SMOOTH FINITE SPLITTINGS OF AZUMAYA ALGEBRAS OVER SURFACES

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Introduction

Let k be an algebraically closed field of characteristic zero, X a quasi-projective smooth surface over k and $\mathcal{A}$ an Azumaya algebra over X of rank n^2. We construct a smooth irreducible quasi-projective surface Y and a flat finite map $\pi_Y : Y \to X$ of degree n such that $\pi^*\mathcal{A}$ is trivial in the Brauer group $\mathrm{Br}(Y)$. We further show that the Galois closure of Y over X is a smooth irreducible quasi-projective surface Z and that the Galois group of $k(Z)$ over $k(X)$ is the symmetric group $\mathcal{S}_n$.

The smooth finite splitting $Y \to X$ was announced, for k of arbitrary characteristic, by Artin and de Jong [dJ], but no proof seems to have been published.

The splitting $Y \to X$ that we construct is locally of the form

$$\mathrm{Spec}\big(\mathcal{O}_{X,x}[T]/(P(T))\big)$$

where $P(T)$ is the characterisitic polynomial of a section of $\mathcal{A}$. This leads to a very easy construction of a deformation of Y into a union of copies of X, like the one in Lemma 5.1 of [dJ]. From this deformation, following the arguments in [dJ], we deduce a splitting criterionfor $\mathcal{A}$. For the use of these results in the proof of de Jong's theorem we refer to [CT].

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1. The characteristic polynomial of the generic matrix

In this section we suppose that k is an algebraically closed field, of arbitrary charactersitic. We denote by $\mathrm{Sing}(X)$ the singular locus of a given scheme X.

Let

$$A_n = \frac{k[X_{11}, X_{12}, \ldots, X_{nn}][T]}{\big(P(T)\big)}$$

where $P(T)$ is the characteristic polynomial of the generic matrix (X_{ij}) with $1 \le i, j \le n$. Let $Y_n = \mathrm{Spec}(A_n)$. We study the singular locus of Y_n.

Lemma 1.1. *Let $\beta = \operatorname{diag}(B_1, \ldots, B_m)$ be a matrix consisting of m cyclic Jordan blocks*

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \lambda_i & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \lambda_i & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_i & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \lambda_i \end{pmatrix}$$

with distinct eigenvalues λ_i. Then, for any i, the scheme Y_n is smooth at (β, λ_i).

Proof. We denote by I_n the identity matrix of size n. Developing the determinant of $(X_{ij}) - T \cdot \mathrm{I}_n$ along the first column we get

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \cdots + X_{n,1}P_n(T)$$

where the polynomials P_i are the cofactors of the first column. Let k_i be the size of B_i. We see that $P_{k_1}(T)(B, \lambda_1)$ is (up to sign) the determinant of a matrix of the form $\operatorname{diag}(\mathrm{I}_{k_1-1}, B_2 - \lambda_1 \mathrm{I}_{k_2}, \ldots, B_m - \lambda_1 \mathrm{I}_{k_m})$, it being understood that the first block is missing if $k_1 = 1$. Since $\lambda_1 \neq \lambda_i$, this shows that $\partial P(T)/\partial X_{k_1,1} = P_{k_1}(T)$ is not zero at (B, λ_1). Thus Y_n is smooth at (β, λ_1) and the same clearly holds for any other λ_i.

Lemma 1.2. *Every neighbourhood of a matrix α with an eigenvalue $\lambda \neq 0$ contains an invertible semisimple matrix with eigenvalue λ.*

Proof. We may assume that α is in Jordan form. The given neighbourhood of α contains an open set defined by the non-vanishing of a polynomial g in the coordinates of the generic matrix (X_{ij}). We may assume that the diagonal entries of α are $(\lambda, \lambda_2, \ldots, \lambda_n)$. Since $g(\alpha) \neq 0$ we may find values $\lambda'_2, \ldots, \lambda'_n$ all distinct and different from λ and different from 0, such that when we replace λ_i by λ'_i in α we obtain an α' for which $g(\alpha') \neq 0$. This new α' is in the given neighbourhood and is semisimple.

Let Y_n be as before. The injection $k[X_{11}, X_{12}, \ldots, X_{nn}] \to A_n$ induces a finite map $\pi : Y_n \to \mathbb{A}^{n^2}$. The projection $C = \pi(\operatorname{Sing}(Y_n))$ is a closed subscheme of $\mathbb{A}^{n^2}$ and is contained in the ramification locus of π, which is the closed subscheme of $\mathbb{A}^{n^2}$ whose closed points correspond to matrices with at least two equal eigenvalues.

Lemma 1.3. *Let $V \subset \mathbb{A}^{n^2}$ be the set of semisimple invertible matrices with at least two coincident eigenvalues. Then $V \subseteq C$.*

Proof. It suffices to check that any matrix of the form $\beta = \operatorname{diag}(\mu_1, \ldots, \mu_{n-2}, \lambda, \lambda)$ is in C. We show that (β, λ) belongs to $\operatorname{Sing}(Y_n)$. Writing $X_{ii} = \mu_i + X_i$ for $i \leq n-2$, $X_{ii} = \lambda + X_i$ for $i \geq n-1$, $T = \lambda + t$ and $\nu_i = \mu_i - \lambda$ we see that $P(T)$ is the determinant of the matrix

$$\begin{pmatrix} \nu_1 + X_1 & X_{12} & \cdots & X_{1n} \\ X_{2,1} & \nu_2 + X_2 & \cdots & X_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & X_{n-1} - t & X_{n-1,n} \\ \cdots & \cdots & X_{n,n-1} & X_n - t \end{pmatrix}$$

and it is clear that it does not contain any linear term in X_i,X_{ij} or t. Thus the variety it defines is singular at the origin, which corresponds to the point (β, λ) in the previous coordinates.

Lemma 1.4. *Let $W \subset M_n(k)$ be the set of all semisimple invertible matrices with at least $n-1$ distinct eigenvalues. Then W is open and dense in $M_n(k)$.*

Proof. The set of all semisimple invertible matrices is open and dense in $M_n(k)$. We claim that matrices having at least $n-1$ distinct eigenvalues is open in $M_n(k)$. In fact this set is the inverse image under the eigenvalue map $M_n \to \mathbb{A}^n/\mathcal{S}_n$ of the complement of the closed set of points with three equal coordinates. Hence W is open and clearly non empty.

By 1.4 the set $U = W \cap C$ of all semisimple invertible matrices with exactly two equal eigenvalues is open in C.

Lemma 1.5. *The set U is dense in C.*

Proof. Let (β, λ) be a point of $\mathrm{Sing}(Y_n)$. By 1.1, β, which we may assume to be in Jordan canonical form, contains at least two cyclic Jordan blocks with the same eigenvalue. We write $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_r)$ with the β_i's cyclic Jordan blocks of size s_i and β_1, β_2 having the same eigenvalue λ. Suppose that β is in the open set defined by $f \neq 0$ for some polynomial function f in the entries X_{ij} of the generic $n \times n$ matrix. Let $\widetilde{\beta} = \mathrm{diag}(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_r)$ be a matrix where each $\widetilde{\beta}_i$ has the same size as β_i and the same off-diagonal entries. Suppose further that $\widetilde{\beta}$ has $n-1$ distinct eigenvalues, with $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ retaining the eigenvalue λ. Then $\widetilde{\beta}$ is semisimple and, for a general $\widetilde{\beta}$, $f(\widetilde{\beta}) \neq 0$.

For example, if

$$\beta = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

then

$$\widetilde{\beta} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

with λ, λ_1, λ_2, λ_3 distinct.

Corollary 1.6. *The dimension of C is equal to the dimension of U.*

Lemma 1.7. *The dimension of U is $n^2 - 3$.*

Proof. Let $\Sigma_{n-1} \subset (k^*)^{n-1}/\mathcal{S}_{n-1}$ be the set of all $\{\lambda, \lambda_3, \ldots, \lambda_n\}$ consisting of $n-1$ distinct elements of k^*. Clearly Σ_{n-1} has dimension $n-1$. Mapping each matrix in U to the set of its eigenvalues we obtain a surjective map $p : U \to \Sigma_{n-1}$. The linear group $GL_n(k)$ acts transitively on each fiber of p and the stabilizer of the matrix $\mathrm{diag}(\lambda, \lambda, \lambda_3, \ldots, \lambda_n)$ is

$GL_2(k) \times (k^*)^{n-2}$. Hence the dimension of U is $\dim\big(GL_n(k)\big) - \dim\big(GL_2(k) \times (k^*)^{n-2}\big) + \dim(\Sigma_{n-1}) = n^2 - (4 + n - 2) + n - 1 = n^2 - 3$.

Corollary 1.8. *The closed set* $\mathrm{Sing}(Y_n)$ *is of codimension* 3.

Proof. The closure of U is $C = \pi(\mathrm{Sing}(Y_n))$ and π is a finite map.

2. Finite smooth splittings

Let X be a smooth quasi-projective surface over an algebraically closed field k, and $\mathcal{A}$ an Azumaya algebra of degree n over X. Let $K = k(X)$ be the field of rational functions of X and $\mathcal{A}_K$ the generic fibre of $\mathcal{A}$. We do not assume that $\mathcal{A}_K$ is a division ring.

Lemma 2.1. *There exists an element σ in $\mathcal{A}_K$ whose characteristic polynomial is irreducible, separable and has Galois group $\mathcal{S}_n$.*

Proof. Let $\sigma_1, \dots, \sigma_m$ be a K-basis of $\mathcal{A}_K$ (m being equal to n^2). Let $K \subset L$ be a separable finite extension of K such that $\mathcal{A}_K \otimes_K L = M_n(L)$. Let $X_1, \dots, X_m$ be indeterminates and $\widetilde{\sigma} = X_1\sigma_1 + \cdots + X_m\sigma_m$. After an L-linear change of variables the characteristic polynomial $P_{\widetilde{\sigma}}(T)$ of $\widetilde{\sigma}$ is the characteristic polynomial of the generic matrix, hence it is irreducible and separable over $L(X_1, \dots, X_m)$, and has Galois group $\mathcal{S}_n$. Since it is defined over $K(X_1, \dots, X_m)$ it has the same properties over this smaller field. By Hilbert's irreducibility theorem (see for instance [FJ], Prop. 16.1.5) there exist $\xi_1, \dots, \xi_m$ in K such that the charactersitic polynomial of $\sigma = \xi_1\sigma_1 + \cdots + \xi_m\sigma_m$ is irreducible, separable, with Galois group $\mathcal{S}_n$.

We fix a smooth embedding of X in a projective space. If d is sufficiently large, the twisted sheaf $\mathcal{A}(d)$ is generated by global sections $s_1, \dots s_{N-1}$ and, for some global section f of $\mathcal{O}_X(d)$ and σ as in Lemma 1, $s_N = \sigma f$ is a global section of $A(d)$. We set $\mathcal{L} = \mathcal{O}_X(d)$.

Let s be any global section of $\mathcal{A}(d) = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$. Choose an arbitrary affine nonempty open set $U \subset X$ over which $\mathcal{L}$ is principal: $\mathcal{L}_{|U} = \mathcal{O}_U f$ for some $f \in \mathcal{L}(U)$. Then $sf^{-1} \in \mathcal{A}(U)$, which is an Azumaya algebra over $\mathcal{O}_X(U)$. Let

$$P_{f,U}(T) = T^n + b_1T^{n-1} + \cdots + b_n$$

with $b_1, \dots, b_n \in k[U]$ be the characteristic polynomial of sf^{-1}. We define $J_{f,U}$ as the ideal of

$$Sym(\mathcal{L}^{-1}|_U) = \mathcal{O}_U \oplus \mathcal{L}^{-1}|_U \oplus \mathcal{L}^{-2}|_U \oplus \cdots = \mathcal{O}_U \oplus \mathcal{O}_U f^{-1} \oplus \mathcal{O}_U f^{-2} \oplus \cdots$$

generated by $f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n$.

Lemma 2.2. *Let Λ be an Azumaya algebra of rank n^2 over a ring R. For any $\alpha \in \Lambda$ and any $c \in R$, the characteristic polynomial $P_\alpha(T)$ of α satisfies the relation $c^nP_\alpha(T) = P_{c\alpha}(cT)$.*

Proof. It immediately follows from the split case $\Lambda = M_n(R)$.

Lemma 2.3. *The ideal $J_{f,U}$ does not depend on the choice of f.*

Proof. We apply 2.2 with $f = ug$ for some other generator g of $\mathcal{L}|_U$ and u invertible on U. (We note that the suffixes f or g stand for the elements s/f, s/g in the algebra). We have

$$P_{g,U}(T) = P_{u^{-1}f,U}(T) = u^n P_{f,U}(u^{-1}T) = T^n + ub_1T^{n-1} + \cdots + u^n b_n \ .$$

Thus the ideal $J_{g,U}$ is generated by

$$g^{-n} \oplus b_1 u g^{-(n-1)} \oplus \cdots \oplus u^n b_n = u^n(f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n) \ .$$

and coincides therefore with $J_{f,U}$.

Patching the ideals $J_{f,U}$ over a suitable affine covering of X yields a global ideal J_s of $Sym(\mathcal{L}^{-1})$ that only depends on the section s. We call J_s *the characteristic ideal of s.*

The ideal J_s defines a closed subscheme Y_s of $\mathrm{Spec}\left(Sym(\mathcal{L}^{-1})\right)$ which is clearly finite and flat over X.

To simplify notation, if $s = \lambda_1 s_1 + \cdots + \lambda_N s_N$ we put $\lambda = (\lambda_1, \ldots, \lambda_N) \in k^N$, $J_s = J_\lambda$ and $Y_s = Y_\lambda$. We denote by $\pi_\lambda : Y_\lambda \to X$ the natural map.

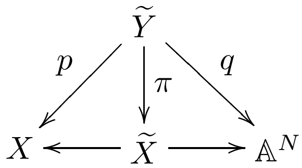
Theorem 2.4. *There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Y_λ is an irreducible quasi-projective surface.*

Before proving this theorem we recall, without proof, two easy lemmas.

Lemma 2.5. *Let $\pi : Y \to X$ be a flat dominant morphism, with X integral. Then Y is reduced if and only if the generic fibre of π is reduced.*

Lemma 2.6. *Let $\pi : Y \to X$ be a flat dominant morphism, with X integral. Then Y is irreducible if and only if the generic fibre of π is irreducible.*

Proof of Theorem 2.4. We set $\mathbb{A}^N = \mathrm{Spec}\,(k[t_1, \ldots, t_N])$ and extend the base to $\widetilde{X} = X \times \mathbb{A}^N$. Let $\widetilde{A}$ and $\widetilde{\mathcal{L}}$ be the inverse images of A and $\mathcal{L}$ under the projection $\pi : \widetilde{X} \to X$. Put $\widetilde{s} = t_1 s_1 + \cdots + t_N s_N$ and let $\widetilde{J}_t(T)$ be the characteristic ideal of $\widetilde{s}$ and $\widetilde{Y}$ the closed subscheme of $\mathrm{Spec}\left(\mathrm{Sym}(\widetilde{\mathcal{L}}^{-1})\right)$ defined by $\widetilde{J}_t(T)$. Look at the diagram



The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}^N$ are flat, hence p and q are flat. We set $\widetilde{Y}_K = \widetilde{Y} \times_X \mathrm{Spec}(K)$ and $q_K : \widetilde{Y}_K \to \mathbb{A}^N_K$ the restriction of q to $\widetilde{Y}_K$. We first note that, by the choice of s_N made above, the fibre $q_K^{-1}(0, \ldots, 0, 1)$ is integral. By Theorem 9.7.7 of [Gr], to prove the theorem it suffices to show that the geometric generic fibre of q is integral. Let Ω be an algebraic closure of $k(t_1, \ldots, t_N)$, $\widetilde{Y}_\Omega = \widetilde{Y} \times_{\mathbb{A}^N} \mathrm{Spec}(\Omega)$ the generic fibre of q, $\widetilde{X}_\Omega = X \times_k \Omega$ and $\pi_\Omega : \widetilde{Y}_\Omega \to \widetilde{X}_\Omega$ the extension of π. Let S be the integral closure of $k[t_1, \ldots, t_N]$ in Ω and $\Lambda = K \otimes_k S$. We set $\widetilde{Y}_\Lambda = \widetilde{Y} \times_{\widetilde{X}} \operatorname{Spec}(\Lambda)$, $\widetilde{X}_\Lambda = \operatorname{Spec}(\Lambda)$ and $\pi_\Lambda : \widetilde{Y}_\Lambda \to \widetilde{X}_\Lambda$ the extension of π. Assume that $\widetilde{Y}_\Omega$ is not integral. Since π_Ω is flat, by 2.5 and 2.6 the generic fibre of π_Ω is not integral. But π_Λ is also flat and has the same generic fibre as π_Ω, hence, again by 2.5 and 2.5, $\widetilde{Y}_\Lambda$ is not integral. The characterstic polynomial $P_{\widetilde{s}/f}(T) \in K[t_1, \ldots, t_N]$ that generates $\widetilde{J}_t(T)$ over a suitable open set of X is clearly separable over $K(t_1, \ldots, t_N)$, hence $\widetilde{Y}_\Lambda$ is reduced by Lemma 2.5. If $\widetilde{Y}_\Lambda$ is not integral, being reduced it has more than one component and since π_Λ is finite and flat, each component maps surjectively onto $\widetilde{X}_\Lambda$ and hence no fibre is integral. Let z be a point of $\widetilde{X}_\Lambda$ over the point $(0, \ldots, 0, 1)$ of $\mathbb{A}^N_K$. Specializing at z we get a contradiction with the irreducibility of $\pi_\Lambda^{-1}(0, \ldots, 0, 1) = \operatorname{Spec}(K) \times_X Y_{(0,\ldots,0,1)}$.

Corollary 2.7. *Let U be as in 2.4. For any $\lambda \in W$ the field $k(Y_\lambda)$ splits $\mathcal{A}_K$.*

Proof. By construction the field $k(Y_\lambda)$ is a maximal subfield of $\mathcal{A}_K$.

We now show that, assuming that k is of characteristic zero, a general fibre is smooth.

Proposition 2.8. *The dimension of* $\operatorname{Sing}(\widetilde{Y})$ *is at most* $N - 1$.

Proof.
We try to determine the singularities of $\widetilde{Y}$ using the following lemma.

Lemma 2.9. *Let $f : Z \to X$ be a flat map of schemes. Suppose that X is regular. If $z \in Z$ is a singular point of Z, then z is a singularity of its fiber $f^{-1}(f(z))$.*

Proof. Let C be the local ring of Z at z and A be the local ring of $f(z)$. By assumption the maximal ideal of A is generated by a regular sequence $(x_1, \ldots, x_m)$. Since f is flat, C is faithfully flat over A and this sequence is still regular as a sequence in C. If z is not a singular point of its fiber, then $C/(x_1, \ldots, x_m)$ is regular and hence its maximal ideal is generated by a regular sequence $(\overline{y}_1, \ldots, \overline{y}_r)$. This implies that the maximal ideal of C is generated by the regular sequence $(x_1, \ldots, x_m, y_1, \ldots, y_r)$, hence C is regular.

By 2.9 the singularities of $\widetilde{Y}$ are contained in the union of the singularities of the fibers of p.

Lemma 2.10. *The singular locus of every fiber $p^{-1}(x)$ of p has codimension 3 in $p^{-1}(x)$.*

Proof. Let $k(x)$ be the residue field of $x \in X$, Ω its algebraic closure and F_x the fiber of p at x. The geometric fibre $\mathcal{A}(\overline{x})$ of $\mathcal{A}$ at x is a matrix algebra $M_n(\Omega)$ and

$$F_{\overline{x}} = \operatorname{Spec}\left(\Omega[t_1, \ldots, t_N][T]/(P_x(T))\right) ,$$

where $P_x(T)$ is the characteristic polynomial of $\overline{s} = \big(t_1 s_1(x) + \cdots + t_N s_N(x)\big)/f(x)$ for some generator f of $\mathcal{L}|_U$, U a neighbourhood of x. Since the sections $s_i(x)/f(x)$ generate $M_n(\Omega)$ over Ω, by a linear change of coordinates we may assume that $\overline{s} = t_1 e_1 + \cdots + t_m e_m$ where $m = n^2$ and $\{e_1, \ldots, e_m\}$ form a basis of $M_n(\Omega)$. Then

$$F_{\overline{x}} = Y_n \times \operatorname{Spec}\left(\Omega[t_{m+1}, \ldots, t_N]\right) .$$

We proved that the singular locus of Y_n has codimension 3, hence the same holds for the singular locus of $F_{\overline{x}}$. For every $x \in X$ the fiber F_x is a finite cover of $\mathbb{A}^N$ and hence the dimension of F_x is N. Let $\mathrm{Sing}(\widetilde{Y})$ be the singular locus of $\widetilde{Y}$. By 2.9, for every $x \in X$, the fiber at x of $p|_{\mathrm{Sing}(\widetilde{Y})} : \mathrm{Sing}(\widetilde{Y}) \to X$ is contained in the singular locus of F_x and has therefore dimension at most $N-3$. Since X is 2-dimensional, the dimension of $\mathrm{Sing}(\widetilde{Y})$ is at most $N-1$.

Theorem 2.11. *There exists a nonempty open set $V \subset k^N$ such that, for any $\lambda \in V$, Y_λ is a smooth integral quasi-projective surface. Further, the pull-back $\pi_\lambda^* \mathcal{A}$ is trivial in* $\mathrm{Br}(Y_\lambda)$.

Proof. Look at $q : \widetilde{Y} \to \mathbb{A}^N$. Since $\mathrm{Sing}(\widetilde{Y})$ is at most $(N-1)$-dimensional, its image $q(\mathrm{Sing}(\widetilde{Y}))$ is contained in a proper closed subset of $\mathbb{A}^N$. Choose an open set $W \subset \mathbb{A}^N$ which does not intersect $q(\mathrm{Sing}(\widetilde{Y}))$ and let $\widetilde{W} = q^{-1}(W)$. We now have a map $q : \widetilde{W} \to W$ of smooth varieties. This map is clearly flat and surjective and therefore, if k is of characteristic zero, it is generically smooth (see [Ha], Ch. III, Corollary 10.7). By definition of generic smoothness there exists a dense open set $U' \subset \mathbb{A}^N$ such that $q^{-1}(U') \to U'$ is smooth. Thus for any $\lambda \in U'$ the fiber $Y_\lambda = q^{-1}(\lambda)$ is smooth. By 2.4, if $\lambda \in U$ then Y_λ is integral, hence for any $\lambda \in V = U \cap U'$ the surface Y_λ is smooth and integral. By 2.7 the field $k(Y_\lambda)$ splits $\mathcal{A}_K$. But Y_λ being smooth, the canonical map $\mathrm{Br}(Y_\lambda) \to \mathrm{Br}(k(Y_\lambda))$ is injective and thus $\pi_\lambda^* \mathcal{A}$ is trivial in $\mathrm{Br}(Y_\lambda)$.

Scholium 2.12 (suggested by D. Saltman)**.** *For any $\lambda \in V$ the $\mathcal{O}_X$-algebra $(\pi_\lambda)_* \mathcal{O}_{Y_\lambda}$ embeds into $\mathcal{A}$ as a smooth locally free maximal commutative subalgebra.*

Proof. Write Y, s and π instead of Y_λ, s_λ and π_λ. Over any sufficiently small affine open set U the line bundle $\mathcal{L}$ is generated by a local section f and $\pi_* \mathcal{O}_Y(U) = k[U][T]/(P_{s/f}(T)$ maps isomorphically onto $k[U][s/f]$, which is a commutative subalgebra of $\mathcal{A}(U)$. It is easy to see that these local isomorphisms patch to give an isomorphism of $(\pi_\lambda)_* \mathcal{O}_{Y_\lambda}$ onto a subsheaf $\mathcal{S}$ of subalgebras of $\mathcal{A}$ locally generated by sections of the form s/f. The generic fibre of $\mathcal{S}$ is a maximal subfield $K(s/f) \simeq k(Y)$ of $\mathcal{A}_K$. Since $\mathcal{S}(U) = k[U][s/f]$ is smooth, it is integrally closed and therefore it is a maximal $k[U]$-order of $K(s/f)$. This shows that it is a maximal commutative subalgebra of $\mathcal{A}(U)$.

Remark. Theorem 2.11 is not true in positive characteristic. Let for instance X be the affine plane $X = \mathrm{Spec}(k[u, v])$ over a field of characteristic $p \neq 0$ and $\mathcal{A}$ the trivial Azumaya algebra $M_2(\mathcal{O}_X)$ over X. Then $\mathcal{A}$ is generated by its global sections

$$s_1 = \begin{pmatrix} 1 & u^p \\ 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and the generic splitting that we denoted $\widetilde{Y}$ is the spectrum of

$$S = k[u, v, t_1, t_2, t_3, t_4][T]/(P(T))$$

where P is the determinant of $T - (t_1 s_1 + t_2 s_2 + t_3 s_3 + t_4 s_4)$. We find

$$P(T) = T^2 - (t_1 + t_4)T + t_1 t_4 - t_2 t_3 - u^p t_1 t_3 \ .$$

The algebra S is smooth over k if and only if P, P', $\partial P/\partial u$ and $\partial P/\partial v$ have no common zero over the algebraic closure of $k(t_1, t_2, t_3, t_4)$. But in fact, eliminating T, u and v we find the equation

$$(t_1 - t_4/2)^2 + t_2 t_3 + u^p t_1 t_3 = 0$$

which is solvable with respect to u.

Nevertheless, a better choice of the twisting $\mathcal{A}(d)$ and of the sections $s_1, \dots, s_N$ might still lead to a proof in the positive characteristic case.

3. Galois splittings

We now construct, for any $\lambda \in k^N$, a Galois covering Z_λ of X with group $G = \mathcal{S}_n$, such that $X = Z_\lambda/G$. Notice that, in general, even if Y_λ is smooth and $Y_\lambda \to X$ is a projective map, the Galois closure of Y_λ is not smooth. Therefore, in order to have Y and Z smooth in the characteristic zero case, we must construct both at the same time. We achieve this by globalizing the construction of the universal splitting algebra of a monic polynomial, which we now recall.

Let R be a commutative ring and $P(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ a monic polynomial with coefficients in R. For $1 \leq i \leq n$ let σ_i be the i-th elementary symmetric function in the n variables $T_1, \dots, T_n$. The universal splitting algebra of $P(T)$ is the quotient S of the polynomial algebra $R[T_1, \dots, T_n]$ by the ideal I generated by the elements

$$\sigma_i(T_1, \dots, T_n) - (-1)^i b_i, \quad 1 \leq i \leq n \ .$$

We denote by $\tau_1, \dots, \tau_n$ the classes modulo I of $T_1, \dots, T_n$. We clearly have

$$P(T) = (T - \tau_1) \cdots (T - \tau_n) \ .$$

The symmetric group $\mathcal{S}_n$ operates on S by permuting $\tau_1, \dots, \tau_n$.

We will use the following properties of S. (For more details and proofs see [Bou] or [EL]).

$P1$. The construction of S commutes with scalar extensions ([EL], 1.9).

$P2$. As an R-module S is free of rank $n!$ ([EL], 1.10).

$P3$. For any commutative R-algebra A and any n-tuple $(a_1, \dots, a_n)$ of elements of A such that $p(T) = (T - a_1) \cdots (T - a_n)$ in $A[T]$ there is a unique R-homomorphism $\varphi : S \to A$ such that $\varphi(\tau_i) = a_i$ ([EL], 1.3) .

$P4$. The subalgebra $R[\tau_n]$ of S is isomorphic to $R[T]/(P(T))$ and S is the universal splitting algebra of $P(T)/(T - \tau_n)$ over $R[\tau_n]$ ([EL], 1.8).

$P5$. If the discriminant of $P(T)$ is a regular element of R, then $S^{\mathcal{S}_n} = R$ ([EL], 2.2).

$P6$. If R is a field and $P(T)$ is separable with Galois group $\mathcal{S}_n$, then S is a Galois extension of R with Galois group $\mathcal{S}_n$.

We now construct Z_λ. Let $U \subset X$ be an affine open set for which $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U f$ for some section f on U . Let $\mathcal{L}$, $s_1, \dots, s_N \in H^0(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L})$ and $s = \lambda_1 s_1 + \cdots \lambda_n s_n$ be as in §2. Let $P_{f,U}(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ be the characteristic polynomial of

$s/f \in \mathcal{A}(U)$. We choose n isomorphic copies $\mathcal{L}_1, \ldots, \mathcal{L}_n$ of $\mathcal{L}$ and for each i, $f_i = f$ the generator of $\mathcal{L}_i|_U$. Consider

$$\mathcal{T} = Sym\big(\mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_n^{-1}\big) \ .$$

Writing $f_i^{-1} f_j^{-1}$ instead of $f_i^{-1} \otimes_{\mathcal{O}_U} f_j^{-1}$ we shall write the restriction of $\mathcal{T}$ to U simply as

$$\bigoplus \mathcal{O}_U f_1^{-i_1} \cdots f_n^{-i_n} \ .$$

Note that $\mathcal{O}_U[T_1, \ldots, T_n]$ is isomorphic to $\mathcal{T}|_U$ under $T_i \mapsto f_i^{-1}$.
We define $\mathcal{J}_{f,U} \subset \mathcal{T}|_U$ as the ideal generated by

$$\sigma_i(f_1^{-1}, \ldots, f_n^{-1}) - (-1)^i b_i \ 1 \le i \le n \ .$$

It corresponds in the polynomial algebra to the ideal generated by

$$\sigma_i(T_1, \ldots, T_n) - (-1)^i b_i \ 1 \le i \le n$$

which defines the universal splitting algebra of $P_{f,U}(T)$. As in the preceding section, it is easy to check that these ideals do not depend on the choice of f and can therefore be patched over the various U's to obtain a global ideal $\mathcal{J}_\lambda \subset \mathcal{T}$.

Let Z_λ be the closed subscheme of $\mathrm{Spec}(\mathcal{T})$ defined by $\mathcal{J}_\lambda$.

Proposition 3.1. *Assume that $\lambda \in k^N$ has been chosen such that $P_{f,U}(T) = P(T)$ is separable and irreducible over K. The symmetric group $\mathcal{S}_n$ acts on Z_λ via its obvious action on $\mathcal{T}$. The quotient $Z_\lambda/\mathcal{S}_n$ coincides with X and Y_λ coincides with the quotient $Z_\lambda/\mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ is the isotropy group of* 1.

Proof. It suffices to deal with the affine case, when S is the universal splitting algebra of $P(T)$ over $R = k[U]$ and show that $S^{\mathcal{S}_n} = R$ and $S^{\mathcal{S}_{n-1}} = R[T]/(P(T))$. Since $P(T)$ is separable over K the first assertion follows from property P6 and the second from properties P3 and P6.

We want to prove the following theorems.

Theorem 3.2. *There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Z_λ is an irreducible quasi-projective surface and the natural map $Z_\lambda \to X$ is a ramified Galois cover with group $\mathcal{S}_n$.*

Theorem 3.3. *Assume that k is of characteristic zero. There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Z_λ is a quasi-projective smooth surface.*

The proofs require some preliminaries. Let X_{ij} with i, j running from 1 to n be indeterminates and write $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ for the characteristic polynomial of the generic matrix (X_{ij}). Let A be the polynomial k-algebra in the X_{ij}. Consider another set $T_1, \ldots, T_n$ of indeterminates and put

$$B_n = A[T_1, \ldots, T_n]/I$$

where I is the ideal generated by all the polynomials $\sigma_i(T_1, \dots, T_n) - (-1)^i a_i$ for $1 \le i \le n$. Let $Z_n = \mathrm{Spec}(B_n)$. We want to determine $\mathrm{Sing}(Z_n)$.

A k-point of Z_n is a pair (α, t) with $\alpha \in M_n(k)$ and $t = (t_1, \dots, t_n) \in k^n$ such that $t_1, \dots, t_n$ are the eigenvalues of α, i.e. the roots of the characteristic polynomial of α, which we write as

$$P(\alpha)(T) = T^n + a_1(\alpha)T^{n-1} + \cdots + a_n(\alpha) \; .$$

Let $\pi : Z_n \to \mathrm{Spec}(A)$ be the first projection and let $S = \pi(\mathrm{Sing}(Z_n))$. We want to compute the dimension of S.

Let (α, t) be a singularity of Z_n. Since no $\sigma_i(T_1, \dots, T_n)$ involves the X_{ij} and no a_j involves the T_i, if we order the X_{ij} lexicographically, the Jacobian matrix of the equations $\sigma_i(T_1, \dots, T_n) - (-1)^i a_i = 0$ is of size $(n^2 + n) \times n$ and looks as follows:

$$J = \begin{pmatrix} \frac{\partial \sigma_1}{\partial T_1} & \cdots & \frac{\partial \sigma_n}{\partial T_1} \\ \vdots & & \vdots \\ \frac{\partial \sigma_1}{\partial T_n} & \cdots & \frac{\partial \sigma_n}{\partial T_n} \\ \frac{\partial a_1}{\partial X_{11}} & \cdots & \frac{\partial a_n}{\partial X_{11}} \\ \vdots & & \vdots \\ \frac{\partial a_1}{\partial X_{nn}} & \cdots & \frac{\partial a_n}{\partial X_{nn}} \end{pmatrix} .$$

By 3.1, π is a finite map and the dimension of Z_n is n^2. The point (α, t) being a singularity of Z_n, the Jacobian criterion implies that the rank of J at (α, t) is at most $n - 1$. Thus, in particular, the determinant δ of the top $n \times n$ block of J must vanish at (α, t). It is well-known (and can be proved by an easy induction on n) that $\delta = \pm \prod_{i<j}(T_i - T_j)$. This shows that α has at least two equal eigenvalues. In other words, denoting by $V(-)$ the vanishing locus of a given set of polynomials, (α, t) belongs to the vanishing locus $V(\delta^2)$ of the discriminant δ^2 of $P(T)$.

Consider now $\mathrm{Sing}(Z_n) \cap V(a_1, \dots, a_n)$. Since $\mathrm{Sing}(Z_n) \subset V(\delta^2)$ we have

$$\mathrm{Sing}(Z_n \cap V(a_1, \dots, a_n)) = \mathrm{Sing}(Z_n \cap V(\delta^2, a_1, \dots, a_n)) \; .$$

But the vanishing of $a_1, \dots, a_{n-1}$ and δ^2 already implies the vanishing of a_n; in fact, if $T^n - a_n$ has a multiple root, then $a_n = 0$ (we are in characteristic 0). Thus

$$\mathrm{Sing}(Z_n) \cap V(a_1, \dots, a_{n-1}) = \mathrm{Sing}(Z_n) \cap V(a_1, \dots, a_n)$$

and therefore $\dim(\mathrm{Sing}(Z_n)) \le \dim(\mathrm{Sing}(Z_n) \cap V(a_1, \dots, a_n)) + n - 1$. The set $V(a_1, \dots, a_n)$ is the set $\mathcal{N}$ of nilpotent matrices. On the other hand, the bottom block of the Jacobian matrix must have rank at most $n - 1$, which means that α is a singular point of $\mathcal{N}$. This shows that $\mathrm{Sing}(Z_n) \cap \mathcal{N} \subseteq \mathrm{Sing}(\mathcal{N})$ and from the previous inequality we obtain the next result.

Lemma 3.4. *The dimension of* $\mathrm{Sing}(Z_n)$ *is at most* $\dim(\mathrm{Sing}(\mathcal{N})) + n - 1$.

We now compute the dimension of $\mathrm{Sing}(\mathcal{N})$. As pointed out by George McNinch, our computation could be deduced from results already in the literature (see for instance [Ja], §7) but we prefer to be as self-contained as possible. We begin with the computation of the dimension of $\mathcal{N}$.

Proposition 3.5. *Let* $\mathcal{N} \subset M_n$ *denote the variety of nilpotent matrices. Then the dimension of* $\mathcal{N}$ *is* $n^2 - n$.

Proof. Since $\mathcal{N}$ is defined by the ideal $(a_1, \dots, a_n)$ of $A = k[X_{11}, X_{12}, \dots, X_{nn}]$, it suffices to show that this ideal has height n. Let I be the ideal generated by $(a_1, \dots a_n, X_{ij} \mid i \neq j)$. We claim that this ideal has height n^2. The ring A/I is isomorphic to $k[X_{11}, X_{2,2}, \dots, X_{nn}]/J$ where J is the ideal generated by elementary symmetric functions $\sigma_1, \dots, \sigma_n$ in X_{ii}. Since $k[X_{11}, \dots, X_{nn}]$ is finite over $k[\sigma_1, \dots, \sigma_n]$, the ideal J has height n in $k[X_{11}, \dots, X_{nn}]$. Hence I is supported only at closed points. Since the a_i are homogeneous, it follows that the ideal $(a_1, \dots, a_n)$ has height n.

Lemma 3.6. *A nilpotent matrix* α *whose Jordan form consists of only one cyclic block is not a singularity of* $\mathcal{N}$. *More precisely, the determinant of* $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$ *is not zero at* α.

Proof. Let A be as before and $P(T) = T^n + a_1 T^{n-1} + \dots + a_n$ the characteristic polynomial of the generic matrix (X_{ij}). The variety of nilpotent matrices is $\mathcal{N} = V(a_1, \dots, a_n)$. We show that at

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

the jacobian matrix $\left(\frac{\partial a_i}{\partial X_{ij}}\right)$ has rank n. We compute the $n \times n$ matrix $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$. The derivative of a_i by X_{j1} is the coefficient of T^{n-i} in $\frac{\partial P(T)}{\partial X_{j1}}$. Developing the determinant of $(X_{ij}) - T\mathrm{I}_n$ along the first column we find

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \dots + X_{n,1}P_n(T)$$

where $P_i(T)$ is the determinant of an $(n-1) \times (n-1)$ matrix M_i. At $(X_{ij}) = \alpha$ we find

$$M_i(\alpha) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

with

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \end{pmatrix}$$

of size $j-1$ and

$$B_2 = \begin{pmatrix} -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & -T & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -T \end{pmatrix}$$

of size $n-j$. Thus $P_j(T) = \pm T^{n-j}$ and $\frac{\partial a_i}{\partial X_{j1}}(\alpha)$ is ± 1 for $j = i$ and zero otherwise. This proves the lemma.

Lemma 3.7. *The set $\mathcal{N}_2$ of nilpotent matrices whose Jordan form has exactly two cyclic blocks are dense in the set of nilpotent matrices whose Jordan form has two or more blocks.*

Proof. Let $\alpha = \operatorname{diag}(B_1, B_2, \dots, B_m)$ be a nilpotent matrix which we can assume to be in Jordan form with blocks $B_1, \dots, B_m$, $m \geq 3$. Let $g \neq 0$ with $g \in A$ define a neighbourhood of α. We can find constants $\epsilon_2, \dots, \epsilon_{m-1}$ such that replacing the zeros between the superdiagonals of B_2 and B_3, between the superdiagonals B_3 and B_4 and so on, by the ϵ_i we obtain a matrix α' such that $g(\alpha') \neq 0$. Clearly α' has two cyclic blocks.

Lemma 3.8. *If $\alpha \in \mathcal{N}$ has a Jordan form with two or more cyclic blocks, then α is a singularity of $\mathcal{N}$.*

Proof. We may assume that α is in Jordan form and can be written as $\operatorname{diag}(B_1, B_2, \dots, B_m)$ where $m \geq 2$, each B_i is a cyclic Jordan block, B_1 is of size p and B_2 of size q. We can write the generic matrix as $(X_{ij}) = (\alpha + Y_{ij})$. Then $\frac{\partial a_i}{\partial X_{ij}}(\alpha) = \frac{\partial a_i}{\partial Y_{ij}}(0)$. But in the matrix $\alpha + (Y_{ij})$ the p-th line and the $(p+q)$-th line are linear homogeneous in the Y_{ij}, hence developing the determinant of $\alpha + (Y_{ij})$ along these two lines we see that $a_n(Y_{ij} \mid 1 \leq i, j \leq n)$ has no constant and no linear term. This shows that all the derivatives $\frac{\partial a_n}{\partial Y_{ij}}$ vanish at the origin and therefore the Jacobian matrix $\frac{\partial a_i}{\partial Y_{ij}}$ cannot be of rank n.

Corollary 3.9. *The set $\mathcal{N}_2$ is dense in* $\operatorname{Sing}(\mathcal{N})$

The set $\mathcal{N}_2$ is the union of the $GL_n(k)$-orbits $S_{p,q}$ of all the matrices of the form $\beta = \operatorname{diag}(B_p, B_q)$ where B_p is the nilpotent cyclic Jordan block of size p and B_q the nilpotent cyclic Jordan block of size $q = n-p$. In particular, it is the finite union of the constructible sets $S_{p,q}$. The dimension of $S_{p,q}$ is $n^2 - s$ where s is the dimension of the isotropy group of β

Lemma 3.10. *For $n \geq 3$ the dimension of the isotropy group of* $\operatorname{diag}(B_p, B_q)$ *is $n + 2\min(p,q)$. In particular it is always at least $n+2$.*

Proof. Let $\Gamma \subset GL_n(K)$ be the isotropy group of $\beta = \operatorname{diag}(B_p, B_q)$. Let

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of Γ, written with blocks A, B, C, D of suitable sizes. The condition $\gamma\beta\gamma^{-1} = \beta$ is equivalent to the conditions

$$AB_p = B_pA\ ,\ DB_q = B_qD\ ,\ BB_q = B_pB\ ,\ CB_p = B_qC\ .$$

We compute the dimension of the linear subspace Γ_0 of $M_n(K)$ consisting of matrices that satisfy the four conditions above.

An explicit matrix computation shows that the first condition gives

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_{p-1} & a_p \\ 0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{p-2} & a_{p-1} \\ 0 & 0 & a_1 & \cdot & \cdot & \cdot & a_{p-3} & a_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_1 & a_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_1 \end{pmatrix}$$

A similar result holds for D, hence the matrices $\operatorname{diag}(A, D)$ in Γ_0 span a linear space of dimension $p + q = n$.

Assume now that $p \leq q$. An explicit computation shows that the third condition gives

$$B = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & b_{p-1} & b_p \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{p-2} & b_{p-1} \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & b_1 & \cdot & \cdot & \cdot & b_{p-3} & b_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & b_1 \end{pmatrix}$$

A similar result holds for C, hence, when $p \leq q$ the dimension of Γ_0 is $n + p + p = n + 2\min(p, q)$ and clearly this is also the dimension (as a variety) of Γ.

Proposition 3.11. *For $n \geq 3$ the dimension of $\operatorname{Sing}(\mathcal{N})$ is $n^2 - n - 2$.*

Proof. By 3.9 and 3.10, $\dim(\operatorname{Sing}(\mathcal{N})) = \dim(\mathcal{N}_2) = n^2 - \min_{p,q}(\dim(S_{p,q}))$. The isotropy group of minimal dimension is $S_{1,n-1}$ which has dimension $n + 2$. Thus $\dim(\mathcal{N}_2) = n^2 - (n + 2)$.

Theorem 3.12. *For $n \geq 3$ the dimension of $\operatorname{Sing}(Z_n)$ is at most $n^2 - 3$*

Proof. This immediately follows from 3.4 and 3.11.

Proof of Theorem 3.2. It suffices to show that for a general λ the fibre Z_λ is irreducible. We extend the base to $\widetilde{X} = X \times \mathbb{A}^N$ where $\mathbb{A}^N = \operatorname{Spec}(k[t_1, \ldots, t_N])$ and define $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_i$ for $1 \leq i \leq n$ as the inverse images of $\mathcal{A}$, $\mathcal{L}$ and the $\mathcal{L}_i$'s under the projection $\pi : \widetilde{X} \to X$. Repeating the construction of $\mathcal{J}_\lambda$ we obtain an ideal $\mathcal{J}_t$, where $t = (t_1, \ldots, t_N)$, which specializes to $\mathcal{J}_\lambda$ when we specialize t to λ. The scheme $\widetilde{Z}$ is the closed subscheme of

$$\operatorname{Spec}(\widetilde{\mathcal{T}}) = \operatorname{Spec}\big(Sym\big(\widetilde{\mathcal{L}_1}^{-1} \oplus \cdots \oplus \widetilde{\mathcal{L}_n}^{-1}\big)\big)$$

defined by $\mathcal{J}_t$.

Look at the diagram

$$\begin{array}{ccccc} & & \widetilde{Z} & & \\ & \swarrow^{p} & \downarrow{\pi} & \searrow^{q} & \\ X & \longleftarrow & X \times \mathbb{A}^N & \longrightarrow & \mathbb{A}^N \end{array}$$

The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}^N$ are flat, hence p and q are flat. As in the previous section we set $\widetilde{Z}_K = \widetilde{Z} \times_X \mathrm{Spec}(K)$ and $q_K : \widetilde{Z}_K \to \mathbb{A}^N_K$ the restriction of q to $\widetilde{Z}_K$.

We first note that, by the choice of s_N made above, the fibre $q_K^{-1}(0, \ldots, 0, 1)$ is integral. In fact, by construction, its coordinate algebra is the universal splitting algebra of the characteristic polynomial $P_{s_N/f}(T)$ of s_N/f. Since the Galois group of $P_{s_N/f}(T)$ is $\mathcal{S}_n$, its universal splitting algebra, by property P6, is a field . We can now complete the proof exactly as we did in the proof of Theorem 2.4. By Theorem 9.7.7 of [Gr], it suffices to show that the geometric generic fibre of q is integral. Let Ω, S, Λ and $\widetilde{X}_\Lambda$ be as in section 2 and define $\widetilde{Z}_\Omega$, $\widetilde{Z}_\Lambda$, π_Ω and π_Λ as we did there for $\widetilde{Y}_\Omega$ and so on. The proof given in section 2 goes through once we remark that the universal splitting algebra $\widetilde{Z}_\Lambda$ is reduced. This is a special case of the following lemma.

Lemma 3.13. *Let R be a domain, K its field of fractions and $P(T) \in R[T]$ a monic polynomial. Assume that $P(T)$ is separable over K. Then the universal splitting algebra of $P(T)$ over R is reduced.*

Proof. Let S be the universal splitting algebra of $P(T)$ over R. It is a free R-algebra of degree $n!$. The construction of the universal splitting algebra commutes with scalar extensions (property P1), hence $S \otimes_R K$ is the splitting algebra of $P(T)$ over K. Since $P(T)$ is separable over K, it follows immediately from property P4 that $S \otimes_R K$ is étale over K, in particular reduced. By Lemma 2.5 S is reduced too.

Proof of Theorem 3.3. If $n = 2$ then $U = k^N$ and for any $\lambda \in k^N$, $Z_\lambda = Y_\lambda$. We therefore assume that $n \geq 3$. In this case the proof is on similar lines as the proof of Theorem 1.11. By 1.12 the singularities of $\widetilde{Z}$ are contained in the union of the singularities of the fibers of p. Since, by Theorem 3.12, the singularities of the closed fibres of p are at worst in codimension 3, we can argue exactly as in the proof of Theorem 1.11 and conclude that q is generically smooth, from which the assertion of Theorem 3.3 immediately follows.

4. Deformations

We now construct a flat family of surfaces over $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ that deforms the surface Y constructed in §2 into a union of copies of X. As we mentioned in the introduction, this is a crucial step in the proof of the main result of [dJ] (see also [CT]).

Let $\overline{k}(t)$ be an algebraic closure of $k(t)$

Proposition 4.1. *Let X be a smooth projective surface over an algebraically closed field k of characteristic zero and $\mathcal{A}$ an Azumaya algebra over X, of rank n^2. There exists a* *diagram*

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ {\scriptstyle f}\downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

such that

(1) W is a 3*-dimensional integral scheme with $W_{\overline{k(t)}}$ integral,*

(2) the map f is proper,

(3) $W_1 = f^{-1}(1)$ has n irreducible components V_i, each with multiplicity 1 *and such that $g|_{V_i} : V_i \to X$ is an isomorphism for every i,*

(4) W is normal at the generic point of each V_i,

(5) $Y = W_0 = f^{-1}(0)$ is an irreducible smooth projective surface, $g|_Y : Y \to X$ is finite and flat, and $g^*(\mathcal{A})|_Y$ is trivial in $\mathrm{Br}(Y)$*.*

Proof.
We fix a projective embedding of X and choose global sections $s_1, \dots, s_N$ of a suitable twist $\mathcal{A}(d)$, as we did in §2. Let $s = \lambda_1 s_1 + \cdots + \lambda_N s_N$ with $\lambda = (\lambda_1, \dots, \lambda_N) \in k^N$ and, denoting $\mathcal{O}_X(d)$ by $\mathcal{L}$, let $J_s \in Sym(\mathcal{L}^{-1})$ be the characteristic ideal of s defined in §2. Recall that Y_λ is the subscheme of $\mathrm{Spec}\big(Sym(\mathcal{L}^{-1})\big)$ defined by J_s and that locally on any affine open set $U \subset X$ over which $\mathcal{L}|_U$ is generated by a section f, $J_s|_U$ is generated by $P_{f,U}(f^{-1}) = f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n$ where $P_{f,U}(T) = T^n + b_1 T^{(n-1)} + \cdots + b_n$ is the characteristic polynomial of $s/f \in H^0(U, \mathcal{A})$. We choose λ such that Y_λ is irreducible, smooth and splits $\mathcal{A}$. Let $\widehat{X}$ be the scheme $X \times \mathbb{A}^1$, $p : \widehat{X} \to X$ its first projection and t the coordinate on $\mathbb{A}^1$. We put $\widehat{\mathcal{L}} = p^*(\mathcal{L})$ and define an ideal in $Sym(\widehat{\mathcal{L}}^{-1})$ as follows. Let $w_1, \dots, w_n$ be n distinct global sections of $\mathcal{L}$. We choose them in such a way that no function w_i/f over U is a zero of $P_{f,U}(T)$. We denote by $\widehat{U}$ the inverse image of U. For simplicity, we still denote by the same letter a function (or a section of a bundle, or a polynomial, ...) on an open set of X and its extension to $\widehat{X}$. Let $\widehat{I}_{f,U}$ be the ideal of $Sym(\widehat{\mathcal{L}}^{-1}|_{\widehat{U}})$ generated by $Q_{f,U}(t, f^{-1})$ where

$$Q_{f,U}(t,T) = (1-t)P_{f,U} + t(T - w_1/f)\dots(T - w_n/f)\ .$$

If we replace f by another generator g such that $g = uf$ for some invertible function u on U, then, as in 2.3, we see that $\widehat{I}_{f,U} = \widehat{I}_{g,U}$. Therefore these ideals patch over X and give rise to an ideal $\widehat{I}_s$ of $Sym(\widehat{\mathcal{L}}^{-1})$. We define W as the closed subscheme of $\mathrm{Spec}\big(Sym(\widehat{\mathcal{L}}^{-1})\big)$ defined by $\widehat{I}_s$.

The composite

$$W \to \mathrm{Spec}\big(Sym(\widehat{\mathcal{L}}^{-1})\big) \to X$$

defines a map $g : W \to X$ and the second projection defines a map $f : W \to \mathbb{A}^1$.

Property (2) follows from the fact that W is finite, hence proper over $\widehat{X}$ which is proper over $\mathbb{A}^1$.

The fibre W_1 is locally the spectrum of $R[T]/((T-w_1/f)\dots(T-w_n/f))$ whose irreducible components $\mathrm{Spec}\big(R[T]/(T-w_i/f)\big)$ have multiplicity 1 and map isomorphically onto $\mathrm{Spec}(R)$ under g. This proves (3).

To show (4) let $\mathfrak{p}_i$ be the generic point of V_i and $U=\mathrm{Spec}(R)$ a suitable affine open set such that its inverse image in W contains $\mathfrak{p}_i$. Then, locally at $\mathfrak{p}_i$, W is the spectrum of

$$S=\big(R[T,t]/\big((1-t)P_{f,U}(T)+t(T-h_1)\dots(T-h_n)\big)\big)_{\mathfrak{p}_i}$$

with $h_i=w_i/f$. Since $T-h_i$ and $1-t$ are in $\mathfrak{p}_i$ we have $\mathfrak{p}_i=(T-h_i,1-t)$. We assumed that $P(h_i)\neq 0$ in $K=S/\mathfrak{p}_i$, hence $\mathfrak{p}_iS$ is generated by $T-h_i$. This proves that W is normal at the generic point of V_i.

The properties in (5) are clear from the construction of W.

To prove property (1) we observe that $f^{-1}(0)=Y$ is integral and that the polynomial defining $k(W)$ over $k(X\times_k\mathbb{A}^1_k)$ is separable, hence the integrality of the geometric generic fibre of f can be proved as we did in §2 for $\widetilde{Y}\to\mathbb{A}^N_k$.

5. A splitting criterion

We now show that the flat family of Proposition 4.1 can be used to show the triviality of an Azumaya algebra.

Proposition 5.1. *Let X be an integral projective d-dimensional variety over an algebraically closed field k and $\mathcal{A}$ an Azumaya algebra over X, of rank n^2. Assume that the characteristic of k is zero or a prime that does not divide n. Fix an element $\eta\in H^2(X,\mu_n)$ which maps to $[\mathcal{A}]\in{}_n\mathrm{Br}(X)\subset H^2(X,\mathbb{G}_m)$. Suppose that there exists a diagram*

$$\begin{array}{ccc} W & \xrightarrow{\;g\;} & X \\ {\scriptstyle f}\downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

with $\mathbb{A}^1=\mathrm{Spec}(k[t])$ and such that

(1) W is a $(d+1)$-dimensional integral scheme with $W_{\overline{k(t)}}$ integral,

(2) the map f is proper,

(3) $W_1=f^{-1}(1)$ has n irreducible components V_i, each with multiplicity 1 and such that $g|_{V_i}:V_i\to X$ is a birational isomorphism for every i,

(4) W is normal at the generic point of each V_i,

(5) $g^*(\eta)|_{W_0}=0$ in $H^2(W_0,\mu_n)$.

Then $\mathcal{A}_{k(X)}$ is a matrix algebra over $k(X)$.

Proof. Let R be the local ring of $\mathbb{A}^1$ at $t=0$ and R^h its henselization. Let $g_h: W\times_{\mathbb{A}^1}\mathrm{Spec}(R^h)\to X$ be the composite map $W\times_{\mathbb{A}^1}\mathrm{Spec}(R^h)\to W\xrightarrow{g}X$. The element $g_h^*(\eta)\in H^2(W\times_{\mathbb{A}^1}\mathrm{Spec}(R^h))$ maps to zero in $H^2(W_0,\mu_n)$. By proper base change ([Mi], Ch.

VI, 2.7), $g_h^*(\eta) = 0$, hence there exists a finite tale map $C_0 \xrightarrow{\alpha} \mathbb{A}^1$ of a curve onto a neighbourhood of 0, such that if $g_{C_0} : W \times_{\mathbb{A}^1} C_0 \to X$ denotes the restriction of g_h, then $g_{C_0}^*(\eta) = 0$. We extend $\alpha : C_0 \to \mathbb{A}^1$ to an $\alpha : C_1 \to \mathbb{A}^1$ such that the point $t = 1$ is the image of a rational point of C_1. Such a point exists, since k is algebraically closed. Since $W_{\overline{k(t)}}$ is integral, the scheme $W \times_{\mathbb{A}^1} C_1$ is integral, with generic point $\mathrm{Spec}(k(W \times_{\mathbb{A}^1} C_0))$. The class $g_{C_1}^*(\eta) \in H^2(W \times_{\mathbb{A}^1} C_1, \mu_n)$ is generically zero. Since by (3) each V_i occurs with multiplicity 1 in the fibre of 1 and by (4) W is normal at the generic point of V_i, $t-1$ generates the maximal ideal of the discrete valuation ring $\mathcal{O}_{W,V_i}$. Let $1' \in C_1$ be a rational point such that $\alpha(1') = 1$. Then $V_i \times 1' \simeq V_i$ is an irreducible component of the fibre of $1'$. Let S be its local ring in $k(W \times_{\mathbb{A}^1} C_1)$. The maximal ideal of S is generated by a local parameter of C_1 at $1'$, hence S is a discrete valuation ring with quotient field $k(W \times_{\mathbb{A}^1} C_1)$ and the map $H^2(S, \mu_n) \to H^2(k(W \times_{\mathbb{A}^1} C_1), \mu_n)$ is injective. Thus $g_{C_1}^*(\eta)$ restricts to zero in $H^2(S, \mu_n)$ and specializes to zero in

$$H^2(\kappa(V_i \times \{1'\}), \mu_n) = H^2(\kappa(V_i), \mu_n) = H^2(k(X), \mu_n)$$

under the map g. The composite map $k(X) \to \kappa(V_i) \xrightarrow{g} k(X)$ being the identity, we have $\eta_{k(X)} = 0$.

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