_I$. Now let C be the closed chamber of Y, fixed by $P(k)$ and C_I its face fixed by P_I (for proper subsets I of $\mathbb{N}_l$), then Y can be described as $G(k)/P(k) \times C$ with identifications. Borel–Serre provide $G(k)/P(k) = (G/P)(k)$ with the k–analytic topology, so it is compact, C with its simplicial topology and Y by the quotient topology, defined by $G(k) \times C \longrightarrow G(k)/P(k) \times C \xrightarrow{\lambda} Y$.

2.2 In the next step we use the Bruhat–decomposition $G(k) = \coprod_{w\in W} P(k)wP(k)$, which can be made unique by the refinement $G(k) = \coprod_{w\in W} U_w(k)wP(k)$, where U_w is a connected subgroup of the unipotent radical U of P; for the only element w_0 of maximal length in W one has $U_{w_0} = U$. Thus $C(w) := P(k)wP(k)/P(k)$ is a principal homogenous space over $U_w(k)$, therefore isomorphic to k^{d_w} with $d_w = \dim U_w$, especially $C(w_0) \simeq k^{d_0}$, $d_0 = \dim U$. Using this decomposition and enumeration of the Weyl group $W = \{w_i | 1 \le i \le N\}$, compatible with the length function (i.e. $l(w_i) \le l(w_j)$ for $i < j$) Borel–Serre construct a filtration of the building as follows: For any $w_m \in W$ define $I_m := \{i \in \mathbb{N}_l | l(w_m s_i) > l(w_m)\}$ for $s_i \in S$, thus $I_1 = \mathbb{N}_l$ (for $w_1 = \mathrm{id}$), $I_N = \emptyset$ and $I_m \notin \{\mathbb{N}_l, \emptyset\}$ for $1 < i < N$ and setting $L_m := \bigcup_{I\subset I_m} C_I^\circ$ (open faces, in particular $C_\emptyset^\circ = C^\circ$), $L'_m := \bigcup_{I\not\subset I_m} C_I$, we see that $w_m L'_m$ is the union of all codimension–1–faces of $w_m C$, which also belong to a chamber which has smaller distance to C than $w_m C$. With $E_m := \bigcup_{1\le i\le m} C(w_i)$, $Y_m := \lambda(E_m \times C)$ we obtain a filtration of $Y = \bigcup_{1\le i\le N} Y_i$ with $Y_i \subset Y_j$ for $i < j$ and

Lemma 1: *a)* $Y_m \backslash Y_{m-1} = \coprod_{I\subset I_m} \left(\prod_I C(w_m) \times C_I^\circ\right)$, *such that* $Y_m\backslash Y_{m-1}$ *is homeomorphic to* $C(w_m) \times L_m$.

b) $\widetilde{H}^i(Y;M) = H_c^i(Y \setminus Y_{N-1};M)$. *($M$ is a $\mathbb{Z}$–module, $\widetilde{H}^*$ the reduced cohomology and H_c^* cohomology with compact supports.)*

c) $H_c^i(Y \setminus Y_{N-1};M) = H_c^{l-1}(C^\circ;M) \otimes H_c^{i-(l-1)}(C(w_N);M)$ *which is* 0 *for all* $i \ne l-1$, *so* $H^i(Y;M)$ *vanishes for all* $i \ne l-1$ *and* $\widetilde{H}^{l-1}(Y;M) \simeq \mathbb{Z} \otimes H_c^0(C(w_N);M)$.

Remark. It is enough to prove this lemma and the following proposition for $M = \mathbb{Z}$: The results for $\mathbb{Z}$ show that the cohomology modules are 0 or free; therefore the universal coefficient theorem gives the result for a $\mathbb{Z}$-module M by tensoring (cf. [Br1], 0.8). Moreover this is also true for the set of locally constant functions with compact support over a totally disconnected space X: $C_c^\infty(X;M) = C_c^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} M$ (see [BS], lemma 2.2).

Proof. b) For $1 < m < N$ we have $I_m \notin \{\emptyset, \mathbb{N}_l\}$, which implies that L'_m is contractible, as is $C = L_m \,\dot\cup\, L'_m$ and the exact cohomology sequence for C mod L'_m gives $H_c^*(L_m;\mathbb{Z}) = 0$ and with a) and Künneth's formula $H_c^*(Y_m \setminus Y_{m-1};\mathbb{Z}) = 0$. Now $Y_1 = C$ is contractible and by induction on m we get b).

c) We already know that $Y \setminus Y_{N-1} = Y_N \setminus Y_{N-1} = C(w_N) \times C^{\circ}$, and C° is isomorphic to $\mathbb{R}^{l-1}$, oriented by the enumeration of its codimension–1–faces, thus $H^i_c(C^{\circ};\mathbb{Z}) = 0$ for $i \neq l-1$ and $H^{l-1}_c(C^{\circ};\mathbb{Z}) = \mathbb{Z}$. The Künneth–formula implies $H^i_c(Y \setminus Y_{N-1};\mathbb{Z}) = H^{l-1}_c(C^{\circ};\mathbb{Z}) \otimes H^{i-(l-1)}_c(C(w_N);\mathbb{Z})$, but $C(w_N) = P(k)w_N P(k)/P(k) \simeq k^{d_0}$ is a totally disconnected space, therefore 0–dimensional and has zero cohomology in all dimensions but 0. $\square$

2.3 It remains to describe the cohomology for the top dimension. The building Y is the union of all apartments containing C; call this set $\mathcal{A}$ and A_0 the standard apartment, fixed by $Z(T) \subset P$, so $A_0 \in \mathcal{A}$. The correspondence $g \mapsto gA_0$ defines a bijection between $P(k)/Z(T)(k) \simeq U(k)$ and $\mathcal{A}$, and also $C(w_N) = U(k)w_N P(k)/P(k) \approx U(k)$ is in 1–1–correspondence with $\mathcal{A}$: consider $C(w_N)$ as the set of chambers in Y, which are opposite to C: $\mathcal{A} \longleftrightarrow U(k) \longleftrightarrow C(w_N)$.

Part c) of the lemma also shows that $\widetilde{H}^{l-1}(Y;M) = H^0_c(\mathcal{A};M)$ is isomorphic to $C^\infty_c(\mathcal{A};M)$. This isomorphism has an explicit description: The Coxeter–complex Σ_A of each $A \in \mathcal{A}$ is homeomorphic to the sphere $\mathbb{S}^{l-1}$ with orientation, given by the enumeration (Σ_A is a labelled complex). The cycle $\sum_{w\in W}(-1)^{l(w)}wC$ defines for $l \geq 2$ a class $[A]$ in the homology group $H_{l-1}(Y;M)$ — the "fundamental class of the oriented sphere". A given element $h \in H^{l-1}_c(Y;M)$ has by restriction to $H^{l-1}_c(A;M)$ a value $h([A])$ and by definition of cohomology $h([A])$ is uniquely determined by its value on the opposite chamber uw_NC for $u \in U(k)$, if $A = uA_0$. So we can summarize as

Proposition 1:
$$\widetilde{H}^i(Y;M) = \begin{cases} 0 & \textit{for } i \neq l-1 \\ H^0_c(\mathcal{A};M) \simeq C^\infty_c(U(k);M) & \textit{for } i = l-1 \end{cases}$$

Analyzing the proof above, it is obvious, that it also works for subcomplexes Y' of Y, which are unions of apartments containing a fixed chamber C; especially if $Y' = \bigcup_{u\in U'} uA_0$, where U' is a subgroup of $U(k)$.

Corollary: *If U' is a subgroup of $U(k)$, U the unipotent radical of the minimal k–parabolic group P of a semi–simple algebraic k–group G, k a local nonarchimedean field, C the chamber of the Tits–building fixed by P, A_0 an apartment containing C and $Y' = \bigcup_{u\in U'} uA_0$, then*

$$\widetilde{H}^i_c(Y';M) \simeq \begin{cases} 0 & \textit{for } i \neq l-1 \\ C^\infty_c(U'(k);M) & \textit{for } i = l-1 \end{cases}$$

3 Cohomology of affine buildings

3.1 Let X be the affine Bruhat–Tits–building, defined by the group G and the valuation on k. We may suppose that G is semi–simple and simply connected (cf. [BS], 4.1), so G is a direct product of almost simple groups G_j ($1 \leq j \leq m$) of k–rank > 0 and a simply connected group G_0 of k–rank 0, such that $G_0(k)$ is compact. Then the double Tits–system has also a product structure, the Tits–building Y is the join of the buildings Y_j of G_j and X is a polysimplicial complex.

The apartments of X are in 1–1–correspondence with those of Y; denote by A_0 the standard apartment in X, stabilized by $N(T)$ ($T \subset P$ as in section 2) and choose an origin O in A_0 as a special point and a sector (simplicial cone) D with vertex O and direction C, C the standard chamber in Y, fixed by P (for buildings cf. [BT], [Br2] and [R]).

A main part of [BS] contains the construction of a compactification of X by adding Y as a boundary at infinity, such that the induced compactification of any apartment of X is the classical one for an affine space by the sphere of half–lines. $Z := X \amalg Y$ is shown to have a topology, which makes Z compact and contractible and the action of $G(k)$ on Z continuous. The contraction of X to O is geodesic, which means along half–lines in an apartment.

3.2 In this section we are interested in finite groups Γ_x, stabilizing a vertex $x \in D \subseteq A_0$ and contained in $U(k)$, $U = \operatorname{rad} P$; as we already know (see 2.3), these elements define bijectively apartments A of X, such that $\overline{A}$ contains C in Y and moreover fix pointwise the sector D_x with direction C and vertex x (if we assume that x is special — which can be done by changing from x to a neighbour x' with $\Gamma_x \subset \Gamma_{x'}$ — D_x is given by $\{x' \in A_0 | \alpha(x') \geq \alpha(x) \forall \alpha \in \Delta\}$).

We consider the subcomplex X' of X, defined by $X' = \bigcup_{\gamma \in \Gamma_x} \gamma A_0$ and denote by $Z' := X' \cup \partial X'$, where $Y' := \partial X'$ ist the boundary of X' in Z, so Z' is compact. We now remove a collar from Z', in order to get a complex, which is finite modulo Γ_x: This can be done by retraction along half–lines with vertex x and — since Γ_x is finite — in such a way that different apartments γA_0 remain different. We obtain a complex X'_c, contained in X, homoemorphic to Z', whose boundary Y'_c is homeomorphic to $\partial X'$ and $X'_c \cap D_x$ is still contained in all apartements γA_0 for $\gamma \in \Gamma_x$. Observe that the sectors γD (D with vertex O and $\partial D = C$) do not coincide, but all have the sector D_x in common, which has the same dimension and so D_x has the same set of directions as all γD, which implies that $\partial \gamma D = \partial D_x = C$ for all γ. For any apartment γA_0 the intersection $\gamma A_0 \cap Y'_c$ is homoemorphic to $\gamma A_0 \cap Y$, which is a spherical Coxeter–complex and all these complexes contain the chamber C.

Therefore we can apply the Borel–Serre–method for computing the cohomology of Y'_c in the same way as for Y and use the corollary of proposition 1 (notice that Y is the join $Y_1 * \cdots * Y_m$ and one has the formula $\dim Y = \sum_{j=1}^m (\dim Y_j + 1) - 1 = \sum_{j=1}^m \dim X_j - 1 = \sum_{j=1}^m \operatorname{rk}_k G_j - 1$, and $l = \sum_{j=1}^m \operatorname{rk}_k G_j = \operatorname{rk} G$), so we get

Lemma 2: $\widetilde{H}_c^i(Y'_c; M) = 0$ *for* $i \neq l-1$,

$$\widetilde{H}_c^{l-1}(Y'_c; M) \simeq C_c^\infty(\Gamma_x; M).$$

3.3 The exact sequence of cohomology for X'_c modulo Y'_c implies (cf. [BS] thm. 5.6)

Proposition 2: $H_c^i(X'_c; M) = 0$ *for* $i \neq l$ *and*

$$H_c^l(X'_c; M) \simeq C_c^\infty(\Gamma_x; M)$$

for a $\mathbb{Z}$-module M; if M is free, then also $H_c^l(X'_c; M)$ is a free $\mathbb{Z}$–module.

4 S-arithmetic groups

4.1 Let F be a function field, i.e. $[F : \mathbb{F}_q(t)] < \infty$, $q = p^m$, $p = \operatorname{char} F$, S a finite non–empty set of places of F, F_v the completion of F with respect to $v \in S$, G a connected semi–simple algebraic F–group of rank $r > 0$, r_v the F_v–rank of G, $L = \prod_{v\in S} G(F_v)$, X_v the Bruhat–Tits–building of $G(F_v)$ with dimension $d_v = r_v$, $X = \prod_{v\in S} X_v$ with $\dim X = \sum_{v\in S} d_v =: d$.

We consider S–arithmetic subgroups Γ, which are discrete in L. It is well known, that Γ contains a congruence subgroup Γ_0 of finite index, which has only p–torsion, but we need a little bit more and have to use for this purpose

4.2 (Reduction theory) We use the same notations as in section 2, but with respect to F instead of k: P is a minimal F–parabolic subgroup of G, $P = Z(T) \rtimes U$, $\Delta = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots of T, that defines P. We give a list of properties in the version of Harder (see [H1], cf. also [H2], [B2] and for the 1–dimensional case [S2], II.2).

(i) There is a constant c_1, such that for any $x \in X$, there exists a minimal P with numerical invariants $\nu_i(P, x) \geq c_1$ for $1 \leq i \leq r$ ([H1], Satz 2.3.2): "x is reduced with respect to P".

(ii) There is a constant $c_2 \geq c_1$, such that for x reduced with respect to P and P' and $\nu_i(P, x) \geq c_2$ $\forall i \in I' \subseteq \Delta$, P' is also contained in P_I for $I = \Delta \setminus I'$ and P_I is uniquely determined ([H1], Satz 2.3.3): "x is close to P_I".

(iii) For each constant $c' \geq c_1$ the set

$$X_0 := \{x \in X \mid c_1 \leq \nu_i(P, x) \leq c' \text{ for all } i \in \Delta \text{ and all } P \text{ s.t. } x \text{ is } P\text{-reduced}\}$$

is compact modulo Γ (cf. [H1], Satz 2.2.2): "compactness criterion".

(iv) The number of Γ–conjugation classes of parabolic groups is finite ([B2], Satz 8): "finiteness of class number".

(v) There exists a constant $c_3 \geq c_2$ (depending on Γ), such that for a P–reduced $x \in X$ one has $U_I(F_v) = (U_I(F_v) \cap \operatorname{stab}_G x) \cdot (U_I(F_v) \cap \Gamma)$ for all $v \in S$ (cf. [H2], 1.4.5): "x is very close to P_I, $U_I = \operatorname{rad} P_I$".

4.3 ¿From these properties above we deduce the following proposition; for subgroups $H \subseteq G$ we shorten the notation: $F_S := \prod_{v\in S} F_v$ and $H(F_S) := \prod_{v\in S} H(F_v)$.

Proposition 3: *For each S–arithmetic group $\Gamma \subset G(F_S)$, there exists a congruence subgroup Γ_0, finitely many F–parabolic subgroups $P_1, \dots, P_{h_0}$ of G and a set V of representatives for vertices of X/Γ_0, such that for all $x \in V$ the stabilizers $(\Gamma_0)_x$ are contained in one of the groups $[R_u(P_j)](F_S)$, $R_u(P_j)$ the unipotent radical of P_j. Moreover, those $x \in V$ which are close to P_j are contained in an apartment, having P_j and a fixed opposite P_j^{op} in its boundary at infinity.*

Proof. a) $X_0 := \{x \in X \mid c_1 \leq \nu_i(P, x) \leq c_2$ for all $i \in \Delta$ and all P s.t. x is P-reduced$\}$ is compact modulo Γ by (iii), thus there exist finitely many vertices in X, which represent the vertices of X_0 and have finite stabilizers in Γ. Therefore we can find a congruence subgroup Γ_1 of Γ, which has trivial stabilizers for all $x \in X_0$.

b) Now we consider those $x \in X$, which are close to at least one parabolic group Q (but not to $Q' \subset Q$), which means that some $\nu_i(P,x) \geq c_2$ for $P \subseteq Q$. Up to conjugation with Γ_1 we may assume that P belongs to a finite set $\{P_1, \ldots, P_h\}$ by (iv) and $Q = (P_j)_I$ for some j and $I \subseteq \Delta_j$, so there are only finitely many such groups Q. We write $Q = H \ltimes U_Q$ with $H = Z(T')$, $T' \subseteq T_j$, the maximal split torus of P_j and $U_Q = R_u(Q) \subseteq R_u(P_j)$. We want to "split up" Γ_1: There is a subgroup Γ_2 of finite index in Γ_1, such that $\Gamma_2 = (\Gamma_1 \cap H(F_S)) \ltimes (\Gamma_1 \cap U_Q(F_S))$ has finite index in Γ (cf. [Bo], 1.7 and 8.12, which is also valid for function fields, see [B1]). Now H is either of F–rank 0, if $I = \emptyset$ or the numerical invariants $\nu_i(H \cap P, x)$ are bounded by c_2 for all $i \in \Delta \setminus I = I'$ (the set of simple roots of H!), so either $H(F_S)/\Gamma_1 \cap H(F_S)$ is compact or by the same idea as in a) we find a congruence subgroup Γ_3 of Γ_2, whose stabilizer of $x \in X$ has a trivial semi–simple component in $[\Gamma_3 \cap H(F_S)]$, so $(\Gamma_3)_x \subseteq U_Q(F_S) \subseteq [R_u(P_j)](F_S)$. We can repeat this process for the finite set of groups Q and finally obtain a congruence subgroup Γ_0 of Γ, whose stabilizers have only unipotent elements; a set of representatives can be found in the unipotent radicals $[R_u(P_j)](F_S)$, but now for a bigger set $j = 1, \ldots, h_0$ for the Γ_0–conjugates of minimal parabolic groups.

At last poperty (v) of 4.2 shows, that we can choose modulo Γ_0 a fixed opposite group P_j^{op} of P_j for all j and this implies the second assertion of proposition 3 – by making Γ_0 smaller and h_0 bigger if necessary. □

5 Cohomology of S-arithmetic groups

5.1 Stuhler uses in [St] the following spectral sequence

$$(*) \qquad H^r(X/\Gamma, H^s(\Gamma_\sigma; M)) \Longrightarrow H^{r+s}(\Gamma; M)$$

which was introduced by Serre (for a foundation cf. [Br1], VII.5 and VII.7). Stuhler dealt with $\Gamma = \mathrm{SL}_2(\mathcal{O}_S)$, X a product of $|S|$ trees, the Bruhat–Tits–buildings of $\mathrm{SL}_2(F_v)$, $v \in S$ and $M = \mathbb{F}_p$. In our context Γ will be a S–arithmetic subgroup of $G(F)$ and X the polysimplicial product of the Bruhat–Tits–buildings X_v for $G(F_v)$, $v \in S$.

5.2 For the application of the results in section 2 and 3 we have to observe that a minimal parabolic F–subgroup P is in general not minimal over F_v: Denote by Q a minimal F_v–parabolic group, contained in P for some $v \in S$. We have $P = HT \ltimes R_u(P)$ with a maximal split F–torus T and a semi–simple group H of F–rank 0; therefore $H(F_S)/H(F_S) \cap \Gamma$ is compact. There may be infinitely many $Q \subset P$, but since we are interested in vertices x of a set V of representatives for X/Γ with $\Gamma_x \subset [R_u(P)](F)$, where P is one of the minimal parabolic F–subgroups of proposition 3, the compactness above tells us, that we have to consider only finitely many Q's. Moreover $R_u(P) \subseteq R_u(Q)$ (as groups over F_v), but for $R_u(Q) = R_u(P) \cdot U'$, the subgroup U' of H cannot contain unipotent elements of Γ, so $[R_u(P)](F_v) \cap \Gamma = [R_u(Q)](F_v) \cap \Gamma$.

5.3 The building Y_v at infinity of X_v has a chamber C_Q, fixed by $Q(F_v)$ and C_Q has a side–simplex C_P, fixed by $P(F_v)$ ($C_P = (C_Q)_I$ for $P = Q_I$ in section 2). Consider now a finite stabilizer group $\Gamma_x \subset [R_u(P)](F)$ for $x = (x_v) \in \prod_{v\in S} X_v = X$. In any apartment A of X_v, containing x_v, the group Γ_x fixes a cone in X_v, which is the union of sectors, which all have the "side–direction" Q in common. The construction of the subcomplexes X' and X'_c in section 3 can now be realized simultaneously in all buildings X_v with the same group Γ_x, the stabilizer of a vertex $x = (x_v) \in X = \prod_{v\in S} X_v$ in Γ, which is diagonally imbedded in $\prod_{v\in S} G(F_v)$. We start with a fixed apartment A_0 in X_v and define $X'_v = \bigcup_{\gamma\in\Gamma_x} \gamma A_0$; $\overline{A}_0 \cap Y_v$ may contain several chambers C_Q for $Q \subset P$, but the equation $[R_u(Q)](F_v) \cap \Gamma = [R_u(P)](F_v) \cap \Gamma$ shows that X'_v depends only on P. On the other hand we can use any such chamber C_Q for the computation of the cohomology as in section 2.

5.4 We fix now a vertex $x = (x_v)$ in the polysimplicial complex X and denote by Γ_x its stabilizer in Γ, assuming that Γ is the congruence subgroup of an S–arithmetic group, which has the properties of proposition 3. With this group Γ_x construct the subcomplexes $(X'_v)_c$ as in section 3 for all $v \in S$ and define $X'_c = \prod_{v\in S} (X'_v)_c$.

Proposition 2 gives us the cohomology of $(X'_v)_c$ and the Künneth–formula implies

Proposition 4: $H^i_c(X'_c; M) = 0$ *for* $i \neq d$,

$$H^d_c(X'_c; M) \simeq \bigotimes_{v\in S} H^{d_v}_c((X'_v)_c; M)$$

for $d = \sum_{v\in S} d_v$, $d_v = \dim X_v = \mathrm{rk}_{F_v} G$ *and a* $\mathbb{Z}$*-module* M.

This proposition allows to deduce the cohomology of the stabilizer groups.

Lemma 3: $H^i(\Gamma_x; \mathbb{Z}[\Gamma_x]) = 0$ *for* $i \neq d$,
$H^d(\Gamma_x; \mathbb{Z}[\Gamma_x]) \simeq C^\infty_c(\Gamma_x; \mathbb{Z})$.

Proof. We use the isomorphism $H^*(\Gamma_x; \mathbb{Z}[\Gamma_x]) \simeq H^*(X'_c, \mathbb{Z})$ (see [Br1], VIII.7, ex. 4), which is valid, since X'_c is contractible and has only finitely many cells mod Γ_x (and of course finite isotropy groups). For the precise description of $H^d(\Gamma_x; \mathbb{Z}[\Gamma_x])$, we have to remember Lemma 1, which says together with the corollary to proposition 1, that $H^{d_v}_c((X'_v)_c; M) \simeq \widetilde{H}^{d_v-1}_c((Y'_v)_c; M) = H^{d_v-1}_c(C^\circ_v; M) \otimes H^0(\Gamma_x; M)$, where C°_v denotes the interior of a chamber C_v in Y_v. For X'_c we must consider the chamber $C = *_{v\in S} C_v$ (join) in $Y = *_{v\in S} Y_v$ and so we obtain $H^d(X'_c; M) \simeq H^{d-1}_c(C^\circ; M) \otimes H^0(\Gamma_x; M)$, where the first factor is isomorphic to $\mathbb{Z}$ (since $C^\circ \simeq \mathbb{R}^{d-1}$) and the second one to $C^\infty_c(\Gamma_x; M)$: Set $M = \mathbb{Z}$. □

Corollary: a) $H^i(\Gamma_x; \mathbb{Z}[\Gamma]) = 0$ *for* $i \neq d$,
$H^d(\Gamma_x; \mathbb{Z}) \simeq C^\infty_c(\Gamma_x; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma_x]} \mathbb{Z}[\Gamma] \simeq C^\infty_c(\Gamma_x; \mathbb{Z}[\Gamma])$.

b) $H^i(\Gamma_x; \mathbb{F}_p) = 0$ *for* $i \neq d$,
$H^d(\Gamma_x; \mathbb{F}_p) \simeq C^\infty_c(\Gamma_x; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma_x]} \mathbb{F}_p \simeq C^\infty_c(\Gamma_x; \mathbb{F}_p)$.

Proof. For the second parts see [Br1], VIII.6.8, since Γ_x is of type *FP* and $\mathrm{cd}\,\Gamma_x = d$. □

5.5 For the application of the spectral sequence we have to consider the stabilizers of polysimplices σ, but Γ_σ is always the intersection of some Γ_x and we notice that the results above remain valid for Γ_σ. On the other side we have to replace the trivial Γ_σ–module $\mathbb{Z}$ by the "oriented module" $\mathbb{Z}_\sigma$, on which Γ_σ acts with a sign. Then it is true, that the spectral sequence $H^r(X/\Gamma; H^s(\Gamma_\sigma; \mathbb{Z}_\sigma \otimes M))$ converges to $H^{r+s}(\Gamma; M)$ (see [S1] 1.6, especially remark 1). The corollary implies, that the spectral sequences degenerate to isomorphisms, so we obtain

Theorem 1: *For a S–arithmetic subgroup Γ of $G(F)$, G a semi–simple algebraic F–group with $\mathrm{rk}_F\, G > 0$, there exists a subgroup Γ_0 of finite index, which has only p–torsion, such that for the Γ_0–modules $M = \mathbb{Z}[\Gamma_0]$ or $M = \mathbb{F}_p$ (with trivial Γ_0-action) and the Bruhat–Tits–building $X = \prod_{v\in S} X_v$ we have*

(i) $H^i(\Gamma_0; M) = 0$ for $i \neq d = \sum_{v\in S} \mathrm{rk}_{F_v} G$,

(ii) $H^d(\Gamma_0; M) \simeq H^0(X/\Gamma_0; H^d(\Gamma_0)_\sigma; M)$,

in particular $H^d(\Gamma_0; \mathbb{Z}[\Gamma_0])$ is free.

Remark. For $\mathrm{rk}_F\, G = 0$, the "cocompact case", see [BS], thm. 6.2.

6 Finiteness properties of S-arithmetic groups

6.1 In this section we are interested in the finiteness properties FP_n or F_n (see [Br1], VIII.5 and [Br2], VII.2). In contrast to the number field case, where all S–arithmetic subgroups of reductive groups are of type F_∞ (in the general case it is only true for $S = S_\infty$), there exist many counter–examples over function fields and a conjecture says, that for almost simple groups G with $\mathrm{rk}_F\, G > 0$ Γ is of type F_{d-1}, but not F_d, where again $d = \sum_{v\in S} \mathrm{rk}_{F_v} G$. This was proved for $\Gamma = \mathrm{SL}_2(\mathcal{O}_S)$ by Stuhler (see [St]), for the classical cases $d = 1$ (finite generation) and $d = 2$ (finite presentation)) (see [B3]), for $\Gamma = \mathrm{SL}_n(\mathbb{F}_q[t])$ by Abels and Abramenko under the additional assumption, that q is big enough with respect to n and by the second author also for classical almost simple groups over the polynomial ring and with analogous growth conditions (see [A] and [Ab]) and finally the positive result for Chevalley–groups and arithmetic rings with $|S| = 1$ without such a condition in [B4].

6.2 There is a necessary cohomological condition for a group Γ to be of type FP_n: For a ring R, which is a $\mathbb{Z}$–module of finite type, the homology and cohomology groups $H_k(\Gamma; R)$ and $H^k(\Gamma; R)$ have to be finitely generated for $0 \le k \le n$ (see [Bi], prop. 2.15). Stuhler used this criterion for $R = \mathbb{F}_p$ for the group $\mathrm{SL}_2(\mathcal{O}_S)$, to show that it is not of type $FP_{|S|}$ (see [St], section 4). We shall generalize his idea for the congruence subgroup Γ_0 of Γ (from proposition 3) in almost simple groups G, using

$$H^d(\Gamma_0; \mathbb{F}_p) \simeq H^0(X/\Gamma_0; H^d((\Gamma_0)_\sigma; \mathbb{F}_p)) \simeq H^0(X/\Gamma_0; C_c^\infty((\Gamma_0)_\sigma; \mathbb{F}_p))$$

(see theorem 1 and the corollary to lemma 3). We prove that the $\mathbb{F}_p$–dimension of this vector–space goes to infinity for an infinite sequence of vertices $\sigma = x$ in a set V_P of representatives for X/Γ_0 being very close to a fixed parabolic F–subgroup $P = P_j$ for some $j \in \{1, \ldots, h_0\}$ (see 4.2 (v) and proposition 3). We may also assume that these vertices are contained in poly–apartements on which a fixed maximal F–split torus T_0 of P acts. Denote by a_0 the highest F–root on T_0, that defines a subgroup U_0 of the unipotent radical U of P. Consider a_0 as a linear form with integral values on all these apartements.

By Riemann–Roch $U_0(x) := U_0(F_S) \cap (\Gamma_0)_x$ is for $x = (x_v) \in X$ a $\mathbb{F}_q$–vector–space. If $a_0(x_v)$ increases by 1 for some v, all others remaining constant, $\dim_{\mathbb{F}_q} U_{0,x}$ increases by $d_v = \dim_{\mathbb{F}_q} \overline{F}_v$ ($\overline{F}_v$ the residue field of F_v), if U_0 is a 1–dimensional root group and otherwise has moreover to be multiplicated by $\dim_F U_0(F)$. Thus it seems reasonable to define the filtration of V_P by $A_0(x) = \sum_{v \in S} d_v \cdot a_0(x_v)$, A_0 is Γ–invariant by the product–formula for valuations. The whole set of representatives for X/Γ_0 consists of a finite complex and finitely many sets, containing the V_{P_j}.

6.3 For the computation of $H^0(X/\Gamma_0; C_c^\infty((\Gamma_0)_x; \mathbb{F}_p))$ we must satisfy the condition for 0–cocycles, which means concretely that the values of a function on the vertices x and x' of an edge have to coincide on $(\Gamma_0)_x \cap (\Gamma_0)_{x'}$. We have to consider several cases:

a) If $\mathrm{rk}_F\, G = \mathrm{rk}_{F_v}\, G = 1$ for all $v \in S$ we have a cubic complex X (the prototype is Stuhler's SL_2–example) and $U_0 = U_a$ or $U_0 = U_{2a}$. We consider a cube C, whose vertices x and y have maximal resp. minimal A_0–value; the neighbours of x' being x_i with $a_0[(x_i)_{v_i}] = a_0[(x)_{v_i}] - 1$ and $a_0[(x_i)_{v_k}] = a_0[(x)_{v_k}]$ for all $k \neq i$, setting $S = \{v_1, \ldots, v_s\}$, $i, k \in \{1, \ldots, s\}$.

   Unfortunately $U_0(x) = \bigcup_{i=1}^{s} U_0(x_i)$, thus all functions on $U_0(x)$ are uniquely determined by their values on the subgroups $U_0(x_i)$. But functions on the vertices x_{ij}, whose a_0-values differ from those on x for two indices by 1 have different extensions to functions on x_i and x_j. This can be done for all pairs (i, j), which implies that the $\mathbb{F}_q$–dimension of $U_0(y)$ is smaller than that of $U_0(x)$ and so we obtain more functions on $(\Gamma_0)_x$ than on $(\Gamma_0)_y$ and H^0 becomes larger (cf. [St], prop. 2 and lemma 2). One should observe, that many vertices close to P are Γ–equivalent, since the torus $T_0(F_s) \cap \Gamma_0$ acts on these points as a free abelian group of rank $|S| - 1$. But the function A_0 is Γ–invariant and obviously $A_0(y) < A_0(x)$, so x and y are different vertices of V_P.

b) If G is a Chevalley–group of $\mathrm{rk}_F\, G = \mathrm{rk}_{F_v}\, G \geq 2$ for all $v \in S$ than in a fixed building X_v the group $U_0(x) = U_0(F_v) \cap (\Gamma_0)_x$ is larger than $U_0(x')$ for all neighbours x' of x, so also $(\Gamma_0)_x$ is larger than the union $\bigcup_{x'} (\Gamma_0)_{x'}$ and we obtain more cochains. For $|S| > 1$ we use the same technique as in a).

c) If $\mathrm{rk}_F\, G < \mathrm{rk}_{F_v}\, G$ both situations may occur in X_v: $U_0(x) \subset \bigcup_{x'} U_0(x')$ and $U_0(x) = \bigcup_{x'} U_0(x')$ (where x' runs over all neighbours of x). Simple examples are groups G of type $B_{2,1}$ ($G = \mathrm{SO}_5$ with Witt–index 1) or type $A_{3,1}$ ($G = \mathrm{SL}_2(D)$, D central division algebra of degree 2 over F). For the second case the same trick as in a) works: We have to go one A_0–level deeper to a common neighbour y of x'_1 and x'_2. For $|S| > 1$ use again the procedure of case a).

If the values of A_0 go to infinity we find in any case a sequence of vertices x with increasing $\mathbb{F}_p$–dimension of $C_c^\infty((\Gamma_0)_x; \mathbb{F}_p)$, for which also the $\mathbb{F}_p$–dimension of H^0 goes to infinity.

6.4 With the criterion of 6.2 we proved, that Γ cannot be of type F_d — for a convenient subgroup of finite index, but this does not change the F_d–property (see [Br1], VIII.5.1) – thus we have

Theorem 2: *A S–arithmetic subgroup of an almost simple algebraic group G with $\mathrm{rk}_F\, G > 0$ over a function field F with $d = \sum_{v \in S} \mathrm{rk}_{F_v}\, G$ cannot be of type FP_d or F_d.*

Remarks. 1. The direct application of the finiteness criterion to semi–simple groups could provide too large upper bounds, because cohomology groups may vanish if other tensor–factors in Künneth's formula are zero. Thus for semi–simple groups the bound in theorem 2 is given by the minimal d of its simple factors (assuming G simply connected we have a direct product: see [B3] 2.6 c).

2. There exists another proof of theorem 2 by K.U. Bux and K. Wortman (see [BW]), which does not compute cohomology but constructs a sequence of cycles in homology.

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