PROJECTIVE PUSH-FORWARDS IN THE WITT THEORY OF ALGEBRAIC VARIETIES

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Abstract. We define push-forwards along projective morphisms in the Witt theory of smooth quasi-projective varieties over a field. We prove that they have standard properties such as functoriality, compatibility with pull-backs and projection formulas.

1. Introduction

Let k be a field with $char k \neq 2$ and Sm_k denote the category of smooth quasi-projective varieties over k. We consider the Witt theory of such varieties developed by P. Balmer [Ba1-3], which the reader is supposed to be familiar with.

1.1. Twisted pull-backs of line bundles. For $X \in Sm_k$, let $\omega_X = \bigwedge^{\dim X} \Omega_X$ denote the canonical sheaf of X. For a morphism $f : Y \to X$ in Sm_k, let $\omega_f = \omega_{Y/X} = \omega_Y \otimes f^*\omega_X^\vee$ be the relative canonical sheaf; clearly we have $\omega_{fg} \cong g^*\omega_f \otimes \omega_g$ (canonically) for any composable f and g. For a line bundle L on X, we introduce the *twisted pull-back* $L^f = f^*L \otimes \omega_f$; one checks that $(L^f)^g \cong L^{fg}$ canonically.

1.2. The general objective. Our objective is to construct push-forwards along projective morphisms in Witt theory: for every equi-codimensional projective $f : Y \to X$ in Sm_k and a line bundle L on X, we define maps

$$f_* : W^q(Y; L^f) \to W^{q+c}(X; L)\,,$$

where $c = \dim X - \dim Y$ is the codimension of f. We also define push-forwards with support

$$f_* = f_*^{S,T} : W^q_S(Y; L^f) \to W^{q+c}_T(X; L)\,,$$

where S and T are closed subschemes in Y and X respectively, not necessarily smooth, satisfying $S \subset f^{-1}(T)$. The following properties of push-forwards are established:

(a) functoriality (Section 4.5);

(b) compatibility with pull-backs in transversal squares (change of base, Section 4.7);

(c) compatibility with the product structure (projection formulas, Section 4.11).

Push-forwards will be also referred to as trace maps or trace operators.

Key words and phrases. Witt theory of algebraic varieties, projective morphism, push-forward, Gysin operator, Thom isomorphism, deformation to the normal bundle.

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1.3. The closed embedding case. In the case of a closed embedding $i : Y \hookrightarrow X$ one has $\omega_{Y/X} \cong \det N$ (canonically), where $N = N_{X/Y}$ is the normal bundle of i, and the respective push-forward takes the form

$$i_* : W^q(Y; L_Y \otimes \det N) \to W^{q+c}(X; L)\,,$$

where $L_Y = i^*L$. Such push-forwards, referred to as Gysin operators, were constructed in [Ne1] in terms of Thom (dévissage) isomorphisms and deformations to the normal cone. For the reader's convenience we survey on our construction of Gysin operators in Section 2.

1.4. The case of projections. In the present paper we first define push-forwards along projections of the form $p : X \times \mathbb{P}^n \to X$. Here, the situation essentially differs from the respective part of the work of I. Panin and A. Smirnov on push-forwards in *oriented theories* [PS][PS1][SP]. Recall that in an oriented theory, there are two approaches to the traces of projections: the one being based on residues [PS, Section 4.3.2], the other on the use of the cobordism ring MU and the respective formal group law [PS, Section 4.3.1]. In the Witt theory, however, there is no need to apply such advanced techniques. For the situation in W is simpler: for even n, the Gysin operator along a constant section $X \hookrightarrow X \times \mathbb{P}^n$ proves to be an isomorphism, and we can define p_* as its inverse. In this part we use our computation of the Witt groups of projective bundles performed in [Ne2]; the same groups were calculated by a different method by C. Walter, see [W].

The case of an odd n can then be reduced to this one. The proofs of the expected properties of the p_*'s can be obtained on the basis of the properties of Gysin operators proved in [Ne1]. This is done in Section 3.

1.5. The general case. In Section 4 we proceed to the case of an arbitrary projective morphism $f : Y \to X$. Factoring such a morphism as $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p} X$, where i is a closed embedding, we define f_* as $p_* i_*$. We prove that the result does not depend on the choice of a factorization and establish the standard properties of projective push-forwards.

S. Gille introduced push-forwards (transfers) for coherent Witt groups of commutative rings with dualizing complexes [G1]. This yields transfers for the usual Witt groups if we consider finite morphisms of regular rings of finite Krull dimension. (More on coherent Witt groups can be found in [G2].) B. Calmès and J. Hornbostel constructed push-forwards along proper morphisms of smooth varieties by using dualities and adjunctions in derived categories and also working in the coherent Witt theory, see [CH]. This is quite different from our work in which we avoid any use of triangulated or derived categories. Nor do we use the coherent Witt theory. Our approach is based on the general cohomology theory properties of Witt groups and is as geometric as possible. We follow the guidelines of I. Panin and A. Smirnov given in [PS] and [SP] for *oriented* theories. However we have to considerably modify their machinery to the case of Witt theory which is not orientable in the sense of their work.

Acknowledgements. A part of this work was done at the University of Bielefeld and supported by the SFB-701, to which I express my gratitude. I am indebted to Stefan Gille and Ivan Panin for their interest in my work and various help.

2. Review of Gysin operators

In this section we remind the reader how the push-forwards along closed embeddings are defined in [Ne1]. We provide a brief account of the properties of such push-forwards which are also referred to as *Gysin operators*.

2.1. Definition. Let $i : Y \hookrightarrow X$ be a codimension c closed embedding of smooth varieties and let $p : N_{X/Y} \to Y$ denote the normal vector bundle to Y in X. Let L be a line bundle on X and $L_Y = i^*L$ its restriction to Y.

(i) We define the Gysin operator

$$i_* : W^q(Y; L_Y \otimes \det N_{X/Y}) \to W^{q+c}(X; L)$$

as the composition

$$W^q(Y; L_Y \otimes \det N) \xrightarrow{th(N)} W^{q+c}_Y(N; p^*L_Y) \xrightarrow{d(X,Y)} W^{q+c}_Y(X; L) \to W^{q+c}(X; L)\,,$$

where $N = N_{X/Y}$, $th(N)$ and $d(X, Y)$ are the Thom and the deformation to the normal cone isomorphisms, respectively (see [Ne1, Sections 2 and 3]), and the last arrow is an extension of support.

The Thom (dévissage) isomorphisms in Witt theory were also considered by S. Gille, see [G3]. A general reference on the deformations to the normal cone is Fulton's book [Fu, Chapter 4]; see also [PS][PS1][SP] in the context of oriented cohomology theories.

(ii) If T is a closed subscheme in X and S is a closed subscheme in Y such that $S \subset T_Y = T \cap Y$, then we define the *Gysin map with support*

$$i_* = i^{S,T}_* : W^q_S(Y; L_Y \otimes \det N_{X/Y}) \to W^{q+c}_T(X; L)$$

by

$$W^q_S(Y; L_Y \otimes \det N) \xrightarrow{th_S(N)} W^{q+c}_S(N; p^*L_Y) \xrightarrow{d_S(X,Y)} W^{q+c}_S(X; L) \to W^{q+c}_T(X; L)\,.$$

Here S and T do not need to be smooth.

(iii) If $S = T = Y$, we will write i^Y_* for the map $i^{Y,Y}_* : W^q(Y; L_Y \otimes \det N_{X/Y}) \to W^{q+c}_Y(X; L)$ which is an isomorphism (the first two steps in (i)).

(iv) Observe that if $i : Y \to X$ is an isomorphism, then $i_* = (i^{-1})^*$, both on the Witt groups with and without support.

(v) In this section we use the more explicit [Ne1]-notation $L_Y \otimes \det N_{X/Y}$ to denote the twist on Y. In Section 3 we'll switch to the notation introduced in 1.1, which is shorter and more functorial, and will write L^i for the same.

Gysin operators enjoy the following properties.

2.2. Functoriality. (See [Ne1, Proposition 5.1].)

(a) $(id_X)_* = id_{W^q_T(X;L)}$

(b) Let $Z \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$ be equicodimensional closed embeddings of smooth quasiprojective varieties, and let $r = codim\, i$, $s = codim\, j$, $t = codim\,(ji) = r + s$. Let L be a line bundle on X and L_Y, L_Z its restrictions to Y and Z. Then the diagram

$$\begin{tikzcd} W^{q+r}(Y; L_Y \otimes detN_{X/Y}) \arrow[r, "j_*"] & W^{q+t}(X; L) \\ W^q(Z; L_Z \otimes detN_{X/Z}) \arrow[u, "i_*"] \arrow[ur, "(ji)_*"'] & \end{tikzcd}$$

commutes. Moreover, if $R \subset S \subset T$ are compatible closed subvarieties in Z, Y, X respectively, then the diagram

$$\begin{tikzcd} W_S^{q+r}(Y; L_Y \otimes detN_{X/Y}) \arrow[r, "j_*^{S,T}"] & W_T^{q+t}(X; L) \\ W_R^q(Z; L_Z \otimes detN_{X/Z}) \arrow[u, "i_*^{R,S}"] \arrow[ur, "(ji)_*^{R,T}"'] & \end{tikzcd}$$

commutes. (The natural isomorphism $detN_{X/Z} \cong detN_{Y/Z} \otimes det(N_{X/Y}|_Z)$ is involved implicitly in the definition of i_* in both diagrams.)

2.3. Compatibility with extensions of support. (See [Ne1, Section 4.2].)

Given closed embeddings

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & Y \arrow[d] \\ T' \arrow[r] & T \arrow[r] & X \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} W_{S'}^q(Y; L_Y \otimes \det N) \arrow[r, "i_*^{S',T'}"] \arrow[d] & W_{T'}^{q+c}(X; L) \arrow[d] \\ W_S^q(Y; L_Y \otimes \det N) \arrow[r, "i_*^{S,T}"] & W_T^{q+c}(X; L) \end{tikzcd}$$

Here the vertical maps are extensions of support.

2.4. Transversal base changes. (See [Ne1, Section 6.1].)

Consider a transversal square of the form

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "\phi_Y"'] & X' \arrow[d, "\phi_X"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

in Sm_k in which i and (consequently) i' are closed embeddings of codimension c. This means that it is cartesian in Sm_k and the natural map $N_{X'/Y'} \to \phi_Y^* N_{X/Y}$ is an isomorphism of vector bundles, see Def. 3.5 in [Ne1]. Let L be a line bundle on X. Let S and T be compatible closed subschemes in Y and X, see 2.1(ii), and S' and T' be their pullbacks to Y' and X' respectively. Then with the same notation, the diagram commutes

$$\begin{array}{ccc} W^q_S(Y; L_Y \otimes \det N) & \xrightarrow{i_*} & W^{q+c}_T(X; L) \\ \downarrow{\scriptstyle \phi_Y^*} & & \downarrow{\scriptstyle \phi^*} \\ W^q_{S'}(Y'; (f^*L)_{Y'} \otimes \det N') & \xrightarrow{i'_*} & W^{q+c}_{T'}(X'; f^*L) \end{array}$$

The condition of transversality $\phi_Y^* N \cong N'$ guarantees that the twists in the left groups agree with respect to ϕ_Y^*.

2.5. Additivity formula. (See [Ne1, Section 6.2].)

Let Y_1, Y_2 and X be smooth varieties, $Y = Y_1 \amalg Y_2$, and let $i : Y \hookrightarrow X$ be a closed embedding (equicodimensional, of codimension c). Denote $j_r : Y_r \to Y$ the natural embedding and let $i_r = i \circ j_r : Y_r \hookrightarrow X$ for $i = 1, 2$. Let L be a line bundle on X and L_Y, L_{Y_i} denote its restrictions to Y and Y_i, respectively. Then the diagram commutes:

$$\begin{tikzcd}[column sep=small] W^q(Y; L_Y \otimes det N_{X/Y}) \arrow[rr, "i_*"] \arrow[dr, "(j_1^*, j_2^*)"'] & & W^{q+c}(X; L) \\ & W^q(Y_1; L_{Y_1} \otimes det N_{X/Y_1}) \oplus W^q(Y_2; L_{Y_2} \otimes det N_{X/Y_2}) \arrow[ur, "((i_1)_*, (i_2)_*)"'] & \end{tikzcd}$$

which can be expressed by the simple formula

$$i_* = (i_1)_* \circ j_1^* + (i_2)_* \circ j_2^* \,.$$

If furthermore S_1, S_2, and T are closed subvarieties in Y_1, Y_2, and X, respectively, not necessarily smooth, $S = S_1 \amalg S_2$ and $i(S) \subset T$, then we have additivity with support:

$$i_*^{S,T} = (i_1)_*^{S_1,T} \circ j_1^* + (i_2)_*^{S_2,T} \circ j_2^* \,.$$

The case of several components follows by induction.

2.6. Smooth divisor case. (See [Ne1, Section 6.3].) Let X be a smooth variety, L a line bundle on X, and $s : \mathcal{O}_X \to L$ a global section of L transversal to the zero section. Denote by $D = D(s)$ the smooth divisor on X given by the zeros of s and $i : D \hookrightarrow X$ the inclusion. Then $N_{X/D} \cong L_D$ and we can consider the push-forward map $i_* : W^0(D) \to W^1(X; L^\vee)$, where $W^0(D) = W^0(D; \mathcal{O}_D)$. With this notation we have

$$i_*(1) = 0 \,.$$

2.7. Projection formulas. (See [Ne1, Section 6.4] for the proof of the projection formulas and [GN] for the definition and properties of the product structure on the Witt groups introduced by S. Gille and the author.)

Let $i : Y \hookrightarrow X$ be a codimension c closed embedding of smooth varieties. Let L and L' be line bundles over X, and let $\alpha \in W^{q'}(X; L')$ and $\beta \in W^q(Y; L_Y \otimes \det N)$. Then

$$i_*(i^*\alpha \star \beta) = \alpha \star i_*\beta \text{ and } i_*(\beta \star i^*\alpha) = (-1)^{cq'} i_*\beta \star \alpha$$

in $W^{q'+q+c}(X; L \otimes L')$.

If, moreover, S and T are compatible closed subschemes in Y and X, see 4.1(ii), T' is another closed subscheme in X, $\alpha \in W^{q'}_{T'}(X; L')$ and $\beta \in W^q_S(Y; L_Y \otimes \det N)$, then the same formulas hold in $W^{q'+q+c}_{T\cap T'}(X; L \otimes L')$.

3. Traces of projections

In this section we define traces

$$p_* = p_*^T : W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \to W^q_T(X; L) \tag{3.1}$$

along projections of the form $p = p_X^{(n)} : X \times \mathbb{P}^n \to X$ and prove their properties. Here T is a closed subscheme of X, not necessarily smooth, and

$$L^{(n)} = L^{p_X^{(n)}} = (p_X^{(n)})^* L \otimes \omega_{p_X^{(n)}}$$

is the twisted pull-back of L as defined in Section 1.1.

If $S \subset T \times \mathbb{P}^n$ is another closed subscheme of $X \times \mathbb{P}^n$, we can combine (3.1) with the extension of support and get the trace operator

$$p_* = p_*^{S,T} : W^{q+n}_S(X \times \mathbb{P}^n; L^{(n)}) \to W^q_T(X; L)\,.$$

The properties of the operators p_*^T can be easily generalized to the $p_*^{S,T}$, which is left to the reader.

Definitions. For $a \in \mathbb{P}^n(k)$ denote $i = i^{(n)}_{X,a} : X \hookrightarrow X \times \mathbb{P}^n$ the embedding $x \mapsto (x, a)$. As $p \circ i = \mathrm{id}_X$, we have a canonical isomorphism $(L^{(n)})^i \cong L$ and can consider the Gysin operator

$$i_* : W^q_T(X; L) \to W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)})\,. \tag{3.2}$$

Lemma 3.1. *Let n be even.*

(i) The operator (3.2) is an isomorphism.

(ii) It does not depend on the choice of a point $a \in \mathbb{P}^n(k)$, i.e., if $a' \in \mathbb{P}^n(k)$ is another point, then $(i^{(n)}_{X,a})_* = (i^{(n)}_{X,a'})_*$.

Proof. (i) By Definition 2.1(ii), a Gysin operator is the composition of three maps two of which are always isomorphisms. Thus it suffices to show that the extension of support map

$$W^{q+n}_{T\times a}(X \times \mathbb{P}^n; L^{(n)}) \to W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)})$$

is an isomorphism. This map fits into the localization sequence

$$\ldots \to W^{q+n}_{T\times a}(X\times\mathbb{P}^n;L^{(n)}) \to W^{q+n}_{T\times\mathbb{P}^n}(X\times\mathbb{P}^n;L^{(n)}) \to$$
$$\to W^{q+n}_{T\times(\mathbb{P}^n-a)}(X\times\mathbb{P}^n - T\times a; L^{(n)}|_{X\times\mathbb{P}^n-T\times a}) \to \ldots$$

By excision

$$W^{q+n}_{T\times(\mathbb{P}^n-a)}(X\times\mathbb{P}^n - T\times a; L^{(n)}|_{X\times\mathbb{P}^n-T\times a}) \cong W^{q+n}_{T\times(\mathbb{P}^n-a)}(X\times(\mathbb{P}^n-a); L^{(n)}|_{X\times(\mathbb{P}^n-a)})\,.$$

Considering $\mathbb{P}^n - a$ as a line bundle over $\mathbb{P}^{n-1}$, we get by homotopy invariance

$$W^{q+n}_{T\times(\mathbb{P}^n-a)}(X\times(\mathbb{P}^n-a);L^{(n)}|_{X\times(\mathbb{P}^n-a)}) \cong$$
$$W^{q+n}_{T\times\mathbb{P}^{n-1}}(X\times\mathbb{P}^{n-1};L^{(n)}|_{X\times\mathbb{P}^{n-1}})\,.$$

Here we consider $\mathbb{P}^{n-1}$ as a linear subspace in $\mathbb{P}^n$ not meeting a. Let $U = X - T$ and consider the localization sequence

$$\ldots \to W^{q+n-1}(U\times\mathbb{P}^{n-1};L^{(n)}|_{U\times\mathbb{P}^{n-1}}) \to W^{q+n}_{T\times\mathbb{P}^{n-1}}(X\times\mathbb{P}^{n-1};L^{(n)}|_{X\times\mathbb{P}^{n-1}}) \to$$
$$\to W^{q+n}(X\times\mathbb{P}^{n-1};L^{(n)}|_{X\times\mathbb{P}^{n-1}}) \to \ldots$$

As $L^{(n)} \cong (p^{(n)}_X)^*L\otimes\mathcal{O}(-n-1)$ on $X\times\mathbb{P}^n$, we have

$$L^{(n)}|_{X\times\mathbb{P}^{n-1}} \cong (p^{(n-1)}_X)^*L\otimes\mathcal{O}(-n-1) \text{ and } L^{(n)}|_{U\times\mathbb{P}^{n-1}} \cong (p^{(n-1)}_U)^*L\otimes\mathcal{O}(-n-1)\,.$$

Hence the side groups vanish by [Ne2, Cor. 4.2] since n is even. Thus the middle group vanishes as well, which proves (i).

(ii) There exists an automorphism α of $\mathbb{P}^n$ given by a matrix of $SL_{n+1}(k)$ which takes a to a'. Thus $(i^{(n)}_{X,a'})_* = (1_X\times\alpha)_*\circ(i^{(n)}_{X,a})_*$, and it suffices to show that $(1_X\times\alpha)_* = \mathrm{id}$ on $W^q(X\times\mathbb{P}^n;-)$ or, equivalently, that $(1_X\times\alpha)^* = \mathrm{id}$. (Recall that $(f^{-1})_* = f^*$ for an isomorphism f.) Clearly we can assume that α is an elementary matrix. The following argument works for any theory satisfying homotopy invariance. Consider the automorphism $\tilde\alpha$ of $\mathbb{P}^n\times\mathbb{A}^1$ such that $\tilde\alpha_0 = \mathrm{id}_{\mathbb{P}^n}$ and $\tilde\alpha_1 = \alpha$. (If $\alpha = e_{i,j}(\lambda)$, then $\tilde\alpha_t = e_{i,j}(t\lambda)$.) Denote by i_0 and i_1 the embeddings $\mathbb{P}^n \hookrightarrow \mathbb{P}^n\times\mathbb{A}^1$ given by $y\mapsto(y,0)$ and $y\mapsto(y,1)$ respectively, and $\pi:\mathbb{P}^n\times\mathbb{A}^1\to\mathbb{P}^n$ the projection. Since $\pi\circ i_0 = \pi\circ i_1 = \mathrm{id}_{\mathbb{P}^n}$ and π^* is an isomorphism by homotopy invariance, we have $i_0^* = i_1^*$ which is an isomorphism as well.

Applying pull-backs to the diagram

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "i_0"] \arrow[d, "id"'] & \mathbb{P}^n\times\mathbb{A}^1 \arrow[d, "\tilde\alpha"] & \mathbb{P}^n \arrow[l, "i_1"'] \arrow[d, "\alpha"] \\
\mathbb{P}^n \arrow[r, "i_0"] & \mathbb{P}^n\times\mathbb{A}^1 & \mathbb{P}^n \arrow[l, "i_1"']
\end{tikzcd}$$

proves the assertion.■

Definition 3.2. Let X be a smooth quasi-projective variety over k, $T \hookrightarrow X$ be a closed subvariety, not necessarily smooth, and let $p = p_X^{(n)} : X \times \mathbb{P}^n \to X$ denote the projection.

(i) If n is even, we define

$$p_* = (p_X^{(n)})_* : W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \to W^q_T(X; L)$$

as the inverse to the Gysin operator i_*,

$$(p_X^{(n)})_* = (i_{X,a}^{(n)})_*^{-1}.$$

By Lemma 3.1, this does not depend on the choice of a point $a \in \mathbb{P}^n(k)$.

(ii) If n is odd, $p_* = (p_X^{(n)})_*$ is defined as the composition

$$W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \xrightarrow{(j_X^{(n,n+1)})_*} W^{q+n+1}_{T\times\mathbb{P}^{n+1}}(X \times \mathbb{P}^{n+1}; L^{(n+1)}) \xrightarrow{(p_X^{(n+1)})_*} W^q_T(X; L).$$

Thus

$$(p_X^{(n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* = (i_{X,a}^{(n+1)})_*^{-1} \circ (j_X^{(n,n+1)})_*,$$

where $j^{(n,n+1)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ is a k-linear embedding and $j_X^{(n,n+1)} = 1_X \times j^{(n,n+1)}$. Observe that $(j_X^{(n,n+1)})_*$ does not depend on the choice of such a linear embedding. For, every two such embeddings can be connected by an SL-automorphism of $\mathbb{P}^{n+1}$, and the same argument as in Lemma 3.1, (ii) applies.

Properties of push-forwards along projections. As our definition of the traces of projections is stated in terms of Gysin maps, it is not a surprice that basic properties of the operators p_* can be deduced from the properties of Gysin operators.

3.3. Composition. Consider the diagram

$$\begin{tikzcd} X \times \mathbb{P}^m \times \mathbb{P}^n \arrow[r, "p^{(n)}_{X\times\mathbb{P}^m}"] \arrow[d, "p^{(m)}_{X\times\mathbb{P}^n}"'] & X \times \mathbb{P}^m \arrow[d, "p_X^{(m)}"] \\ X \times \mathbb{P}^n \arrow[r, "p_X^{(n)}"] & X \end{tikzcd}$$

by means of which we introduce notation for the obvious projections, and also let $p_X^{(m,n)}$: $X \times \mathbb{P}^m \times \mathbb{P}^n \to X$ denote the diagonal projection and let $L^{(m,n)} = L^{p_X^{(m,n)}}$ for a line bundle L on X. Then for any q the following diagram commutes:

$$\begin{tikzcd} W^{q+m+n}_{T\times\mathbb{P}^m\times\mathbb{P}^n}(X \times \mathbb{P}^m \times \mathbb{P}^n; L^{(m,n)}) \arrow[r, "(p^{(n)}_{X\times\mathbb{P}^m})_*"] \arrow[d, "(p^{(m)}_{X\times\mathbb{P}^n})_*"'] & W^{q+m}_{T\times\mathbb{P}^m}(X \times \mathbb{P}^m; L^{(m)}) \arrow[d, "(p_X^{(m)})_*"] \\ W^{q+n}_{T\times\mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \arrow[r, "(p_X^{(n)})_*"] & W^q_T(X; L) \end{tikzcd} \tag{3.3}$$

Proof. Case 1: m, n even. Choose $a \in \mathbb{P}^m(k), b \in \mathbb{P}^n(k)$. All the four arrows in (3.3) are the isomorphisms inverse to the Gysin operators along the embeddings $i^{(m)}_{X,a}, i^{(n)}_{X,b}, i^{(m)}_{X\times\mathbb{P}^n,a}, i^{(n)}_{X\times\mathbb{P}^m,b}$, respectively. As Gysin maps are functorial, see 2.2, the diagram consisting of the i_*'s commutes, which proves the assertion.

Case 2: m odd, n even. Consider the diagram

$$\begin{tikzcd}
 & & W^{q+(m+1)+n} \arrow[rrr, "(p^{(n)}_{X\times\mathbb{P}^{m+1}})_*"] \arrow[ddl, "(p^{(m+1)}_{X\times\mathbb{P}^n})_*"'] & & & W^{q+m+1} \arrow[ddl, "(p^{(m+1)}_X)_*"] \\
W^{q+m+n} \arrow[urr, "(j^{m,m+1})_*"] \arrow[rrrr, "(p^{(n)}_{X\times\mathbb{P}^m})_*"] \arrow[dr, "(p^{(m)}_{X\times\mathbb{P}^n})_*"'] & & & & W^{q+m} \arrow[ur, "(j^{m,m+1})_*"] \arrow[d, "(p^{(m)}_X)_*"'] & \\
 & W^{q+n} \arrow[rrr, "(p^{(n)}_X)_*"'] & & & W^q &
\end{tikzcd}$$

We leave it to the reader to add spaces, supports and twists accordingly. The side (triangular) faces commute by the definition of $(p^{(m)}_{\_})_*$ for odd m, see Definition 3.2(ii). The back face commutes by Case 1. The horizonal $(p^{(n)}_{\_})_*$'s are the isomorphisms inverse to the respective $(i^{(n)}_{\_})_*$'s. Thus the upper face commutes due to the compatibility of the i_*'s and j_*'s, see 2.2. This implies the commutativity of the front face.

Case 3: m and n odd - is left to the reader as an exercise of the same type.

3.4. Base change. Given $\phi : X' \to X$ in Sm_k, denote

$$\phi^{(n)} = \phi \times 1_{\mathbb{P}^n} : X' \times \mathbb{P}^n \to X \times \mathbb{P}^n\,,$$

and let $p^X_n : X \times \mathbb{P}^n \to \mathbb{P}^n$ and $p^{X'}_n : X' \times \mathbb{P}^n \to \mathbb{P}^n$ denote the projections. As $p^{X'}_n = p^X_n \circ \phi^{(n)}$, we have

$$\omega_{X'\times\mathbb{P}^n/X'} \cong (p^{X'}_n)^*\omega_{\mathbb{P}^n} \cong (\phi^{(n)})^*(p^X_n)^*\omega_{\mathbb{P}^n} \cong (\phi^{(n)})^*\omega_{X\times\mathbb{P}^n/X}\,, \tag{3.4}$$

all the isomorphisms being canonical. The resulting isomorphism $\omega_{X'\times\mathbb{P}^n/X'} \cong (\phi^{(n)})^*\omega_{X\times\mathbb{P}^n/X}$ reflects the fact that the square

$$\begin{tikzcd}
X' \times \mathbb{P}^n \arrow[r, "\phi^{(n)}"] \arrow[d, "p^{(n)}_{X'}"'] & X \times \mathbb{P}^n \arrow[d, "p^{(n)}_X"] \\
X' \arrow[r, "\phi"] & X
\end{tikzcd}$$

is transversal; we refer to 4.7 for a discussion of general transversal squares, which is unnecessary in this trivial case.

Given a line bundle L on X, it follows from (3.4) that

$$(\phi^* L)^{(n)} = (p_{X'}^{(n)})^* \phi^* L \otimes \omega_{X' \times \mathbb{P}^n / X'} \cong (\phi^{(n)})^* (p_X^{(n)})^* L \otimes (\phi^{(n)})^* \omega_{X \times \mathbb{P}^n / X} \cong (\phi^{(n)})^* L^{(n)} ,$$

where all the isomorphisms are canonical. Thus the upper arrow $(\phi^{(n)})^*$ in the diagram

$$\begin{tikzcd}
W^{q+n}_{\phi^{-1}T \times \mathbb{P}^n}(X' \times \mathbb{P}^n; (\phi^* L)^{(n)}) \arrow[d, "(p_{X'}^{(n)})_*"'] & W^{q+n}_{T \times \mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \arrow[l, "(\phi^{(n)})^*"'] \arrow[d, "(p_X^{(n)})_*"] \\
W^q_{\phi^{-1}T}(X'; \phi^* L) & W^q_T(X; L) \arrow[l, "\phi^*"']
\end{tikzcd}$$

has a correct target. We claim that the diagram commutes for any closed T in X.

Proof. Let $i_{X'} : X' \to X' \times \mathbb{P}^n$, $i_X : X \to X \times \mathbb{P}^n$ be constant sections given by the same k-point of $\mathbb{P}^n$. Then

$$\begin{tikzcd}
X' \times \mathbb{P}^n \arrow[r, "\phi^{(n)}"] & X \times \mathbb{P}^n \\
X' \arrow[u, "i_{X'}"] \arrow[r, "\phi"] & X \arrow[u, "i_X"']
\end{tikzcd}$$

is a transversal square in the sense of [Ne1; Def. 3.5], so $(i_{X'})_* \circ \phi^* = (\phi^{(n)})^* \circ (i_X)_*$ by 2.4. Thus for n even, $\phi^* \circ (i_X)_*^{-1} = (i_{X'})_*^{-1} \circ (\phi^{(n)})^*$, which proves the property in this case.

If n is odd, then

$$\begin{aligned}
\phi^* \circ (p_X^{(n)})_* = \phi^* \circ (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* &= (p_{X'}^{(n+1)})_* \circ (\phi^{(n+1)})^* \circ (j_X^{(n,n+1)})_* \\
&= (p_{X'}^{(n+1)})_* \circ (j_{X'}^{(n,n+1)})_* \circ (\phi^{(n)})^* = (p_{X'}^{(n)})_* \circ (\phi^{(n)})^* .
\end{aligned}$$

Here the third equation is true by 2.4 since the square

$$\begin{tikzcd}
X' \times \mathbb{P}^n \arrow[r, "\phi^{(n)}"] \arrow[d, "j_{X'}^{(n,n+1)}"'] & X \times \mathbb{P}^n \arrow[d, "j_X^{(n,n+1)}"] \\
X' \times \mathbb{P}^{n+1} \arrow[r, "\phi^{(n+1)}"] & X \times \mathbb{P}^{n+1}
\end{tikzcd}$$

is transversal.

3.5. Compatibility with linear embeddings. Let $j = j^{(m,n)} : \mathbb{P}^m \to \mathbb{P}^n$ be a k-linear embedding and denote $j_X^{(m,n)} = 1_X \times j^{(m,n)} : X \times \mathbb{P}^m \to X \times \mathbb{P}^n$. Then $(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(m)})_*$, *i.e.*, the following diagram commutes:

$$\begin{tikzcd}
W^{q+m}_{T \times \mathbb{P}^m}(X \times \mathbb{P}^m; L^{(m)}) \arrow[rr, "(j_X^{(m,n)})_*"] \arrow[dr, "(p_X^{(m)})_*"'] & & W^{q+n}_{T \times \mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \arrow[dl, "(p_X^{(n)})_*"] \\
& W^q_T(X; L) &
\end{tikzcd}$$

Proof. Choose constant embeddings $i_X^{(m)}: X \to X \times \mathbb{P}^m$ and $i_X^{(n)}: X \to X \times \mathbb{P}^n$ compatible with $j_X^{(m,n)}$, *i.e.*, so that $i_X^{(n)} = j_X^{(m,n)} \circ i_X^{(m)}$.

Case 1: m and n even. By 2.2 we have $(i_X^{(n)})_* = (j_X^{(m,n)})_* \circ (i_X^{(m)})_*$, the assertion follows.

Case 2: m even, n odd. Consider the diagram

$$\begin{tikzcd} X \times \mathbb{P}^m \arrow[r, "j_X^{(m,n)}"] \arrow[dr, "p_X^{(m)}"'] & X \times \mathbb{P}^n \arrow[r, "j_X^{(n,n+1)}"] \arrow[d, "p_X^{(n)}"'] & X \times \mathbb{P}^{n+1} \arrow[dl, "p_X^{(n+1)}"] \\ & X & \end{tikzcd}$$

Take the respective Witt groups and get

$$(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(m,n+1)})_* = (p_X^{(m)})_* \,.$$

The latter is true by Case 1 and $(j_X^{(m,n+1)})_* = (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_*$ by 2.2.

Case 3: m odd, n even. Consider the diagram

$$\begin{tikzcd} X \times \mathbb{P}^m \arrow[r, "j_X^{(m,m+1)}"] \arrow[dr, "p_X^{(m)}"'] & X \times \mathbb{P}^{m+1} \arrow[r, "j_X^{(m+1,n)}"] \arrow[d, "p_X^{(m+1)}"'] & X \times \mathbb{P}^{n} \arrow[dl, "p_X^{(n)}"] \\ & X & \end{tikzcd}$$

and get

$$(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n)})_* \circ (j_X^{(m+1,n)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m)})_* \,.$$

Case 4: If m and n are odd, then

$$\begin{aligned}(p_X^{(n)})_* \circ (j_X^{(m,n)})_* &= (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(m,n+1)})_* \\ &= (p_X^{(n+1)})_* \circ (j_X^{(m+1,n+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m)})_* \,.\end{aligned}$$

3.6. Compatibility with Gysin operators. Let $i: Y \hookrightarrow X$ be a codimension c closed embedding of smooth varieties. Let S and T be closed subschemes in Y and X, respectively, such that $S \subset T \cap Y$. Then the diagram commutes:

$$\begin{tikzcd} W^{q+n}_{S \times \mathbb{P}^n}(Y \times \mathbb{P}^n; (L^i)^{(n)}) \arrow[r, "(i^{(n)})_*"] \arrow[d, "(p_Y^{(n)})_*"'] & W^{q+n+c}_{T \times \mathbb{P}^n}(X \times \mathbb{P}^n; L^{(n)}) \arrow[d, "(p_X^{(n)})_*"] \\ W^q_S(Y; L^i) \arrow[r, "i_*"] & W^{q+c}_T(X; L) \end{tikzcd}$$

Proof. (i) $i_* \circ (p_Y^{(n)})_* = (p_X^{(n)})_* \circ (i^{(n)})_*$ amounts to $(i_{X,a}^{(n)})_* \circ i_* = (i^{(n)})_* \circ (i_{Y,a}^{(n)})_*$ if n is even. The latter is true by 2.2.

(ii) If n is odd, then

$$\begin{aligned} i_* \circ (p_Y^{(n)})_* = i_* \circ (p_Y^{(n+1)})_* \circ (j_Y^{(n,n+1)})_* &= (p_X^{(n+1)})_* \circ (i^{(n+1)})_* \circ (j_Y^{(n,n+1)})_* \\ &= (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (i^{(n)})_* = (p_X^{(n)})_* \circ (i^{(n)})_* \,. \end{aligned}$$

3.7. Projection formulas. Let T and T' be closed subvarieties in a smooth X and let L and L' be line bundles on X. Then in $W^{q'+q}_{T'\cap T}(X; L'\otimes L)$ we have

$$p_*(p^*\gamma \star \delta) = \gamma \star p_*\delta \qquad \text{and} \qquad p_*(\delta \star p^*\gamma) = (-1)^{nq'} p_*\delta \star \gamma$$

for any $\gamma \in W^{q'}_{T'}(X; L')$ and $\delta \in W^{q+n}_{T\times\mathbb{P}^n}(X\times\mathbb{P}^n; L^{(n)})$. Here $p = p^{(n)}_X$. Observe that $p^*L'\otimes L^{(n)} \cong (L'\otimes L)^{(n)}$, *i.e.*, the twists agree; clearly $f^*L'\otimes L^f \cong (L'\otimes L)^f$ canonically for any $f: Y\to X$.

Proof. Case 1: n even. Put $\alpha = p^*\gamma \in W^{q'}_{T'\times\mathbb{P}^n}(X\times\mathbb{P}^n; p^*L')$ and $\beta = p_*\delta = (i_*)^{-1}\delta \in W^q_t(X; L)$, where $i = i^{(n)}_{X,a}$. Then $\gamma = i^*\alpha$ and $\delta = i_*\beta$. Applying $p_* = (i_*)^{-1}$ to the projection formulas of 2.7 proves the result.

Case 2: n odd. We have

$$\begin{aligned}
(p^{(n)}_X)_*((p^{(n)}_X)^*\gamma\star\delta) &= (p^{(n+1)}_X)_*(j^{(n,n+1)}_X)_*((j^{(n,n+1)}_X)^*(p^{(n+1)}_X)^*\gamma\star\delta) && \text{Def. 3.2(ii)}\\
&= (p^{(n+1)}_X)_*((p^{(n+1)}_X)^*\gamma\star(j^{(n,n+1)}_X)_*\delta) && \text{2.7 for } j^{(n,n+1)}_X\\
&= \gamma\star(p^{(n+1)}_X)_*(j^{(n,n+1)}_X)_*\delta = \gamma\star(p^{(n)}_X)_*\delta && \text{Case 1}\,.
\end{aligned}$$

The second formula can be reduced to Case 1 the same way.

3.8. The section property. Let $s: X\to X\times\mathbb{P}^n$ be a k-rational section of $p^{(n)}_X$. Clearly, $s = (1_X, f)$, where $f = p^X_n\circ s: X\to\mathbb{P}^n$ and p^X_n denotes the projection $X\times\mathbb{P}^n\to\mathbb{P}^n$. According to 1.1, $(L^{(n)})^s\cong L$ canonically, and we can consider the Gysin operator

$$s_*: W^q_T(X; L)\to W^{q+n}_{T\times\mathbb{P}^n}(X\times\mathbb{P}^n; L^{(n)})\,.$$

Proposition 3.9. $(p^{(n)}_X)_*\circ s_* = id$ *on* $W^q_T(X; L)$.

Corollary 3.10. $s_* = (i^{(n)}_{X,a})_*$ *if* n *is even.*

Lemma 3.11. $(p^{(n)}_X)_*s_*(1) = 1$ *in* $W^0(X;\mathcal{O}_X)$.

Proof of the proposition modulo the lemma. Let $\alpha\in W^q_T(X; L)$. As $s^*\circ(p^{(n)}_X)^* = (p^{(n)}_X\circ s)^* = \mathrm{id}$, we have

$$\begin{aligned}
(p^{(n)}_X)_*s_*(\alpha) &= (p^{(n)}_X)_*s_*(s^*(p^{(n)}_X)^*(\alpha)\star 1)\\
&= (p^{(n)}_X)_*((p^{(n)}_X)^*(\alpha)\star s_*(1)) && \text{by the projection formula 2.7}\\
&= \alpha\star(p^{(n)}_X)_*s_*(1) && \text{by the projection formula 3.7}\\
&= \alpha\star 1 = \alpha && \text{by Lemma 3.11}
\end{aligned}$$

The proof of the lemma occupies the rest of the section.

3.12. The trace of diagonal. Consider the diagonal embedding $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ and let $p_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ denote the projection to the ith factor. As $p_1 \circ \Delta = \mathrm{id}_{\mathbb{P}^n}$, $(L^{p_1})^{\Delta} \cong L$ canonically for a line bundle L on $\mathbb{P}^n$, and we can consider the trace

$$\Delta_* : W^q(\mathbb{P}^n; L) \to W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1})\,.$$

For $a \in \mathbb{P}^n(k)$, let $i_a = i^{(n)}_{\mathbb{P}^n,a} : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$, $x \mapsto (x, a)$. Then $(L^{p_1})^{i_a} \cong L$ canonically, and we can equally consider the trace

$$(i_a)_* : W^q(\mathbb{P}^n; L) \to W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1})\,.$$

Lemma 3.13. *$\Delta_* = (i_a)_*$ if n is even.*

Proof. We can assume that $L \cong \mathcal{O}(l)$ for some $l \in \mathbb{Z}$.

Case 1: l is odd. Denote also by a the embedding $pt \hookrightarrow \mathbb{P}^n$, $pt \mapsto a$, where $pt = \operatorname{Spec} k$. Then $\Delta \circ a = i_a \circ a$ and $\Delta_* \circ a_* = (i_a)_* \circ a_*$ by 2.2. As l is odd, the trace

$$a_* : W^{q-n}(pt; L^a) \to W^q(\mathbb{P}^n; L)$$

is an isomorphism by the same argument as in the proof of Lemma 3.1(i), which proves the assertion in this case.

Case 2: l is even. Denote i'_a the embedding $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$, $x \mapsto (a, x)$, and consider the diagram of embeddings

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "i_a"'] & \mathbb{P}^n \times \mathbb{P}^n \\ pt \arrow[u, "a"] \arrow[r, "a"'] & \mathbb{P}^n \arrow[u, "i'_a"'] \end{tikzcd}$$

which is transversal for either choice of the top arrow. By 2.4 we get the diagram

$$\begin{tikzcd} W^q(\mathbb{P}^n; L) \arrow[r, shift left, "\Delta_*"] \arrow[r, shift right, "(i_a)_*"'] \arrow[d, "a^*"'] & W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1}) \arrow[d, "(i'_a)^*"] \\ W^q(pt; a^*L) \arrow[r, "a_*"'] & W^{q+n}(\mathbb{P}^n; L') \end{tikzcd}$$

which commutes for either choice of the top arrow; here $L' = (i'_a)^* L^{p_1}$. The pull-back $(i'_a)^*$ is an isomorphism by [Ne2, Section 3] since $L^{p_1} \cong p_2^* L' \otimes p_1^* \mathcal{O}(l)$ and l is even, whence $\Delta_* = (i_a)_*$.

■

By Definition 3.2(i) we get

Corollary 3.14. *$(p_1)_* \circ \Delta_* = id$ if n is even.*

Proof of Lemma 3.11 for even n. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & X \times \mathbb{P}^n \arrow[r, "p_X"] \arrow[d, "f\times 1"] & X \arrow[d, "f"] \\ \mathbb{P}^n \arrow[r, "\Delta"] & \mathbb{P}^n \times \mathbb{P}^n \arrow[r, "p_1"] & \mathbb{P}^n \end{tikzcd}$$

The first square is transversal, thus we can apply the property 2.4 of Gysin operators. Using 3.4 for the second square, we get

$$(p_X)_* \circ s_* \circ f^* = f^* \circ (p_1)_* \circ \Delta_* = f^*$$

by Corollary 3.14. This equation is true on $W^q(\mathbb{P}^n; L)$ with arbitrary q and L, the twists on X, $X \times \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^n$ should be chosen accordingly. Applying it to the identity element in $W^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n})$, we get $(p_X)_* s_*(1) = 1$, which proves Lemma 3.11 in this case.

■

Proof of Lemma 3.11 for odd n. Let us now write $s^{(n)}$ for s and let $s^{(n+1)} = j_X^{(n,n+1)} \circ s^{(n)}$. Clearly $s^{(n+1)}$ is a section for $p_X^{(n+1)}$. Applying W to the diagram

$$\begin{tikzcd} X \arrow[r, "s^{(n)}"] \arrow[d, "1"'] & X \times \mathbb{P}^n \arrow[r, "p_X^{(n)}"] \arrow[d, "j_X^{(n,n+1)}"] & X \arrow[d, "1"] \\ X \arrow[r, "s^{(n+1)}"] & X \times \mathbb{P}^{n+1} \arrow[r, "p_X^{(n+1)}"] & X \end{tikzcd}$$

completes the proof. Here we consider $W^0(X; \mathcal{O}_X)$ in the corners with the respective shifts and twists in the middle and use 2.2 and Definition 3.2(ii).

■

4. Push-forwards along projective morphisms

4.1. Definition of projective push-forwards. Now that we have push-forwards along projections and closed embeddings, we are prepared to deal with arbitrary projective morphisms. Given such a morphism $f : Y \to X$ of pure codimension c, one can represent it as a composition $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p} X$, where $p = p_X^{(n)}$ and i is a closed embedding of codimension $c + n$. For a given line bundle L over X and closed $S \subset Y$ and $T \subset X$ with $S \subset f^{-1}(T)$, we define the push-forward

$$f_* = f_*^{S,T} : W_S^q(Y; L^f) \to W_T^{q+c}(X; L)$$

as the composition

$$W_S^q(Y; L^f) \xrightarrow{i_*} W_{T\times\mathbb{P}^n}^{q+c+n}(X \times \mathbb{P}^n; L^{(n)}) \xrightarrow{p_*} W_T^{q+c}(X; L)\,. \tag{4.1}$$

Proposition 4.2. *The trace map $f_* : W^q_S(Y; L^f) \to W^{q+c}_T(X; L)$ does not depend on the choice of a factorization $f = p \circ i$.*

Proof. We will first prove the assertion in the following special case. Let $i : Y \hookrightarrow X$ be a closed embedding. We can write it as $i = p^{(0)}_X \circ i^{(0)}$, where $i^{(0)} : Y \to X \times \mathbb{P}^0$ is given by $y \mapsto (i(y), pt)$ and $p^{(0)}_X : X \times \mathbb{P}^0 \to X$ is the (identity) projection. As $(p^{(0)}_X)_* = \mathrm{id}$, the composition in (4.1) yields the Gysin operator i_*. On the other hand, one can choose an embedding $\tilde{i} : Y \hookrightarrow X \times \mathbb{P}^n$ covering i, *i.e.*, satisfying $p^{(n)}_X \circ \tilde{i} = i$, and apply (4.1).

Lemma 4.3. $i_* = (p^{(n)}_X)_* \circ \tilde{i}_*$.

Proof of the lemma. Consider the diagram

$$\begin{tikzcd}
Y \times \mathbb{P}^n \arrow[r, "i^{(n)}"] \arrow[d, shift right, "p^{(n)}_Y"'] & X \times \mathbb{P}^n \arrow[d, "p^{(n)}_X"] \\
Y \arrow[u, shift right, dashed, "s"'] \arrow[r, "i"] \arrow[ur, "\tilde{i}"] & X
\end{tikzcd}$$

where $i^{(n)} = i \times 1$. Let $s : Y \to Y \times \mathbb{P}^n$ be the section of $p^{(n)}_Y$ determined by $i^{(n)} \circ s = \tilde{i}$. Then

$$\begin{aligned}
(p^{(n)}_X)_* \circ \tilde{i}_* &= (p^{(n)}_X)_* \circ i^{(n)}_* \circ s_* && \text{functoriality of Gysin maps, Section 2.2} \\
&= i_* \circ (p^{(n)}_Y)_* \circ s_* && \text{compatibility of Gysins and } p_*\text{'s, Section 3.6} \\
&= i_* && \text{by Proposition 3.9}\,.
\end{aligned}$$

The lemma is proved.■

Now suppose we have $f = p \circ i = p' \circ i'$ with $Y \xrightarrow{i'} X \times \mathbb{P}^m \xrightarrow{p'} X$, where $p = p^{(n)}_X$ and $p' = p^{(m)}_X$. Let $I : Y \to X \times \mathbb{P}^n \times \mathbb{P}^m$ be the unique embedding satisfying $p^{(n)}_{X \times \mathbb{P}^m} \circ I = i'$ and $p^{(m)}_{X \times \mathbb{P}^n} \circ I = i$. We have

$$\begin{aligned}
p'_* \circ i'_* &= p'_* \circ (p^{(n)}_{X \times \mathbb{P}^m})_* \circ I_* && \text{by Lemma 4.3} \\
&= p_* \circ (p^{(m)}_{X \times \mathbb{P}^n})_* \circ I_* && \text{by Section 3.3} \\
&= p_* \circ i_* && \text{by Lemma 4.3}\,.
\end{aligned}$$

Thus we have proved that f_* is well defined.■

Properties of projective push-forwards.

4.4. The general definition agrees with special cases.

(i) If f is a closed embedding, then the trace f_* obtained as in Section 4.1 coincides with the Gysin operator defined in [Ne1], see Section 2.1.

(ii) If f is a projection of the form $p^{(n)}_X$, then the trace f_* defined in 4.1 agrees with the trace of $p^{(n)}_X$ given by Definition 3.2. This is straightforward.

4.5. Functoriality. (i) $(\mathrm{id}_X)_* = \mathrm{id}_{W^q_T(X;L)}$.

(ii) If $Z \xrightarrow{g} Y \xrightarrow{f} X$ are projective morphisms, $R \subset Z$, $S \subset Y$ and $T \subset X$ are compatible closed subvarieties, and L is a line bundle on X, then the diagram commutes:

$$\begin{tikzcd} W^{q+r}_S(Y;L^f) \arrow[r, "f_*^{S,T}"] & W^{q+t}_T(X;L) \\ W^q_R(Z;L^{fg}) \arrow[u, "g_*^{R,S}"] \arrow[ur, "(fg)_*^{R,T}"'] & \end{tikzcd}$$

Here $r = codim\, g$, $s = codim\, f$, $t = r + s$.

Proof. The first assertion is straightforward.

Proof of (ii). We will write the proof in terms of the maps involved; we leave it to the reader to write down the Witt groups that are the domains and codomains of such maps. We won't write the supports in the notation for push-forwards.

Let $Z \xrightarrow{i_g} Y \times \mathbb{P}^m \xrightarrow{p_Y^{(m)}} Y$ and $Y \xrightarrow{i_f} X \times \mathbb{P}^n \xrightarrow{p_X^{(n)}} X$ be factorizations for g and f. Clearly we can assume that m and n are even. The diagram

$$\begin{tikzcd} Y \times \mathbb{P}^m \arrow[r, "i_f^{(m)}"] \arrow[d, "p_Y^{(m)}"'] & X \times \mathbb{P}^n \times \mathbb{P}^m \arrow[d, "p^{(m)}_{X\times\mathbb{P}^n}"] \\ Y \arrow[r, "i_f"] & X \times \mathbb{P}^n \end{tikzcd}$$

is of the type considered in 3.6. Thus $(i_f)_* \circ (p_Y^{(m)})_* = (p^{(m)}_{X\times\mathbb{P}^n})_* \circ (i_f^{(m)})_*$ and

$$\begin{aligned} f_*g_* = (p_X^{(n)})_*(i_f)_*(p_Y^{(m)})_*(i_g)_* &= (p_X^{(n)})_*(p^{(m)}_{X\times\mathbb{P}^n})_*(i_f^{(m)})_*(i_g)_* \\ &= (p_X^{(n)})_*(p^{(m)}_{X\times\mathbb{P}^n})_*(i_f^{(m)} i_g)_*\,, \end{aligned}$$

the latter by Section 2.2.

It now suffices to prove the following

Lemma 4.6. *Let $h : Z \to X$ be a projective morphism decomposed as*

$$Z \xrightarrow{I} X \times \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{p_X^{(n,m)}} X$$

with even m and n, where I is a closed embedding and $p_X^{(n,m)} = p_X^{(n)} \circ p^{(m)}_{X\times\mathbb{P}^n} = p_X^{(m)} \circ p^{(n)}_{X\times\mathbb{P}^m}$. Then $h_* = (p_X^{(n,m)})_* \circ I_*$, where $(p_X^{(n,m)})_*$ is defined as either of the compositions $(p_X^{(n)})_*(p^{(m)}_{X\times\mathbb{P}^n})_*$ or $(p_X^{(m)})_*(p^{(n)}_{X\times\mathbb{P}^m})_*$, which are equal according to Section 3.3.

Now put $h = fg$, $I = i_f^{(m)} i_g$ and get $f_*g_* = (fg)_*$.

Proof of the lemma. Choose a closed embedding $j : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ with N sufficiently large even and let $J = \mathrm{id}_X \times j : X \times \mathbb{P}^n \times \mathbb{P}^m \to X \times \mathbb{P}^N$. Choose $a \in \mathbb{P}^n(k)\,, b \in \mathbb{P}^m(k)$ and let $c = j(a,b) \in \mathbb{P}^N(k)$. Then $J \circ i^{(m)}_{X\times\mathbb{P}^n,b} \circ i^{(n)}_{X,a} = i^{(N)}_{X,c}$, hence by 2.2

$$J_* \circ (i^{(m)}_{X\times\mathbb{P}^n,b})_* \circ (i^{(n)}_{X,a})_* = (i^{(N)}_{X,c})_* \,.$$

As m, n and N are even, all the i_*'s are invertible and we get by Definition 3.2(i)

$$(p^{(N)}_X)_* \circ J_* = (p^{(n)}_X)_* \circ (p^{(m)}_{X\times\mathbb{P}^n})_* = (p^{(n,m)}_X)_* \,.$$

Thus

$$(p^{(n,m)}_X)_* I_* = (p^{(N)}_X)_* J_* I_* = (p^{(N)}_X)_* (JI)_* = h_* \,,$$

where the last equation is true by Proposition 4.2 since $J \circ I$ is a closed embedding.■

Remark. (a) Lemma 4.6 can be viewed as a generalization of Proposition 4.2, for we get the latter if $m = 0$ or $n = 0$. It yields more flexibility in factoring projective morphisms to obtain their push-forwards. The obvious generalization with multiple projective spaces is also true.

(b) The assumption that m, n and N are even is not really necessary. For our purposes it suffices to prove the lemma under this assumption; the general case is left to the reader.

4.7. Transversal base changes.

Definition 4.8. A square of the form

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle \phi_Y}\downarrow & & \downarrow{\scriptstyle \phi_X} \\ Y & \xrightarrow{f} & X \end{array} \tag{4.2}$$

with f projective is called transversal in Sm_k if it is cartesian and the induced square of vector bundles

$$\begin{array}{ccc} T_{Y'} & \longrightarrow & (f')^* T_{X'} \\ \downarrow & & \downarrow \\ \phi_Y^* T_Y & \longrightarrow & (f\phi_Y)^* T_X \end{array}$$

is bicartesian in the category of vector bundles on Y'. Equivalently, one can require that the sequence

$$0 \to T_{Y'} \xrightarrow{(d\phi_Y, -df')} \phi_Y^* T_Y \oplus (f')^* T_{X'} \xrightarrow{(df, d\phi_X)} (f\phi_Y)^* T_X \to 0 \tag{4.3}$$

is exact, cf. [Me, Axiom (iv) in Section 2] or [LM].

If f is factored as $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p_X^{(n)}} X$ with i a closed embedding, then pulling back along ϕ_X we get a factorization of the entire diagram:

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & X' \times \mathbb{P}^n & \xrightarrow{p_{X'}^{(n)}} & X' \\ {\scriptstyle \phi_Y}\big\downarrow & & \big\downarrow{\scriptstyle \phi_X^{(n)}} & & \big\downarrow{\scriptstyle \phi_X} \\ Y & \xrightarrow{i} & X \times \mathbb{P}^n & \xrightarrow{p_X^{(n)}} & X \end{array} \tag{4.4}$$

In these terms, the transversality of (4.2) is equivalent to the transversality of the left square in (4.4) in the sense of Section 2.4. (The right square is always transversal.) The latter is stated in terms of normal bundles: the natural map $N_{i'} \to \phi_Y^* N_i$ is required to be an isomorphism.

Lemma 4.9. *If (4.2) is transversal and L is a line bundle on X, then the line bundles $\phi_Y^* L^f$ and $(\phi_X^* L)^{f'}$ on Y' can be identified canonically.*

Proof. The dual of (4.3) provides a natural isomorphism

$$\det(\phi_Y^*\Omega_Y \oplus (f')^*\Omega_{X'}) \cong \det \Omega_{Y'} \otimes \det (f\phi_Y)^*\Omega_X$$

which yields a natural isomorphism $\phi_Y^*\omega_f \cong \omega_{f'}$. (Compare to (3.4) where the same was verified in the case of projections by a more explicit computation.) It follows that

$$(\phi_X^* L)^{f'} = (f')^*\phi_X^* L \otimes \omega_{f'} \cong \phi_Y^* f^* L \otimes \phi_Y^*\omega_f \cong \phi_Y^*(f^*L \otimes \omega_f) = \phi_Y^* L^f \, .$$

The lemma is proved.■

As we have transversal base changes for projections and closed embeddings, see Sections 2.4 and 3.4, we can derive the same property for arbitrary projective morphisms by using (4.4). Lemma 4.9 guarantees that the twists agree.

Proposition 4.10. *Suppose that f and (therefore) f' are projective equicodimensional morphisms of codimension c in a transversal square of the form (4.2). Let S, T, S', T' be compatible closed subschemes in Y, X, Y', X', respectively. Then the diagram commutes:*

$$\begin{array}{ccc} W^q_{S'}(Y'; \phi_Y^* L^f) & \xrightarrow{f'_*} & W^{q+c}_{T'}(X'; \phi_X^* L) \\ {\scriptstyle \phi_Y^*}\big\downarrow & & \big\downarrow{\scriptstyle \phi_X^*} \\ W^q_S(Y; L^f) & \xrightarrow{f_*} & W^{q+c}_T(X; L) \end{array}$$

4.11. Projection formulas. Let $f : Y \to X$ be a projective morphism of codimension c, T and T' be closed subschemes in X and $S \subset f^{-1}(T)$ a closed subscheme in Y, and let L and L' be line bundles on X. Then for any $\alpha \in W^{q'}_{T'}(X; L')$ and $\beta \in W^q_S(Y; L^f)$ we have

$$f_*(f^*\alpha \star \beta) = \alpha \star f_*\beta \text{ and } f_*(\beta \star f^*\alpha) = (-1)^{cq'} f_*\beta \star \alpha$$

in $W^{q'+q+c}_{T \cap T'}(X; L \otimes L')$. We get these formulas by factoring f and applying the respective formulas of Sections 2.7 and 3.7.

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