TAME GROUP ACTIONS ON CENTRAL SIMPLE ALGEBRAS

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Abstract. We study actions of linear algebraic groups on finite-dimensional central simple algebras. We describe the fixed algebra for a broad class of such actions.

1. Introduction

Let k be an algebraically closed base field of characteristic zero, K/k be a finitely generated field extension, A/K be a finite-dimensional central simple algebra and G be a linear algebraic group defined over k. Suppose G acts on A by k-algebra automorphisms. We would like to describe the fixed ring A^G for this action. In particular, we would like to know under what conditions A^G is semisimple, simple or commutative, under what conditions the center of A^G (or all of A^G) is contained in K, etc.

The goal of this paper is to describe, in terms of geometric data, the finite-dimensional algebras B/F that can occur as fixed algebras A^G/K^G for a particular type of G-action on A which we call *tame*. We give a precise definition of tame actions in Section 2, and we show in Section 4 that many naturally occurring actions belong to this class. To every tame action one associates a subgroup S of PGL_n (defined up to conjugacy), which we call the *associated stabilizer*. If the associated stabilizer S is reductive, we say that the action is *very tame*.

To state our first main result, let K be a field extension of k. As usual, we shall say that an algebra A/K is a *form* of A_0/k if the two become isomorphic after extension of scalars, i.e., $A \otimes_K L \simeq A_0 \otimes_k L$ for some field extension $k \subset K \subset L$.

1.1. **Theorem.** *Consider a tame action of a linear algebraic group G on a central simple algebra A of degree n with center K, with associated stabilizer subgroup $S \subset \mathrm{PGL}_n$. Then the fixed algebra A^G/K^G is a form of M_n^S/k.*

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Here $S \subset \mathrm{PGL}_n$ is acting on the matrix algebra M_n by conjugation, and $\mathrm{M}_n^S = (\mathrm{M}_n)^S$ denotes the fixed subalgebra for this action.

In the case where the G-action on A is very tame, one can describe M_n^S (and thus A^G) more explicitly. In this case the associated stabilizer subgroup $S \subset \mathrm{PGL}_n$ is, by definition, reductive, and hence so is the preimage S^* of S in GL_n. We shall view the inclusion $\phi\colon S^* \hookrightarrow \mathrm{GL}_n$ as an n-dimensional linear representation of S^*.

1.2. **Corollary.** *Consider a very tame action of a linear algebraic group G on a central simple algebra A of degree n with center K, and let ϕ be as above. Then*

(a) *A^G is semisimple.*
(b) *A^G is commutative if and only if ϕ is multiplicity-free (i.e., is a sum of distinct irreducible representations).*
(c) *$A^G \subset K$ if and only if ϕ is irreducible.*
(d) *$\mathrm{Z}(A^G) \subset K$ if and only if ϕ is a power of an irreducible representation.*

For a more detailed description of A^G, in terms of the irreducible decomposition of ϕ, see Section 8.

We now return to the setting of Theorem 1.1. Let G be a linear algebraic group, A be a central simple algebra of degree n and ψ be a tame action of G on A, with associated stabilizer $S \subset \mathrm{PGL}_n$. Since Theorem 1.1 asserts that A^G/K^G is a form of M_n^S/k, it is natural to ask which forms of M_n^S/k can occur in this way. Note that here n and $S \subset \mathrm{PGL}_n$ are fixed, but we are allowing A, G and ψ to vary.

Our answer to this question will be stated in terms of Galois cohomology. If G is an algebraic group defined over k, and K/k is a field extension, we write $H^1(K, G)$ for the Galois cohomology set $H^1(\mathrm{Gal}(K), G(K^{sep}))$, where K^{sep} is the separable closure of K and $\mathrm{Gal}(K) = \mathrm{Gal}(K^{sep}/K)$ acts on $G(K^{sep})$ in the natural way; cf. [S_1, Chapter II].

For any field extension F/k, the F-forms of a k-algebra R are in a natural 1-1 correspondence with $H^1(F, \mathrm{Aut}(R))$; see [KMRT, §29.B] or [S_2, Proposition X.4]. Here, $\mathrm{Aut}(R)$ is the algebraic group of algebra automorphisms of R. If B is an F-form of R, we will write $[B]$ for the element of $H^1(F, \mathrm{Aut}(R))$ corresponding to B.

The normalizer $N = N_{\mathrm{PGL}_n}(S)$ naturally acts on M_n^S (by conjugation). Since S acts trivially, this action gives rise to a homomorphism $\alpha\colon N/S \to \mathrm{Aut}(\mathrm{M}_n^S)$ of algebraic groups.

1.3. **Theorem.** *Let $S \subset \mathrm{PGL}_n$ be a closed subgroup and B be an F-form of M_n^S/k. Then the following are equivalent.*

(a) *There exists a tame action of a linear algebraic group G on a central simple algebra A of degree n with center K, with associated stabilizer S, such that A^G/K^G is isomorphic to B/F (over k).*

(b) $[B]$ *lies in the image of the natural map*

$$\alpha_* \colon H^1(F, N/S) \longrightarrow H^1(F, \mathrm{Aut}(\mathrm{M}_n^S))\,.$$

The rest of this paper is structured as follows. In Section 2, we introduce tame and very tame actions. In Section 3 we prove a preparatory lemma and explain how it is used later on. Sections 4 and 5 are intended to motivate the notions of tame and very tame actions. In Sections 7, 8 and 9 we prove Theorem 1.1, Corollary 1.2 and Theorem 1.3 along with some generalizations. In the last section we illustrate Theorem 1.3 with an example. This example shows, in particular, that every finite-dimensional semisimple algebra (over a finitely generated field extension of k) occurs as A^G/K^G for some very tame action of a linear algebraic group G on some central simple algebra A with center K; see Corollary 10.4.

Notation and Terminology. We shall work over an algebraically closed base field k of characteristic zero. All fields will be assumed to be finitely generated extensions of k, and all central simple algebras to be finite-dimensional over their centers. All algebraic varieties, algebraic groups, group actions, morphisms, rational maps, etc., are assumed to be defined over k. By a point of an algebraic variety we shall always mean a k-point.

We denote by M_n the algebra of $n \times n$ matrices over k. If S is a subgroup of PGL_n, we denote by M_n^S the fixed algebra for the natural action of S on M_n. Throughout, G will be a linear algebraic group. We shall refer to an algebraic variety X endowed with a regular G-action as a G-variety. We will say that X (or the G-action on X) is generically free if $\mathrm{Stab}_G(x) = \{1\}$ for $x \in X$ in general position.

2. Tame actions on central simple algebras

Tame actions are a special class of *geometric* actions. To review the latter concept, we need to recall the relationship between central simple algebras of degree n and irreducible, generically free PGL_n-varieties.

If X is a PGL_n-variety, we shall denote the algebra of PGL_n-invariant rational maps $X \dashrightarrow \mathrm{M}_n$ by $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. The algebra structure in $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is naturally induced from M_n, i.e., given $a, b \colon X \dashrightarrow \mathrm{M}_n$, we define $a + b$ and ab by $(a+b)(x) = a(x) + b(x)$ and $ab(x) = a(x)b(x)$ for $x \in X$ in general position. If the PGL_n-action on X is generically free, then $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n, with center $k(X)^{\mathrm{PGL}_n}$; cf. [Re, Lemmas 8.5 and 9.1]. In this case we will sometimes denote this algebra by $k_n(X)$. As this notation suggests, $k_n(X)$ shares many properties with the function field $k(X)$ (see [RV_2]); in particular, the two coincide if $n = 1$. Under our hypotheses, every central simple algebra of degree n is isomorphic to $k_n(X)$ for some irreducible generically free PGL_n-variety X; see [RV_2, Theorem 1.2].

Suppose that X is a $G \times \mathrm{PGL}_n$-variety, and that the PGL_n-action on X is generically free. Then the G-action on X naturally induces a G-action on

$k_n(X)$; see [RV$_3$]. Following [RV$_3$] we will say that the action of an algebraic group G on a central simple algebra A is *geometric* if A is G-equivariantly isomorphic to $k_n(X)$ for some $G \times \mathrm{PGL}_n$-variety X as above. (In particular, that this is only possible if the center $\mathrm{Z}(A)$ is a finitely generated field extension of k.) The $G \times \mathrm{PGL}_n$-variety X is then called the *associated variety* for the G-action on A. This associated variety is unique up to birational isomorphism of $G \times \mathrm{PGL}_n$-varieties, see [RV$_3$, Corollary 3.2].

Note that the class of geometric actions on central simple algebras is rather broad. It includes, in particular, all "algebraic" actions; see [RV$_3$, Theorem 5.3]. Roughly speaking, an action of G on A is algebraic if there is a G-invariant order R in A such that G acts regularly ("rationally") on R. For details, see [RV$_3$].

Consider a geometric action ψ of a linear algebraic group G on a central simple algebra A, with associated $G \times \mathrm{PGL}_n$-variety X. Given $x \in X$, denote by S_x the projection of $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subset G \times \mathrm{PGL}_n$ to PGL_n.

2.1. **Definition.** (a) We shall call the action ψ *tame* if there is a dense open subset $U \subset X$ such that S_x is conjugate to S_y in PGL_n for every $x, y \in U$. In this case we shall refer to S_x ($x \in U$) as the *associated stabilizer*[1] for ψ; the associated stabilizer is defined up to conjugacy. (b) We shall call the action ψ *very tame* if it is tame and if its associated stabilizer is reductive.

See Lemma 4.1 for an alternate description of tame and very tame actions. There exist non-tame geometric actions, and tame actions that are not very tame (see Proposition 5.1). However, informally speaking, many interesting actions are geometric and very tame; in particular, all geometric actions on division algebras are very tame. For details, see Section 4.

3. An exact sequence of stabilizer groups

3.1. **Lemma.** *Let Γ be a linear algebraic group, N a closed normal subgroup, and $\pi\colon \Gamma \longrightarrow \Gamma/N$ the natural projection. Suppose X is a Γ-variety, Y is a birational model for X/N (as a Γ/N-variety), and $q\colon X \dashrightarrow Y$ is the rational quotient map. Then for $x \in X$ in general position, the sequence of stabilizer subgroups*

$$1 \longrightarrow \mathrm{Stab}_N(x) \longrightarrow \mathrm{Stab}_\Gamma(x) \xrightarrow{\pi} \mathrm{Stab}_{\Gamma/N}(q(x)) \longrightarrow 1$$

is exact.

Proof. The sequence is clearly exact at $\mathrm{Stab}_N(x)$ and $\mathrm{Stab}_\Gamma(x)$, so we only need to show that the map

$$\mathrm{Stab}_\Gamma(x) \longrightarrow \mathrm{Stab}_{\Gamma/N}(q(x)),$$

which we denote by π, is surjective. By a theorem of Rosenlicht (see [Ro$_1$], [Ro$_2$] or [PV, Proposition 2.5]) there is a dense Γ-invariant open subset

[1] For $x \in X$ in general position, S_x is the stabilizer of the image of x in the rational quotient X/G, see Remark 3.2(b).

$X_0 \subset X$ such that $q^{-1}(q(x)) = N \cdot x$ for every $x \in X_0$. Thus for $x \in X_0$, $g \in \Gamma$,

$$\begin{aligned} \pi(g) \in G/N \text{ stabilizes } q(x) &\Longleftrightarrow g(x) \in N \cdot x \\ &\Longleftrightarrow n^{-1}g \in \mathrm{Stab}_\Gamma(x) \text{ for some } n \in N \\ &\Longleftrightarrow \pi(g) = \pi(n^{-1}g) \in \pi(\mathrm{Stab}_\Gamma(x))\,, \end{aligned}$$

as claimed. $\square$

3.2. **Remark.** In the sequel we will mostly apply Lemma 3.1 in a situation, where $\Gamma = G \times \mathrm{PGL}_n$ and X is a Γ-variety with a generically free PGL_n-action. There are two natural choices of N here, $N = G$ and $N = \mathrm{PGL}_n$ (or, more precisely, $N = G \times \{1\}$ and $N = \{1\} \times \mathrm{PGL}_n$). Thus we will consider two rational quotient maps:

$$\begin{tikzcd} & X \arrow[dl, dashed] \arrow[dr, dashed] & \\ Y = X/G & & Z = X/\mathrm{PGL}_n\,. \end{tikzcd}$$

(a) Taking $N = \mathrm{PGL}_n$ in Lemma 3.1 and remembering that the PGL_n-action is generically free, we see that the natural projection $G \times \mathrm{PGL}_n \longrightarrow G$ restricts to an isomorphism

$$\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x) \xrightarrow{\simeq} \mathrm{Stab}_G(z)\,,$$

where $x \in X$ is in general position and z is the image of x in X/PGL_n; cf. [RV_3, Lemma 7.1].

(b) On the other hand, taking $N = G$ and denoting by π the natural projection $G \times \mathrm{PGL}_n \longrightarrow \mathrm{PGL}_n$ we obtain

$$\mathrm{Stab}_{\mathrm{PGL}_n}(y) = \pi(\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)) \simeq \mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)/\,\mathrm{Stab}_G(x)\,,$$

where $x \in X$ is in general position and y is the image of x in X/G, cf. [RV_4, Lemma 2.3].

4. An alternative description of tame and very tame actions, and some examples

It follows from Remark 3.2(b) that if ψ is a geometric G-action on a central simple algebra A, with associated $G \times \mathrm{PGL}_n$-variety X, then ψ is tame if and only if the PGL_n-action on X/G has a *stabilizer in general position* $S \subset \mathrm{PGL}_n$, in the sense of [PV, p. 228]. Recall that this means that $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ is conjugate to S for a general point $y \in X/G$, and that S is only well defined up to conjugacy in PGL_n. Recall also that if ψ is tame, we called S the *associated stabilizer* for ψ; see Definition 2.1.

Note that the stabilizer in general position is known to exist under certain mild assumptions on the action; cf. [PV, §7]. Informally speaking, this means that "most" geometric actions on central simple algebras are tame; we illustrate this point by the examples below. In particular, if $\mathrm{Stab}(y)$ is reductive for general $y \in Y$ then, by a theorem of Richardson [Ri, Theorem 9.3.1], Y has a stabilizer in general position and thus the action ψ is very tame. For later use, we record some of these remarks as a lemma.

4.1. **Lemma.** *Consider a geometric action of a linear algebraic group G on a central simple algebra A with associated $G \times \mathrm{PGL}_n$-variety X.*

(a) *The action of G on A is tame if and only if the PGL_n-action on X/G has a stabilizer in general position.*
(b) *The action of G on A is very tame if and only if $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ is reductive for $y \in X/G$ in general position.* □

4.2. **Example.** A geometric action ψ on a central simple algebra A is very tame if

(a) A is a division algebra, or
(b) the connected component G^0 of G is a torus (this includes the case where G is finite), or
(c) $\mathrm{Stab}_G(z)$ is reductive for $z \in X/\mathrm{PGL}_n$ in general position, or
(d) G is reductive and the G-action on A is inner, or
(e) A has a G-invariant maximal étale subalgebra.

Here by an étale subalgebra of A we mean a subalgebra of A which is a (finite) direct sum of finite field extensions of the center of A. (Since we are working in characteristic zero, these field extensions are necessarily separable.)

Proof. Using Richardson's theorem (see the remarks before Lemma 4.1), part (a) follows from [RV_4, Section 3], so we will focus on the other parts. Let X be the associated $G \times \mathrm{PGL}_n$-variety, let $x \in X$ be a point in general position, and let y and z be the images of x in $Y = X/G$ and $Z = X/\mathrm{PGL}_n$, respectively, as in Remark 3.2. By Lemma 4.1, it suffices to show that $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ is reductive for $y \in Y$ in general position.

(b) For $x \in X$ in general position, $\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x)$ is isomorphic to a closed subgroup of G and is thus reductive; see Remark 3.2(a). Hence,

$$\mathrm{Stab}_{\mathrm{PGL}_n}(y) \simeq \mathrm{Stab}_{G \times \mathrm{PGL}_n}(x)/\,\mathrm{Stab}_G(x)$$

is also reductive for $y \in Y$ in general position; cf. Remark 3.2(b).

(c) Let $x \in X$ in general position. Since $\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x) \simeq \mathrm{Stab}_G(z)$, it is a reductive group, and so is its image $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$. The desired conclusion follows from Lemma 4.1.

(d) Here G acts trivially on $\mathrm{Z}(A) = k(X/\mathrm{PGL}_n)$ and hence, on X/PGL_n. Thus $\mathrm{Stab}_G(z) = G$ is reductive for every $z \in X/\mathrm{PGL}_n$, and part (c) applies.

(e) By [RV_3, Theorem 1.5(b)] there exists a maximal torus $T \subset \mathrm{PGL}_n$ (depending on x) such that $\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x) \subset G \times \mathrm{N}(T)$ where $\mathrm{N}(T)$ denotes the normalizer of T in PGL_n. Thus the image $\pi(\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x))$ of $\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x)$ under the natural projection $\pi \colon G \times \mathrm{PGL}_n \longrightarrow \mathrm{PGL}_n$ consists of semisimple elements and, consequently, is reductive. The desired assertion now follows from the fact that $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ and $\pi(\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x))$ are equal; see Remark 3.2(b). $\square$

5. Examples of wild actions

The purpose of this section is to show that Definition 2.1 is not vacuous.

5.1. **Proposition.** (a) *There exists a geometric action of an algebraic group on a central simple algebra which is tame but not very tame.*

(b) *There exists a geometric action of an algebraic group on a central simple algebra which is not tame.*

Our proof of Proposition 5.1 will rely on the following lemma.

5.2. **Lemma.** *Let G be an algebraic group and H a closed subgroup. Then for every H-variety Y (not necessarily generically free) there exists a $G\times H$-variety X such that*

(i) *the action of $H=\{1\}\times H$ on X is generically free, and*
(ii) *X/G is birationally isomorphic to Y (as an H-variety).*

Proof. Set $X = G\times Y$, with G acting on the first factor by translations on the left and H acting by $h\colon (g,y)\mapsto (gh^{-1}, hy)$. The G-action and the H-action on X commute and thus define a $G\times H$-action. Conditions (i) and (ii) are now easy to check. $\square$

Proof of Proposition 5.1. (a) In view of Lemma 4.1, we need to construct a $G\times\mathrm{PGL}_n$-variety X, with PGL_n acting generically freely, and such that the PGL_n-action on the rational quotient variety X/G has a non-reductive stabilizer in general position. By Lemma 5.2 (with $G=H=\mathrm{PGL}_n$) it suffices to construct a PGL_n-variety Y with a non-reductive stabilizer in general position. To construct such a Y, take any non-reductive subgroup $S\subset\mathrm{PGL}_n$ and let Y be the homogeneous space PGL_n/S.

(b) Arguing as in part (a), we see that it suffices to construct a PGL_n-variety Y which does not have a stabilizer in general position. Richardson [Ri, §12.4] showed that (for suitably chosen n) there exists an SL_n-variety Z with the following property: for every Zariski open subset $U\subset Z$ there are infinitely many stabilizers $\mathrm{Stab}_{\mathrm{SL}_n}(z)$, $z\in U$, with pairwise non-isomorphic Lie algebras. Clearly such an SL_n-variety Z cannot have a stabilizer in general position. We will now construct a PGL_n-variety Y with no stabilizer in general position by modifying this example. (Note that in [Ri] the variety Z is only constructed over the field $\mathbb{C}$ of the complex numbers. However, by the Lefschetz principle this construction can be reproduced over any algebraically closed base field k of characteristic zero.)

Consider the rational quotient map $\pi\colon Z \dashrightarrow Y = Z/\mu_n$, for the action of the center μ_n of SL_n on Z. Recall that (after choosing a suitable birational model for Y), the SL_n-action on Z descends to a PGL_n-action on Y. In fact, this is exactly the situation we considered in the setting of Lemma 3.1

(with $\Gamma = \mathrm{SL}_n$ and $N = \mu_n$); this lemma tells us that, after replacing Y by a dense open subset Y_0, and Z by $\pi^{-1}(Y_0)$,

$$\mathrm{Stab}_{\mathrm{PGL}_n}(y) \simeq \mathrm{Stab}_{\mathrm{SL}_n}(z)/\operatorname{Stab}_{\mu_n}(z)$$

for every $z \in Z$ and $y = \pi(z)$. Since $\mathrm{Stab}_{\mu_n}(z)$ is a finite group we see that $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ and $\mathrm{Stab}_{\mathrm{SL}_n}(z)$ have isomorphic Lie algebras. This shows that for every open subset V of Y there are infinitely many stabilizers $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$, $y \in V$, with pairwise non-isomorphic Lie algebras. In particular, the PGL_n-action on Y has no stabilizer in general position. □

6. Homogeneous fiber spaces

In this section we briefly recall the definition and basic properties of homogeneous fiber spaces. Let $\varphi\colon H \to G$ be a homomorphism of linear algebraic groups (defined over k) and Z be an H-variety. The *homogeneous fiber space* $X = G *_H Z$ is defined as the rational quotient of $G \times Z$ for the action of $\{1\} \times H$, where $G \times H$ acts on $G \times Z$ via $(g, h) \cdot (\tilde{g}, z) = (g\tilde{g}\varphi(h)^{-1}, hz)$. As usual, we choose a model for this rational quotient, so that the G-action on $G \times Z$ descends to a regular action on this model. Note that $\{1_G\} \times Z$ projects to an H-subvariety of X birationally isomorphic to $Z/\ker(\varphi)$.

It is easy to see that $G*_H Z$ is birationally isomorphic to $G*_{\varphi(H)}(Z/\ker(\varphi))$ as a G-variety; thus in carrying out this construction we may assume that H is a subgroup of G and φ is the inclusion map. For a more detailed discussion of homogeneous fiber spaces in this setting and further references, we refer the reader to [RV_1, §3].

We also recall that a G-variety X is called *primitive* if G transitively permutes the irreducible components of X or, equivalently, if $k(X)^G$ is a field; see [Re, Lemma 2.2(b)].

Our goal in this section is to prove the following variant of [LPR, Lemma 2.17], which will be repeatedly used in the sequel.

6.1. **Lemma.** *Let $\varphi\colon H \to G$ be a homomorphism of linear algebraic groups (defined over k), Z be an H-variety and R be a finite-dimensional k-algebra. Let X be the homogeneous fiber space $G *_H Z$. Assume further that G acts on R via k-algebra automorphisms (i.e., via a morphism $G \longrightarrow \mathrm{Aut}(R)$ of algebraic groups). Then*

(a) (*cf.* [PV, p. 161]) *$k(X)^G$ and $k(Z)^H$ are isomorphic as k-algebras. In particular, X is primitive if and only if Z is primitive.*
(b) *If X and Z are primitive, the algebras $\mathrm{RMaps}_G(X, R)/k(X)^G$ and $\mathrm{RMaps}_H(Z, R)/k(Z)^H$ are isomorphic.*

Here, as usual, $\mathrm{RMaps}_H(Z, R)$ denotes the $k(Z)^H$-algebra of H-equivariant rational maps $Z \dashrightarrow R$.

Proof. Let $f\colon X = G *_H Z \dashrightarrow R$ be a G-equivariant rational map. The indeterminacy locus of such a map is a G-invariant subvariety of X. Let $\chi\colon Z \dashrightarrow X$ be the rational map sending $z \in Z$ to the image of $(1_G, z) \in$

$G \times Z$ in $X = G *_H Z$. Since $G \cdot \chi(Z)$ is dense in X, $\chi(Z)$ cannot be contained in the indeterminacy locus of f. Thus the map

$$\psi\colon \operatorname{RMaps}_G(X, R) \to \operatorname{RMaps}_H(Z, R)$$

given by $\psi(f) = f \circ \chi$ is a well-defined k-algebra homomorphism. Since $G \cdot \chi(Z)$ is dense in X, ψ is injective.

It is easy to see that ψ has an inverse. Indeed, given H-equivariant rational map $\lambda\colon Z \dashrightarrow R$, we define a G-equivariant map $\lambda'\colon G \times Z \dashrightarrow R$ by $(g, z) \mapsto g\lambda(z)$. Since $\lambda'(g\varphi(h)^{-1}, hz) = \lambda'(g, z)$ for every $g \in G$ and $z \in Z$, and $X = (G \times Z)/H$, the universal property of rational quotients tells us that λ' descends to a G-equivariant rational map $f\colon X \dashrightarrow R$. Moreover, by our construction $\psi(f) = f \circ \chi = \lambda$, so that the map $\operatorname{RMaps}_H(Z, R) \to \operatorname{RMaps}_G(X, R)$ taking λ to f is the inverse of ψ.

We are now ready to complete the proof of Lemma 6.1. To prove part (a), we apply the above construction in the case where $R = k$, with trivial G-action. Here $\operatorname{RMaps}_G(X, k) = k(X)^G$, $\operatorname{RMaps}_H(Z, k) = k(Z)^H$, and ψ is the desired isomorphism of k-algebras. To prove part (b), note that if we identify elements of $k(X)^G$ (respectively, $k(Z)^H$) with G-equivariant rational maps $X \dashrightarrow R$ (respectively, H-equivariant rational maps $Z \dashrightarrow R$) whose image is contained in $k \cdot 1_R$ then ψ is an algebra isomorphism between $\operatorname{RMaps}_G(X, R)/k(X)^G$ and $\operatorname{RMaps}_H(Z, R)/k(Z)^H$. □

7. Proof of Theorem 1.1

We begin with a simple observation. Let X be the associated $G \times \mathrm{PGL}_n$-variety for the G-action on A. Let Y be a PGL_n-variety, which is a birational model for X/G. Then by the universal property of rational quotients (see, e.g, [PV, §2.3-2.4], [Re, Remark 2.4] or [RV$_3$, Remark 6.1]),

$$A^G = \operatorname{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)^G \simeq \operatorname{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)\,. \tag{7.1}$$

Moreover,

$$\mathrm{Z}(A)^G = (k(X)^{\mathrm{PGL}_n})^G = k(X)^{G \times \mathrm{PGL}_n} \simeq k(Y)^{\mathrm{PGL}_n}\,.$$

Thus Theorem 1.1 is a special case of the following result (applied to the conjugation action of $\Gamma = \mathrm{PGL}_n$ on $R = \mathrm{M}_n$).

7.2. **Proposition.** *Let R be a finite-dimensional k-algebra, and Γ a linear algebraic group, acting on R by k-algebra automorphisms (i.e., via a morphism $\Gamma \longrightarrow \operatorname{Aut}(R)$ of algebraic groups). If a primitive Γ-variety Y has a stabilizer $S \subset \Gamma$ in general position then* $\operatorname{RMaps}_\Gamma(Y, R)/k(Y)^\Gamma$ *is a form of R^S/k.*

Here we view $\operatorname{RMaps}_\Gamma(Y, R)$ as a K-algebra, where $K = k(Y)^\Gamma$. Note that K is a field since Y is a primitive Γ-variety (the latter means that Γ transitively permutes the irreducible components of Y).

Proof. We will proceed in two steps.

Step 1. Suppose first that $S = \{1\}$, i.e., that the Γ-action on Y is generically free. Let Y_0 be an irreducible component of Y, and let $\phi\colon \Gamma \times Y_0 \longrightarrow Y$ be the natural Γ-equivariant map, where Γ acts on itself by left multiplication and trivially on Y_0. Then ϕ induces an injective homomorphism $\mathrm{RMaps}_\Gamma(Y, R) \hookrightarrow \mathrm{RMaps}_\Gamma(\Gamma \times Y_0, R)$ via $f \mapsto f \circ \phi$. Clearly Γ-equivariant rational maps from $\Gamma \times Y_0$ to R are in 1-1 correspondence with rational maps from Y_0 to R. In other words,

$$\mathrm{RMaps}_\Gamma(Y, R) \hookrightarrow \mathrm{RMaps}_\Gamma(\Gamma \times Y_0, R) \simeq \mathrm{RMaps}(Y_0, R) \simeq R \otimes_k k(Y_0)\,.$$

On the other hand, by [Re, Lemma 7.4(b)], $\dim_K \mathrm{RMaps}_\Gamma(Y, R) = \dim_k(R)$. By [Re, Lemma 7.4(a)], the induced map

$$\mathrm{RMaps}_\Gamma(Y, R) \otimes_K k(Y_0) \to \mathrm{RMaps}_\Gamma(\Gamma \times Y_0, R)$$

is injective, so an isomorphism. That is, $\mathrm{RMaps}_\Gamma(Y, R)$ is a form of R. This completes the proof of Proposition 7.2 in the case that $S = \{1\}$.

Step 2. We will now reduce the general case to the case considered in Step 1. Let Z be the union of the components of Y^S of maximal dimension, and let $N = N_\Gamma(S)$ be the normalizer of S in Γ. Note that Z is an N-variety. By [RV$_1$, Lemma 3.2], Y is birationally isomorphic to $\Gamma *_N Z$, as a Γ-variety. By Lemma 6.1, Z is primitive since Y is, and

$$\mathrm{RMaps}_\Gamma(Y, R) \simeq \mathrm{RMaps}_N(Z, R) \quad \text{and} \quad k(Y)^\Gamma \simeq k(Z)^N\,. \tag{7.3}$$

Thus it suffices to show that $\mathrm{RMaps}_N(Z, R)/k(Z)^N$ is a form of R^S/k.

Recall that S acts trivially on Z (because Z is, by definition, a subset of Y^S); hence, the image of every N-equivariant map from Z to R will actually lie in R^S. In other words,

$$\mathrm{RMaps}_N(Z, R) = \mathrm{RMaps}_N(Z, R^S) = \mathrm{RMaps}_{N/S}(Z, R^S)\,. \tag{7.4}$$

Moreover, $k(Z)^N = k(Z)^{N/S}$. We now show that the N/S-action on Z is generically free; the desired conclusion will then follow from the result of Step 1. Equivalently, we want to show that $\mathrm{Stab}_N(z) = S$ for $z \in Z$ in general position. Clearly, $S \subset \mathrm{Stab}_N(z)$ for any $z \in Z \subset Y^S$. To prove the opposite inclusion, it is enough to show that $S = \mathrm{Stab}_\Gamma(z)$ for $z \in Z$ in general position. Indeed, by the definition of S, $\mathrm{Stab}_\Gamma(y)$ is conjugate to S for $y \in Y$ in general position. Since ΓZ is dense in Y, $\mathrm{Stab}_\Gamma(z)$ is conjugate to S for $z \in Z$ in general position. Since we know that $S \subset \mathrm{Stab}_\Gamma(z)$ we conclude that $S = \mathrm{Stab}_\Gamma(z)$ for $z \in Z$ in general position, as desired. □

We conclude this section with a simple corollary of Theorem 1.1.

7.5. **Corollary.** *Consider a tame action of G on a central simple algebra A with associated stabilizer $S \subset \mathrm{PGL}_n$. Let N be the normalizer of S in PGL_n. If A^G is a division algebra then $\mathrm{M}_n^N = k$.*

Proof. Let X be the associated $G \times \mathrm{PGL}_n$-variety. Set $Y = X/G$, and denote by Z the union of the irreducible components of Y^S of maximal dimension. Recall from the proof of Theorem 1.1 that

$$A^G \simeq \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \simeq \mathrm{RMaps}_N(Z, \mathrm{M}_n^S)\,.$$

Now observe that if $m \in \mathrm{M}_n^N$, then the constant map $f_m\colon Z \longrightarrow \mathrm{M}_n^S$, given by $f_m(z) = m$ for every $z \in Z$, is N-equivariant. Thus f_m may be viewed as an element of A^G, and $m \mapsto f_m$ defines an embedding $\mathrm{M}_n^N \hookrightarrow A^G$. Since we are assuming that A^G is a division algebra, this implies that M_n^N cannot have zero divisors. On the other hand, since M_n^N is a finite-dimensional k-algebra and k is algebraically closed, this is only possible if $\mathrm{M}_n^N = k$. □

8. Proof of Corollary 1.2

In this section we spell out what Theorem 1.1 says in the case where the G-action on A is very tame. Here the associated stabilizer subgroup $S \subset \mathrm{PGL}_n$ is reductive, and hence so is the preimage S^* of S in GL_n. Thus the natural n-dimensional linear representation $\phi\colon S^* \hookrightarrow \mathrm{GL}_n$ decomposes as a direct sum of irreducibles. Suppose ϕ has irreducible components $\phi_1, \dots, \phi_r$ of dimensions $d_1, \dots, d_r$, occurring with multiplicities $e_1, \dots, e_r$, respectively. Note that $d_1e_1 + \dots + d_re_r = n$. Our main result is the following proposition. Corollary 1.2 is an immediate consequence of parts (a), (b) and (c).

8.1. **Proposition.** (a) *A^G is a form of $\mathrm{M}_{e_1}(k) \times \dots \times \mathrm{M}_{e_r}(k)$. In particular, A^G is semisimple.*
(b) *The maximal degree of an element of A^G over K is $e_1 + \dots + e_r$.*
(c) *The maximal degree of an element of $\mathrm{Z}(A^G)$ over K is r.*
(d) *If A^G is a simple algebra, then the normalizer $N(S^*)$ of S^* in GL_n transitively permutes the (isomorphism types of) the irreducible representations $\phi_1, \dots, \phi_r$ of S^* occurring in ϕ. In particular, in this case, $d_1 = \dots = d_r$ and $e_1 = \dots = e_r$.*

Proof. (a) By a corollary of Schur's Lemma (see, e.g., [FD, Proposition 1.8]),

$$\mathrm{M}_n^S = (\mathrm{M}_n)^{S^*} = \mathrm{End}_{S^*}(k^n) \simeq \mathrm{M}_{e_1}(k) \times \dots \times \mathrm{M}_{e_r}(k)\,.$$

Part (a) now follows from Theorem 1.1.

(b) The degree of $x \in A^G$ over K is $\dim_K(K[x])$. In view of part (a), the maximal degree of an element of A^G over K is the same as the maximal degree of an element of $\mathrm{M}_{e_1}(k) \times \dots \times \mathrm{M}_{e_r}(k)$ over k. The latter is easily seen to be $e_1 + \dots + e_r$.

(c) Once again, in view of part (a), we only need to show that the maximal degree of a central element of $\mathrm{M}_{e_1}(k) \times \dots \times \mathrm{M}_{e_r}(k)$ over k is r. The center of $\mathrm{M}_{e_1}(k) \times \dots \times \mathrm{M}_{e_r}(k)$ is isomorphic, as a k-algebra, to $k \times \dots \times k$ (r times), and part (c) follows.

(d) Let V_i be the unique S^*-subrepresentation of k^n isomorphic to $\phi_i^{e_i}$. The normalizer $N(S^*)$ clearly permutes the subspaces V_i (and hence, the representations ϕ_i); we want to show that this permutation action is transitive. Assume the contrary: say, $\{\phi_1, \dots, \phi_s\}$ is an $N(S^*)$-orbit for some $s < r$. Then $W_1 = V_1 \oplus \cdots \oplus V_s$ and $W_2 = V_{s+1} \oplus \cdots \oplus V_r$ are complementary $N(S^*)$-invariant subspaces of k^n and

$$\mathrm{M}_n^S = \mathrm{End}_{S^*}(k^n) \simeq \mathrm{End}_{S^*}(W_1) \oplus \mathrm{End}_{S^*}(W_2)\,,$$

where $\simeq$ denotes an $N(S)$-equivariant isomorphism of k-algebras. Now let X be an associate $G \times \mathrm{PGL}_n$-variety for the action of G on A, $Y = X/G$, and Z be the union of components of maximal dimension of Y^S, as in the proof of Proposition 7.2. Then,

$$\begin{aligned} A^G &\overset{(7.1)}{=} \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \overset{(7.3),(7.4)}{=} \mathrm{RMaps}_{N(S)}(Z, \mathrm{M}_n^S) \\ &= \mathrm{RMaps}_{N(S)}(Z, \mathrm{End}_{S^*}(W_1)) \oplus \mathrm{RMaps}_{N(S)}(Z, \mathrm{End}_{S^*}(W_2))\,. \end{aligned}$$

This shows that A^G is not simple. $\square$

9. Proof of Theorem 1.3

We begin with some preliminaries on Galois cohomology. Let F/k be a finitely generated field extension and Γ/k be a linear algebraic group. Let Γ-$\mathrm{Var}(F)$ be the set of generically free Γ-varieties X with $k(X)^\Gamma = F$. (Since $k(X)^\Gamma$ is a field, X is necessarily primitive, i.e., Γ transitively permutes the irreducible components of X; see Section 6.) Recall that $H^1(F, \Gamma)$ is in a natural bijective correspondence with Γ-$\mathrm{Var}(F)$; see, e.g., [P, (1.3)]. We can thus identify $H^1(F, \Gamma)$ with Γ-$\mathrm{Var}(F)$. If $\alpha\colon \Gamma \to \Gamma'$ is a homomorphism of algebraic groups then the induced map $\alpha_*\colon H^1(F, \Gamma) \to H^1(F, \Gamma')$ is given by $Z \mapsto \Gamma' *_\Gamma Z$. (This follows from [P, Theorem 1.3.3(b)]; see also [KMRT, Proposition 28.16].)

If Γ is the automorphism group of some finite-dimensional k-algebra R then, as we remarked before the statement of Theorem 1.3, elements of $H^1(F, \Gamma)$ are also in a natural bijective correspondence with the set of F-forms of R. These two interpretations of $H^1(F, \Gamma)$ can be related as follows: the algebra corresponding to the Γ-variety W is $B = \mathrm{RMaps}_\Gamma(W, R)$; cf. [Re, Proposition 8.6]. Here, as usual, RMaps_Γ stands for Γ-equivariant rational maps.

We are now ready to proceed with the proof of Theorem 1.3. We fix $n \geq 1$ and a subgroup $S \subset \mathrm{PGL}_n$. For notational convenience, set $H = \mathrm{Aut}(\mathrm{M}_n^S)$. Consider the following diagram:

$$\begin{array}{ccccc} H^1(F, N/S) & \xrightarrow{\alpha_*} & H^1(F, H) & & \\ \parallel & & \parallel & & \\ N/S\text{-}\mathrm{Var}(F) & \longrightarrow & H\text{-}\mathrm{Var}(F) & \xrightarrow{\simeq} & \{F\text{-forms of } \mathrm{M}_n^S\} \\ & & & & \\ Z & \longmapsto & H *_{N/S} Z & & \\ & & & & \\ & & W & \longmapsto & \mathrm{RMaps}_H(W, \mathrm{M}_n^S) \end{array}$$

This diagram shows that the image of the map $\alpha_*\colon H^1(F, N/S) \to H^1(F, H)$ consist of all classes in $H^1(F, H)$ corresponding to algebras B/F of the form

$$B = \mathrm{RMaps}_H(H *_{N/S} Z, \mathrm{M}_n^S)\,,$$

where $F = k(H *_{N/S} Z)^H$ and Z is a primitive N/S-variety. But

$$\mathrm{RMaps}_H(H *_{N/S} Z, \mathrm{M}_n^S) \simeq \mathrm{RMaps}_{N/S}(Z, \mathrm{M}_n^S)$$

and $k(H *_{N/S} Z)^H \simeq k(Z)^{N/S}$; see Lemma 6.1. Consequently, $[B]$ lies in the image of α_* if and only if there exists a generically free primitive N/S-variety Z such that $B \simeq \mathrm{RMaps}_{N/S}(Z, \mathrm{M}_n^S)$ and $F \simeq k(Z)^{N/S}$, i.e., if and only if condition (c) of the following lemma is satisfied.

We will say that an F-algebra B is a *fixed algebra with associated stabilizer* $S \subset \mathrm{PGL}_n$ if there exists a tame action of an algebraic group G on a central simple algebra A/K of degree n with associated stabilizer S such that $B \simeq A^G$ and $F \simeq K^G$. In other words, B/F is a fixed algebra with associated stabilizer S if it satisfies condition (a) of Theorem 1.3. Hence Theorem 1.3 follows from the following result:

9.1. **Lemma.** *Let B be an F-algebra, S a subgroup of PGL_n, and $N = N_{\mathrm{PGL}_n}(S)$. The following are equivalent:*

(a) *B/F is a fixed algebra with associated stabilizer S.*
(b) *There exists an irreducible PGL_n-variety Y with stabilizer S in general position such that $B \simeq \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ and $F \simeq k(Y)^{\mathrm{PGL}_n}$.*
(c) *There exists a generically free primitive N/S-variety Z such that $B \simeq \mathrm{RMaps}_{N/S}(Z, \mathrm{M}_n^S)$ and $F \simeq k(Z)^{N/S}$.*

The isomorphisms in the various parts of the lemma are compatible: the second is always a restriction of the first.

Proof. (a) $\Longrightarrow$ (b): By definition the algebras B/F satisfying condition (a) of Theorem 1.3 are those of the form $B = \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)^G$, where $F = k(X)^{G\times \mathrm{PGL}_n}$. Here X is an irreducible $G\times\mathrm{PGL}_n$-variety, such that the PGL_n-action on X is generically free, and the projection of $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)$ to PGL_n is conjugate to S for $x \in X$ in general position. To write B and F as in (b), take $Y = X/G$; see (7.1).

(b) $\Longrightarrow$ (c): Take Z to be the union of the components of Y^S of maximal dimension, as in Step 2 of the proof of Proposition 7.2.

(c) $\Longrightarrow$ (b): Take Y to be the homogeneous fiber space $\mathrm{PGL}_n *_N Z$, and use Lemma 6.1 and (7.4). Note that Y is irreducible since $k(Y)^{\mathrm{PGL}_n}$ is a field and PGL_n is connected. A simple computation shows that S is the stabilizer in general position for the PGL_n-action on Y.

(b) $\Longrightarrow$ (a): Use Lemma 5.2, with $H = \mathrm{PGL}_n$ to reconstruct X from Y. Note that here G can be any irreducible group containing PGL_n.

This completes the proof of Lemma 9.1 and thus of Theorem 1.3. $\square$

10. An example

We will now illustrate Theorem 1.3 with an example. Fix an integer $n \geq 1$ and write $n = d_1 e_1 + \cdots + d_r e_r$ for some positive integers $d_1, e_1, \dots, d_r, e_r$. A partition of n of this form naturally arises when an n-dimensional representation is decomposed as a direct sum of irreducibles of dimensions $d_1, \dots, d_r$ occurring with multiplicities $e_1, \dots, e_r$, respectively. (In particular, the integers d_i and e_i came up in this way at the beginning of Section 8.) For this reason $\tau = [(d_1, e_1), \dots, (d_r, e_r)]$ is often referred to as a "representation type"; cf., e.g., [LBP].

Choose vector spaces $V_1, W_1, \dots, V_r, W_r$, so that $\dim V_i = d_i$ and $\dim W_i = e_i$. Write

$$k^n = (V_1 \otimes W_1) \oplus \cdots \oplus (V_r \otimes W_r)$$

and let $S_\tau^* \simeq \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r}$ be the subgroup

$$S_\tau^* = (\mathrm{GL}(V_1) \otimes I_{e_1}) \oplus \cdots \oplus (\mathrm{GL}(V_r) \otimes I_{e_r}) \subset \mathrm{GL}_n \, .$$

Let N_τ^* be the normalizer of S_τ^* in GL_n. Denote the image of S_τ^* under the natural projection $\mathrm{GL}_n \to \mathrm{PGL}_n$ by S_τ and the normalizer of S_τ in PGL_n by N_τ. The purpose of this section is to spell out what Theorem 1.3 tells us for $S = S_\tau \subset \mathrm{PGL}_n$. Note that

$$\mathrm{M}_n^{S_\tau} = (I_{d_1} \otimes \mathrm{End}(W_1)) \oplus \cdots \oplus (I_{d_r} \otimes \mathrm{End}(W_r)) \simeq \mathrm{M}_{e_1} \times \cdots \times \mathrm{M}_{e_r} \, .$$

Our goal is to determine which forms of $\mathrm{M}_n^{S_\tau}$ occur as fixed algebras with associated stabilizer S_τ.

Note that reordering the pairs (d_i, e_i) will have the effect of replacing S_τ by a conjugate subgroup of PGL_n. Since the group S in Theorems 1.1 and 1.3 is only defined up to conjugacy, we are free to order them any way we want. In particular, if there are t distinct numbers $f_1, \dots, f_t$ among $e_1, \dots, e_r$, occurring with multiplicities $r_1, \dots, r_t$, respectively, we will, for notational convenience, renumber the pairs (d_i, e_i) so that τ assumes the form

$$\tau = [(d_{11}, f_1), \dots, (d_{1r_1}, f_1), \dots, (d_{t1}, f_t), \dots, (d_{tr_t}, f_t)] \, .$$

Suppose that for each fixed i, exactly l_i numbers occur among $d_{i1}, \dots, d_{ir_i}$, with multiplicities $r_{i1}, r_{i2}, \dots, r_{il_i}$. (We order the pairs $(d_{i1}, f_i), \dots, (d_{ir_i}, f_i)$ so that the first r_{i1} of the d_{ij}s are the same, the next r_{i2} are the same, etc.) Clearly $r_i = r_{i1} + \cdots + r_{il_i}$,

$$\mathrm{M}_n^{S_\tau} \simeq \mathrm{M}_{e_1} \times \cdots \times \mathrm{M}_{e_r} \simeq \mathrm{M}_{f_1}^{r_1} \times \cdots \times \mathrm{M}_{f_t}^{r_t} \, .$$

and

$$\mathrm{Aut}(\mathrm{M}_n^{S_\tau}) \simeq \prod_{i=1}^{t} \mathrm{Aut}(\mathrm{M}_{f_i}^{r_i}) \, .$$

Note that $\mathrm{Aut}(\mathrm{M}_f^r)$ is the semidirect product $\mathrm{PGL}_f^r \rtimes \mathrm{Sym}_r$, where PGL_f^r is the group of inner automorphisms of M_f^r, and Sym_r acts by permuting the r factors of M_f.

10.1. **Lemma.** *The natural homomorphism* $\alpha\colon N_\tau/S_\tau \to \mathrm{Aut}(\mathrm{M}_n^{S_\tau})$ *corresponds to the inclusion map*

$$\prod_{i=1}^{t}\Big(\prod_{j=1}^{l_i}\mathrm{Aut}(\mathrm{M}_{f_i}^{r_{ij}})\Big) \hookrightarrow \prod_{i=1}^{t}\mathrm{Aut}(\mathrm{M}_{f_i}^{r_i})\,.$$

(*Recall that* $r_i = r_{i1} + \cdots + r_{il_i}$ *for each* i.)

Note that this lemma says that α embeds N_τ/S_τ in $\mathrm{Aut}(\mathrm{M}_n^{S_\tau})$ as a subgroup of finite index. In particular, the image of α contains the connected component

$$\mathrm{PGL}_{e_1} \times \cdots \times \mathrm{PGL}_{e_r} = \mathrm{PGL}_{f_1}^{r_1} \times \cdots \times \mathrm{PGL}_{f_t}^{r_t}$$

of $\mathrm{Aut}(\mathrm{M}_n^{S_\tau})$.

Proof. First of all, note that we may replace N_τ/S_τ by N_τ^*/S_τ^*; these two groups are isomorphic and act the same way on $\mathrm{M}_n^{S_\tau}$.

We claim that the homomorphism $\alpha\colon N_\tau^*/S_\tau^* \to \mathrm{Aut}(\mathrm{M}_n^{S_\tau})$ is injective. Indeed, the kernel of the action of N_τ^* on $\mathrm{M}_n^{S_\tau}$ is the centralizer of

$$\mathrm{M}_n^{S_\tau} = I_{d_1} \otimes \mathrm{End}(W_1) \times \cdots \times I_{d_r} \otimes \mathrm{End}(W_r)$$

in GL_n. Using Schur's Lemma, as in the proof of Proposition 8.1(a), we see that this centralizer is

$$S_\tau^* = \mathrm{GL}(V_1) \otimes I_{e_1} \times \cdots \times \mathrm{GL}(V_r) \otimes I_{e_r}\,,$$

and the claim follows. Thus α is injective, and we only need to determine which automorphisms of $\mathrm{Aut}(\mathrm{M}_n^{S_\tau}) \simeq \mathrm{M}_{e_1} \times \cdots \times \mathrm{M}_{e_r}$ can be realized as conjugation by some $g \in N_\tau^*$. Taking $g = 1\otimes a_1 + \cdots + 1 \otimes a_r$ for $a_i \in \mathrm{GL}(W_i)$, we see that every element of

$$\mathrm{PGL}_{e_1} \times \cdots \times \mathrm{PGL}_{e_r} = \mathrm{PGL}_{f_1}^{r_1} \times \cdots \times \mathrm{PGL}_{f_t}^{r_t}$$

lies in the image of α.

Let us now view k^n as an n-dimensional representation of the group $S_\tau^* \simeq \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r}$, as we did at the beginning of this section. Any $g \in N_\tau^*$ permutes the isomorphism types of the irreducible representations $V_1, \ldots, V_r$ of S_τ^*; it also permutes the isotypical components $V_1 \otimes W_1, \ldots, V_r \otimes W_r$ of this representation (as subspaces of k^n). If g maps the isotypical component $V_i \otimes W_i$ to the isotypical component $V_j \otimes W_j$ then their dimensions d_ie_i and d_je_j have to be the same. The dimensions d_i and d_j of the underlying irreducible representations V_i and V_j also have to be the same. In other words, this is only possible if $(d_i, e_i) = (d_j, e_j)$.

Conversely, if $(d_i, e_i) = (d_j, e_j)$ then a suitably chosen permutation matrix $g \in N_\tau^*$, will interchange $V_i \otimes W_i$ and $V_j \otimes W_j$ and preserve every other isotypical component $V_m \otimes W_m$ for $m \neq i, j$. These permutation matrices generate the finite subgroup $\prod_{i=1}^t \prod_{j=1}^{l_i} \mathrm{Sym}_{r_{ij}}$ of $\mathrm{Aut}(\mathrm{M}_n^{S_\tau})$.

Now suppose $g \in N_\tau^*$. After composing any g with a product of permutation matrices as above, we arrive at a $g_0 \in N_\tau^*$ which preserves every isotypical component $V_i \otimes W_i$. Conjugating by such a g_0 will preserve every factor $I_{d_i} \otimes \mathrm{End}(W_i) \simeq \mathrm{M}_{e_i}$ of $\mathrm{Aut}(\mathrm{M}_n^{S_\tau})$ and thus will lie in $\mathrm{PGL}_{f_1}^{r_1} \times \cdots \times \mathrm{PGL}_{f_t}^{r_t}$. We conclude that the image of α is generated by

$$\mathrm{PGL}_{f_1}^{r_1} \times \cdots \times \mathrm{PGL}_{f_t}^{r_t} \text{ and } \prod_{i=1}^{t} \prod_{j=1}^{l_i} \mathrm{Sym}_{r_{ij}}$$

and the lemma follows. □

10.2. **Proposition.** *B/F is a fixed algebra with associated stabilizer S_τ if and only if B is F-isomorphic to the direct product*

$$(B_{11} \times \cdots \times B_{1l_1}) \times \cdots \times (B_{t1} \times \cdots \times B_{tl_t})$$

where B_{ij}/F is a form of $\mathrm{M}_{f_i}^{r_{ij}}/k$.

Proof. Recall that $H^1(F, \mathrm{Aut}(R)) = \{F\text{-forms of } R\}$. Now suppose $R_1, \ldots, R_m$ are k-algebras, $R = R_1 \times \cdots \times R_m$, $G_i = \mathrm{Aut}(R_i)$ and $G = \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_m)$. Then the natural inclusion $G_1 \times \cdots \times G_r \hookrightarrow \mathrm{Aut}(R)$ induces a morphism from

$$H^1(F, G_m \times \cdots \times G_m) \simeq H^1(F, G_1) \times \cdots \times H^1(F, G_m)$$

to $H^1(F, \mathrm{Aut}(R))$ which maps $(A_1, \ldots, A_m)$ to $A = A_1 \times \cdots \times A_m$. Here A_i is an F-form of R_i (and hence, R is an F-form of A).

In the language of G-varieties, the first isomorphism says that X is a generically free primitive G-variety if and only if X is birationally isomorphic (as a G-variety) to the fiber product $X_1 \times_Y \cdots \times_Y X_m$, where X_i is a primitive generically free G_i-variety such that $X_i/G_i \simeq Y$ is a model for F (Y is the same for each i). Note that X_1 can be recovered from X as the rational quotient $X/(G_2 \times \cdots \times G_m)$; similarly for $X_2, \ldots, X_m$. The second map is given by the natural isomorphism

$$\mathrm{RMaps}_G(X, R) \simeq \mathrm{RMaps}_{G_1}(X_1, R_1) \times \cdots \times \mathrm{RMaps}_{G_m}(X_m, R_m)$$

of F-algebras.

We now apply this to the homomorphism $\alpha\colon N_\tau/S_\tau \to \mathrm{Aut}(\mathrm{M}_n^{S_\tau})$ described in Lemma 10.1, and the proposition follows from Theorem 1.3. □

We conclude with two corollaries of Proposition 10.2.

10.3. **Corollary.** *Let $\tau = [(d_1, e_1), \ldots, (d_r, e_r)]$. Then the following are equivalent.*

(a) *$d_i = d_j$ whenever $e_i = e_j$.*
(b) *Every form B/F of $\mathrm{M}_n^{S_\tau}$ appears as a fixed algebra with associated stabilizer S_τ.*

Proof. (a) is equivalent to $l_i = 1$ for every $i = 1, \ldots, t$. The implication (a) $\Rightarrow$ (b) is now an immediate consequence of the proposition. To show that (b) $\Longrightarrow$ (a), consider an F-algebra $B = D_1 \times \cdots \times D_t$, where D_i is a division algebra of degree f_i such that $[\mathrm{Z}(D_i) : F] = r_i$. Here $\mathrm{Z}(D_i)$ denotes the center of D_i. (For example, we can take $F = k(a_1, b_1, \ldots, a_t, b_t, c)$, where $a_1, b_2, \ldots, a_t, b_t, c$ are independent variables, and D_i is the symbol algebra $(a_i, b_i)_{f_i}$ over $\mathrm{Z}(D_i) = F(\sqrt[r_i]{c})$.) The algebra $B = D_1 \times \cdots \times D_t$ is then a form of

$$\mathrm{M}_n^{S_\tau} = \mathrm{M}_{f_1}^{r_1} \times \cdots \times \mathrm{M}_{f_t}^{r_t} .$$

On the other hand, if (b) holds, Proposition 10.2 tells us that B can be written as a direct product of $l_1 + \cdots + l_t$ F-algebras. But $B = D_1 \times \cdots \times D_t$ clearly cannot be written as a non-trivial direct product of more than t F-algebras. Thus $l_1 + \cdots + l_t \leq t$, i.e., $l_1 = \cdots = l_t = 1$, and (a) follows. $\square$

10.4. **Corollary.** *Every finite-dimensional semisimple algebra B/F (where F is a finitely generated field extension of k) appears as A^G/K^G for some very tame geometric action of a linear algebraic group G on a central simple algebra A with center K.*

Note that here we make no assumption on the degree of A or on the associated stabilizer. However, the proof will show that the associated stabilizer can always be taken to be the reductive group S_τ, where $\tau = [(1, e_1), \ldots, (1, e_r)]$ for suitable positive integers $e_1, \ldots, e_r$.

Proof. Every semisimple algebra is a form of $\mathcal{A} = \mathrm{M}_{e_1} \times \cdots \times \mathrm{M}_{e_r}$ for some positive integers $e_1, \ldots, e_r$. Now set $n = e_1 + \cdots + e_r$, $d_1 = \cdots = d_r = 1$ and $\tau = [(1, e_1), \ldots, (1, e_r)]$. Then $\mathrm{M}_n^{S_\tau} \simeq \mathcal{A}$ and the conditions of part (a) of Corollary 10.3 are satisfied for this choice of τ. Hence, every form of $\mathcal{A}$ appears as a fixed algebra (with associated stabilizer S_τ). $\square$

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