On Voevodsky's algebraic K-theory spectrum BGL

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Abstract

Under a certain normalization assumption we prove that the $\mathbf{P}^1$-spectrum BGL of Voevodsky which represents algebraic K-theory is unique over $\mathrm{Spec}(\mathbb{Z})$. Following an idea of Voevodsky, we equip the $\mathbf{P}^1$-spectrum BGL with the structure of a commutative ring $\mathbf{P}^1$-spectrum in the motivic stable homotopy category. Furthermore, we prove that under a certain normalization assumption this ring structure is unique over $\mathrm{Spec}(\mathbb{Z})$. For an arbitrary Noetherian base scheme S we pull this structure back to get a distinguished monoidal structure on BGL.

1   Preliminaries

We refer to the Appendix A for the basic terminology, notation, constructions, definitions, results concerning homotopy theory. For a scheme S we write $\mathbf{M}(S)$, $\mathbf{M}_\bullet(S)$, $\mathrm{H}^{\mathrm{cm}}_\bullet(S)$ and $\mathrm{SH}(S)$ for the category of motivic spaces, the category of pointed motivic spaces, the pointed motivic homotopy category and the stable motivic homotopy category over S. These categories are equipped with symmetric monoidal structures. In particular, a symmetric monoidal structure $(\wedge, \mathbb{I})$ is constructed on the motivic stable homotopy category and its basic properties are proved. This structure is used intensively over the present text.

Let S be a regular scheme, and let $K^0(S)$ denote the Grothendieck group of vector bundles over S. Morel and Voevodsky proved in [MV, Thm. 4.3.13] that the Thomason-Trobaugh K-theory [TT] is represented in the pointed motivic homotopy category $\mathrm{H}_\bullet(S)$ by the space $\mathbb{Z} \times \mathrm{Gr}$ pointed by $(0, x_0)$. Here Gr is the union of the finite Grassmann varieties $\bigcup_{n=0}^\infty \mathrm{Gr}(n, 2n)$, considered as motivic spaces. There is an element $\tau_\infty - \infty \in K^0(\mathbb{Z} \times \mathrm{Gr})$ which corresponds to the identity morphism $\mathrm{id} \colon \mathbb{Z} \times \mathrm{Gr} \to \mathbb{Z} \times \mathrm{Gr}$. It follows that there exists a unique morphism

$$\mu_\otimes \colon (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \to \mathbb{Z} \times \mathrm{Gr}$$

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in $H^{cm}_\bullet(S)$ such that the composition $(\mathbb{Z} \times \text{Gr}) \times (\mathbb{Z} \times \text{Gr}) \to (\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr}) \xrightarrow{\mu_\otimes} \mathbb{Z} \times \text{Gr}$ represents the element $(\tau_\infty - \infty) \otimes (\tau_\infty - \infty)$ in $K^0\big((\mathbb{Z} \times \text{Gr}) \times (\mathbb{Z} \times \text{Gr})\big)$ (see Cor. B.1.2). Let $e_\otimes : S^0 \to \mathbb{Z} \times \text{Gr}$ be the map which corresponds to the point $(1, x_0) \in \mathbb{Z} \times \text{Gr}$. The triple

$$(\mathbb{Z} \times \text{Gr}, \mu_\otimes, e_\otimes) \tag{1}$$

is a commutative monoid in $H^{cm}_\bullet(S)$.

Using this fact Voevodsky constructed in [V1] a $\mathbf{P}^1$-spectrum

$$\text{BGL} = (\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots)$$

with structure maps $e_i \colon \mathcal{K}_i \wedge \mathbf{P}^1 \to \mathcal{K}_{i+1}$ such that

1. there is a motivic weak equivalence $w \colon \mathbb{Z} \times \text{Gr} \to \mathcal{K}_0$, and for all i one has $\mathcal{K}_i = \mathcal{K}_0$ and $e_i = e_0$,

2. the morphism

$$\mathbb{Z} \times \text{Gr} \times \mathbf{P}^1 \xrightarrow{\text{can}} (\mathbb{Z} \times \text{Gr}) \wedge \mathbf{P}^1 \xrightarrow{w \wedge \mathbf{P}^1} K_i \wedge \mathbf{P}^1 \xrightarrow{e_i} \mathcal{K}_{i+1} \xrightarrow{w^{-1}} \mathbb{Z} \times \text{Gr}$$

   in $H_\bullet(S)$ represents the element $(\tau_\infty - \infty) \otimes ([\mathcal{O}(-1)] - [\mathcal{O}]) \in K^0(\mathbb{Z} \times \text{Gr} \times \mathbf{P}^1)$,

3. and the adjoint $\mathcal{K}_i \to \Omega_{\mathbf{P}^1}(\mathcal{K}_{i+1})$ of e_i is a motivic weak equivalence.

With this spectrum in hand given a smooth X over S we may identify $K^0(X)$ with $\text{BGL}^{2i,i}(X)$ as follows

$$K^0(X) = \text{Hom}_{H_\bullet(S)}(X_+, \mathbb{Z} \times \text{Gr}) = \text{Hom}_{H_\bullet(S)}(X_+, \mathcal{K}_i) = \text{BGL}^{2i,i}(X) \tag{2}$$

Our first aim is to recall Voevodsky construction to show that this spectrum is essentially unique. This has also been obtained in [R]. *Our second and more important aim is to give a commutative monoidal structure to the $\mathbf{P}^1$-spectrum* BGL *which respects the naive multiplicative structure on the functor* $X \mapsto \text{BGL}^{2*,*}(X)$. To be more precise, we construct a product

$$\mu_{\text{BGL}} : \text{BGL} \wedge \text{BGL} \to \text{BGL} \tag{3}$$

in the stable motivic homotopy category $\text{SH}(S)$ such that for any $X \in \mathcal{S}m/S$ the diagram

$$
\begin{array}{ccc}
K^0(X) \times K^0(X) & \xrightarrow{\quad\otimes\quad} & K^0(X) \\
{\scriptstyle\cong}\Big\downarrow & & \Big\downarrow{\scriptstyle\cong} \\
\text{BGL}^{2i,i}(X) \times \text{BGL}^{2j,j}(X) & \xrightarrow{\mu_{\text{BGL}}} & \text{BGL}^{2(i+j),i+j}(X)
\end{array}
$$

commutes. We show that *there is a unique product* $\mu_{\text{BGL}} \in \text{Hom}_{\text{SH}(\mathbb{Z})}(\text{BGL} \wedge \text{BGL}, \text{BGL})$ satisfying this property. This induces a product $\mu_{\text{BGL}} \in \text{Hom}_{\text{SH}(S)}(\text{BGL} \wedge \text{BGL}, \text{BGL})$ for an arbitrary regular scheme S by pull-back along the structural morphism $S \to \text{Spec}(\mathbb{Z})$. As well, we show that the product is associative, commutative and unital. The resulting multiplicative structure on the bigraded theory $\text{BGL}^{*,*}$ coincides with the Waldhausen multiplicative structure on the Thomason-Trobaugh K-theory, as we prove in Proposition 1.2.2.

1.1 Voevodsky's K-theory spectrum BGL

The basic definitions, constructions and model structures used in the text are given in the Appendix A. Let S be a Noetherian finite-dimensional scheme. A *motivic space over S* is a simplicial presheaf on the site $\mathcal{S}m/S$ of smooth quasi-projective S-schemes. A *pointed motivic space over S* is a pointed simplicial presheaf on the site $\mathcal{S}m/S$. Let $S^n = S^{n,0}$ denote the n-fold smash product of the constant simplicial presheaf $\Delta^1/\partial\Delta^1$ with itself. We write $\mathbf{M}_\bullet(S)$ for the category of pointed motivic spaces over S. A *closed motivic model structure* $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ is constructed in A.3.9. The adjective "closed" refers to the fact that closed embeddings in $\mathcal{S}m/S$ are forced to become cofibrations. The resulting homotopy category $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ is called *the motivic homotopy category* A.3.9 of S. By A.3.11 it is equivalent to the Morel-Voevodsky $\mathbf{A}^1$-homotopy category [MV], and we may drop the superscript in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ for convenience. The closed motivic model structure has the properties that

1. for any closed S-point $x_0 \colon S \hookrightarrow X$ in a smooth S-scheme, the pointed motivic space (X, x_0) is cofibrant in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$,

2. a morphism $f \colon S \to S'$ of base schemes induces a left Quillen functor $f^* \colon \mathbf{M}_\bullet^{\mathrm{cm}}(S') \to \mathbf{M}_\bullet^{\mathrm{cm}}(S)$, and

3. taking complex points is a left Quillen functor $\mathbf{R}_\mathbb{C} \colon \mathbf{M}_\bullet^{\mathrm{cm}}(\mathbb{C}) \to \mathbf{Top}_\bullet$.

Conditions 2 and 3 do not hold for the Morel-Voevodsky model structure, condition 1 fails for the so-called projective model structure [DRØ]. For a morphism $f \colon A \to B$ of pointed motivic spaces we will write $[f]$ for the class of f in $\mathrm{H}_\bullet(S)$.

   We will consider $\mathbf{P}^1$ as a pointed space over S pointed by $\infty \colon S \hookrightarrow \mathbf{P}^1$. A $\mathbf{P}^1$-*spectrum* E over S consists of a sequence $E_0, E_1, \ldots$ of pointed motivic spaces over S, together with structure maps $\sigma_n \colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$. Let $\mathrm{SH}(S)$ denote the homotopy category of $\mathbf{P}^1$-spectra, as described in A.5. By Theorem A.5.6 it is canonically equivalent to the motivic stable homotopy category constructed in [V1] and [J]. As we will see below there exists an essentially unique $\mathbf{P}^1$-spectrum BGL over $S = \mathrm{Spec}(\mathbb{Z})$ satisfying properties 1 to 2 from Section 1. In the following, we will construct BGL in a slightly different way than Voevodsky did originally in [V1]. In order to achieve this, we begin with a description of the known monoidal structure on the Thomason-Trobaugh K-theory [TT].

1.2 A construction of BGL

Let S be a regular scheme. For every S-scheme X consider the category $\mathrm{Big}(X)$ of big vector bundles over X (see for instance [FS] for the definition and basic properties). The assignments $X \mapsto \mathrm{Big}(X)$ and $(f \colon Y \to X) \mapsto f^* \colon \mathrm{Big}(X) \to \mathrm{Big}(Y)$ form a functor from schemes to the category of small categories. The reason is that there is an equality $(f \circ g)^* = g^* \circ f^*$, not just a unique natural isomorphism. In what follows we will always consider the Thomason-Trobaugh K-theory space of X as the space obtained by applying Waldhausen's $\mathcal{S}_\bullet$-construction [W] applied to the category $\mathrm{Big}(X)$ rather than

to the category $\mathrm{Vect}(X)$ of usual vector bundles on X. This has the advantage that the assignment taking an S-scheme X to the Thomason-Trobaugh K-theory space of X becomes a functor on the category of S-schemes, and in particular a pointed motivic space over S.

Let $\mathbb{K}^{\mathrm{TT}}$ be the pointed motivic space defined in A.3.4. It has the properties that it is fibrant in $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$ and that $\mathbb{K}^{\mathrm{TT}}(X)$ is naturally weakly equivalent to the Waldhausen K-theory space associated to the category of big vector bundles on X. For $X \in \mathcal{S}m/S$ and S regular, $\mathbb{K}^{\mathrm{TT}}(X)$ is thus a Kan simplicial set having the Thomason-Trobaugh K-theory groups $K^{\mathrm{TT}}_*(X)$ as its homotopy groups. Note that since X is regular, these K-theory groups coincide with Quillen's higher K-theory groups. It follows immediately from the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(S^{p,0} \wedge X_+, \mathbb{K}^{\mathrm{TT}}) \cong \mathrm{Hom}_{\mathrm{H}_\bullet}\big(S^p, \mathbb{K}^{\mathrm{TT}}(X)\big) = K^{\mathrm{TT}}_p(X) \qquad (4)$$

that $\mathbb{K}^{\mathrm{TT}}$ regarded as an object in the motivic homotopy category $\mathrm{H}_\bullet(S)$ (see A.3.9) represents the Thomason-Trobaugh K-theory on $\mathcal{S}m/S$. Using a result of Morel and Voevodsky we describe now a commutative monoidal structure on $\mathbb{K}^{\mathrm{TT}}$ as follows.

Let (Gr, x_0) and $\big(\mathbb{Z} \times \mathrm{Gr}, (0, x_0)\big)$ be the pointed motivic spaces described in Section 1.

Lemma 1.2.1. *The canonical map*

$$\mathrm{Hom}_{\mathrm{H}^{\mathrm{cm}}_\bullet(S)}(\mathrm{Gr}, \mathbb{K}^{\mathrm{TT}}) \to \varprojlim \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(\mathrm{Gr}(n, 2n), \mathbb{K}^{\mathrm{TT}}) = \varprojlim K^{\mathrm{TT}}_0\big(\mathrm{Gr}(n, 2n)\big)$$

is an isomorphism. A similar statement holds for the pointed motivic spaces $\mathbb{Z} \times \mathrm{Gr}$ and $(\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr})$.

Proof. The pointed motivic spaces $(\mathrm{Gr}(n, 2n), x_0)$ are closed cofibrant and the inclusions $(\mathrm{Gr}(n, 2n), x_0) \xrightarrow{i_n} (\mathrm{Gr}(n+1, 2n+2), x_0)$ are closed cofibrations. Thus the colimit Gr is a model for the homotopy colimit. Since we already know that $\mathbb{K}^{\mathrm{TT}}$ is weakly equivalent to the zeroth term in a fibrant $\mathbf{P}^1$-spectrum BGL, there is an adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{H}^{\mathrm{cm}}_\bullet(S)}(A, \mathbb{K}^{\mathrm{TT}}) \cong \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty_{\mathbf{P}^1} A, \mathrm{BGL}).$$

It follows from Lemma A.5.10 that there is an exact sequence of the form

$$0 \to \varprojlim{}^1 [S^{1,0} \wedge \mathrm{Gr}(n, 2n), \mathbb{K}^{\mathrm{TT}}] \to [\mathrm{Gr}, \mathbb{K}^{\mathrm{TT}}] \to \varprojlim [\mathrm{Gr}(n, 2n), \mathbb{K}^{\mathrm{TT}}] \to 0$$

It remains to check that the $\varprojlim^1$-group vanishes. To prove this, note that by (4) there is an identification

$$\varprojlim{}^1 [S^{1,0} \wedge \mathrm{Gr}(n, 2n), \mathbb{K}^{\mathrm{TT}}] = \varprojlim{}^1 K^{\mathrm{TT}}_1(\mathrm{Gr}(n, 2n)).$$

Lemma B.1.4 implies that $\varprojlim^1 K^{\mathrm{TT}}_1(\mathrm{Gr}(n, 2n)) = \varprojlim^1 K^{\mathrm{TT}}_1(S) \otimes_{K^{\mathrm{TT}}_0(S)} K^{\mathrm{TT}}_0\big(\mathrm{Gr}(n, 2n)\big)$, and the connecting homomorphisms in this tower are given by the tensor products $\mathrm{id} \otimes i^*_n$, where $i^*_n \colon K^{\mathrm{TT}}_0\big(\mathrm{Gr}(n+1, 2n+2)\big) \to K^{\mathrm{TT}}_0\big(\mathrm{Gr}(n, 2n)\big)$. It is known that i^*_n is surjective for all $n \geq 0$. Thus the $\varprojlim^1$-group vanishes. The other cases are handled similarly. $\quad\square$

For an integer n let $[-n, n] = \{a \in \mathbb{Z} \colon -n \leq a \leq n\}$. ¿From now on we will identify $K_0^{\mathrm{TT}}(\mathbb{Z} \times \mathrm{Gr})$ with the group $\varprojlim K_0^{\mathrm{TT}}\big([-n, n] \times \mathrm{Gr}(n, 2n)\big)$. Following Lemma 1.2.1 we will identify $\mathrm{Hom}_{\mathrm{H}_\bullet^{\mathrm{cm}}(S)}(\mathbb{Z} \times \mathrm{Gr}, \mathbb{K}^{\mathrm{TT}})$ with $K_0^{\mathrm{TT}}(\mathbb{Z} \times \mathrm{Gr})$.

Let τ_n be the tautological vector bundle over $\mathrm{Gr}(n, 2n)$, and let m be the trivial vector bundle of rank m. Let $\tau_\infty - \infty \in K_0^{\mathrm{TT}}(\mathbb{Z} \times \mathrm{Gr})$ be the unique element such that for each integer $m \geq 0$ its restriction to the subspace

$$\{m\} \times \mathrm{Gr}(n, 2n) \subseteq \mathbb{Z} \times \mathrm{Gr}$$

coincides with the element $[\tau_n] - n + m \in K_0^{\mathrm{TT}}(\mathrm{Gr}(n, 2n))$. Let $f \colon \mathbb{Z} \times \mathrm{Gr} \to \mathbb{K}^{\mathrm{TT}}$ be the unique morphism in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ representing $\tau_\infty - \infty$. By [MV, Thm. 4.3.13] the morphism $f \colon \mathbb{Z} \times \mathrm{Gr} \to \mathbb{K}^{\mathrm{TT}}$ is an isomorphism in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$.

The space $\mathbb{Z} \times \mathrm{Gr}$ is closed cofibrant and the space $\mathbb{K}^{\mathrm{TT}}$ is closed motivic fibrant. Thus by Quillen's Theorem A.2.1 there is a motivic weak equivalence $\mathbb{Z} \times \mathrm{Gr} \xrightarrow{w} \mathbb{K}^{\mathrm{TT}}$ in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ such that $[w] \in \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(\mathbb{Z} \times \mathrm{Gr}, \mathbb{K}^{\mathrm{TT}}) = K_0^{\mathrm{TT}}(\mathbb{Z} \times \mathrm{Gr})$ coincides with the element $\tau_\infty - \infty \in K_0^{\mathrm{TT}}(\mathbb{Z} \times \mathrm{Gr})$. Note that w is unique up to a simplicial homotopy.

Set $K_0^{\mathrm{TT}}\big((\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr})\big) := \varprojlim K_0^{\mathrm{TT}}\big(([-n, n] \times \mathrm{Gr}(n, 2n)) \times ([-n, n] \times \mathrm{Gr}(n, 2n))\big)$. Following 1.2.1, identify $\mathrm{Hom}_{\mathrm{H}_\bullet(S)}\big((\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr}), \mathbb{K}^{\mathrm{TT}}\big)$ with $K_0^{\mathrm{TT}}\big((\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr})\big)$. The space $\mathbb{Z} \times \mathrm{Gr}$ pointed by $(0, x_0)$ is *a pointed S-cellular scheme* (see B.1 for the definition of a pointed S-cellular scheme). The element $\tau_\infty - \infty$ vanishes at the S-point $(0, x_0)$. Corollary B.1.2 and Lemma 1.2.1 show that there exists a unique morphism

$$\mu^{\mathrm{TT}} \in \mathrm{Hom}_{\mathrm{H}_\bullet(S)}(\mathbb{K}^{\mathrm{TT}} \wedge \mathbb{K}^{\mathrm{TT}}, \mathbb{K}^{\mathrm{TT}}) \tag{5}$$

such that the composite morphism in $\mathrm{H}_\bullet(S)$

$$(\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr}) \to (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \xrightarrow{w \wedge w} \mathbb{K}^{\mathrm{TT}} \wedge \mathbb{K}^{\mathrm{TT}} \xrightarrow{\mu^{\mathrm{TT}}} \mathbb{K}^{\mathrm{TT}} \tag{6}$$

represents the class $(\tau_\infty - \infty) \otimes (\tau_\infty - \infty)$ in $K_0^{\mathrm{TT}}((\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr}))$. In fact, $w \wedge w$ is a weak equivalence in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ by Theorem A.3.9. Now define e^{TT} as the composite morphism $S^0 \xrightarrow{e_\otimes} \mathbb{Z} \times \mathrm{Gr} \xrightarrow{w} \mathbb{K}^{\mathrm{TT}}$. in $\mathbf{M}_\bullet(S)$. Clearly the triple

$$\big(\mathbb{K}^{\mathrm{TT}}, \mu^{\mathrm{TT}}, [e^{\mathrm{TT}}]\big) \tag{7}$$

is a commutative monoidal structure on $\mathbb{K}^{\mathrm{TT}}$ in $\mathrm{H}_\bullet(S)$.

Proposition 1.2.2. *The commutative monoidal structure (7) on $\mathbb{K}^{\mathrm{TT}}$ regarded as an object in $\mathrm{H}_\bullet(S)$ coincides with the structure defined by Waldhausen in [W].*

A sketch of the proof will be given in Appendix B. Now we continue our exposition.

Lemma 1.2.3. *There is a trivial cofibration $i \colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}$ in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$ and a trivial fibration $p \colon \mathcal{K} \to \mathbb{K}^{\mathrm{TT}}$ such that $w = p \circ i$. In particular, the motivic space $\mathcal{K}$ is cofibrant and fibrant in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$.*

Proof. This follows from factoring w as a cofibration followed by a trivial fibration in the model category $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$. $\qquad\square$

Choose and fix i and p as required by the Lemma. The morphisms $[w]$, $[i]$ and $[p]$ in $\mathrm{H}^{cm}_{\bullet}(S)$ are isomorphisms. The structure (7) of a commutative monoid on $\mathbb{K}^{TT}$ in $\mathrm{H}_{\bullet}(S)$ induces via the isomorphism $[w]$ (resp. $[p]$) the structure of a commutative monoid $(\mathbb{Z} \times \mathrm{Gr}, \mu^{Gr}, e^{Gr})$ (resp. $(\mathcal{K}, \bar{\mu}, \bar{e})$) on the motivic space $\mathbb{Z} \times \mathrm{Gr}$ (resp. $\mathcal{K}$) in $\mathrm{H}^{cm}_{\bullet}(S)$ such that $[i]$ and $[p]$ are monoid isomorphisms. The space $\mathcal{K}$ is closed motivic fibrant and closed cofibrant, thus the morphisms $\bar{\mu}$ and $\bar{e}$ in $\mathrm{H}^{cm}_{\bullet}(S)$ can be represented by certain morphisms in $\mathbf{M}_{\bullet}(S)$ by Theorem A.2.1. We will do this as follows.

Let $\mu \colon (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \to \mathcal{K}$ be a morphism in $\mathbf{M}_{\bullet}(S)$ representing the morphism $[i \circ \mu^{Gr}]$ in $\mathrm{H}^{cm}_{\bullet}(S)$. There exists such a μ since the space $(\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr})$ is cofibrant and the space $\mathcal{K}$ is fibrant in $\mathbf{M}^{cm}_{\bullet}(S)$. There exists a unique (up to a homotopy) morphism $\mu^{\mathcal{K}} \colon \mathcal{K} \wedge \mathcal{K} \to \mathcal{K}$ in $\mathbf{M}_{\bullet}(S)$ such that $[\mu^{\mathcal{K}} \circ (i \wedge i)] = [\mu]$ in $\mathrm{H}^{cm}_{\bullet}(S)$. Here one uses that the smash product of cofibrant objects in $\mathbf{M}^{cm}_{\bullet}(S)$, and in particular $\mathcal{K} \wedge \mathcal{K}$, is cofibrant by A.3.9. Define $e^{\mathcal{K}}$ in $\mathbf{M}_{\bullet}(S)$ as the composition $S^0 \xrightarrow{e_{\otimes}} \mathbb{Z} \times \mathrm{Gr} \xrightarrow{i} \mathcal{K}$. The maps $\mu^{\mathcal{K}}$ and $e^{\mathcal{K}}$ represent the classes $\bar{\mu}$ and $\bar{e}$ respectively. So

$$(\mathcal{K}, [\mu^{\mathcal{K}}], [e^{\mathcal{K}}]) \tag{8}$$

is a commutative monoid in $\mathrm{H}^{cm}_{\bullet}(S)$ which coincides with $(\mathcal{K}, \bar{\mu}, \bar{e})$.

Remark 1.2.4. The functors $\mathrm{Hom}_{\mathrm{H}^{cm}_{\bullet}(S)}(-, \mathcal{K})$ and $\mathrm{Hom}_{\mathrm{H}^{cm}_{\bullet}(S)}(-, \mathbb{K}^{TT})$ are isomorphic via the isomorphism sending g to $[p] \circ g$. Under this identification the class $[i]$ of i represents the element $\tau_{\infty} - \infty \in K_0^{TT}(\mathbb{Z} \times \mathrm{Gr})$ since $w = p \circ i$ and $[w] \in [\mathbb{Z} \times \mathrm{Gr}, \mathbb{K}^{TT}]$ represents $\tau_{\infty} - \infty$.

The composition of the inclusion $\mathbf{P}^1 = \mathrm{Gr}(1, 2) \hookrightarrow \mathrm{Gr}$ and the map i is denoted $b \colon \mathbf{P}^1 \to \mathcal{K}$. This map represents the class $[\mathcal{O}(-1)] - [\mathcal{O}]$ in the kernel of the homomorphism $\infty^* \colon K^0(\mathbf{P}^1) \to K^0(k)$ induced by $\infty \colon \mathrm{Spec}(k) \to \mathbf{P}^1$. Consider the map of pointed motivic spaces given by the composition

$$\epsilon \colon \mathcal{K} \wedge \mathbf{P}^1 \xrightarrow{\mathrm{id} \wedge b} \mathcal{K} \wedge \mathcal{K} \xrightarrow{\mu^{\mathcal{K}}} \mathcal{K}. \tag{9}$$

Definition 1.2.5. *Define* BGL *as the* $\mathbf{P}^1$*-spectrum of the form* $(\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots)$ *with* $\mathcal{K}_i = \mathcal{K}$ *for all* i *and with the structure maps* $e_i \colon \mathcal{K}_i \wedge \mathbf{P}^1 \to \mathcal{K}_{i+1}$ *equal to the map* $\epsilon \colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$.

It will be proved in Section 1.5 that in the case $S = \mathrm{Spec}(\mathbb{Z})$ this spectrum is essentially unique. In the next section, we will construct a monoidal structure on BGL regarded as an object in the stable homotopy category $\mathrm{SH}(S)$. In the case of $S = \mathrm{Spec}(\mathbb{Z})$ such a monoidal structure is unique. Pulling it back via the structural morphism $S' \to \mathrm{Spec}(\mathbb{Z})$ we get a monoidal structure on BGL in $\mathrm{SH}(S')$ for an arbitrary Noetherian finite-dimensional base scheme S'.

To complete this section we prove that BGL is an $\Omega_{\mathbf{P}^1}$-spectrum which represents the Thomason-Trobaugh K-theory on Sm/S. For $X \in Sm/S$ we abbreviate $\mathrm{BGL}(X_+)$ as $\mathrm{BGL}(X)$, which forces us to write $\mathrm{BGL}(X, x_0)$ for a pointed S-scheme (X, x_0).

Lemma 1.2.6. *The spectrum* BGL *is an* $\Omega_{\mathbf{P}^1}$*-spectrum.*

Proof. For $Y \in \mathcal{S}m/S$ and a closed subscheme $Z \hookrightarrow Y$ write $K_Z^n(Y)$ for the n-th Thomason-Trobaugh K-group of perfect complexes on Y with support on Z. It may be obtained as the n-th homotopy group of the homotopy fiber of the map $\mathbb{K}^{\mathrm{TT}}(Y) \to \mathbb{K}^{\mathrm{TT}}(X \smallsetminus Z)$. For each smooth X over S the map

$$K^n(X) = [S^{n,0} \wedge X_+, \mathcal{K}_i] \to [S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_i \wedge \mathbf{P}^1]$$
$$\to [S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_{i+1}] \cong K_{X \times \{\infty\}}^n(X \times \mathbf{P}^1)$$

induced by the structure map e_i coincides with the multiplication by the class $[\mathcal{O}(-1)] - [\mathcal{O}]$ in $K_{\{\infty\}}^0(\mathbf{P}^1)$. This multiplication is known to be an isomorphism for the Thomason-Trobaugh K-groups, by the projective bundle theorem [TT, Thm. 4.1] for $X \times \mathbf{P}^1$. Whence the Lemma. $\square$

Corollary 1.2.7. *For each pointed motivic space A over S the adjunction map*

$$\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(A, \mathcal{K}_0) \to \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma_{\mathbf{P}^1}^\infty A, \mathrm{BGL})$$

is an isomorphism. In particular, for every smooth scheme X over S and each closed subscheme Z in X one has $K_{Z,p}^{\mathrm{TT}}(X) = \mathrm{BGL}^{-p,0}\big(X/(X \smallsetminus Z)\big)$. The family of these isomorphisms form an isomorphism $Ad: K_*^{\mathrm{TT}} \to \mathrm{BGL}^{-*,0}$ of cohomology theories on the category $\mathcal{S}m\mathcal{O}p/S$ in the sense of [PS1]. Moreover the adjunction map $[A, \mathcal{K}_i] \to [\Sigma_{\mathbf{P}^1}^\infty(A)(-i), \mathrm{BGL}]$ is an isomorphism. In particular, for every smooth scheme X over S and each closed subscheme Z in X one has $K_{Z,p}^{\mathrm{TT}}(X) = \mathrm{BGL}^{2i-p,i}(X/(X-Z))$.

The family of pairings $\mathcal{K}_i \wedge \mathcal{K}_j \xrightarrow{m_{ij}} \mathcal{K}_{i+j}$ with $m_{ij} = \mu^{\mathcal{K}}$ defines a family of pairings

$$\cup : \mathrm{BGL}^{p,i}(A) \otimes \mathrm{BGL}^{q,j}(B) \to \mathrm{BGL}^{p+i,q+j}(A \wedge B) \tag{10}$$

for pointed motivic spaces A and B. We will refer to the latter as the *naive product structure* on the functor $\mathrm{BGL}^{*,*}$ on the category $\mathbf{M}_*(S)$. With this naive product in hand one has the following

Corollary 1.2.8. *The isomorphism $Ad: K_*^{\mathrm{TT}} \to \mathrm{BGL}^{-*,0}$ of cohomology theories on $\mathcal{S}m\mathcal{O}p/S$ is an isomorphism of ring cohomology theories in the sense of [PS1].*

1.3 The Bott element

The aim of this Section is to construct an element $\beta \in \mathrm{BGL}^{2,1}(\mathrm{Spec}(k))$, to show that it is invertible and to check that for any pointed motivic space A one has

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] = \mathrm{BGL}^{*,*}(A)$$

(the Laurent polynomials over $\mathrm{BGL}^{*,0}(A)$). We will use the naive product structure on BGL described just above Corollary 1.2.8.

Definition 1.3.1. *Set $\beta := [S^0 \xrightarrow{e_{\mathcal{K}}} \mathcal{K} = \mathcal{K}_1] \in \mathrm{BGL}^{2,1}(\mathrm{Spec}(k))$, where $e_{\mathcal{K}}$ is the unit of the monoid $\mathcal{K}$ (see (8)).*

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Lemma 1.3.2. *Let $b\colon \mathbf{P}^1 \hookrightarrow \mathcal{K}$ be the map described just above (9). It represents the element $[\mathcal{O}(-1)] - [\mathcal{O}]$ in $\mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty) = \mathrm{Ker}\big(\infty^*\colon K^0(\mathbf{P}^1) \to K^0(k)\big)$. There is a relation*

$$\beta \cup \big([\mathcal{O}(-1)] - [\mathcal{O}]\big) = \Sigma_{\mathbf{P}^1}(1) \in \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty), \tag{11}$$

where $\Sigma_{\mathbf{P}^1}$ is the suspension isomorphism and $1 \in \mathrm{BGL}^{0,0}(\mathrm{Spec}(k))$ is the unit. There is another relation

$$\beta \cup \big([\mathcal{O}(1)] - [\mathcal{O}]\big) = -\Sigma_{\mathbf{P}^1}(1) \in \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty). \tag{12}$$

Proof. The element $\Sigma_{\mathbf{P}^1}(1)$ is represented by the map

$$S^0 \wedge \mathbf{P}^1 \xrightarrow{e_{\mathcal{K}} \wedge id} \mathcal{K}_0 \wedge \mathbf{P}^1 \xrightarrow{id \wedge b} \mathcal{K}_0 \wedge \mathcal{K}_1 \xrightarrow{m_{01}} \mathcal{K}_1,$$

where m_{01} is from Theorem 2.2.1. The element $\beta \cup \big([\mathcal{O}(-1)] - [\mathcal{O}]\big)$ is represented by the map

$$S^0 \wedge \mathbf{P}^1 \xrightarrow{e_{\mathcal{K}} \wedge b} \mathcal{K}_1 \wedge \mathcal{K}_0 \xrightarrow{m_{01}} \mathcal{K}_1.$$

Since $\mathcal{K}_0 = \mathcal{K}_1 = \mathcal{K}$ one has $(id \wedge b) \circ (e_{\mathcal{K}} \wedge id) = e_{\mathcal{K}} \wedge b$. This implies the relation (11). Relation (12) follows from the first one since $[\mathcal{O}(1)] - [\mathcal{O}] = -[\mathcal{O}(-1)] + [\mathcal{O}]$ in $K^0(\mathbf{P}^1)$. $\qquad\square$

Lemma 1.3.3. *Let $u \in \mathrm{BGL}^{-2,-1}(\mathrm{Spec}(k))$ be the unique element such that $\Sigma_{\mathbf{P}^1}(u) = [\mathcal{O}(1)] - [\mathcal{O}]$ in $\mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty)$. Then $\beta \cup u = 1$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{BGL}^{2,1}(\mathrm{Spec}(k)) \otimes \mathrm{BGL}^{0,0}(\mathbf{P}^1, \infty) & \xrightarrow{\;\cup\;} & \mathrm{BGL}^{2,1}(\mathbf{P}^1, \infty) \\
{\scriptstyle id \otimes \Sigma_{\mathbf{P}^1}} \uparrow & & \uparrow {\scriptstyle \Sigma_{\mathbf{P}^1}} \\
\mathrm{BGL}^{2,1}(\mathrm{Spec}(k)) \otimes \mathrm{BGL}^{-2,-1}(\mathrm{Spec}(k)) & \xrightarrow{\;\cup\;} & \mathrm{BGL}^{0,0}(\mathrm{Spec}(k)).
\end{array}
$$

Now the Lemma follows from the relation (12). $\qquad\square$

Definition 1.3.4. *For $\mathbf{P}^1$-spectra E and F set $E^{\mathrm{alg}}(F) = \oplus_{-\infty}^{+\infty} E^{2i,i}(F)$.*

Proposition 1.3.5. *For every pointed motivic space A the map*

$$\mathrm{BGL}^{*,0}(A) \otimes \mathrm{BGL}^{\mathrm{alg}}(\mathrm{Spec}(k)) \to \mathrm{BGL}^{*,*}(A) \tag{13}$$

given by $a \otimes b \mapsto a \cup b$ is a ring isomorphism and $\mathrm{BGL}^{\mathrm{alg}}(\mathrm{Spec}(k)) = \mathbb{Z}[\beta, \beta^{-1}]$ is the Laurent polynomial ring. One can rewrite this ring isomorphism as

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] = \mathrm{BGL}^{*,*}(A) \tag{14}$$

Proof. In fact, $\mathrm{BGL}^{*,0}(A) \xrightarrow{\cup \beta} \mathrm{BGL}^{*+2,1}(A)$ is an isomorphism since β is invertible. Since $\mathrm{BGL}^{0,0}(\mathrm{Spec}(k)) = K^0(\mathrm{Spec}(k)) = \mathbb{Z}$ the map (13) is a ring isomorphism. $\qquad\square$

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Using the isomorphism $Ad: K_*^{\text{TT}} \to \text{BGL}^{-*,0}$ of ring cohomology theories from Corollary 1.2.8 we get the following statement.

Corollary 1.3.6. *For every $X \in \mathcal{S}m/S$ and every closed subscheme $Z \hookrightarrow X$ one has*

$$K_{Z,-*}^{\text{TT}}(X)[\beta,\beta^{-1}] = \text{BGL}_Z^{*,*}(X). \tag{15}$$

The family of these isomorphisms form an isomorphism of the ring cohomology theories on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1]. As well, there is an isomorphism

$$K_{Z,-*}^{\text{TT}}(X) = \text{BGL}_Z^{*,*}(X)/(\beta+1)\text{BGL}_Z^{*,*}(X). \tag{16}$$

The family of these isomorphisms form an isomorphism of the ring cohomology theories on $\mathcal{S}m\mathcal{O}p$ in the same sense.

Remark 1.3.7. The identification (18) of $\text{BGL}^{-1,0}(X)$ with $\text{BGL}^{2i-1,i}(X)$ coincides with the Bott periodicity isomorphism $\text{BGL}^{-1,0}(X) \xrightarrow{\cup\beta^i} \text{BGL}^{2i-1,i}(X)$.

1.4 $\text{BGL}^{*,*}$ as an oriented ring cohomology theory

The restriction of the functor $\text{BGL}^{*,*}$ to the category $\mathcal{S}m\mathcal{O}p$ is a ring cohomology theory. In this section we promote it to an *oriented ring cohomology theory in the sense of* [PS1]. For that it suffices to construct a Chern structure on $\text{BGL}^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1]. Let $\mathbf{P}^\infty$ be the motivic space pointed by $\infty \in \mathbf{P}^1 \hookrightarrow \mathbf{P}^\infty$. A functor isomorphism $\text{Hom}_{\mathbf{H}_\bullet(S)}(-,\mathbf{P}^\infty) \cong \text{Pic}(-)$ on the category $\mathcal{S}m/S$ is constructed in [MV, Prop. 4.3.8]. That functor isomorphism takes the homotopy class of the canonical map $\mathbf{P}_+^\infty \to (\mathbf{P}^\infty,\infty)$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. Consider the element

$$c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \text{BGL}^{2,1}(\mathbf{P}^\infty,\infty).$$

For a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in \text{BGL}^{2,1}(X)$, where $f_L \in \text{Hom}_{\mathbf{H}_\bullet(S)}(X_+,\mathbf{P}^\infty)$ is the morphism corresponding to the class $[L]$ of L in the group $\text{Pic}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $\text{BGL}^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in \text{BGL}^{2,1}(\mathbf{P}^1)$ by (12). With that Chern structure, $\text{BGL}^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1].

If the Thomason-Trobaugh K-theory is equipped with the Chern structure given by $L/X \mapsto [\mathcal{O}] - [L^\vee] \in K_0^{\text{TT}}(X)$, then the isomorphism (16) of the ring cohomology theories respects the orientations.

1.5 Uniqueness of BGL

We prove in this Section that over $S = \text{Spec}(\mathbb{Z})$ the spectrum BGL is essentially unique regarded as an object in the stable homotopy category $\text{SH}(S)$. This has also been obtained in [R]. By [MV, Thm. 4.3.13] and Corollary B.1.2 there exists a unique morphism

$$\epsilon: (\mathbb{Z} \times \text{Gr}) \wedge \mathbf{P}^1 \to \mathbb{Z} \times \text{Gr}$$

in $H_\bullet(S)$ such that the composite morphism $(\mathbb{Z} \times \mathrm{Gr}) \times \mathbf{P}^1 \to (\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1 \xrightarrow{\epsilon} \mathbb{Z} \times \mathrm{Gr}$ represents the element $(\tau_\infty - \infty) \otimes \big([\mathcal{O}(-1)] - [\mathcal{O}]\big) \in K^0\big((\mathbb{Z} \times \mathrm{Gr}) \times \mathbf{P}^1\big)$. Let

$$\mathbb{Z} \times \mathrm{Gr} \xrightarrow{i} \mathcal{K}$$

be a motivic weak equivalence and let $e \colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ be a morphism in $\mathbf{M}_\bullet(S)$ such that the diagram

$$
\begin{array}{ccc}
(\mathbb{Z} \times \mathrm{Gr}) \wedge \mathbf{P}^1 & \xrightarrow{\ \epsilon\ } & \mathbb{Z} \times \mathrm{Gr} \\
{\scriptstyle i \wedge \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle i} \\
\mathcal{K} \wedge \mathbf{P}^1 & \xrightarrow{\ \ e\ \ } & \mathcal{K}
\end{array}
$$

commutes in $H_\bullet^{\mathrm{cm}}(S)$. Let $i' \colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}'$ be another motivic weak equivalence and let $e' \colon \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$ be a morphism in $\mathbf{M}_\bullet(S)$ such that $e' \circ (i' \wedge \mathrm{id}) = i' \circ \epsilon$ in $H_\bullet^{\mathrm{cm}}(S)$.

Definition 1.5.1. *Define* BGL *as the* $\mathbf{P}^1$*-spectrum of the form* $(\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \dots)$ *with* $\mathcal{K}_i = \mathcal{K}$ *for all i, and with the structure maps* $e_i \colon \mathcal{K}_i \wedge \mathbf{P}^1 \to \mathcal{K}_{i+1}$ *equal to the map* $\epsilon \colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$. *Define* BGL$'$ *in the same way using* $\mathcal{K}'$ *and* e'.

Proposition 1.5.2. *There exists a unique morphism* $\theta \colon \mathrm{BGL} \to \mathrm{BGL}'$ *in* $\mathrm{SH}(S)$ *such that for every integer $i \geq 0$ the diagram*

$$
\begin{array}{ccc}
\Sigma_{\mathbf{P}^1}^\infty \mathcal{K}_i(-i) & \xrightarrow{\ u_i\ } & \mathrm{BGL} \\
{\scriptstyle \Sigma_{\mathbf{P}^1}^\infty \phi_i(-i)}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\
\Sigma_{\mathbf{P}^1}^\infty \mathcal{K}_i'(-i) & \xrightarrow{\ u_i'\ } & \mathrm{BGL}'
\end{array}
$$

commutes, where $\phi_i = i' \circ i^{-1} \in [\mathcal{K}_i, \mathcal{K}_i']_{H_\bullet(S)}$ *and* u_i, u_i' *are the canonical morphisms. Similarly, there exists a unique morphism* $\theta' \colon \mathrm{BGL}' \to \mathrm{BGL}$ *in* $\mathrm{SH}(S)$ *such that for every integer $i \geq 0$ the diagram*

$$
\begin{array}{ccc}
\Sigma_{\mathbf{P}^1}^\infty \mathcal{K}_i'(-i) & \xrightarrow{\ u_i'\ } & \mathrm{BGL}' \\
{\scriptstyle \Sigma_{\mathbf{P}^1}^\infty \phi_i'(-i)}\big\downarrow & & \big\downarrow{\scriptstyle \theta'} \\
\Sigma_{\mathbf{P}^1}^\infty \mathcal{K}_i(-i) & \xrightarrow{\ u_i\ } & \mathrm{BGL}
\end{array}
$$

commutes, where $\theta_i = i \circ (i')^{-1} \in [\mathcal{K}_i', \mathcal{K}_i]_{H_\bullet(S)}$.

Proof. Consider the exact sequence

$$0 \to {\varprojlim}^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i') \to \mathrm{BGL}^{0,0}(\mathrm{BGL}') \to \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}_i') \to 0$$

from Lemma A.5.10. The family of elements $(u_i \circ \Sigma_{\mathbf{P}^1}^\infty \theta_i'(-i))$ is an element of the group $\varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}_i')$. Thus there exists the required morphism θ'. To prove its uniqueness, observe that the $\varprojlim^1$-group vanishes by Proposition 1.7.1. Whence $\mathrm{BGL}^{0,0}(\mathrm{BGL}') = \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}_i')$ and θ' is indeed unique. By symmetry there also exists a unique morphism θ with the required property. $\square$

Proposition 1.5.3. *The morphism* $\theta\colon \mathrm{BGL} \to \mathrm{BGL}'$ *is the inverse of* $\theta'\colon \mathrm{BGL}' \to \mathrm{BGL}$ *in* $\mathrm{SH}(S)$, *and in particular an isomorphism.*

Proof. The composite morphism $\theta' \circ \theta\colon \mathrm{BGL} \to \mathrm{BGL}$ has the property that for every integer $i \geq 0$ the diagram

$$
\begin{array}{ccc}
\Sigma^{\infty}_{\mathbf{P}^1} \mathcal{K}_i(-i) & \xrightarrow{\ u_i\ } & \mathrm{BGL} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \theta'\circ\theta} \\
\Sigma^{\infty}_{\mathbf{P}^1} \mathcal{K}_i(-i) & \xrightarrow{\ u_i\ } & \mathrm{BGL}'
\end{array}
$$

commutes. However, the identity morphism $\mathrm{id}\colon \mathrm{BGL} \to \mathrm{BGL}$ has the same property. Thus $\theta' \circ \theta = \mathrm{id}$. by the uniqueness in Proposition 1.5.2, and similarly $\theta \circ \theta' = \mathrm{id}$. $\qquad\square$

Remark 1.5.4. The isomorphisms θ and θ' are monoid isomorphisms provided that BGL and BGL' are equipped with the monoidal structures given by Theorem 2.2.1.

1.6 Preliminary computations I

In this section we prepare for the next section, in which we show that certain $\varprojlim^1$-groups vanish. Let BGL be the $\mathbf{P}^1$-spectrum defined in 1.2.5. We will identify in this section the functors $\mathrm{BGL}^{0,0}$ and $\mathrm{BGL}^{2i,i}$ on the category $\mathrm{H}_{\bullet}(S)$ as follows:

$$
\mathrm{BGL}^{0,0}(A) \cong \mathrm{Hom}_{\mathrm{H}_{\bullet}(S)}(A, \mathcal{K}_0) = \mathrm{Hom}_{\mathrm{H}_{\bullet}(S)}(A, \mathcal{K}_0) \cong \mathrm{BGL}^{2i,i}(A). \tag{17}
$$

Similarly,

$$
\mathrm{BGL}^{-1,0}(A) \cong \mathrm{Hom}_{\mathrm{H}_{\bullet}(S)}(S^{1,0} \wedge A, \mathcal{K}_0) = \mathrm{Hom}_{\mathrm{H}_{\bullet}(S)}(S^{1,0} \wedge A, \mathcal{K}_i) \cong \mathrm{BGL}^{2i-1,i}(A). \tag{18}
$$

These identifications respect the naive product structure (10) on the functor $\mathrm{BGL}^{*,*}$. In particular, the following diagram commutes for every pointed motivic space A over S.

$$
\begin{array}{ccc}
\mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{0,0}(A) & \longrightarrow & \mathrm{BGL}^{-1,0}(A) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathrm{BGL}^{-1,0}(S) \otimes \mathrm{BGL}^{2i,i}(A) & \longrightarrow & \mathrm{BGL}^{2i-1,i}(A)
\end{array} \tag{19}
$$

Lemma 1.6.1. *Let S be a regular scheme. For every integer i the map*

$$
\mathrm{BGL}^{-1,0}(S) \otimes_{\mathrm{BGL}^{0,0}(S)} \mathrm{BGL}^{2i,i}(\mathcal{K}) \to \mathrm{BGL}^{2i-1,i}(\mathcal{K})
$$

induced by the naive product structure is an isomorphism. The same holds if $\mathcal{K} \wedge \mathbf{P}^1$ replaces $\mathcal{K}$.

Proof. The commutativity of the diagram (19) shows that it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \mathrm{Gr}$ since the map $i\colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K} = \mathcal{K}$ is a weak equivalence. The functor isomorphism $K_*^{\mathrm{TT}} \to \mathrm{BGL}^{-*,0}$

is a ring cohomology isomorphism by Corollary 1.2.8. Thus it remains to check that the map

$$K_1^{\mathrm{TT}}(S) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(\mathrm{Gr}) \to K_1^{\mathrm{TT}}(\mathrm{Gr})$$

is an isomorphism. This is well-known. See Lemma B.1.4) for a proof. The assertion concerning $\mathcal{K} \wedge \mathbf{P}^1$ is proved similarly using Lemma B.1.3 instead. $\qquad\square$

To state the next lemma, consider the scheme morphism $f\colon S = \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{C})$, the pull-back functor $f^*\colon \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}(\mathbb{C})$ described in Proposition A.7.4, and the topological realization functor $\mathbf{R}_{\mathbb{C}}\colon \mathrm{SH}(\mathbb{C}) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$ described in Section A.7. Set $r = \mathbf{R}_{\mathbb{C}} \circ f^*\colon \mathrm{SH}(S) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$. The functor r will be called for short the realization functor below in this Section.

Lemma 1.6.2. *Let $\mathbb{B}\mathrm{U}$ be the periodic complex K-theory $\mathbb{C}\mathbf{P}^1$-spectrum with terms $\mathbb{Z}\times\mathrm{BU}$. There is a termwise zigzag weak equivalence $\mathbb{B}\mathrm{U} \leftarrow E \to r\mathrm{BGL}$ of $\mathbb{C}\mathbf{P}^1$-spectra.*

Proof. This follows from A.7.3 and the fact that Grassmann varieties pull back. $\qquad\square$

Lemma 1.6.3. *Let $S = \mathrm{Spec}(\mathbb{Z})$ and let $r\colon \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}_{\mathbb{C}\mathbf{P}^1}$ be the topological realization functor. Then for every integer i the realization homomorphism $\mathrm{BGL}^{2i,i}(\mathcal{K}) \to (r\mathrm{BGL})^{2i}(r\mathcal{K})$ is an isomorphism.*

Proof. Clearly it suffices to prove the case $i = 0$. We may replace the pointed motivic space $\mathcal{K}_i$ with $\mathbb{Z} \times \mathrm{Gr}$ as in the proof of Lemma 1.6.1. It remains to check that the topological realization homomorphism $\mathrm{BGL}^{0,0}(\mathrm{Gr}) \to (r\mathrm{BGL})^0(\mathrm{Gr})$ is an isomorphism.

Since $\mathrm{Gr}(n, 2n)$ is a smooth cellular S-scheme, Lemma 1.6.4 below implies that the map $\mathrm{BGL}^{0,0}(\mathrm{Gr}(n, 2n)) \to (r\mathrm{BGL})^0(\mathrm{Gr}(n, 2n))$ is an isomorphism for every n. To conclude the statement for $\mathrm{Gr} = \cup\mathrm{Gr}(n,2n)$, use the short exact sequence from Lemma A.5.10. In the resulting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim{}^1 \mathrm{BGL}^{-1,0}\big(\mathrm{Gr}(n,2n)\big) & \longrightarrow & \mathrm{BGL}^{0,0}(\mathrm{Gr}) & \longrightarrow & \varprojlim \mathrm{BGL}^{0,0}\big(\mathrm{Gr}(n,2n)\big) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim{}^1 r\mathrm{BGL}^{-1,0}\big(r\mathrm{Gr}(n,2n)\big) & \longrightarrow & r\mathrm{BGL}^{0,0}(r\mathrm{Gr}) & \longrightarrow & \varprojlim r\mathrm{BGL}^{0,0}\big(r\mathrm{Gr}(n,2n)\big) & \longrightarrow & 0
\end{array}
$$

the map on the right hand side is then an isomorphism. Furthermore one concludes from [Sw, Thm. 16.32] that $\varprojlim{}^1 r\mathrm{BGL}^{-1,0}\big(r\mathrm{Gr}(n,2n)\big) = \varprojlim{}^1 K^1_{\mathrm{top}}\big(r\mathrm{Gr}(n,2n)\big) = 0$. On the other hand $\varprojlim{}^1 \mathrm{BGL}^{-1,0}\big(\mathrm{Gr}(n,2n)\big) = 0$, as we already mentioned in the proof of 1.2.1. The result follows. $\qquad\square$

Lemma 1.6.4. *Let $X \in \mathcal{S}m/S$, where $S = \mathrm{Spec}(\mathbb{Z})$, and let $X_0 \subset X_1 \subset \cdots \subset X_n = X$ be a filtration by closed subsets such that for every integer $i \geq 0$ the S-scheme $X_i - X_{i-1}$ is isomorphic to a disjoint union of several copies of the affine space $\mathbf{A}^i_S$. The map $\mathrm{BGL}^{0,0}(X) \to (r\mathrm{BGL})^0(rX)$ is an isomorphism.*

Proof. Consider the class $\mathcal{R}$ of $\mathbf{P}^1$-spectra E such that $\mathrm{BGL}^{0,0}(E) \to r\mathrm{BGL}^0(rE)$ is an isomorphism. It contains $S^{0,0}$ because in this case we obtain the isomorphism $\mathrm{BGL}^{0,0}(S^{0,0}) \cong K^0(\mathbb{Z}) \cong \mathbb{Z} \cong K^0_{\mathrm{top}}(S^0$ which identifies the class of an algebraic resp. complex topological vector bundle over $\mathrm{Spec}(\mathbb{Z})$ resp. $\bullet$ with its rank. The Bott periodicity isomorphisms for BGL and rBGL which are compatible by A.7.3 imply that $S^{2m,m} \in \mathcal{R}$ for all $m \in \mathbb{Z}$. Finally, if $E \to F \to G \to S^{1,0} \wedge E$ is a distinguished triangle in $\mathrm{SH}(S)$ such that E and G are in $\mathcal{R}$, then so is F.

For $i \geq 0$ write $U^i := X \smallsetminus X_i$, so that U^i is an open subset of U^{i-1}. In particular we have $U^n = \emptyset$ and $U^{-1} = X$. The closed subscheme $X_i \smallsetminus X_{i-1} = X_i \cap U^{i-1} \hookrightarrow U^{i-1}$ is isomorphic to a disjoint union m_i copies of affine spaces $\mathbf{A}^i$, and is in particular smooth over S. Furthermore the normal bundle is trivial. The homotopy purity theorem [MV, 3.2.29] supplies a distinguished triangle

$$\Sigma^\infty_{\mathbf{P}^1} U^i_+ \to \Sigma^\infty_{\mathbf{P}^1} U^{i-1}_+ \to \Sigma^\infty_{\mathbf{P}^1} U^{i-1}/U^i \cong \vee^{m_i}_{j=1} S^{2(n-i),(n-i)}$$

of $\mathbf{P}^1$-spectra. Since $\mathcal{R}$ contains $\Sigma^\infty_{\mathbf{P}^1} U^n = \bullet$ we obtain inductively that $\mathcal{R}$ contains $\Sigma^\infty_{\mathbf{P}^1} U^{-1} = \Sigma^\infty_{\mathbf{P}^1} X_+$. $\qquad\square$

Lemma 1.6.5. *Let $\mathbb{B}^0\mathrm{U}$ be the sub-spectrum of $\mathbb{B}\mathrm{U}$ with the n-th term equal to the connected component BU of the topological space $\mathbb{Z} \times \mathrm{BU}$ containing the basepoint $\bullet$. The inclusion $\mathbb{B}^0\mathrm{U} \to \mathbb{B}\mathrm{U}$ is a weak equivalence of $\mathbb{C}\mathbf{P}^1$-spectra.*

Proof. One has to check that the inclusion induces an isomorphism on stable homotopy groups. This follows because the structure map $(\mathbb{Z} \times BU) \wedge \mathbb{C}\mathbf{P}^1 \to \mathbb{Z} \times BU$ factors over $\{0\} \times BU$. $\qquad\square$

Lemma 1.6.6. *There exists a sub-spectrum $\mathbb{B}^f\mathrm{U}$ of the $\mathbb{C}\mathbf{P}^1$-spectrum $\mathbb{B}\mathrm{U}$ with the n-th term $\mathrm{Gr}\big(b(n), 2b(n)\big)$ such that the inclusion $\mathbb{B}^f\mathrm{U} \to \mathbb{B}\mathrm{U}$ is a stable equivalence.*

Proof. The sequence $b(n)$ will be constructed such that $b(n) \geq 2n+1$. Set $b(0) = 1$. We may assume that the structure map $e_0 \colon \mathrm{BU} \wedge \mathbb{C}\mathbf{P}^1 \to \mathrm{BU}$ is cellular. Since $r\mathrm{Gr}(b(0), 2b(0)) \wedge \mathbb{C}\mathbf{P}^1$ is a finite cell complex, it lands in a Grassmannian $r\mathrm{Gr}(b(1), 2b(1))$ for some integer $b(1) \geq 2 \cdot 1 + 1$. Continuing this process produces the required sequence of $b(n)$'s. The inclusions induce an isomorphism $\mathrm{colim}_{n \geq 0} \mathrm{Gr}\big(b(n), 2b(n)\big) \cong \mathrm{Gr}$.

To observe that the inclusion $j \colon \mathbb{B}^f\mathrm{U} \to \mathbb{B}\mathrm{U}$ is then a stable equivalence, recall that the number of $2k$-cells in $\mathrm{Gr}(n,m)$ is given by the number of partitions of k into at most n each of which is $\geq m - n$. In particular, the $2k$-skeleton of BU coincides with the $2k$-skeleton of $r\mathrm{Gr}(k, 2k)$. To prove the surjectivity of $\pi_i(j)$ choose an element $\alpha \in \pi_i\mathbb{B}\mathrm{U}$. It is represented by a cellular map $a \colon S^{i+2m} \to \mathrm{BU}$ for some m with $i + 2m \geq 0$. We may choose m such that $m \geq i$. Thus a lands in $r\mathrm{Gr}(b(m), 2b(m))$ and gives rise to an element in $\pi_i\mathbb{B}^f\mathrm{U}$ mapping to α. To prove the injectivity of $\pi_i(j)$, choose an element $\alpha \in \pi_i\mathbb{B}^f\mathrm{U}$ such that $\pi_i(j)(\alpha) = 0$. We may represent α by some map $a \colon \pi_{i+2m}(j)\mathrm{Gr}(b(m), 2b(m))$ for some m with $i + 2m > 0$ and $m \geq i$. The composition

$$S^{i+2m} \to a\mathrm{Gr}(b(m), 2b(m)) \hookrightarrow \mathrm{BU}$$

is nullhomotopic since $\pi_{i+2m}\mathbb{B}U \cong \pi_i\mathbb{B}U$ via the homomorphism induced by the structure map. The nullhomotopy may be chosen to be cellular and thus lands in $\mathrm{Gr}(b(m), 2b(m))$. This completes the proof. $\qquad\square$

1.7 Vanishing of certain groups I

Consider the stable equivalence $\mathrm{hocolim}_{i\geq 0}\, \Sigma^\infty_{\mathbf{P}^1}\mathcal{K}_i(-i) \cong \mathrm{BGL}$ (see (21)) and the respecting short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) \to \mathrm{BGL}^{0,0}(\mathrm{BGL}) \to \varprojlim \mathrm{BGL}^{2i,i}(\mathcal{K}_i) \to 0$$

We prove in this section the following result

Proposition 1.7.1. *Let $S = \mathrm{Spec}(\mathbb{Z})$, then $\varprojlim{}^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) = 0$.*

Proof. The connecting homomorphism in the tower of groups for the $\varprojlim{}^1$-term is the composite map

$$\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) \xleftarrow{\Sigma^{-1}_{\mathbf{P}^1}} \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_i \wedge \mathbf{P}^1) \xleftarrow{e_i^*} \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_{i+1})$$

where $\Sigma^{-1}_{\mathbf{P}^1}$ is the inverse to the $\mathbf{P}^1$-suspension isomorphism and e_i^* is the pull-back induced by the structure map e_i. Set $A = \mathrm{BGL}^{-1,0}(S)$ and consider the diagram

$$\mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) \xleftarrow{\Sigma^{-1}_{\mathbf{P}^1}} \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_i \wedge \mathbf{P}^1) \xleftarrow{e_i^*} \mathrm{BGL}^{2i+1,i+1}(\mathcal{K}_{i+1})$$
$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$
$$A \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}_i) \xleftarrow{\mathrm{id}\otimes\Sigma^{-1}_{\mathbf{P}^1}} A \otimes \mathrm{BGL}^{2(i+1),i+1}(\mathcal{K}_i \wedge \mathbf{P}^1) \xleftarrow{\mathrm{id}\otimes e_i^*} A \otimes \mathrm{BGL}^{2(i+1),i+1}(\mathcal{K}_{i+1})$$

where the vertical arrows are induced by the naive product structure on the functor $\mathrm{BGL}^{*,*}$. Clearly it commutes. Since S is regular, the vertical arrows are isomorphisms by Lemma 1.6.1. It follows that $\varprojlim{}^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i) = \varprojlim{}^1 (A \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}_i))$ where in the last tower of groups the connecting maps are $\mathrm{id} \otimes (\Sigma^{-1}_{\mathbf{P}^1} \circ e_i^*)$. It remains to prove the following assertion.

Claim. $\varprojlim{}^1 (A \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}_i)) = 0$

Since $S = \mathrm{Spec}(\mathbb{Z})$ one gets $A = \mathrm{BGL}^{-1,0}(S) = K_1^{\mathrm{TT}}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Thus $A \otimes \mathrm{BGL}^{2i,i}(\mathcal{K}_i) = \mathrm{BGL}^{2i,i}(\mathcal{K}_i)/m\mathrm{BGL}^{2i,i}(\mathcal{K}_i)$ with $m = 2$ and the connecting maps in the tower are just the mod-m reduction of the maps $\Sigma^{-1}_{\mathbf{P}^1} \circ e_i^*$. Now a chain of isomorphisms completes the proof of the Claim.

$$\varprojlim{}^1 \mathrm{BGL}^{2i,i}(\mathcal{K}_i)/m \cong \varprojlim{}^1 (r\mathrm{BGL})^{2i}(r\mathcal{K}_i)/m \cong \varprojlim{}^1 \mathbb{B}U^{2i}(r\mathcal{K}_i)/m$$
$$\cong \varprojlim{}^1 \mathbb{B}U^{2i}(\mathbb{Z} \times BU)/m \cong \varprojlim{}^1 \mathbb{B}U^{2i}(\mathbb{Z} \times BU; \mathbb{Z}/m)$$
$$\cong K^1_{\mathrm{top}}(\mathbb{B}U; \mathbb{Z}/m) \cong K^1_{\mathrm{top}}(\mathbb{B}^0U; \mathbb{Z}/m)$$
$$\cong \varprojlim{}^1 K^{2i}_{\mathrm{top}}\big(\mathrm{Gr}(b(i), 2b(i)); \mathbb{Z}/m\big) = 0$$

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The first isomorphism follows from Lemma 1.6.3. The second isomorphism is induced by the levelwise weak equivalence $\mathbb{B}U \simeq r\mathrm{BGL}$ mentioned in Lemma 1.6.2. The third isomorphism is induced by the image of the weak equivalence $\mathcal{K}_i \simeq \mathbb{Z} \times \mathrm{Gr}$ under topological realization. The forth and fifth isomorphism hold since $\mathbb{B}U^{2i+1}(\mathbb{Z} \times \mathrm{BU}) = 0$. The sixth isomorphism is induced by the stable equivalence $\mathbb{B}^0U \simeq \mathbb{B}U$ from Lemma 1.6.5, the seventh one is induced by the stable equivalence $\mathbb{B}^fU \simeq \mathbb{B}^0U$ from Lemma 1.6.6. The last one holds since all groups in the tower are finite. $\square$

2 Smash-product, pull-backs, topological realization

In this section we construct a smash-product $\wedge$ of $\mathbf{P}^1$-spectra, check its basic properties, consider its behavior with respect to pull-back and realization functors. We follow here an idea of Voevodsky [V1, Comments to Thm. 5.6] and use results of Jardine [J].

2.1 The smash product

Definition 2.1.1. *Let* $V \colon \mathrm{SH}(S) \to \mathrm{SH}^{\Sigma}(S)$ *and* $U \colon \mathrm{SH}^{\Sigma}(S) \to \mathrm{SH}(S)$ *be the equivalence described in Theorem A.6.4. For a pair of* $\mathbf{P}^1$-*spectra* E *and* F *set*

$$E \wedge F := U(VE \wedge VF)$$

as in Remark A.6.5.

Proposition 2.1.2. *Let* S *be a Noetherian finite-dimensional base scheme. The smash-product of* $\mathbf{P}^1$-*spectra over* S *induces a closed symmetric monoidal structure* $(\wedge, \mathbb{I})$ *on the motivic stable homotopy category* $\mathrm{SH}(S)$ *having the properties required by Theorem 5.6 of Voevodsky congress talk [V1]:*

1. *There is a canonical isomorphism* $E \wedge \Sigma^{\infty}_{\mathbf{P}^1} A \cong (E_i \wedge A, e_i \wedge \mathrm{id})$ *for every pointed motivic space* A *and every* $\mathbf{P}^1$-*spectrum.*

2. *There is a canonical isomorphism* $(\oplus E_{\alpha}) \wedge F \cong \oplus(E_{\alpha} \wedge F)$ *for* $\mathbf{P}^1$-*spectra* E_i, F.

3. *Smashing with a* $\mathbf{P}^1$-*spectrum preserves distinguished triangles. To be more precise, if* $E \xrightarrow{f} F \to \mathrm{cone}(f) \xrightarrow{\epsilon} E[1]$ *is a distinguished triangle and* G *is a* $\mathbf{P}^1$-*spectrum, the sequence* $E \wedge G \xrightarrow{f} F \wedge G \to \mathrm{cone}(f) \wedge G \xrightarrow{\epsilon} E \wedge G[1]$ *is a distinguished triangle, where the last morphism is the composition of* $\epsilon \wedge \mathrm{id}_G$ *with the canonical isomorphism* $E[1] \wedge G \to (E \wedge F)[1]$.

Proof. Follows from Remark A.6.5 and Theorem A.6.4. $\square$

Lemma 2.1.3. *Let* $E = \mathrm{hocolim}_{i \geq 0} E_i$ *is a sequential homotopy colimit of* $\mathbf{P}^1$-*spectra. For every* $\mathbf{P}^1$-*spectrum* F *there is an exact sequence of abelian groups*

$$0 \to \varprojlim{}^1 F^{p-1,q}(E_i) \to F^{p,q}(E) \to \varprojlim F^{p,q}(E_i) \to 0. \tag{20}$$

Proof. This is Lemma A.5.10. □

By Lemma A.5.9, any $\mathbf{P}^1$-spectrum E can be expressed as the homotopy colimit

$$\text{hocolim } \Sigma^\infty_{\mathbf{P}^1} E_i(-i) \cong E. \tag{21}$$

Corollary 2.1.4. *For two $\mathbf{P}^1$-spectra E and F there is a canonical short exact sequence*

$$0 \to \varprojlim{}^1 F^{p+2i-1,q+i}(E_i) \to F^{p,q}(E) \to \varprojlim F^{p+2i,q+i}(E_i) \to 0. \tag{22}$$

Corollary 2.1.5. *For a pair of spectra E and F and each spectrum G one has a canonical exact sequence of the form*

$$0 \to \varprojlim{}^1 G^{p+4i-1,q+2i}(E_i \wedge F_i) \to G^{p,q}(E \wedge F) \to \varprojlim G^{p+4i,q+2i}(E_i \wedge E_i) \to 0. \tag{23}$$

Proof. For a pair of spectra E and F one has a canonical isomorphism of the form

$$\text{hocolim } (\Sigma^\infty_{\mathbf{P}^1}(E_i \wedge F_i)(-2i)) \cong E \wedge F \tag{24}$$

as deduced in Lemma A.6.8. The result follows from Corollary 2.1.4. □

2.2 A monoidal structure on BGL

For a $\mathbf{P}^1$-spectrum E and an integer $i \geq 0$ $u_i \colon \Sigma^\infty_{\mathbf{P}^1} E_i(-i) \to E$ denotes the canonical map. The aim of this section is to prove the following

Theorem 2.2.1. *Consider the family of pairings $\mathcal{K}_i \wedge \mathcal{K}_j \xrightarrow{m_{ij}} \mathcal{K}_{i+j}$ with $m_{ij} = \mu^{\mathcal{K}}$. For $S = \text{Spec}(\mathbb{Z})$ there is a unique morphism $\mu_{\text{BGL}} \colon \text{BGL} \wedge \text{BGL} \to \text{BGL}$ in the motivic stable homotopy category $\text{SH}(S)$ such that for every i the diagram*

$$
\begin{array}{ccc}
\Sigma^\infty_{\mathbf{P}^1} \mathcal{K}_i(-i) \wedge \Sigma^\infty_{\mathbf{P}^1} \mathcal{K}_i(-i) & \xrightarrow{\Sigma^\infty(m_{ii})} & \Sigma^\infty_{\mathbf{P}^1} \mathcal{K}_{2i}(-2i) \\
\downarrow{\scriptstyle u_i \wedge u_i} & & \downarrow{\scriptstyle u_{2i}} \\
\text{BGL} \wedge \text{BGL} & \xrightarrow{\mu_{\text{BGL}}} & \text{BGL}
\end{array}
$$

commutes. Let $e_{\text{BGL}} \colon \mathbb{I} \to \text{BGL}$ in $\text{SH}(S)$ be adjoint to the unit $e_{\mathcal{K}} \colon S^{0,0} \to \mathcal{K}$. Then

$$(\text{BGL}, \mu_{\text{BGL}}, e_{\text{BGL}})$$

is a commutative monoid in $\text{SH}(S)$.

Proof. The morphism μ_{BGL} we are looking for is an element of the group $\text{BGL}^{0,0}(\text{BGL} \wedge \text{BGL})$. This group fits in the exact sequence

$$0 \to \varprojlim{}^1 \text{BGL}^{4i-1,2i}(\mathcal{K}_i \wedge \mathcal{K}_i) \to \text{BGL}^{0,0}(\text{BGL} \wedge \text{BGL}) \to \varprojlim \text{BGL}^{4i,2i}(\mathcal{K}_i \wedge \mathcal{K}_i) \to 0$$

by Corollary 2.1.5. The family of elements $\{u_{2i} \circ \Sigma^\infty(m_{ii})\}$ is an element of the $\varprojlim$ group. The $\varprojlim^1$ group vanishes by Proposition 2.4.1 below, whence there exist a unique element

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μ_{BGL} whose image in the $\varprojlim$ group coincides with the element $\{u_{2i} \circ \Sigma^\infty(m_{ii})\}$. That morphism μ_{BGL} is the required one.

In fact, the identities $u_{2i} \circ \Sigma^\infty(m_{ii}) = \mu_{\mathrm{BGL}} \circ (u_i \wedge u_i)$ hold by the very construction of μ_{BGL}. The operation μ_{BGL} is associative because the group $\varprojlim^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathcal{K}_i)$ vanishes by Proposition 2.5.1). That μ_{BGL} is commutative follows from the vanishing of the group $\varprojlim^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}_i \wedge \mathcal{K}_i)$ (see Proposition 2.4.1). The fact that e_{BGL} is a two-sided unit for the multiplication μ_{BGL} follows by Proposition 1.7.1, which shows that the group $\varprojlim^1 \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i)$ vanishes. $\qquad\square$

Definition 2.2.2. *Let S be a regular scheme, with structural morphism $f\colon S \to \mathrm{Spec}(\mathbb{Z})$. Let $f^*\colon \mathrm{SH}(\mathbb{Z}) \to \mathrm{SH}(S)$ be the strict symmetric monoidal pull-back functor from A.7.4. Set*

$$\mu_{\mathrm{BGL}}^S := f^*(\mathrm{BGL}) \wedge f^*(\mathrm{BGL}) \xrightarrow{\mathrm{can}} f^*(\mathrm{BGL} \wedge \mathrm{BGL}) \xrightarrow{f^*(\mu_{\mathrm{BGL}})} f^*(\mathrm{BGL})$$

$$e_{\mathrm{BGL}}^S := S^0 \xrightarrow{\mathrm{can}} f^*(S^0) \xrightarrow{f^*(e_{\mathrm{BGL}})} f^*(\mathrm{BGL})$$

and $\mathrm{BGL}^S = f^*(\mathrm{BGL})$. Then $(\mathrm{BGL}^S, \mu_{\mathrm{BGL}}^S, e_{\mathrm{BGL}}^S)$ is a commutative monoid in $\mathrm{SH}(S)$.

We will sometimes refer to a monoid in $\mathrm{SH}(S)$ as a $\mathbf{P}^1$-*ring spectrum*.

Corollary 2.2.3. *The multiplicative structure on the functor $\mathrm{BGL}^{*,*}$ induced by the pairing μ_{BGL}^S and the unit e_{BGL}^S coincides with the naive product structure (10).*

Proof. Follows from Theorem 2.2.1. $\qquad\square$

Corollary 2.2.4. *The functor isomorphism $[X, \mathcal{K}_0] \to [\Sigma_{\mathbf{P}^1}^\infty(X), \mathrm{BGL}]$ respects the multiplicative structures on both sides. The isomorphism $Ad\colon K_*^{\mathrm{TT}} \to \mathrm{BGL}^{-*,0}$ of cohomology theories on $\mathcal{S}m\mathcal{O}p/S$ is an isomorphism of ring cohomology theories in the sense of [PS1].*

Proof. Follows from Theorem 2.2.1. $\qquad\square$

2.3 Preliminary computations II

We will identify in this section the functors $\mathrm{BGL}^{0,0}$ and $\mathrm{BGL}^{2i,i}$, $\mathrm{BGL}^{-1,0}$ and $\mathrm{BGL}^{2i-1,i}$ on the motivic unstable category $\mathrm{H}_\bullet(S)$ as in Section 1.6.

Lemma 2.3.1. *Let S be a regular base scheme. For every integer i the map*

$$\mathrm{BGL}^{-1,0}(S) \otimes_{\mathrm{BGL}^{0,0}(S)} \mathrm{BGL}^{2i,i}(\mathcal{K}_i \wedge \mathcal{K}_i) \to \mathrm{BGL}^{2i-1,i}(\mathcal{K}_i \wedge \mathcal{K}_i)$$

induced by the naive product structure is an isomorphism. The same holds if we replace $\mathcal{K}_i \wedge \mathcal{K}_i$ by $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1$.

Proof. Since diagram (19) commutes, it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}_i$ with $\mathbb{Z} \times \mathrm{Gr}$ since the map $i\colon \mathbb{Z} \times \mathrm{Gr} \to \mathcal{K}_i = \mathcal{K}$ is

a motivic weak equivalence. The functor isomorphism $\mathbb{K}_*^{\mathrm{TT}} \to \mathrm{BGL}^{-*,0}$ is an isomorphism of ring cohomology theories. Thus it remains to check that the map

$$K_1^{\mathrm{TT}}(S) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(\mathrm{Gr} \wedge \mathrm{Gr}) \to K_1^{\mathrm{TT}}(\mathrm{Gr} \wedge \mathrm{Gr})$$

is an isomorphism. This holds by Lemma B.1.3. The case of $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbf{P}^1$ is proved by the same arguments. $\qquad\square$

Lemma 2.3.2. *Let $S = \mathrm{Spec}(\mathbb{Z})$ and let $r \colon \mathrm{SH}(S) \to \mathrm{SH}_{\mathbb{CP}^1}$ be the topological realization functor. Then for every integer i the homomorphism $\mathrm{BGL}^{2i,i}(\mathcal{K} \wedge \mathcal{K}) \to (r\mathrm{BGL})^{2i}\big(r(\mathcal{K} \wedge \mathcal{K})\big) \cong (r\mathrm{BGL})^{2i}(r\mathcal{K} \wedge r\mathcal{K})$ is an isomorphism.*

Proof. Since the Bott periodicity isomorphisms in the algebraic and complex topological setting correspond by A.7.3, it suffices to consider the case $i = 0$. We may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \mathrm{Gr}$ as in the proof of Lemma 1.6.1. It remains to check that the realization homomorphism

$$\mathrm{BGL}^{0,0}(\mathrm{Gr} \wedge \mathrm{Gr}) \to (r\mathrm{BGL})^0(r(\mathrm{Gr} \wedge \mathrm{Gr})) = (r\mathrm{BGL})^0(r(\mathrm{Gr}) \wedge r(\mathrm{Gr})))$$

is an isomorphism. By Example A.5.8 $\Sigma_{\mathbf{P}^1}^\infty \mathrm{Gr} \wedge \mathrm{Gr}$ is a retract of $\Sigma_{\mathbf{P}^1}^\infty \mathrm{Gr} \times \mathrm{Gr}$ in $\mathrm{SH}(S)$, whence it suffices to consider the topological realization homomorphism for $\mathrm{Gr} \times \mathrm{Gr}$. Since the latter is an increasing union of the cellular S-schemes $\mathrm{Gr}(n, 2n) \times \mathrm{Gr}(m, 2m)$, the result follows with the help of Lemma 1.6.4 as in the proof of Lemma 1.6.3. $\qquad\square$

2.4 Vanishing of certain groups II

Consider the stable equivalence hocolim $\Sigma_{\mathbf{P}^1}^\infty(\mathcal{K} \wedge \mathcal{K})(-2i) \cong \mathrm{BGL} \wedge \mathrm{BGL}$ displayed in (24) and the corresponding short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) \to \mathrm{BGL}^{0,0}(\mathrm{BGL} \wedge \mathrm{BGL}) \to \varprojlim \mathrm{BGL}^{4i,2i}(\mathcal{K} \wedge \mathcal{K}) \to 0$$

from Corollary 2.1.5. We prove in this section the following result

Proposition 2.4.1. *If $S = \mathrm{Spec}(\mathbb{Z})$ then $\varprojlim{}^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) = 0$.*

Proof. For a pointed motivic space A we abbreviate $A \wedge A$ as $A^{\wedge 2}$. The connecting homomorphism in the tower of groups forming the $\varprojlim^1$-term is the composite map

$$\mathrm{BGL}^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) \xleftarrow{(\Sigma \circ \Sigma)^{-1} \circ \mathrm{tw}} \mathrm{BGL}^{4(i+1)-1,2(i+1)}\big((\mathcal{K} \wedge \mathbf{P}^1)^{\wedge 2}\big) \xleftarrow{(\epsilon \wedge \epsilon)^*} \mathrm{BGL}^{4(i+1)-1,2(i+1)}(\mathcal{K}^{\wedge 2})$$

where Σ is the $\mathbf{P}^1$-suspension isomorphism, tw is induced by interchanging the two pointed motivic spaces in the middle of the four-fold smash product, and $\epsilon \colon \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ is the structure map of BGL. Set $A = \mathrm{BGL}^{-1,0}(S)$ and write B for BGL. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{B}^{4i-1,2i}(\mathcal{K}^{\wedge 2}) & \xleftarrow{(\Sigma \circ \Sigma)^{-1} \circ \mathrm{tw}} & \mathrm{B}^{4i+3,2(i+1)}\big((\mathcal{K} \wedge \mathbf{P}^1)^{\wedge 2}\big) & \xleftarrow{(\epsilon \wedge \epsilon)^*} & \mathrm{B}^{4i+3,2i+2}(\mathcal{K}^{\wedge 2}) \\
\uparrow & & \uparrow & & \uparrow \\
A \otimes \mathrm{B}^{4i,2i}(\mathcal{K}^{\wedge 2}) & \xleftarrow{\mathrm{id} \otimes (\Sigma \circ \Sigma)^{-1} \circ \mathrm{tw}} & A \otimes \mathrm{B}^{4i+4,2i+2}\big((\mathcal{K} \wedge \mathbf{P}^1)^{\wedge 2}\big) & \xleftarrow{\mathrm{id} \otimes (\epsilon \wedge \epsilon)^*} & A \otimes \mathrm{B}^{4i+4,2i+2}(\mathcal{K}^{\wedge 2})
\end{array}
$$

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where the vertical arrows are induced by the naive product structure on the functor $\mathrm{BGL}^{*,*}$. Clearly it commutes. The vertical arrows are isomorphisms by Lemma 1.6.1. It follows that $\varprojlim^1 \mathrm{BGL}^{4i-1,2i}(\mathcal{K}^{\wedge 2}) = \varprojlim^1\big(A \otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})\big)$ where in the last tower of groups the connecting maps are $\mathrm{id} \otimes \big((\Sigma \circ \Sigma)_{\mathbf{P}^1}^{-1} \circ \mathrm{tw}\big) \circ (\epsilon \wedge \epsilon)^*)$. It remains to prove the following claim.

Claim. $\varprojlim^1(A \otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})) = 0$.

Since $A = \mathrm{BGL}^{-1,0}(S) = K_1^{\mathrm{TT}}(S) = \mathbb{Z}/2\mathbb{Z}$, there is an isomorphism $A \otimes \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2}) = \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})/m\mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})$ with $m = 2$. The connecting map in the tower are just the mod-m reduction of the maps $(\Sigma \circ \Sigma)^{-1} \circ \mathrm{tw} \circ (\epsilon \wedge \epsilon)^*$. Now a chain of isomorphisms completes the proof of the Claim.

$$
\begin{aligned}
\varprojlim^1 \mathrm{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})/m &\cong \varprojlim^1 (r\mathrm{BGL})^{4i}(r\mathcal{K} \wedge r\mathcal{K})/m \cong \varprojlim^1 \mathbb{B}\mathrm{U}^{4i}(r\mathcal{K} \wedge r\mathcal{K})/m \\
&\cong \varprojlim^1 \mathbb{B}\mathrm{U}^{4i}\big((\mathbb{Z} \times \mathrm{BU}) \wedge (\mathbb{Z} \times \mathrm{BU})\big)/m \\
&\cong \varprojlim^1 \mathbb{B}\mathrm{U}^{4i}\big((\mathbb{Z} \times \mathrm{BU}) \wedge (\mathbb{Z} \times \mathrm{BU}); \mathbb{Z}/m\big) \\
&\cong K_{\mathrm{top}}^1(\mathbb{B}\mathrm{U} \wedge \mathbb{B}\mathrm{U}; \mathbb{Z}/m) \cong K_{\mathrm{top}}^1(\mathbb{B}^0\mathrm{U} \wedge \mathbb{B}^0\mathrm{U}; \mathbb{Z}/m) \\
&\cong \varprojlim^1 K_{\mathrm{top}}^{4i}\big(\mathrm{Gr}(b(i), 2b(i)) \wedge \mathrm{Gr}(b(i), 2b(i)); \mathbb{Z}/m\big) = 0
\end{aligned}
$$

The first isomorphism follows from Lemma 2.3.2. The second isomorphism is induced by the levelwise weak equivalence $\mathbb{B}\mathrm{U} \simeq r\mathrm{BGL}$ mentioned in Lemma 1.6.2. The third isomorphism is induced by the image of the weak equivalence $\mathcal{K}_i \simeq \mathbb{Z} \times \mathrm{Gr}$ under topological realization. The forth and fifth isomorphism hold since $\mathbb{B}\mathrm{U}^{4i+1}(\mathbb{Z} \times \mathrm{BU}) = 0$. The sixth isomorphism is induced by the stable equivalence $\mathbb{B}^0\mathrm{U} \simeq \mathbb{B}\mathrm{U}$ from Lemma 1.6.5, the seventh one is induced by the stable equivalence $\mathbb{B}^f\mathrm{U} \simeq \mathbb{B}^0\mathrm{U}$ from Lemma 1.6.6. The last one holds since all groups in the tower are finite. $\qquad\square$

2.5 Vanishing of certain groups III

Consider the stable equivalence

$$
\mathrm{hocolim}\ \Sigma_{\mathbf{P}^1}^\infty(\mathcal{K} \wedge \mathcal{K} \wedge \mathcal{K})(-3i) \cong \mathrm{BGL} \wedge \mathrm{BGL} \wedge \mathrm{BGL}
$$

from (24) and the induced short exact sequence

$$
0 \to \varprojlim^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K}^{\wedge 3}) \to \mathrm{BGL}^{0,0}(\mathrm{BGL} \wedge \mathrm{BGL} \wedge \mathrm{BGL}) \to \varprojlim \mathrm{BGL}^{8i,4i}(\mathcal{K}^{\wedge 3}) \to 0.
$$

Proposition 2.5.1. *If* $S = \mathrm{Spec}(\mathbb{Z})$ *then* $\varprojlim^1 \mathrm{BGL}^{8i-1,4i}(\mathcal{K} \wedge \mathcal{K} \wedge \mathcal{K}) = 0$.

Proof. This is proved in the same way as Proposition 2.4.1. $\qquad\square$

2.6 BGL as an oriented commutative $\mathbf{P}^1$-ring spectrum

Following Adams and Morel, we define an orientation of a commutative $\mathbf{P}^1$-ring spectrum. However we prefer to use a Thom class rather than a Chern class. Let $\mathbf{P}^\infty = \bigcup \mathbf{P}^n$ be the motivic space pointed by $\infty \in \mathbf{P}^1 \hookrightarrow \mathbf{P}^\infty$. $\mathcal{O}(-1)$ be the tautological line bundle over $\mathbf{P}^\infty$.

It is also known as the Hopf bundle. If $V \to X$ is a vector bundle over $X \in \mathcal{S}m/S$, with zero section $z\colon X \hookrightarrow V$, let $\mathrm{Th}_X(V) = V/(V \smallsetminus z(X))$ be the Thom space of V, considered as a pointed motivic space over S. For example $\mathrm{Th}_X(\mathbf{A}^n_X) \simeq S^{2n,n}$. Define $\mathrm{Th}_{\mathbf{P}^\infty}\big(\mathcal{O}(-1)\big)$ as the obvious colimit of the Thom spaces $\mathrm{Th}_{\mathbf{P}^n}\big(\mathcal{O}(-1)\big)$.

Definition 2.6.1. *Let E be a commutative $\mathbf{P}^1$-ring spectrum. An orientation of E is an element $th \in E^{2,1}(\mathrm{Th}_{\mathbf{P}^\infty}(\mathcal{O}(-1))) = E^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathrm{Th}(1)) = E^{2,1}(\mathbf{P}^1, \infty)$.*

Remark 2.6.2. Let th be an orientation of E. Set $c := z^*(th) \in E^{2,1}(\mathbf{P}^\infty)$. It is proved in [PY, Prop. 6.5.1] that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. The class $th(\mathcal{O}(-1)) \in E^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ given by (26) coincides with the element th (see [PS1, Thm.3.5]). Thus another possible definition of an orientation of E is the following.

Definition 2.6.3. *Let E be a commutative ring $\mathbf{P}^1$-spectrum. An orientation of E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$ (of course the element c should be regarded as the first Chern class of the Hopf bundle $\mathcal{O}(-1)$ on $\mathbf{P}^\infty$).*

Remark 2.6.4. Let c be an orientation of E. Consider $th(\mathcal{O}(-1)) \in E^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ given by (26) and set $th = th(\mathcal{O}(-1))$. It is straightforward to check that $th|_{\mathrm{Th}(1)} = \Sigma_{\mathbf{P}^1}(1)$. Thus th is an orientation of E. Clearly $c = z^*(th) \in E^{2,1}(\mathbf{P}^\infty)$, whence the two definitions of orientations of E are equivalent.

Example 2.6.5. Set $c^K = (-\beta) \cup \big([\mathcal{O}] - [\mathcal{O}(1)]\big) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (12) shows that c^K is an orientation of BGL. Consider $th(\mathcal{O}(-1)) \in \mathrm{BGL}^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ given by (26) and set $th^K = th(\mathcal{O}(-1))$. The class th^K is the same orientation of BGL.

The orientation of BGL described in Example 2.6.5 has the following property. The map (16)

$$\mathrm{BGL}^{*,*} \to K^{\mathrm{TT}}_*$$

which takes β to -1 is an oriented morphism of oriented cohomology theories, provided that K^{TT}_* is oriented via the Chern structure $L/X \mapsto [\mathcal{O}] - [L^{-1}] \in K_0(X)$.

2.7 $\mathrm{BGL}^{*,*}$ as an oriented ring cohomology theory

An oriented $\mathbf{P}^1$-ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1, Defn.3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m\mathcal{O}p$ is a ring cohomology theory. By [PS1, Th.3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn.3.2]. The functor isomorphism $\mathrm{Hom}_{\mathrm{H}_\bullet(S)}(-, \mathbf{P}^\infty) \to \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Thm. 4.3.8] takes the class of the canonical map $\mathbf{P}^\infty_+ \to \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. Now for a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$, where the morphism $f_L\colon X_+ \to \mathbf{P}^\infty$ in $\mathrm{H}_\bullet(S)$ corresponds to the class $[L]$ of L in the group $\mathrm{Pic}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure

on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\operatorname{rank}(V),\operatorname{rank}(V)}\big(\operatorname{Th}_X(V)\big)$ on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. Now for a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W. Set

$$\bar{t}h(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{25}$$

It follows from [PS1, Cor. 3.18] that the support extension map

$$E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big) \to E^{2r,r}\big(\mathbf{P}(W)\big)$$

is injective and $\bar{t}h(E) \in E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big)$. Set

$$th(E) = j^*(\bar{t}h(E)) \in E^{2r,r}\big(\operatorname{Th}_X(V)\big), \tag{26}$$

where $j \colon \operatorname{Th}_X(V) \to \mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS1, Thm. 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.7.1. *For a rank r vector bundle $p \colon V \to X$ on $X \in \mathcal{S}m/S$ the map*

$$- \cup th(V) \colon E^{*,*}(X) \to E^{*+2r,*+r}\big(\operatorname{Th}_X(V)\big)$$

is an isomorphism of the two-sided $E^{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $\big(- \cup th(V)\big) \circ p^*$.

Proof. See [PS1, Defn. 3.32.(4)]. $\qquad\qquad\square$

A Motivic homotopy theory

The aim of this section is to present details on the model structures we use to perform homotopical calculations. Our reference on model structures is [Ho]. For the convenience of the reader who is not familiar with model structures, we recall the basic features and purposes of the theory below, after discussing categorical prerequisites.

A.1 Categories of motivic spaces

Let S be a Noetherian separated scheme of finite Krull dimension (*base scheme* for short). The category of smooth quasi-projective S-schemes is denoted $\mathcal{S}m/S$. A smooth morphism is always of finite type. In particular, $\mathcal{S}m/S$ is equivalent to a small category.

The category of compactly generated topological spaces is denoted **Top**, the category of simplicial sets is denotes **sSet**. The set of n-simplices in K is K_n.

Definition A.1.1. *A motivic space over S is a functor $A\colon \mathcal{S}m/S^{\mathrm{op}} \to \mathbf{sSet}$. The category of motivic spaces over S is denoted $\mathbf{M}(S)$.*

For $X \in \mathcal{S}m/S$ the motivic space sending $Y \in \mathcal{S}m/S$ to the discrete simplicial set $\mathrm{Hom}_{\mathcal{S}m/S}(Y, X)$ is denoted X as well. More generally, any scheme X over S defines a motivic space X over S. Any simplicial set K defines a constant motivic space K. A pointed motivic space is a pair (A, a_0), where $a_0\colon S \to A$. Usually the basepoint will be omitted from the notation. The resulting category is denoted $\mathbf{M}_\bullet(S)$.

Definition A.1.2. *A morphism $f\colon S \to S'$ of base schemes defines the functor*

$$f_*\colon M_\bullet(S) \to M_\bullet(S')$$

sending A to $(Y \to S') \mapsto A(S \times_{S'} Y)$. Left Kan extension produces a left adjoint $f^*\colon M_\bullet(S') \to M_\bullet(S)$ of f_*.

If A is a motivic space, let A_+ denote the pointed motivic space $(A \coprod S, i)$, where $i\colon S \to A \coprod S$ is the canonical inclusion. The category $\mathbf{M}_\bullet(S)$ is closed symmetric monoidal, with smash product $A \wedge B$ defined by the sectionwise smash product

$$\big(A \wedge B\big)(X) := A(X) \wedge B(X) \tag{27}$$

and with internal hom $\underline{\mathrm{Hom}}_{M_\bullet(S)}(A, B)$ defined by

$$\underline{\mathrm{Hom}}_{M_\bullet(S)}(A, B)(x\colon X \to S)_n := \mathrm{Hom}_{\mathbf{M}_\bullet(S)}(A \wedge \Delta^n_+, x_* x^* B). \tag{28}$$

In particular, $M_\bullet(S)$ is also enriched over the category of pointed simplicial sets, with enrichment $\mathbf{sSet}_\bullet(A, B) := \underline{\mathrm{Hom}}_{M_\bullet(S)}(A, B)(S)$. The *mapping cylinder* of a map $f\colon A \to B$ is the pushout of the diagram

$$
\begin{array}{ccc}
A \wedge \partial\Delta^1_+ \xrightarrow{\;\cong\;} A \coprod A \xrightarrow{\mathrm{id}_A \coprod f} A \coprod B\, . \\
\downarrow{\scriptstyle\rho} \qquad\qquad\qquad\qquad\qquad\quad \downarrow \\
A \wedge \Delta^1_+ \xrightarrow{\qquad\qquad\qquad\qquad} \mathrm{Cyl}(f)
\end{array}
\tag{29}
$$

The composition of the canonical maps $A \hookrightarrow \mathrm{Cyl}(f) \to B$ is f.

The *pushout product* of two maps $f\colon A \to C$ and $g\colon B \to D$ of motivic spaces over S is the map $f \sqcup g\colon A \wedge D \cup_{A \wedge B} C \wedge B \to C \wedge D$ induced by the commutative diagram

$$
\begin{array}{ccc}
A \wedge B & \longrightarrow & A \wedge D \\
\downarrow & & \downarrow \\
C \wedge B & \longrightarrow & C \wedge D.
\end{array}
\tag{30}
$$

The functor $f^*\colon M_\bullet(S') \to M_\bullet(S)$ is strict symmetric monoidal, since so is the pullback functor sending $X \in \mathcal{S}m_{S'}$ to $S \times_{S'} X \in \mathcal{S}m/S$. This ends the categorical considerations.

A.2 Model categories

The basic purpose of a model structure is to give a framework for the construction of a homotopy category. Suppose $w\mathcal{C}$ is a class of morphisms in a bicomplete $\mathcal{C}$ one wants to be invertible. Call them weak equivalences. One can define the homotopy "category" of the pair $(\mathcal{C}, w\mathcal{C})$ to be the universal "functor" $\Gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ such that every weak equivalence is mapped to an isomorphism. In general, this homotopy "category" may not be a category: it has hom-classes, but not necessarily hom-sets. If one requires the existence of two auxiliary classes of morphisms $f\mathcal{C}$ (the fibrations) and $c\mathcal{C}$ (the cofibrations), together with certain compatibility axioms, one does get a homotopy category $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ and an explicit description of the hom-sets in it.

Theorem A.2.1 (Quillen)**.** *Let $(w\mathcal{C}, f\mathcal{C}, c\mathcal{C})$ be a model structure on a bicomplete category $\mathcal{C}$. Then the homotopy category $\Gamma\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ exists and is the identity on objects. The set of morphisms in $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$ from ΓA to ΓB is the set of morphisms in $\mathcal{C}$ from A to B modulo a homotopy equivalence relation, provided that $\emptyset \to A$ is a cofibration and $B \to *$ is a fibration.*

Here $\emptyset$ is the initial object and $*$ is the terminal object in $\mathcal{C}$. An object A resp. B as in Theorem A.2.1 is called *cofibrant* resp. *fibrant*. Every object ΓA in the homotopy category is isomorphic to an object ΓC, where C is both fibrant and cofibrant. A (co)fibration which is also a weak equivalence is usually called a *trivial* or *acyclic* (co)fibration.

To describe the standard way to construct model structures on a bicomplete category, one needs a definition.

Definition A.2.2. *Let $f\colon A \to B$ and $g\colon C \to D$ be morphisms in $\mathcal{C}$. If every commutative diagram*

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \longrightarrow & D
\end{array}
$$

admits a morphism $h\colon B \to C$ such that the resulting diagram

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
{\scriptstyle f}\downarrow & \nearrow^{h} & \downarrow{\scriptstyle g} \\
B & \longrightarrow & D
\end{array}
$$

commutes (a lift *for short), then f has the left lifting property with respect to g, and g has the right lifting property with respect to f.*

Here is the standard way of constructing a model structure on a given bicomplete category. Choose the class of weak equivalences such that it contains all identities, is closed under retracts and satisfies the two-out-of-three axiom. Pick a set I (the *generating cofibrations*) and define a cofibration to be a morphism which is a retract of a transfinite composition of cobase changes of morphisms in I. Pick a set J (the *generating acyclic*

cofibrations) of weak equivalences which are also cofibrations and define the fibrations to be those morphisms which have the right lifting property with respect to every morphism in J. Some technical conditions have to be fulfilled in order to conclude that this indeed is a model structure, which is then called *cofibrantly generated*. See [Ho, Theorem 2.1.19].

Example A.2.3. In **Top**, let the weak equivalences be the weak homotopy equivalences, and set

$$I = \{\partial D^n \hookrightarrow D^n\}_{n \geq 0} \quad J = \{D^n \times \{0\} \hookrightarrow D^n \times I\}_{n \geq 0}.$$

Then the fibrations are precisely the Serre fibrations, and the cofibrations are retracts of generalized cell complexes ("generalized" refers to the fact that cells do not have to be attached in order of dimension). In **sSet**, let the weak equivalences be those maps which map to (weak) homotopy equivalences under geometric realization. Set

$$I = \{\partial \Delta^n \hookrightarrow \Delta^n\}_{n \geq 0} \quad J = \{\Lambda^n_j \hookrightarrow \Delta^n\}_{n \geq 1, 0 \leq j \leq n}$$

where Λ^n_j is the sub-simplicial set of $\partial \Delta^n$ obtained by removing the j-th face. Then the fibrations are precisely the Kan fibrations, and the cofibrations are the inclusions.

Example A.2.4. For the purpose of this paper, model structures on presheaf categories $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{sSet})$ with values in simplicial sets are relevant. There is a canonical one, due to Quillen, which is usually referred to as the *projective* model structure. It has as weak equivalences those morphisms $f \colon A \to B$ such that $f(c) \colon A(c) \to B(c)$ is a weak equivalence for every $c \in \mathrm{Ob}\,\mathcal{C}$ (the *objectwise* or *sectionwise* weak equivalences). Set

$$I = \{\mathrm{Hom}_{\mathcal{S}}(-, c) \times (\partial \Delta^n \hookrightarrow \Delta^n)\}_{n \geq 0} \quad J = \{\mathrm{Hom}_{\mathcal{S}}(-, c) \times (\Lambda^n_j \hookrightarrow \Delta^n)\}_{n \geq 1, 0 \leq j \leq n}$$

so that by adjointness, the fibrations are precisely the sectionwise Kan fibrations. There is another one with the same weak equivalences, due to Heller [He], such that the cofibrations are precisely the injective morphisms (whence the name *injective* model structure).The description of J involves the cardinality of the set of morphisms in $\mathcal{C}$ and is not explicit. Neither is the characterization of the fibrations.

The morphisms of model categories are called Quillen functors. A Quillen functor of model categories $\mathcal{M} \to \mathcal{N}$ is an adjoint pair $(F, G) \colon \mathcal{M} \to \mathcal{N}$ such that F preserves cofibrations and G preserves fibrations. This condition ensures that (F, G) induces an adjoint pair on homotopy categories $(\mathcal{L}F, \mathcal{R}G)$, where $\mathcal{L}F$ is the total left derived functor of F. A Quillen functor is a Quillen equivalence if the total left derived is an equivalence. For example, geometric realization is a strict symmetric monoidal left Quillen equivalence $|-| \colon \mathbf{sSet} \to \mathbf{Top}$, and similarly in the pointed setting.

If a model category has a closed symmetric monoidal structure as well, one has the following statement.

Theorem A.2.5 (Quillen)**.** *Let $\mathcal{C}$ be a bicomplete category with a model structure. Suppose that $(\mathcal{C}, \otimes, \mathbb{I})$ is closed symmetric monoidal. Suppose further that these structures are compatible in the following sense:*

- *The pushout product of two cofibrations is a cofibration, and*

- *the pushout product of an acyclic cofibration with a cofibration is an acyclic cofibration.*

Then $A \otimes -$ is a left Quillen functor for all cofibrant objects $A \in \mathcal{C}$. In particular, there is an induced (total derived) closed symmetric monoidal structure on $\mathrm{Ho}(\mathcal{C}, w\mathcal{C})$.

One abbreviates the hypotheses of Theorem A.2.5 by saying that $\mathcal{C}$ is a symmetric monoidal model category. This ends our introduction to model category theory.

A.3 Model structures for motivic spaces

To equip $\mathbf{M}_\bullet(S)$ with a model structure suitable for the various requirements (compatibility with base change, taking complex points, finiteness conditions, having the correct motivic homotopy category), we construct a preliminary model structure first. Start with the following construction, which is a special case of the considerations in [I]. Choose any $X \in Sm_S$ and a finite set

$$\{i^j \colon Z^j \hookrightarrow X\}_{j=1}^m$$

of closed embeddings in $\mathcal{S}m/S$. Regarding i_j as a monomorphism of motivic spaces, one may form the categorical union (not the categorical coproduct!) $\star_{j=1}^m i^j \colon \cup_{j=1}^m Z^j \hookrightarrow X$. That is, $\cup_{j=1}^m Z^j$ is the coequalizer in the category of motivic spaces of the diagram

$$\coprod_{j=1}^m Z_j \rightrightarrows \coprod_{j,j'} Z_j \times_X Z_{j'} \tag{31}$$

Call the resulting monomorphism $\star_{j=1}^m i^j \colon \cup_{j=1}^m Z^j \hookrightarrow X$ *acceptable*. The closed embedding $\emptyset \hookrightarrow X$ is acceptable as well. Consider the set Ace of acceptable monomorphisms. Let I_S^c be the set of maps

$$\{i_+ \sqcup (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{i \in \mathrm{Ace}, n \geq 0} \tag{32}$$

and let J_S^c be the set of maps

$$\{i_+ \sqcup (\Lambda_j^n \hookrightarrow \Delta^n)_+\}_{i \in \mathrm{Ace}, n \geq 1, 0 \leq j \leq n} \tag{33}$$

Definition A.3.1. *A map $f \colon A \to B$ in $\mathbf{M}_\bullet(S)$ is a* schemewise weak equivalence *if $f \colon A(X) \to B(X)$ is a weak equivalence of simplicial sets for all $X \in \mathcal{S}m/S$. It is a* closed schemewise fibration *if $f \colon A \to B$ has the right lifting property with respect to J_S^c. It is a* closed cofibration *if it has the left lifting property with respect to all acyclic closed schemewise fibrations (closed schemewise fibrations which are also schemewise weak equivalences).*

Theorem A.3.2. *The classes defined in A.3.1 are a closed symmetric monoidal model structure on $\mathbf{M}_\bullet(S)$, denoted $\mathbf{M}_\bullet^{cs}(S)$. A morphism $f \colon S \to T$ of base schemes induces a strict symmetric monoidal left Quillen functor $f^* \colon \mathbf{M}_\bullet(T) \to \mathbf{M}_\bullet(S)$.*

Proof. The existence of the model structure follows from [I]. The pushout product axiom follows, because the pushout product of two acceptable monomorphism is again acceptable. To conclude the last statement, it suffices to check that f^* maps any map in I_T^c resp. J_T^c to a closed cofibration resp. schemewise weak equivalence. In fact, if $i = \star_{j=1}^m i^j \colon \cup_{j=1}^m Z^j \hookrightarrow X$ is an acceptable monomorphism in $\mathcal{S}m_T$, then $f^*(i)$ is the acceptable monomorphism obtained from the closed embeddings

$$\{T \times_S Z_j \hookrightarrow T \times_S X.\}$$

Because f^* is strict symmetric monoidal and a left adjoint, it preserves the pushout product. Hence f^* even maps the set I_T^c to the set I_S^c, and likewise for J_T^c. The result follows. □

The resulting homotopy category is equivalent – via the identity functor – to the usual homotopy category of the diagram category $\mathbf{M}_\bullet(S)$ (obtained via the projective model structure), since the weak equivalences are just the objectwise ones. The model structure $\mathbf{M}_\bullet^{cs}(S)$ has the advantage that for any S-point $x_0 \colon S \hookrightarrow X$ in a smooth S-scheme, the pair (X, x_0) is closed cofibrant. Not all pointed motivic spaces are closed cofibrant. Let $(-)^{cs} \to \mathrm{Id}_{\mathbf{M}_\bullet(S)}$ denote a cofibrant replacement functor, for example the one obtained from applying the small object argument to I_S^c. That is, the map $A^{cs} \to A$ is a natural closed schemewise fibration and a schemewise weak equivalence, and A^{cs} is closed cofibrant. Dually, let $\mathrm{Id}_{\mathbf{M}_\bullet(S)} \to (-)^{cf}$ denote the fibrant replacement functor obtained by applying the small object argument to J_S^c. The closed schemewise fibrations may be characterized explicitly.

Lemma A.3.3. *A map $f \colon A \to B$ is a closed schemewise fibration if and only if the following two conditions hold.*

1. *$f(X) \colon A(X) \to B(X)$ is a Kan fibration for every $X \in \mathcal{S}m/S$, and*

2. *for every finite set $\{Z_j \hookrightarrow X\}_{j=1}^m$ of closed embeddings in $\mathcal{S}m/S$, the induced map*

$$A(X) \to B(X) \times_{\mathbf{sSet}_\bullet(\cup_{j=1}^m Z_j, B)} \mathbf{sSet}_\bullet(\cup_{j=1}^m Z_j, A)$$

   is a Kan fibration.

Proof. Follows by adjointness from the definition. □

To obtain a motivic model structure, one localizes $\mathbf{M}_\bullet^{cs}(S)$ as follows. Recall that an elementary distinguished square (or simply *Nisnevich square*) is a pullback diagram

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ j\ } & X \end{array}$$

in $\mathcal{S}m/S$, where j is an open embedding and p is an étale morphism inducing an isomorphism $Y - V \cong X - U$ of reduced closed subschemes. Say that a pointed motivic space C is *closed motivic fibrant* if it is closed schemewise fibrant, the map

$$C\big(X \times_S \mathbb{A}^1_S \xrightarrow{\mathrm{pr}} X\big)$$

is a weak equivalence of simplicial sets for every $X \in \mathcal{S}m/S$, the square

$$
\begin{array}{ccc}
C(V) & \longleftarrow & C(Y) \\
\downarrow & & \downarrow \\
C(U) & \longleftarrow & C(X)
\end{array}
$$

is a homotopy pullback square of simplicial sets for every Nisnevich square in $\mathcal{S}m/S$ and $C(\emptyset)$ is contractible.

Example A.3.4. Let $X \mapsto K^{\mathrm{TT}}(X)$ be the pointed motivic space sending $X \in \mathcal{S}m/S$ to the first term of the Waldhausen K-theory spectrum [W] associated to the category of big vector bundles over X [FS], with isomorphisms as weak equivalences. That is,

$$K^{\mathrm{TT}}(X) = \Omega \operatorname{Sing} |w\mathcal{S}_\bullet(\mathrm{Vect}^{\mathrm{big}}(X), \mathrm{iso})|$$

Suppose now that S is regular. Then so is X, and $K^{\mathrm{TT}}(X)$ has the same homotopy type as the first term of the Thomason-Trobaugh K-theory spectrum of X [TT, Cor. 3.9]. It follows that the projection induces a weak equivalene $K^{\mathrm{TT}}(X) \to K^{\mathrm{TT}}(X \times_S \mathcal{A}^1_S)$ [TT, Prop. 6.8]. By [TT, Thm. 10.8] the square

$$
\begin{array}{ccc}
K^{\mathrm{TT}}(X) & \longrightarrow & K^{\mathrm{TT}}(U) \\
{\scriptstyle K^{\mathrm{TT}}(p)}\downarrow & & \downarrow \\
K^{\mathrm{TT}}(Y) & \longrightarrow & K^{\mathrm{TT}}\big(p^{-1}(U)\big)
\end{array}
$$

associated to a Nisnevich square is a homotopy pullback square. Hence K^{TT} is fibrant in the projective motivic model structure on $\mathbf{M}_\bullet(S)$. However, if $i\colon Z \hookrightarrow X$ is a closed embedding in $\mathcal{S}m/S$, the map $K^{\mathrm{TT}}(i)\colon K^{\mathrm{TT}}(X) \to K^{\mathrm{TT}}(Z)$ is not necessarily a Kan fibration. In particular, K^{TT} is not closed schemewise fibrant. Choose a closed cofibration which is also a schemewise weak equivalence $K^{\mathrm{TT}} \to \mathbb{K}^{\mathrm{TT}}$ such that $\mathbb{K}^{\mathrm{TT}}$ is closed schemewise fibrant. It follows immediately that $\mathbb{K}^{\mathrm{TT}}$ is closed motivic fibrant, and that $^{\mathrm{TT}}(X)$ has the homotopy type of the first term of the Thomason-Trobaugh K-theory spectrum of X.

Definition A.3.5. *A map $f\colon A \to B$ is a* motivic weak equivalence *if the map*

$$\mathbf{sSet}_\bullet(f^{cs}, C)\colon \mathbf{sSet}_\bullet(B^{cs}, C) \to \mathbf{sSet}_\bullet(A^{cs}, C)$$

is a weak equivalence of simplicial sets for every closed motivic fibrant C. It is a closed motivic fibration *if it has the right lifting property with respect to all acyclic closed cofibrations (closed cofibrations which are also motivic equivalences).*

Example A.3.6. Suppose that $f\colon A \to B$ is a map in $\mathbf{M}_\bullet(S)$ inducing weak equivalences $x^*f\colon x^*A \to x^*B$ of simplicial sets on all Nisnevich stalks $x^*\colon \mathbf{M}_\bullet(S) \to \mathbf{sSet}$. Then f is a motivic weak equivalence. If $f\colon A \to B$ is an $\mathbf{A}^1$-homotopy equivalence (for example, the projection of a vector bundle), then it is a motivic weak equivalence.

Example A.3.7. The canonical covering of $\mathbf{P}^1$ shows that it is motivic weakly equivalent as a pointed motivic space to the suspension $S^1 \wedge (\mathcal{A}^1 - \{0\}, 1)$, where $S^1 = \Delta^1/\partial\Delta^1$. Set $S^{1,0} := S^1$ and $S^{1,1} := (\mathcal{A}^1 - \{0\}, 1)$, and define

$$S^{p,q} := \left(S^{1,0}\right)^{\wedge p-q} \wedge \left(S^{1,1}\right)^{\wedge q} \quad \text{for} \quad p \geq q \geq 0$$

To generalize the example of $\mathbf{P}^1$, one can show that if $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ is a linear embedding, then $\mathbf{P}^n/\mathbf{P}^{n-1}$ is motivic weakly equivalent to $S^{2n,n}$.

To prove that the classes from Definition A.3.5 are part of a model structure, it is helpful to characterize the closed motivic fibrant objects via a lifting property. Let J_S^{cm} be the union of the set J_S^c from (33) and the set J_S^m of pushout products of maps $(\partial\Delta^n \hookrightarrow \Delta^n)_+$ with maps of the form

$$X_+ \xrightarrow{\quad \text{zero}_+ \quad} (\mathcal{A}_S^1 \times_S X)_+ \tag{34}$$

$$U_+ \cup_{V_+} \mathrm{Cyl}h_+ \longrightarrow \mathrm{Cyl}\big(U_+ \cup_{V_+} \mathrm{Cyl}(h_+) \to X_+\big)$$

$$* \longrightarrow \emptyset_+$$

where h is the open embedding appearing on top of a Nisnevich square

$$\begin{array}{ccc} V & \xrightarrow{\ h\ } & Y \\ \downarrow & & {\scriptstyle p}\big\downarrow \\ U & \longrightarrow & X \end{array}$$

in $\mathcal{S}m/S$.

Lemma A.3.8. *A pointed motivic space C is closed motivic fibrant if and only if the map $C \to *$ has the right lifting property with respect to the set J_S^{cm}.*

Proof. This follows from adjointness, the Yoneda lemma and the construction of J_S^{cm}. $\quad\square$

Theorem A.3.9. *The classes of motivic weak equivalences, closed motivic fibrations and closed cofibrations constitute a symmetric monoidal model structure on $\mathbf{M}_\bullet(S)$, denoted $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$. The resulting homotopy category is denoted $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ and called the pointed motivic unstable homotopy category of S. A morphism $f\colon S \to T$ of base schemes induces a strict symmetric monoidal left Quillen functor $f^*\colon M_\bullet^{\mathrm{cm}}(T) \to M_\bullet^{\mathrm{cm}}(S)$.*

Proof. The existence of the model structure follows by standard Bousfield localization techniques. Here are some details. The problem is that J_S^{cm} might be to small in order to characterize all closed motivic fibrations. Let κ be a regular cardinal strictly bigger than the cardinality of the set of morphisms in $\mathscr{S}m/S$. A motivic space A is κ-bounded if the union

$$\coprod_{n \geq 0, X \in \mathscr{S}m/S} A(X)_n$$

has cardinality $\leq \kappa$. Let J_S^κ be a set of isomorphism classes of acyclic monomorphisms whose target is κ-bounded. One may show that given an acyclic monomorphism $j \colon A \hookrightarrow B$ and a κ-bounded subobject $C \subseteq B$, there exists a κ-bounded subobject $C' \subseteq B$ containing C such that $j^{-1}(C') \hookrightarrow C'$ is an acyclic monomorphism. Via Zorn's lemma, one then gets that a map $f \in \mathbf{M}_\bullet(S)$ has the right lifting property with respect to all acyclic monomorphisms if (and only if) it has the right lifting property with respect to the set J_S^κ. Such a map is in particular a closed motivic fibration. Any given map $f \colon A \to B$ can now be factored (via the small object argument) as an acyclic monomorphism $j \colon A \hookrightarrow C$ followed by a closed motivic fibration. Factoring j as a closed cofibration followed by an acyclic closed schemewise fibration in the model structure of Theorem A.3.2 implies the existence of the model structure.

To prove that the model structure is symmetric monoidal, it suffices – by the corresponding statement A.3.2 for $\mathbf{M}_\bullet^{cs}(S)$ – to check that the pushout product of a generating closed cofibration and an acyclic closed cofibration is again a motivic equivalence. However, from the fact that the injective motivic model structure is symmetric monoidal, one knows that motivic equivalences are closed under smashing with arbitrary motivic spaces [DRØ]. The first sentence is now proven.

Concerning the third sentence, Theorem A.3.2 already implies that f^* preserves closed cofibrations. To prove that f^* is a left Quillen functor, it suffices (by Dugger's lemma [D, Cor. A2]) to check that it maps the set J_T^{cm} to motivic weak equivalences in $\mathbf{M}_\bullet(S)$. One may calculate that $f^*(J_T^{\mathrm{cm}}) = J_S^{\mathrm{cm}}$, whence the statement. $\qquad\square$

The closed motivic model structure is cofibrantly generated. As remarked in the proof of Theorem A.3.9, the set J_S^{cm} is perhaps not big enough to yield a full set of generating trivial cofibrations. By localization theory and Lemma A.3.8, a map with closed motivic fibrant codomain is a closed motivic fibration if and only if it has the right lifting property with respect to J_S^{cm}. Note that the domains and codomains of the maps in J_S^{cm} are closed cofibrant and finitely presentable. The following Lemma is an easy consequence.

Lemma A.3.10. *Motivic equivalences and closed motivic fibrations with closed motivic fibrant codomain are closed under filtered colimits.*

Proof. This follows, since the domains of the maps in I_S^c and J_S^{cm} are finitely presentable. See [DRØ] for the corresponding statement for the projective motivic model structure. $\qquad\square$

Let $\mathcal{M}_\bullet(S)$ be the category of simplicial objects in the category of pointed Nisnevich sheaves on $\mathscr{S}m/S$. The functor mapping a (pointed) presheaf to its associated (pointed)

Nisnevich sheaf determines by degreewise application a functor $a_{Nis} \colon \mathbf{M}_\bullet(S) \to \mathcal{M}_\bullet(S)$. Let $\mathcal{M}_\bullet(S) \xrightarrow{i} \mathbf{M}_\bullet(S)$ be the inclusion functor, the right adjoint of a_{Nis}.

Theorem A.3.11. *The pair (a_{Nis}, i) is a Quillen equivalence to the Morel-Voevodsky model structure. The functor a_{Nis} is strict symmetric monoidal. In particular, the total left derived functor of a_{Nis} is a strict symmetric monoidal equivalence*

$$\mathrm{H}^{\mathrm{cm}}_\bullet(S) \to \mathrm{H}_\bullet(S)$$

to the unstable pointed $\mathbf{A}^1$-homotopy category from [MV].

Proof. Recall that the cofibrations in the Morel-Voevodsky model structure are precisely the monomorphisms. Since every closed cofibration is a monomorphism and Nisnevich sheafification preserves these, a_{Nis} preserves cofibrations. The unit $A \to i(a_{\mathrm{Nis}}(A))$ of the adjunction is an isomorphism on all Nisnevich stalks, hence a motivic weak equivalence by Example A.3.6 for every motivic space A. In particular, a_{Nis} maps schemewise weak equivalences as well as the maps in J^m_S described in (34) to weak equivalences. Let $\mathrm{Id}_{M_\bullet(S)} \to (-)^{\mathrm{fib}}$ be the fibrant replacement functor in $\mathbf{M}^{\mathrm{cm}}_\bullet(S)$ obtained from the small object argument applied to J^{cm}_S. Hence if f is a motivic weak equivalence, then f^{fib} is a schemewise weak equivalence. One concludes that a_{Nis} preserves all weak equivalences, thus is a left Quillen functor. Since the unit $A \to i(a_{\mathrm{Nis}}(A))$ is a motivic weak equivalence for every A, the functor a_{Nis} is a Quillen equivalence. $\square$

Note that a map f in $\mathbf{M}_\bullet(S)$ is a motivic weak equivalence if and only if $a_{\mathrm{Nis}}(f)$ is a weak equivalence in the Morel-Voevodsky model structure on simplicial sheaves. Conversely, a map of simplicial sheaves is a weak equivalence if and only if it is a motivic weak equivalence when considered as a map of motivic spaces.

Remark A.3.12. Starting with the injective model structure on simplicial presheaves mentioned in Example A.2.4, there is a model structure $\mathbf{M}^{im}_\bullet(S)$ on the category of pointed motivic spaces with motivic weak equivalences as weak equivalences and monomorphisms as cofibrations. It has the advantage that every object is cofibrant, but the disadvantage that it does not behave well under base change [MV, ??] or geometric realization (to be defined below). The identity functor is a left Quillen equivalence $\mathrm{Id} \colon \mathbf{M}^{\mathrm{cm}}_\bullet(S) \to \mathbf{M}^{im}_\bullet(S)$, since the homotopy categories coincide.

Further, let $Spc_\bullet(S)$ be the category of pointed Nisnevich sheaves on $\mathcal{S}m/S$. Recall the cosimplicial smooth scheme over S whose value at n is

$$\Delta^n_S = \mathrm{Spec}\big(\mathcal{O}_S[X_0, \dots, X_n]/(\sum_{i=0}^{n} X_i = 1)\big) \tag{35}$$

The functor $Spc_\bullet(S) \to \mathcal{M}_\bullet(S)$ sending A to the simplicial object $\mathrm{Sing}_S(A)_n = A(- \times \Delta^n_S)$ has a left adjoint $|-|_S \colon \mathcal{M}_\bullet(S)$. It maps B to the coend

$$|B|_S = \int_{n \in \Delta} B_n \times \Delta^n_S$$

in the category of pointed Nisnevich sheaves. The following statement is proved in [MV].

Theorem A.3.13 (Morel-Voevodsky)**.** *There is a model structure on the category $Spc_\bullet(S)$ such that the pair $(|-|_S, \mathrm{Sing}_S)$ is a Quillen equivalence to the Morel-Voevodsky model structure. The functor $|-|_S$ is strict symmetric monoidal. In particular, the total left derived functor of $|-|_S$ is a strict symmetric monoidal equivalence from Voevodsky's pointed homotopy category to the unstable pointed $\mathbf{A}^1$-homotopy category*

A.4 Topological realization

In the case where the base scheme is the complex numbers, there is a topological realization functor $\mathbf{R}_\mathbb{C}\colon \mathbf{M}_\bullet^{\mathrm{cm}}(\mathbb{C}) \to \mathbf{Top}_\bullet$ which is a strict symmetric monoidal left Quillen functor. It is defined as follows. If $X \in \mathcal{S}m_\mathbb{C}$, the set $X(\mathbb{C})$ of complex points is a topological space when equipped with the analytic topology. Call this topological space X^{an}. It is a smooth manifold, and in particular a compactly generated topological space. One may view $X \mapsto X^{\mathrm{an}}$ as a functor $\mathcal{S}m_\mathbb{C} \to \mathbf{Top}$. Note that if $i\colon Z \hookrightarrow X$ is a closed embedding in $\mathcal{S}m_\mathbb{C}$, then the resulting map i^{an} is the closed embedding of a smooth submanifold, and in particular a cofibration of compactly generated topological spaces. Every motivic space A is a canonical colimit

$$\operatorname*{colim}_{X\times\Delta^n \to A} X \times \Delta^n \xrightarrow{\cong} A$$

and one defines

$$\mathbf{R}_\mathbb{C}(A) := \operatorname*{colim}_{X\times\Delta^n \to A} X^{\mathrm{an}} \times |\Delta^n| \in \mathbf{Top}.$$

Observe that if A is pointed, then so is $\mathbf{R}_\mathbb{C}(A)$.

Theorem A.4.1. *The functor $\mathbf{R}_\mathbb{C}\colon \mathbf{M}_\bullet^{\mathrm{cm}}(\mathbb{C}) \to \mathbf{Top}_\bullet$ is a strict symmetric monoidal left Quillen functor.*

Proof. The right adjoint of $\mathbf{R}_\mathbb{C}$ maps the compactly generated topological space Z to the motivic space $\mathrm{Sing}_\mathbb{C}(Z)$ which sends $X \in \mathcal{S}m_\mathbb{C}$ to the simplicial set $\mathbf{sSet}_{\mathbf{Top}}(X^{\mathrm{an}}, Z)$. To conclude that $\mathbf{R}_\mathbb{C}$ is strict symmetric monoidal, it suffices to observe that there is a canonical homeomorphism $(X \times Y)^{\mathrm{an}} \cong X^{\mathrm{an}} \times Y^{\mathrm{an}}$, and that geometric realization is strict symmetric monoidal.

Suppose now that $i\colon \cup_{j=1}^m Z_j \hookrightarrow X$ is an acceptable monomorphism. One computes $\mathbf{R}_\mathbb{C}(\cup_{j=1}^m Z_j)$ as the coequalizer

$$\coprod_{j=1}^m Z_j^{\mathrm{an}} \rightrightarrows \coprod_{j,j'} (Z_j \times_X Z_{j'})^{\mathrm{an}}.$$

in $\mathbf{Top}$. Every map $(Z_j \times_X Z_{j'})^{\mathrm{an}} \to Z_j^{\mathrm{an}}$ is a closed embedding of smooth submanifolds of complex projective space. In particular, one may equip Z_j^{an} with a cell complex structure such that $(Z_j \times_X Z_{j'})^{\mathrm{an}}$ is a subcomplex for every j'. Then $\mathbf{R}_\mathbb{C}(\cup_{j=1}^m Z_j)$ is the union of these subcomplexes, and in particular again a subcomplex. It follows that $\mathbf{R}_\mathbb{C}(i)$ is a cofibration of topological spaces. Since $\mathbf{R}_\mathbb{C}$ is compatible with pushout products, it maps the generating closed cofibrations to cofibrations of topological spaces.

To conclude that $\mathbf{R}_\mathbb{C}$ preserves trivial cofibrations as well, it suffices by Dugger's lemma [D, Cor. A2] to check that $\mathbf{R}_\mathbb{C}$ maps every map in $J_\mathbb{C}^m$ to a weak homotopy equivalence. In fact, since the domains and codomains of the maps $\partial\Delta^m \hookrightarrow \Delta^m$ are cofibrant, it suffices to check the latter for the maps in diagram (34). In the first case, one obtains the map $X^{\mathrm{an}} \hookrightarrow (\mathcal{A}_\mathbb{C}^1 \times X)^{\mathrm{an}} \cong \mathbb{R}^2 \times X^{\mathrm{an}}$, in the second case one obtains up to simplicial homotopy equivalence the canonical map $U^{\mathrm{an}} \cup_{p^{-1}(V)^{\mathrm{an}}} Y^{\mathrm{an}} \to X^{\mathrm{an}}$ for a Nisnevich square

$$
\begin{array}{ccc}
V & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle p} \\
U & \longrightarrow & X
\end{array}
$$

This is in fact a homeomorphism of topological spaces. The result follows. $\qquad\square$

Suppose now that $R \hookrightarrow \mathbb{C}$ is a subring of the complex numbers. Let $f\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(R)$ denote the resulting morphism of base schemes. The realization with respect to R (or better f) is defined as the composition

$$
\mathbf{R}_R = \mathbf{R}_\mathbb{C} \circ f^*\colon \mathbf{M}_\bullet(R) \to \mathbf{M}_\bullet(\mathbb{C}) \to \mathbf{Top}_\bullet. \tag{36}
$$

It is a strict symmetric monoidal Quillen functor. The most relevant case is $R = \mathbb{Z}$.

Example A.4.2. The topological realization of the finite Grassmannian $\mathrm{Gr}(m,n)$ (over any base with a complex point) is the complex Grassmannian with the usual topology. Since $\mathbf{R}_\mathbb{C}$ commutes with filtered colimits, $\mathbf{R}_\mathbb{C}(\mathrm{Gr})$ is the infinite complex Grassmannian, which in turn is the classifying space $\mathrm{B}U$ for the infinite unitary group. Because $\mathbf{R}_\mathbb{C}$ is a left Quillen functor, the topological realization of any closed cofibrant motivic space weakly equivalent to Gr (such as $\mathcal{K}$) is homotopy equivalent to $\mathrm{B}U$.

A.5 Spectra

Definition A.5.1. *Let $\mathbf{P}_S^1$ denote the pointed projective line over S. The category $\mathbf{MS}(S)$ of $\mathbf{P}^1$-spectra over S has the following objects. A $\mathbf{P}^1$-spectrum E consists of a sequence $(E_0, E_1, E_2, \ldots)$ of pointed motivic spaces over S, together with structure maps $\sigma_n^E\colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$ for every $n \geq 0$. A map of $\mathbf{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps.*

Example A.5.2. Any pointed motivic space B over S gives rise to a $\mathbf{P}_S^1$-suspension spectrum

$$
\Sigma_{\mathbf{P}^1}^\infty B = (B, B \wedge \mathbf{P}^1, B \wedge \mathbf{P}^1 \wedge \mathbf{P}^1, \ldots)
$$

having identities as structure maps. More generally, let $\mathrm{Fr}_n B$ denote the $\mathbf{P}^1$-spectrum having values

$$
(\mathrm{Fr}_n B)_{n+m} = \begin{cases} B \wedge \mathbf{P}^{1 \wedge m} & m \geq 0 \\ * & m < 0 \end{cases}
$$

and identities as structure maps, except for $\sigma_{n-1}^{\mathrm{Fr}_n B}$. The functor $B \mapsto \mathrm{Fr}_n B$ is left adjoint to the functor sending the $\mathbf{P}^1$-spectrum E to E_n.

Remark A.5.3. In Definition A.5.1, one may replace $\mathbf{P}^1$ by any pointed motivic space A, giving the category $\mathbf{MS}_A(S)$ of A-spectra over S. Essentially the only relevant example for us is when A is weakly equivalent to the pointed projective line $\mathbf{P}^1$. The Thom space $T = \mathcal{A}^1/\mathcal{A}^1 - \{0\}$ of the trivial line bundle over S admits motivic weak equivalences

$$\mathbf{P}^1 \xrightarrow{\sim} \mathbf{P}^1/\mathcal{A}^1 \xleftarrow{\sim} T \tag{37}$$

The motivic space $\mathbf{P}^1$ itself is not always the ideal suspension coordinate. For example, the algebraic cobordism spectrum $\mathbf{MGL}$ naturally comes as a T-spectrum. In order to switch between T-spectra and $\mathbf{P}^1$-spectra, consider the following general construction. A map $\phi\colon A \to B$ induces a functor $\phi^*\colon \mathbf{MS}_B(S) \to \mathbf{MS}_A(S)$ sending the B-spectrum $(E_0, E_1, \dots, \sigma_n^E)$ to the A-spectrum

$$(E_0, E_1, \dots) \quad \text{with structure maps} \quad \sigma_n^{\phi^* E} = \sigma_n^E \circ (E_n \wedge \phi)$$

Its left adjoint $\phi_\sharp$ maps the A-spectrum $(F_0, F_1, \dots, \sigma_n^F)$ to the B-spectrum

$$\left(F_0, B \wedge F_0 \cup_{A \wedge F_0} F_1, B \wedge \left(B \wedge F_0 \cup_{A \wedge F_0} F_1 \right) \cup_{A \wedge F_1} F_2 \right), \dots \right) \tag{38}$$

having the canonical maps as structure maps. Note that for the purpose of constructing a model structure on A-spectra over S, the pointed motivic space A has to be cofibrant in the model structure under consideration.

The next goal is to construct a model structure on $\mathbf{MS}(S)$ having the motivic stable homotopy category as its homotopy category.

Definition A.5.4. *Let* $\Omega_{\mathbf{P}^1} = \underline{\operatorname{Hom}}_{\mathbf{M}_\bullet(S)}(\mathbf{P}^1_S, -)$ *denote the right adjoint of* $\mathbf{P}^1 \wedge -$. *For a* $\mathbf{P}^1$*-spectrum E with structure maps* $\sigma_n^E\colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$, *let* $\omega_n^E\colon E_n \to \Omega_{\mathbf{P}^1} E_{n+1}$ *denote the adjoint structure map. A* $\mathbf{P}^1$*-spectrum E is* closed stably fibrant *if*

- E_n *is closed motivic fibrant for every $n \geq 0$, and*

- $\omega_n^E\colon E_n \to \Omega_{\mathbf{P}^1} E_{n+1}$ *is a motivic weak equivalence for every $n \geq 0$.*

Any $\mathbf{P}^1$-spectrum E admits a closed stably fibrant replacement. First replace E by a levelwise $\mathbf{P}^1$-spectrum E^ℓ as follows. Let $E_0^\ell = E_0^{\mathrm{fib}}$ for a fibrant replacement in $\mathbf{M}_\bullet^{\mathrm{cm}}(S)$. Given $E_n \to E_n^\ell$, set

$$E_{n+1}^\ell := \left(E_n^\ell \wedge \mathbf{P}^1 \cup_{E_n \wedge \mathbf{P}^1} E_{n+1} \right)^{\mathrm{fib}}$$

which yields a levelwise motivic weak equivalence $E \to E^\ell$ of $\mathbf{P}^1$-spectra. To continue, observe that the adjoint structure maps of any $\mathbf{P}^1$-spectrum F may be viewed as a natural transformation

$$q\colon F \to Q(F)$$

where $Q(F)$ is the $\mathbf{P}^1$-spectrum with terms $\Omega_{\mathbf{P}^1} F_1, \Omega_{\mathbf{P}^1} F_2, \dots$ and structure maps

$$\Omega_{\mathbf{P}^1}(\omega_{n+1}^F)\colon \Omega_{\mathbf{P}^1} F_{n+1} \to \Omega_{\mathbf{P}^1}^2 F_{n+2}.$$

Define $Q^\infty(E)$ as the colimit of the sequence $E^\ell \xrightarrow{q} Q(E^\ell) \xrightarrow{Q(q)} Q^2(E^\ell) \longrightarrow \cdots$

Definition A.5.5. *A map $f\colon E \to F$ of $\mathbf{P}^1$-spectra is a* stable equivalence *if the map $Q^\infty(f)_n$ is a weak equivalence for every $n \geq 0$. It is a* closed stable fibration *if $f_n\colon E_n \to F_n$ is a closed motivic fibration and the induced map $E_n \to F_n \times_{Q^\infty(E)_n} Q^\infty(F)_n$ is a motivic weak equivalence for every $n \geq 0$. It is a* closed cofibration *if $f_n\colon E_n \to F_n$ and $F_n \wedge \mathbf{P}^1 \cup_{E_n \wedge \mathbf{P}^1} E_{n+1} \to F_{n+1}$ are closed cofibrations for every $n \geq 0$.*

Theorem A.5.6. *The classes from Definition A.5.5 are a model structure on the category of $\mathbf{P}^1$-spectra, denoted $\mathbf{MS}^{\mathrm{cm}}(S)$. The identity functor on $\mathbf{P}^1$-spectra from Jardine's stable model structure to $\mathbf{MS}^{\mathrm{cm}}(S)$ is a left Quillen equivalence. In particular, the homotopy category $\mathrm{SH}^{\mathrm{cm}}(S) := \mathrm{Ho}\big(\mathbf{MS}^{\mathrm{cm}}(S)\big)$ is equivalent to the motivic stable homotopy category $\mathrm{SH}(S)$.*

Proof. Recall that $\mathbf{P}^1$ is closed cofibrant. The existence of the model structure follows as in [J, Thm. 2.9]. Moreover, the stable equivalences coincide with the ones in [J], because so do the stabilization constructions and the unstable weak equivalences. Since every closed cofibration of motivic spaces is in particular a monomorphism, $\mathrm{Id}_{\mathbf{MS}(S)}$ is a left Quillen equivalence. Note that the closed cofibrations are generated by the set

$$\{\mathrm{Fr}_m\big(X \times \partial\Delta^n \hookrightarrow X \times \Delta^n)_+\big)\}_{m,n \geq 0, X \in \mathcal{S}m/S} \tag{39}$$

One may also describe a set of generating acyclic cofibrations. $\qquad\square$

Remark A.5.7. Note that one may form the smash product of a motivic space A and a $\mathbf{P}^1$-spectrum E by setting $(A \wedge E)_n := A \wedge E_n$ and $\sigma_n^{A \wedge E} := A \wedge \sigma_n^E$. If A is closed cofibrant, $A \wedge -$ is a left Quillen functor by Theorem A.3.9. From Theorem A.5.6 one may deduce that $\mathbf{P}^1 \wedge -\colon \mathbf{MS}(S) \to \mathbf{MS}(S)$ is a left Quillen equivalence. Since $\P \simeq S^{2,1} = S^1 \wedge (\mathcal{A}^1 - \{0\}, 1)$, also $S^1 \wedge -$ is a left Quillen equivalence. In particular, $\mathrm{SH}^{\mathrm{cm}}(S)$ is triangulated, with the total left derived of $S^1 \wedge -$ as the shift functor. The triangles are those which are isomorphic to the image of

$$E \hookrightarrow F \to F/E \leftarrow E \wedge \Delta^1 \cup_E F \to S^1 \wedge E$$

where $E \hookrightarrow F$ is an inclusion of $\mathbf{P}^1$-spectra. As well, one has sphere spectra $S^{p,q} \in \mathrm{SH}^{\mathrm{cm}}(S)$ for all integers $p, q \in \mathbb{Z}$.

Example A.5.8. Since $\mathrm{SH}(S)$ is an additive category, the canonical map $E \vee F \to E \times F$ is a stable equivalence. In the special case of $\mathbf{P}^1$-suspension spectra, the canonical map factors as

$$\Sigma_{\mathbf{P}^1}^\infty A \vee \Sigma_{\mathbf{P}^1}^\infty B \cong \Sigma_{\mathbf{P}^1}^\infty(A \vee B) \to \Sigma_{\mathbf{P}^1}^\infty(A \times B) \to \Sigma_{\mathbf{P}^1}^\infty A \times \Sigma_{\mathbf{P}^1}^\infty B$$

which shows that $\Sigma_{\mathbf{P}^1}^\infty(A \times B)$ contains $\Sigma_{\mathbf{P}^1}^\infty(A \wedge B)$ as a retract in $\mathrm{SH}(S)$. Thus it is even a direct summand. The latter can be deduced as follows. The (reduced) *join* $(A, a_0) * (B, b_0)$ is defined as the pushout in the diagram

$$
\begin{array}{ccc}
A \times B \times \partial\Delta^1 \cup \{a_0\} \times \{b_0\} \times \Delta^1 & \longrightarrow & A \vee B \\
\downarrow & & \downarrow \\
A \times B \times \Delta^1 & \longrightarrow & A * B.
\end{array}
$$

of pointed motivic spaces over S. Attaching $A \wedge (\Delta^1, 0)$ and $B \wedge (\Delta^1, 0)$ to $A * B$ via $A \vee B$ produces a pointed motivic space C which is equipped with a sectionwise weak equivalence $C \to (A \times B) \wedge S^{1,0}$. Collapsing $\{a_0\} * B$ and $A * \{b_0\}$ inside C yields a sectionwise weak equivalence $C \to (A \wedge B \wedge S^{1,0}) \vee (A \wedge S^{1,0}) \vee (B \wedge S^{1,0})$. Since $S^{1,0}$ is invertible in $\mathrm{SH}(S)$, one gets a splitting $\Sigma_{\mathbf{P}^1}^{\infty}(A \times B) \simeq (\Sigma_{\mathbf{P}^1}^{\infty}(A \wedge B)) \vee (\Sigma_{\mathbf{P}^1}^{\infty} A) \vee (\Sigma_{\mathbf{P}^1}^{\infty} B)$ in $\mathrm{SH}(S)$.

For a $\mathbf{P}^1$-spectrum E let $\mathrm{Tr}_n E$ denote the $\mathbf{P}^1$-spectrum with

$$
(\mathrm{Tr}_n E)_m = \begin{cases} E_m & m \leq n \\ E_n \wedge (\mathbf{P}^1)^{\wedge m-n} = (\mathrm{Fr}_n E_n)_m & m \geq n \end{cases}
$$

and with the obvious structure maps. The structure maps of E determine maps $\mathrm{Tr}_n E \to \mathrm{Tr}_{n+1} E$ such that $E = \mathrm{colim}_n \mathrm{Tr}_n E$. Since the canonical map $\mathrm{Fr}_n E_n \to \mathrm{Tr}_n E$ adjoint to the identity $\mathrm{id} \colon E_n \to E_n$ is an identity in all levels $\geq n$, it is in particular a stable equivalence. The identity $\mathrm{id} \colon E_n \wedge (\mathbf{P}^1)^{\wedge n} \to (\mathrm{Fr}_0 E_n)_n$ leads by adjointness to the map

$$
\mathrm{Fr}_n(E_n) \wedge (\mathbf{P}^1)^{\wedge n} \xrightarrow{\cong} \mathrm{Fr}_n\left(E_n \wedge (\mathbf{P}^1)^{\wedge n}\right) \longrightarrow \mathrm{Fr}_0 E_n \tag{40}
$$

and hence to a map $\mathrm{Fr}_n E_n \to \Omega_{\mathbf{P}^1}^n (\mathrm{Fr}_0 E_n)$. Since the map (40) is an isomorphism in all levels $\geq n$, it is a stable equivalence. Because $\Omega_{\mathbf{P}^1}$ is a Quillen equivalence, the map $\mathrm{Fr}_n E_n \to \Omega_{\mathbf{P}^1}^n ((\mathrm{Fr}_0 E_n)^{\mathrm{fib}})$ is a stable equivalence as well if E_n is closed cofibrant. In fact, the condition on E_n can be removed since Fr_n preserves all weak equivalences. This leads to the following statement.

Lemma A.5.9. *Any $\mathbf{P}^1$-spectrum E is the colimit of a natural sequence*

$$
\mathrm{Tr}_0 E \longrightarrow \mathrm{Tr}_1 E \longrightarrow \mathrm{Tr}_2 E \longrightarrow \cdots \tag{41}
$$

of $\mathbf{P}^1$-spectra in which the n-th term is naturally stably equivalent to $\Omega_{\mathbf{P}^1}^n ((\Sigma_{\mathbf{P}^1}^{\infty} E_n)^f)$.

One may use the description in Lemma A.5.9 for computations as follows. Say that a $\mathbf{P}^1$-spectrum F is *finite* if it is stably equivalent to a $\mathbf{P}^1$-spectrum F' such that $* \to E'$ is obtained by attaching finitely many cells from the set (39).

Lemma A.5.10. *Let $D(0) \to D(1) \to D(2) \to \cdots$ be a sequence of $\mathbf{P}^1$-spectra, with colimit $D(\infty)$.*

1. *Suppose that F is a finite $\mathbf{P}^1$-spectrum. The canonical map*

$$
\underset{i \geq 0}{\mathrm{colim}} \, \mathrm{Hom}_{\mathrm{SH}(S)}(F, D(i)) \to \mathrm{Hom}_{\mathrm{SH}(S)}(F, D(\infty))
$$

   is an isomorphism.

2. *For any $\mathbf{P}^1$-spectrum E there is a canonical short exact sequence*

$$
0 \to \underset{i \geq 0}{\lim{}^1} \left[S^{1,0} \wedge D(i), E\right] \to \left[D(\infty), E\right] \to \underset{i \geq 0}{\lim} \left[D(i), E\right] \to 0 \tag{42}
$$

   of abelian groups, where $[-, -]$ denotes $\mathrm{Hom}_{\mathrm{SH}(S)}(-, -)$.

Proof. Observe first that stable equivalences and closed stable fibrations are detected by the functor Q^∞ which is defined in A.5.5 as a sequential colimit. Lemma A.3.10 implies that stable equivalences and closed stable fibrations with closed stably fibrant codomain are closed under filtered colimits. Thus by Theorem A.2.1 one may compute

$$\mathrm{Hom}_{\mathrm{SH}(S)}(F, \operatorname*{colim}_{i \geq 0} D(i)) \cong \mathrm{Hom}_{\mathbf{MS}(S)}(F, \operatorname*{colim}_{i \geq 0} D(i)^{\mathrm{fib}})/\simeq$$

for any cofibrant $\mathbf{P}^1$-spectrum F, where $\simeq$ denotes the equivalence relation "simplicial homotopy". This implies statement 1 because $\mathrm{Hom}_{\mathbf{MS}(S)}(F, -)$ commutes with filtered colimits if $* \to F$ is obtained by attaching finitely many cells.

To prove the second statement, let C be the coequalizer of the diagram

$$\bigvee_{i \geq 0} D(i) \xrightarrow[g]{f} \bigvee_{i \geq 0} \Delta^1_+ \wedge D(i)$$

where f resp. g is defined on the i-th summand $D(i)$ as $D(i) = 1_+ \wedge D(i) \hookrightarrow \Delta^1_+ \wedge D(i)$ resp. $D(i) \to D(i+1) = 0_+ \wedge D(i+1) \hookrightarrow \Delta^1_+ \wedge D(i+1)$. The canonical map $C \to \mathrm{colim}_{i \geq 0} D(i)$ induced by the $\Delta^1_+ \wedge D(i) \to D(i) \to \mathrm{colim}_{i \geq 0} D(i)$ is a weak equivalence. In the stable homotopy category, which is additive, one may take the difference of f and g, and thus describe $\mathrm{colim}_{i \geq 0} D(i)$ via the distinguished triangle

$$\bigvee_{i \geq 0} D(i) \xrightarrow{f-g} \bigvee_{i \geq 0} D(i) \longrightarrow \mathrm{colim}_{i \geq 0} D(i) \longrightarrow \bigvee_{i \geq 0} S^{1,0} \wedge D(i). \qquad (43)$$

Applying $[-, E] := \mathrm{Hom}_{\mathrm{SH}(S)}(-, E)$ to the triangle (43) produces a long exact sequence

$$\cdots \xleftarrow{f^* - g^*} \prod_{i \geq 0} [D(i), E] \longleftarrow [D(\infty), E] \longleftarrow \prod_{i \geq 0} [S^{1,0} \wedge D(i), E] \longleftarrow \cdots$$

which may be split into the short exact sequence

$$0 \longleftarrow \lim_{i \geq 0} [D(i), E] \longleftarrow [D(\infty), E] \longleftarrow \lim^1_{i \geq 0} [S^{1,0} \wedge D(i), E] \longleftarrow 0$$

$\square$

A.6   Symmetric spectra

There seems to be no reasonable (i.e. symmetric monoidal) smash product for $\mathbf{P}^1$-spectra inducing a decent symmetric monoidal smash product on $\mathrm{SH}(S)$. This will be solved as in [HSS] and [J].

Definition A.6.1. *A* symmetric $\mathbf{P}^1$-spectrum E over S *consists of a sequence* $(E_0, E_1, \ldots)$ *of pointed motivic spaces over* S, *together with group actions* $(\Sigma_n)_+ \wedge E_n \to E_n$ *and structure maps* $\sigma^E_n \colon E_n \wedge \mathbf{P}^1 \to E_{n+1}$ *for all* $n \geq 0$. *Iterations of these structure maps are required to be as equivariant as they can, using the permutation action of* Σ_n *on* $(\mathbf{P}^1)^{\wedge n}$. *A map of motivic symmetric* $\mathbf{P}^1$-spectra *is a sequence of maps of pointed motivic spaces which is compatible with all the structure (group actions and structure maps). Call the resulting category* $\mathbf{MSS}(S)$.

Example A.6.2. Analogous to Example A.5.2, the n-th shifted suspension spectrum $\mathrm{Fr}_n^\Sigma A$ of a pointed motivic space A has as values

$$(\mathrm{Fr}_n^\Sigma B)_{m+n} = \begin{cases} \Sigma_{m+n}^+ \wedge_{\Sigma m \times \{1\}} A \wedge (\mathbf{P}^1)^{\wedge m} & m \geq 0 \\ * & m < 0 \end{cases}$$

where the m-th fold smash product $(\mathbf{P}^1)^{\wedge m}$ carries the natural permutation action.

Every symmetric $\mathbf{P}^1$-spectrum determines a $\mathbf{P}^1$-spectrum by forgetting the symmetric group actions. Call the resulting functor $u \colon \mathbf{MSS}(S) \to \mathbf{MS}(S)$. It has a left adjoint v, which is characterized uniquely up to unique isomorphism by the fact that

$$v\big(\mathrm{Fr}_n A\big) = \mathrm{Fr}_n^\Sigma A. \tag{44}$$

The smash product $E \wedge F$ of two symmetric $\mathbf{P}^1$-spectra E and F is constructed as follows. Set $(E \wedge F)_n$ as the coequalizer of the diagram

$$\coprod_{r+1+s=n} \Sigma_n^+ \wedge_{\Sigma_r \times \Sigma_1 \times \Sigma_s} E_r \wedge \mathbf{P}^1 \wedge F_s \xrightarrow[\sigma_r^E \wedge (F_s \circ \mathrm{twist})]{\sigma_r^E \wedge F_s} \coprod_{r+s=n} \Sigma_n^+ \wedge_{\Sigma_r \times \Sigma_s} E_r \wedge F_s \tag{45}$$

where the coequalizer is taken in the category of pointed Σ_n-motivic spaces. The structure map $\sigma_n^{E \wedge F}$ is induced by the structure maps $\sigma_0^F, \ldots, \sigma_n^F$ of F. One may provide natural coherence isomorphisms for associativity, commutativity and unitality, where the unit is $\mathbb{I}_S = (S_+, \mathbf{P}^1, \mathbf{P}^1 \wedge \mathbf{P}^1, \ldots, (\mathbf{P}^1)^{\wedge n}, \ldots)$ with the obvious permutation action and identities as structure maps.

We proceed with the homotopy theory of symmetric $\mathbf{P}^1$-spectra, as in [J].

Definition A.6.3. *A map $\phi \colon E \to F$ of motivic symmetric $\mathbf{P}^1$-spectra is a* levelwise acyclic fibration *if $\phi_n \colon E_n \to F_n$ is an acyclic closed motivic fibration of pointed motivic spaces over S for all $n \geq 0$. A map $\phi \colon E \to F$ of motivic symmetric $\mathbf{P}^1$-spectra is a* closed cofibration *if it has the left lifting property with respect to all levelwise acyclic fibrations. There exists a cofibrant replacement functor $(-)^{\mathrm{cof}} \to \mathrm{Id}_{\mathbf{MSS}(S)}$. A levelwise fibrant motivic symmetric $\mathbf{P}^1$-spectrum E is* closed stably fibrant *if the adjoint $E_n \to \mathbf{M}_X(\P, E_{n+1})$ of the structure map is a weak equivalence for every $n \geq 0$. A map $\phi \colon E \to F$ is a* stable equivalence *if the map*

$$\mathbf{sSet}_{\mathbf{MSS}_X}(\phi^{\mathrm{cof}}, G) \colon \mathbf{sSet}_{\mathbf{MSS}_X}(F^{\mathrm{cof}}, G) \to \mathbf{sSet}_{\mathbf{MSS}_X}(E^{\mathrm{cof}}, G)$$

is a weak equivalence of pointed motivic spaces for all closed stably fibrant motivic symmetric $\mathbf{P}^1$-spectra G. The closed stable fibrations *are then defined by the right lifting property.*

Theorem A.6.4 (Jardine)**.** *The classes of stable equivalences, closed cofibrations and closed stable fibrations from Definition A.6.3 constitute a symmetric monoidal model structure on $\mathbf{MSS}(S)$. The forgetful functor $u \colon \mathbf{MSS}^{\mathrm{cm}}(S) \to \mathbf{MS}^{\mathrm{cm}}(S)$ is a right Quillen equivalence.*

Proof. The proof of the first statement follows as in [J, Thm. 4.15, Prop. 4.19]. Note that the stable equivalences in $\mathbf{MSS}^{\mathrm{cm}}(S)$ and in Jardine's model structure $\mathbf{MSS}^{\mathrm{Jar}}(S)$ coincide, since the unstable weak equivalences do so by Remark A.3.12. In particular, the identity $\mathrm{Id}\colon \mathbf{MSS}^{\mathrm{cm}}(S) \to \mathbf{MSS}^{\mathrm{Jar}}(S)$ is a left Quillen equivalence. The closed cofibrations in $\mathbf{MSS}^{\mathrm{cm}}(S)$ are generated by the inclusions

$$\{\mathrm{Fr}_m^{\Sigma}\big(X \times \partial\Delta^n \hookrightarrow X \times \Delta^n)_+\big)\}_{m,n\geq 0, X\in \mathcal{S}m/S} \tag{46}$$

By formula (44), the left adjoint v of u sends the generating cofibrations to the generating cofibrations. It follows that v is a left Quillen functor. Since $v\colon \mathbf{MS}^{\mathrm{Jar}}(S) \to \mathbf{MSS}^{\mathrm{Jar}}(S)$ is a Quillen equivalence by [J, Thm. 4.31], so is the functor $v\colon \mathbf{MS}^{\mathrm{cm}}(S) \to \mathbf{MSS}^{\mathrm{cm}}(S)$. $\qquad\square$

Remark A.6.5. In particular, the category $\mathrm{Ho}(\mathbf{MS}^{\mathrm{cm}}(S))$ inherits a closed symmetric monoidal product $\wedge$ by setting

$$E \wedge F := \mathcal{R}u\big(\mathcal{L}v(E) \wedge \mathcal{L}v(F)\big)$$

In other words, if E and F are closed cofibrant $\mathbf{P}^1$-spectra, their smash product in $\mathrm{SH}(S)$ is given by the $\mathbf{P}^1$-spectrum $u\big((v(E) \wedge v(F))^{\mathrm{fib}}\big)$. The unit is $\mathcal{R}u\big(\mathcal{L}v(\mathbb{I})\big) \cong \mathbb{I}$ the sphere $\mathbf{P}^1$-spectrum.

Notation A.6.6. *For $\mathbf{P}^1$-spectra E and F over S define the E-cohomology and the E-homology of F as*

$$E^{p,q}(F) = \mathrm{Hom}_{\mathrm{SH}(S)}(F, S^{p,q} \wedge E) \tag{47}$$

$$E_{p,q}(F) = \mathrm{Hom}_{\mathrm{SH}(S)}(S^{p,q}, F \wedge E). \tag{48}$$

In the special case $F = \Sigma_{\mathbf{P}^1}^{\infty} A$, where A is a pointed motivic space over S one writes $E^{p,q}(A)$ and $E_{p,q}(A)$ instead. Sometimes we abbreviate $S^{2n,n} \wedge E$ by $E(n)$.

Remark A.6.7. Since (v, u) is a Quillen adjoint pair of stable model categories the total derived pair respects in particular the triangulated structures. Note that since u preserves all colimits, both $\mathcal{L}v$ and $\mathcal{R}u$ preserve arbitrary coproducts.

Lemma A.6.8. *Let E and F be $\mathbf{P}^1$-spectra. Then $E \wedge F \in \mathrm{SH}(S)$ may be obtained as the sequential colimit of a sequence whose n-th term is stably equivalent to $\Omega_{\mathbf{P}^1}^{2n}(\Sigma_{\mathbf{P}^1}^{\infty} E_n \wedge F_n)^{\mathrm{fib}}$.*

Proof. We may assume that both E and F are closed cofibrant. By Lemma A.5.9 E and F can be expressed as sequential colimits of their truncations. Since v preserves colimits, $v(E)$ is the sequential colimit of the diagram

$$v(\mathrm{Tr}_0 E) = v(\mathrm{Fr}_0 E_0) = \mathrm{Fr}_0^{\Sigma} E_0 \to v(\mathrm{Tr}_1 E) \to \cdots$$

and similarly for F. The stable equivalence $\mathrm{Fr}_n E_n \to \mathrm{Tr}_n E$ of cofibrant $\mathbf{P}^1$-spectra induces a stable equivalence $v(\mathrm{Fr}_n E_n) = \mathrm{Fr}_n^{\Sigma} E_n \to v(\mathrm{Tr}_n E)$. Since smashing with a symmetric $\mathbf{P}^1$-spectrum preserves colimits, one has

$$v(\mathrm{Tr}_m E) \wedge \operatorname*{colim}_n v(\mathrm{Tr}_n F) \cong \operatorname*{colim}_n \big(v(\mathrm{Tr}_m E) \wedge v(\mathrm{Tr}_n F)\big)$$

for every n. It follows that $v(E) \wedge v(F)$ is the filtered colimit of the diagram sending (m, n) to $v(\mathrm{Tr}_m E) \wedge v(\mathrm{Tr}_n F)$. Since the diagonal is a final subcategory in $\mathbb{N} \times \mathbb{N}$, there is a canonical isomorphism $\mathrm{colim}_n v(\mathrm{Tr}_n E) \wedge v(\mathrm{Tr}_n F) \cong v(E) \wedge v(F)$. Theorem A.6.4 says that $\mathbf{MSS}^{\mathrm{cm}}(S)$ is symmetric monoidal, thus the canonical map

$$\mathrm{Fr}_{2n}^{\Sigma}(E_n \wedge F_n) \cong \mathrm{Fr}_n^{\Sigma} E_n \wedge_n^{\Sigma} E_n \to v(\mathrm{Tr}_n E) \wedge v(\mathrm{Tr}_n F) \tag{49}$$

is a stable equivalence. There is a canonical map $\mathrm{Fr}_{2n}^{\Sigma}(E_n \wedge F_n) \to \Omega_{\mathbf{P}^1}^{2n} \mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n)$ which is adjoint to the unit $E_n \wedge F_n \to \Omega_{\mathbf{P}^1}^{2n}(\mathbf{P}^1)^{\wedge 2n} \wedge (E_n \wedge F_n)$. As in the case of $\mathbf{P}^1$-spectra, the map

$$\mathrm{Fr}_{2n}^{\Sigma}(E_n \wedge F_n) \to \Omega_{\mathbf{P}^1}^{2n} \mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n) \to \Omega_{\mathbf{P}^1}^{2n} \left(\mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n) \right)^{\mathrm{fib}}$$

is a stable equivalence. It follows that $v(E) \wedge v(F)$ is the colimit of a sequence whose n-th term is stably equivalent to $\Omega_{\mathbf{P}^1}^{2n} \left(\mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n) \right)^{\mathrm{fib}}$. Hence it also follows that a fibrant replacement of $v(E) \wedge v(F)$ may be obtained as the colimit of a sequence of closed stably fibrant symmetric $\mathbf{P}^1$-spectra whose n-th term is stably equivalent to $\Omega_{\mathbf{P}^1}^{2n} \left(\mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n) \right)^{\mathrm{fib}}$. Since the forgetful functor u preserves colimits, stable equivalences of closed stably fibrant symmetric $\mathbf{P}^1$-spectra and $\Omega_{\mathbf{P}^1}$, the $\mathbf{P}^1$-spectrum $u\left(v(E) \wedge v(F) \right)$ is the colimit of a sequence of $\mathbf{P}^1$-spectra whose n-th term is stably equivalent to $\Omega_{\mathbf{P}^1}^{2n} u\left(\left(\mathrm{Fr}_0^{\Sigma}(E_n \wedge F_n) \right)^{\mathrm{fib}} \right)$. The map

$$\mathrm{Fr}_0(E_n \wedge F_n) \to u\left(\left(v\mathrm{Fr}_0(E_n \wedge F_n) \right)^{\mathrm{fib}} \right)$$

is a stable equivalence because (v, u) is a Quillen equivalence A.6.4, whence the result. $\square$

As in the case of non-symmetric spectra, one may change the suspension coordinate A.5.3. If A is a pointed motivic space over S, let $\mathbf{MSS}_A(S)$ denote the category of symmetric A-spectra over S.

Lemma A.6.9. *A map $A \to B$ in $\mathbf{M}_\bullet(S)$ induces a strict symmetric monoidal functor $\mathbf{MSS}_A(S) \to \mathbf{MSS}_B(S)$ having a right adjoint. If the map is a motivic weak equivalence of closed cofibrant pointed motivic spaces, this pair is a Quillen equivalence.*

Proof. This is quite formal. For a proof consider [Ho2, Thm. 9.4]. $\square$

Because the change of suspension coordinate functors are lax symmetric monoidal, they preserve (commutative) monoid objects, that is, (commutative) symmetric ring spectra.

A.7 Stable topological realization

Let $\mathrm{Sp} = \mathrm{Sp}(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ be the category of $\mathbb{C}\mathbf{P}^1$-spectra (in $\mathbf{Top}$). An object in Sp is thus a sequence of pointed compactly generated topological spaces $E_0, E_1, \ldots$ with structure maps $E_n \wedge \mathbb{C}\mathbf{P}^1 \to E_{n+1}$. The model structure on Sp is obtained as follows: Cofibrations are generated by

$$\{\mathrm{Fr}_m^{\mathbf{Top}}(|\partial \Delta^n \hookrightarrow \Delta^n|_+)\}_{m,n \geq 0}$$

so that every $\mathbb{C}\mathbf{P}^1$-spectrum E has a cofibrant replacement $E^{\mathrm{cof}} \to E$ mapping to E via a levelwise acyclic Serre fibration. For any $\mathbb{C}\mathbf{P}^1$-spectrum E and any $n \in \mathbb{Z}$ let

$$\pi_n E := \operatorname*{colim}_{2m+n \geq 0,\, m \geq 0} \pi_{n+2m} E_m \to \pi_{n+2m+2} E_m \wedge \mathbb{C}\mathbf{P}^1 \to \pi_{n+2(m+1)} E_{1+m} \to \cdots$$

be the n-th stable homotopy group of E. Note that homotopy groups of non-degenerately based compactly generated topological spaces commute with filtered colimits. If E is cofibrant, E_n is in particular non-degenerately based for all $n \geq 0$. A map $f \colon E \to F$ of $\mathbb{C}\mathbf{P}^1$-spectra is a *stable equivalence* if the induced map $\pi_n f \colon \pi_n E^{\mathrm{cof}} \to \pi_n F^{\mathrm{cof}}$ is an isomorphism for all $n \in \mathbb{Z}$. It is a *stable fibration* if it has the right lifting property with respect to all stable acyclic cofibrations.

Similarly, one may form the category $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ of symmetric $\mathbb{C}\mathbf{P}^1$-spectra in $\mathbf{Top}$. Cofibrations are generated by

$$\{\mathrm{Fr}_m^{\mathbf{Top},\Sigma}(|\partial\Delta^n \hookrightarrow \Delta^n|_+)\}_{m,n \geq 0}$$

and a symmetric $\mathbb{C}\mathbf{P}^1$-spectrum is *stably fibrant* if its underlying $\mathbb{C}\mathbf{P}^1$-spectrum is stably fibrant. A map $f \colon E \to F$ of symmetric $\mathbb{C}\mathbf{P}^1$-spectra is a *stable equivalence* if the induced map $\mathbf{sSet}_{\mathrm{Sp}^\Sigma}(f^{\mathrm{cof}}, G)$ of simplicial sets of maps is an isomorphism for all stably fibrant symmetric $\mathbb{C}\mathbf{P}^1$-spectra G. It is a *stable fibration* if it has the right lifting property with respect to all stable acyclic cofibrations.

Theorem A.7.1. *Stable equivalences, stable fibrations and cofibrations form (symmetric monoidal) model structures on the categories of (symmetric) $\mathbb{C}\mathbf{P}^1$-spectra in $\mathbf{Top}$. The functor forgetting the symmetric group actions is a right Quillen equivalence. There is a zig-zag of strict symmetric monoidal left Quillen functors connecting $\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$ and $\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1)$. In particular, the homotopy category of (symmetric) $\mathbb{C}\mathbf{P}^1$-spectra is equivalent as a closed symmetric monoidal and triangulated category to the stable homotopy category.*

Proof. The statement about the model structures follows as in [HSS] if one replaces S^1 by $\mathbb{C}\mathbf{P}^1$ and simplicial sets by compactly generated topological spaces. The same holds for the statement about the functor forgetting the symmetric group actions. To construct the zig-zag, consider the corresponding stable model structure on the category of symmetric S^1-spectra in the category $\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1)$, which is isomorphic as a symmetric monoidal model category to the category of symmetric $\mathbb{C}\mathbf{P}^1$-spectra in the category $\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1)$ of topological symmetric S^1-spectra. The suspension spectrum functors give a zig-zag

$$\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1) \to \mathrm{Sp}^\Sigma\big(\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1), S^1\big) \cong \mathrm{Sp}^\Sigma\big(\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1)\mathbb{C}\mathbf{P}^1\big) \leftarrow \mathrm{Sp}^\Sigma(\mathbf{Top}, S^1) \tag{50}$$

of strict symmetric monoidal left Quillen functors. Since $\mathbb{C}\mathbf{P}^1 \wedge -$ is a left Quillen equivalence on the left hand side in the zig-zag (50) and $S^1 \wedge S^1 \cong \mathbb{C}\mathbf{P}^1$, $S^1 \wedge -$ is a left Quillen equivalence on the left hand side as well. By [Ho2, Theorem 9.1], the arrow pointing to the right in the zig-zag (50) is a Quillen equivalence. A similar argument works for the arrow on the right hand side, which completes the proof. $\qquad\square$

Given a $\mathbf{P}^1$-spectrum E over $\mathbb{C}$, one gets a $\mathbb{C}\mathbf{P}^1$-spectrum $\mathbf{R}_\mathbb{C}(E) = (\mathbf{R}_\mathbb{C}E_0, \mathbf{R}_\mathbb{C}E_1, \dots)$ with structure maps $\mathbf{R}_\mathbb{C}(E_n) \wedge \mathbf{P}^1 \cong \mathbf{R}_\mathbb{C}(E_n \wedge \mathbf{P}^1) \to \mathbf{R}_\mathbb{C}(E_{n+1})$. The right adjoint for the resulting functor $\mathbf{R}_\mathbb{C} \colon \mathbf{MS}(\mathbb{C}) \to \mathrm{Sp}$ is also obtained by a levelwise application of $\mathrm{Sing}_\mathbb{C}$. The same works for symmetric $\mathbf{P}^1$-spectra over $\mathbb{C}$.

Theorem A.7.2. *The functors $\mathbf{R}_\mathbb{C} \colon \mathbf{MS}(\mathbb{C}) \to \mathrm{Sp}$ and $\mathbf{R}_\mathbb{C} \colon \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^\Sigma$ are left Quillen functors, the latter being strict symmetric monoidal.*

Proof. Since the diagrams

$$
\begin{array}{ccccc}
\mathbf{MSS}(\mathbb{C}) & \xrightarrow{\ u\ } & \mathbf{MS}(\mathbb{C}) & \xrightarrow{E \mapsto E_n} & \mathbf{M}_\bullet(\mathbb{C}) \\
{\scriptstyle \mathrm{Sing}_\mathbb{C}} \downarrow & & {\scriptstyle \mathrm{Sing}_\mathbb{C}} \downarrow & & \downarrow {\scriptstyle \mathrm{Sing}_\mathbb{C}} \\
\mathrm{Sp}^\Sigma & \xrightarrow{\ u\ } & \mathrm{Sp} & \xrightarrow{E \mapsto E_n} & \mathbf{Top}_\bullet
\end{array}
$$

commute, $\mathbf{R}_\mathbb{C}$ preserves the generating cofibrations by A.4.1. Then Dugger's Lemma [D, Cor. A.2] implies that $\mathbf{R}_\mathbb{C}$ is a left Quillen functor, because $\mathrm{Sing}_\mathbb{C}$ preserves weak equivalences and fibrations between fibrant objects. The fact that $\mathbf{R}_\mathbb{C} \colon \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^\Sigma$ is strict symmetric monoidal follows from the definition of the smash product (45). $\qquad\square$

Example A.7.3. Let BGL be the $\mathbf{P}^1$-spectrum over $\mathbb{C}$ constructed in 1.2. Its n-th term is a closed cofibrant pointed motivic space $\mathcal{K}$ weakly equivalent to $\mathbb{Z} \times \mathrm{Gr}$. Thus the n-th term of $\mathbf{R}_\mathbb{C}(\mathrm{BGL})$ is weakly equivalent to $\mathrm{B}U$. To show that the $\mathbb{C}\mathbf{P}^1$-spectrum $\mathbf{R}_\mathbb{C}(\mathrm{BGL})$ is the one representing complex K-theory, it suffices to check that the structure map $\mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ realizes to the structure map $\mathrm{B}U \wedge \mathbb{C}\mathbf{P}^1 \to \mathrm{B}U$ of complex K-theory. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{H}_\bullet(\mathbb{C})}(\mathbb{Z} \times \mathrm{Gr}, \mathbb{Z} \times \mathrm{Gr}) & \xrightarrow{\ \cong\ } & K_0^{\mathrm{alg}}(\mathbb{Z} \times \mathrm{Gr}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_\bullet)}\big(\mathbf{R}_\mathbb{C}(\mathbb{Z} \times \mathrm{Gr}), \mathbf{R}_\mathbb{C}(\mathbb{Z} \times \mathrm{Gr})\big) & \xrightarrow{\ \cong\ } & K_0^{\mathrm{top}}\big(\mathbf{R}_\mathbb{C}(\mathbb{Z} \times \mathrm{Gr})\big)
\end{array}
$$

where the vertical map on the left hand side is induced by $\mathbf{R}_\mathbb{C}$ and the vertical map on the right hand side is induced by the passage from algebraic to topological complex vector bundles. The latter is a ring homomorphism, and in particular preserves the unit, which is the image of the identity map under the horizontal isomorphisms. By naturality, it follows that the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{H}_\bullet(\mathbb{C})}(A, \mathbb{Z} \times \mathrm{Gr}) & \xrightarrow{\ \cong\ } & K_0^{\mathrm{alg}}(\mathbb{Z} \times \mathrm{Gr}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_\bullet)}\big(\mathbf{R}_\mathbb{C}(A), \mathbf{R}_\mathbb{C}(\mathbb{Z} \times \mathrm{Gr})\big) & \xrightarrow{\ \cong\ } & K_0^{\mathrm{top}}\big(\mathbf{R}_\mathbb{C}(\mathbb{Z} \times \mathrm{Gr})\big)
\end{array}
\tag{51}
$$

commutes for every pointed motivic space A over $\mathbb{C}$. In particular, the structure map of BGL which corresponds to $(\tau_\infty - \infty) \otimes ([\mathcal{O}(-1)] - [\mathcal{O}])$ maps to the structure map of the complex K-theory spectrum, since it corresponds to the same bundle, viewed as a topological bundle.

Proposition A.7.4. *A morphism $f\colon S \to S'$ of base schemes induces a strict symmetric monoidal left Quillen functor*

$$f^*\colon \mathbf{MSS}(S') \to \mathbf{MSS}(S)$$

such that $\bigl(f^*(E)\bigr)_n = f^*(E_n)$.

Proof. The structure maps of $f^*(E)$ are defined via the canonical map

$$f^*(E)_n \wedge \mathbf{P}^1_{S'} \cong f^*(E_n) \wedge f^*(\mathbf{P}^1_S) \cong f^*(E_n \overset{f^*(\sigma^E_n)}{\wedge} \mathbf{P}^1_S) \longrightarrow f^*(E_{n+1}) = f^*(E)_{n+1}.$$

It follows that f^* has the functor f_* as right adjoint, where $\bigl(f_*(E)\bigr)_n = f_*(E_n)$ and

$$\sigma^{f_*E}_n = f_*E_n \wedge \mathbf{P}^1_{S'} \longrightarrow f_*E_n \wedge f_*f^*\mathbf{P}^1_{S'} \cong f_*E_n \wedge f_*\mathbf{P}^1_S \longrightarrow f_*(E_n \wedge \mathbf{P}^1_S) \overset{f_*\sigma^E_n}{\longrightarrow} f_*E_{n+1}.$$

Theorem A.3.9 and Dugger's lemma imply that f^* preserves cofibrations and f_* preserves fibrations. Since $f^*\colon \mathbf{M}_\bullet(S') \to \mathbf{M}_\bullet(S)$ is strict symmetric monoidal and preserves all colimits, then so is $f^*\colon \mathbf{MSS}(S') \to \mathbf{MSS}(S)$ by the definition of the smash product (45). $\qquad\square$

In particular, any complex point $f\colon \mathrm{Spec}(\mathbb{C}) \to S$ of a base scheme S induces a strict symmetric monoidal left Quillen functor

$$\mathbf{MSS}(S) \to \mathbf{MSS}(\mathbb{C}) \to \mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}\mathbf{P}^1) \tag{52}$$

to the category of topological $\mathbb{C}\mathbf{P}^1$-spectra.

B   Products in K-theory

One of the aim of the Appendix is to prove Proposition 1.2.2. For that we need Proposition B.1.1 and Corollary B.1.2.

B.1   Cellular schemes

Suppose that S is a regular base scheme. Recall that an S-cellular scheme is an S-scheme X equipped with a filtration $X_0 \subset X_1 \subset \cdots \subset X_n = X$ by closed subsets such that for every integer $i \geq 0$ the S-scheme $X_i \setminus X_{i-1}$ is a disjoint union of several copies of the affine space $\mathbf{A}^i_S$. We do not assume that X is connected. *A pointed S-cellular scheme* is an S-cellular scheme equipped with a closed S-point $x\colon S \hookrightarrow X$ such that $s(X)$ is contained in one of the open cells (a cell which is an open subscheme of X). The examples we are interested in are Grassmannians, projective lines and their products. We need the following proposition which will be proved below. Its proof uses only cup-products of the form $K_i \otimes K_0 \to K_i$. The latter are easily defined without knowing the Waldhausen products since the tensor product with a vector bundle is an exact functor.

Proposition B.1.1. *Let (X, x) and (Y, y) be pointed smooth S-schemes. Then the sequence*

$$0 \to K_0^{\mathrm{TT}}(X \wedge Y) \to K_0^{\mathrm{TT}}(X \times Y) \to K_0^{\mathrm{TT}}(X \vee Y) \to 0$$

is short exact.

Proof. This follows from the isomorphism $K_0^{\mathrm{TT}} \cong \mathrm{BGL}^{0,0}$ and Example A.5.8. □

Corollary B.1.2. *Under the assumptions of Proposition B.1.1 let $a \in K_0^{\mathrm{TT}}(X)$ and $b \in K_0^{\mathrm{TT}}(Y)$ be such that $x^*(a) = 0 = y^*(b)$ in $K_0^{\mathrm{TT}}(S)$. Then the element $a \otimes b \in K_0^{\mathrm{TT}}(X \times Y)$ belongs to the subgroup $K_0^{\mathrm{TT}}(X \wedge Y)$. Furthermore the map*

$$K_0^{\mathrm{TT}}(X/x(S)) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(Y/y(S)) \to K_0^{\mathrm{TT}}(X \wedge Y)$$

is an isomorphism.

Proof. In fact, clearly the pull-back map $K_0^{\mathrm{TT}}(X \vee Y) \to K_0^{\mathrm{TT}}(X) \oplus K_0^{\mathrm{TT}}(Y)$ is injective. Since $a \otimes b$ vanishes on $x(S) \times Y$ and on $X \times x(S)$ it follows that $a \otimes b$ vanishes on $X \vee Y$. Whence $a \otimes b \in K_0^{\mathrm{TT}}(X \wedge Y)$. We skip a proof of the last assertion of the Corollary since we do not need it. □

With the Corollary B.1.2 in hand we are ready to prove Proposition 1.2.2. For that we use below the notation of Section 1.2. Consider the diagram

$$\mathbb{K}^{\mathrm{TT}}(X) \wedge \mathbb{K}^{\mathrm{TT}}(X) = \Omega_s^1(W_1(X)) \wedge \Omega_s^1(W_1(X)) \xrightarrow{m} \Omega_s^2(W_2(X)) \xleftarrow{ad} \mathbb{K}^{\mathrm{TT}}(X) \qquad (53)$$

with the Waldhausen morphisms m and the adjunction weak equivalence ad. The diagram defines a morphism

$$\mathbb{K}^{\mathrm{TT}} \wedge \mathbb{K}^{\mathrm{TT}} \xrightarrow{\mu^W} \mathbb{K}^{\mathrm{TT}} \qquad (54)$$

in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ which is the Waldhausen multiplication on $\mathbb{K}^{\mathrm{TT}}$.

We claim that the morphism μ^{TT} given by (5) coincides with the Waldhausen multiplication μ^W on $\mathbb{K}^{\mathrm{TT}}$. In fact, the Waldhausen multiplication μ^W has the property that for any smooth X over S the pairing $K_0^{\mathrm{TT}}(X) \times K_0^{\mathrm{TT}}(X) \to K_0^{\mathrm{TT}}(X)$ coincides with the one given by the tensor product of vector bundles. Taking $X = \mathbb{Z} \times \mathrm{Gr}$ and using Corollary B.1.2 we see that the composite morphism

$$(\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \xrightarrow{w \wedge w} \mathbb{K}^{\mathrm{TT}} \wedge \mathbb{K}^{\mathrm{TT}} \xrightarrow{\mu^W} \mathbb{K}^{\mathrm{TT}}$$

represents the class $(\tau_\infty - \infty) \otimes (\tau_\infty - \infty) \in K_0^{\mathrm{TT}}((\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}))$. However there exists a unique class in $[\mathbb{K}^{\mathrm{TT}} \wedge \mathbb{K}^{\mathrm{TT}}, \mathbb{K}^{\mathrm{TT}}]_{\mathrm{H}_\bullet(S)}$ with this property since $w \wedge w$ is an isomorphism in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$. Whence $\mu^W = \mu^{\mathrm{TT}}$. Similarly the known unit $S^0 \to \mathbb{K}^{\mathrm{TT}}$ on the Thomason-Trobaugh K-theory coincides in $\mathrm{H}_\bullet^{\mathrm{cm}}(S)$ with e^{TT} defined above. *Whence the monoidal structure (7) on the space $\mathbb{K}^{\mathrm{TT}}$ coincides with the Waldhausen one.* Proposition 1.2.2 is proved.

The following lemma is useful also.

Lemma B.1.3. *Let (X, x) and (Y, y) be pointed smooth S-cellular schemes. Then the map*

$$K_i^{\mathrm{TT}}(S) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(X \wedge Y) \to K_i^{\mathrm{TT}}(X \wedge Y)$$

is an isomorphism and $K_0^{\mathrm{TT}}(X \wedge Y)$ is a projective $K_0^{\mathrm{TT}}(S)$-module.

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
K_i(X \wedge Y) & \xrightarrow{\ \alpha\ } & K_i(X \times Y) & \xrightarrow{\ \beta\ } & K_i(X \vee Y) \\
\Big\uparrow{\scriptstyle\epsilon} & & \Big\uparrow{\scriptstyle\rho} & & \Big\uparrow{\scriptstyle\theta} \\
K_i(S) \otimes K_0(X \wedge Y) & \xrightarrow{\ \gamma\ } & K_i(S) \otimes K_0(X \times Y) & \xrightarrow{\ \delta\ } & K_i(S) \otimes K_0(X \vee Y)
\end{array}
$$

in which K_i is written for K_i^{TT} and the tensor product is taken over $K_0(S)$.

The sequence

$$0 \to K_i(X \wedge Y) \xrightarrow{\alpha} K_i(X \times Y) \xrightarrow{\beta} K_i(X \vee Y) \to 0$$

is short exact as was checked in the proof of Proposition B.1.1. In particular it is short exact for $i = 0$. Now the sequence

$$0 \to K_i(S) \otimes K_0(X \wedge Y) \xrightarrow{\gamma} K_i(S) \otimes K_0(X \times Y) \xrightarrow{\delta} K_i(S) \otimes K_0(X \vee Y) \to 0$$

is short exact since $K_0(X \vee Y)$ is a projective $K_0(S)$-module.

The arrows ρ and θ are isomorphisms by Lemmas B.1.4 and B.1.6 below respectively. Whence ϵ is an isomorphism as well. Finally $K_0(X \wedge Y)$ is a projective $K_0(S)$-module since the sequence $0 \to K_0(X \wedge Y) \to K_0(X \times Y) \to K_0(X \vee Y) \to 0$ is short exact and $K_0(X \times Y)$ and $K_0(X \vee Y)$ are projective $K_0(S)$-modules. $\qquad\square$

Lemma B.1.4. *Let X be a smooth S-cellular scheme. Then the map*

$$K_r^{\mathrm{TT}}(S) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(X) \to K_r^{\mathrm{TT}}(X)$$

is an isomorphism and $K_0^{\mathrm{TT}}(X)$ is a free $K_0^{\mathrm{TT}}(S)$-module of rank equal to the number of cells.

Proof. The Lemma easily follows from a slightly different claim.

Claim B.1.5. *Under the assumption of the Lemma the map of Quillen's K-groups*

$$K_r(S) \otimes_{K_0(S)} K_0'(X_j) \to K_r'(X_j)$$

is an isomorphism and $K_0'(X_j)$ is a free $K_0(S)$-module of the expected rank.

We prove the Claim induction by j. If $j = 0$, then $X = X_0$ is a disjoint union of several copies of the base scheme S. So the Claim is obvious in this case. Assume the Claim holds for all $j \le d$ and prove it for $j = d + 1$. For that consider the subset $X_{d+1} - X_d$. It is a disjoint union of of several copies U_i of the affine space $\mathbf{A}_S^{d+1}$. Each of the copy is an

open subset of X_{d+1}. Thus $X_d^1 := X_d \cup U_1$ is a closed subset of X_{d+1}. Consider the long localization sequence of Quillen's K'-groups [Qu]

$$\cdots \to K_r'(X_d) \xrightarrow{i_*} K_r'(X_d^1) \xrightarrow{j^*} K_r'(U_1) \to \dots$$

where i_* is the push-forward map and j^* is the pull-back map. Let $p : X_1 \to S$ be the structure map. Then $j^* \circ p^*$ is an isomorphism since S is regular. Thus j^* is surjective. So i_* is injective and the sequence splits in short exact sequences (split ones)

$$0 \to K_r'(X_d) \xrightarrow{i_*} K_r'(X_d^1) \xrightarrow{j^*} K_r'(U_1) \to 0.$$

Taking $r = 0$ and using the inductive assumption we see that $K_0'(X_d^1)$ is a free $K_0(S)$-module of the expected rank. Taking once again $r = 0$ and tensoring this sequence with $K_r(S)$ over $K_0(S)$ we get an exact sequence of the form

$$0 \to K_r(S) \otimes_{K_0(S)} K_0'(X_d) \to K_r(S) \otimes_{K_0(S)} K_0'(X_d^1) \to K_r(S) \otimes_{K_0(S)} K_0'(U_1) \to 0.$$

The pairings of the form $K_r(S) \otimes_{K_0(S)} K_0'(V) \to K_r'(V)$ define a morphism of the last short exact sequence to the previous one. Using the inductive assumption we see that the pairing $K_r(S) \otimes_{K_0(S)} K_0'(X_d^1) \to K_r'(X_d^1)$ is an isomorphism. Repeating this procedure several times we get that the pairing $K_r(S) \otimes_{K_0(S)} K_0'(X_{d+1}) \to K_r'(X_{d+1})$ is an isomorphism. The induction runs, whence the Claim and the Lemma. $\square$

Lemma B.1.6. *Let (X, x) and (Y, y) be pointed smooth S-cellular schemes. Then the map*

$$K_i^{\mathrm{TT}}(S) \otimes_{K_0^{\mathrm{TT}}(S)} K_0^{\mathrm{TT}}(X \vee Y) \to K_i^{\mathrm{TT}}(X \vee Y)$$

is an isomorphism and $K_0^{\mathrm{TT}}(X \vee Y)$ is a projective $K_0^{\mathrm{TT}}(S)$-module.

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
K_i(X \vee Y) & \xrightarrow{\ \ \alpha\ \ } & K_i(X) \oplus K_i(Y) & \xrightarrow{\ \ \beta\ \ } & K_i(S) \\
\big\uparrow{\scriptstyle\epsilon} & & \big\uparrow{\scriptstyle\rho} & & \big\uparrow{\scriptstyle\theta} \\
K_i(S) \otimes K_0(X \vee Y) & \xrightarrow{\ \gamma\ } & K_i(S) \otimes (K_0(X) \oplus K_0(Y)) & \xrightarrow{\ \delta\ } & K_i(S) \otimes K_0(S)
\end{array}
$$

in which K_i is written for K_i^{TT} and the tensor product is taken over $K_0(S)$. Its rows are exact, the arrows α and γ are injective, the arrows β and δ are surjective. The arrows ρ and θ are isomorphisms by Lemma B.1.4. Whence ϵ is an isomorphism too. $\square$

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