The u-invariant of the function fields of p-adic curves

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Abstract

The u-invariant of a field is the maximum dimension of ansiotropic quadratic forms over the field. It is an open question whether the u-invariant of function fields of p-aidc curves is 8. In this paper, we answer this question in the affirmative for function fields of non-dyadic p-adic curves.

Introduction

It is an open question ([L], Q. 6.7, Chap XIII) whether every quadratic form in at least nine variables over the function fields of p-adic curves has a non-trivial zero. Equivalently, one may ask whether the u-invariant of such a field is 8. The u-invariant of a field F is defined as the maximal dimension of anisotropic quadratic forms over F. In this paper we answer this question in the affirmative if the p-adic field is non-dyadic.

In ([PS], 4.5), we showed that every quadratic form in eleven variables over the function field of a p-adic curve, $p \neq 2$, has a nontrivial zero. The main ingredients in the proof were the following: Let K be the function field of a p-adic curve X and $p \neq 2$.

1. (Saltman ([S1], 3.4)) Every element in the Galois cohomology group $H^2(K, \mathbf{Z}/2\mathbf{Z})$ is a sum of at most two symbols.

2. (Kato [K], 5.2) The unramified cohomology group $H^3_{\text{nr}}(K/\mathcal{X}, \mathbf{Z}/2\mathbf{Z}(2))$ is zero for a regular projective model $\mathcal{X}$ of K.

If K is as above, we proved ([PS], 3.9) that every element in $H^3(K, \mathbf{Z}/2\mathbf{Z})$ is a symbol of the form $(f) \cdot (g) \cdot (h)$ for some $f, g, h \in K^*$ and f may be chosen to be a value of a given binary form $<a, b>$ over K. If further given $\zeta = (f) \cdot (g) \cdot (h) \in H^3(K, \mathbf{Z}/2\mathbf{Z})$ and a ternary form $< c, d, e >$, one can choose $g', h' \in K^*$ such that $\zeta = (f) \cdot (g') \cdot (h')$ with g' a value of $<c, d, e>$, then, one is led to the conclusion that $u(K) = 8$ (cf 4.3). We in fact prove that such a choice of $g', h' \in K^*$ is possible by proving the following local global principle:

Theorem. Let $K = k(X)$ be the function field of a curve X over a p-adic field k. Let l be a prime not equal to p. Assume that k contains a primitive l^{th} root of unity. Given $\zeta \in H^3(K, \mu_l^{\otimes 2})$ and $\alpha \in H^2(K, \mu_l)$ corresponding to a degree l central division algebra over K, satisfying $\zeta = \alpha \cup (h_v)$ for some $(h_v) \in H^1(K_v, \mu_l)$, for all discrete valuations of K, there exists $(h) \in H^1(K, \mu_l)$ such that $\zeta = \alpha \cup (h)$. In fact one can restrict the hypothesis to discrete valuations of K centered on codimension one points of a regular model $\mathcal{X}$, projective over the ring of integers $\mathcal{O}_k$ of k.

A key ingredient towards the proof of the theorem is a recent result of Saltman ([S3]) where the ramification pattern of prime degree central simple algebras over function fields of p-adic curves is completely described.

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1. Some Preliminaries

In this section we recall a few basic facts from the algebraic theory of quadratic forms and Galois cohomology. We refer the reader to ([C]) and ([Sc]).

Let F be a field and l a prime not equal to the characteristic of F. Let μ_l be the group of l^{th} roots of unity. For $i \geq 1$, let $\mu_l^{\otimes i}$ be the Galois module given by the tensor product of i copies of μ_l. For $n \geq 0$, let $H^n(F, \mu_l^{\otimes i})$ be the n^{th} Galois cohomology group with coefficients in $\mu_l^{\otimes i}$.

We have the Kummer isomorphism $F^*/F^{*^l} \simeq H^1(F, \mu_l)$. For $a \in F^*$, its class in $H^1(F, \mu_l)$ is denoted by (a). If $a_1, \cdots, a_n \in F^*$, the cup product $(a_1) \cdots (a_n) \in H^n(F, \mu_l^{\otimes n})$ is called a *symbol*. We have an isomorphism $H^2(F, \mu_l)$ with the l-torsion subgroup ${}_lBr(F)$ of the Brauer group of F. We define the *index* of an element $\alpha \in H^2(F, \mu_l)$ to be the index of the corresponding central simple algebra in ${}_lBr(F)$.

Suppose F contains all the l^{th} roots of unity. We fix a generator ρ for the cyclic group μ_l and identify the Galois modules $\mu_l^{\otimes i}$ with $\mathbf{Z}/l\mathbf{Z}$. This leads to an identification of $H^n(F, \mu_l^{\otimes m})$ with $H^n(F, \mathbf{Z}/l\mathbf{Z})$. The element in $H^n(F, \mathbf{Z}/l\mathbf{Z})$ corresponding to the symbol $(a_1)\cdots(a_n) \in H^n(F, \mu_l^{\otimes n})$ through this identification is again denoted by $(a_1)\cdots(a_n)$. In particular for $a, b \in F^*$, $(a)\cdot(b) \in H^2(K, \mathbf{Z}/l\mathbf{Z})$ represents the cyclic algebra (a, b) defined by the relations $x^l = a$, $y^l = b$ and $xy = \rho yx$.

Let v be a discrete valuation of F. The residue field of v is denoted by $\kappa(v)$. Suppose $\text{char}(\kappa(v)) \neq l$. Then there is a *residue* homomorphism $\partial_v : H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(v), \mu_l^{\otimes(m-1)})$. Let $\alpha \in H^n(F, \mu_l^{\otimes m})$. We say that α is *unramified* at v if $\partial_v(\alpha) = 0$; otherwise it is said to be *ramified* at v. If F is complete with respect to v, we denote the kernel of ∂_v by $H^n_{\text{nr}}(F, \mu_l^{\otimes m})$. Suppose α is unramified at v. Let $\pi \in K^*$ be a parameter at v and $\zeta = \alpha \cup (\pi) \in H^{n+1}(F, \mu_l^{\otimes(m+1)})$. Let $\overline{\alpha} = \partial_v(\zeta) \in H^n(\kappa(v), \mu_l^{\otimes m})$. The element $\overline{\alpha}$ is independent of the choice of the parameter π and is called the *specialization* of α at v. We say that α *specializes to* $\overline{\alpha}$ at v. The following result is well known.

Lemma 1.1 Let k be a field and l a prime not equal to the characteristic of k. Let K be a complete discrete valuated field with residue field k. If $H^n(k, \mu_l^{\otimes 3}) = 0$ for $n \geq 3$, then $H^3_{\text{nr}}(K, \mu_l^{\otimes 3}) = 0$. Suppose further that every element in $H^2(k, \mu_l^{\otimes 2})$ is a symbol. Then every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol.

Proof. Let R be the ring of integers in K. The Gysin exact sequence in étale cohomology yields an exact sequence (cf. [C], p.21, §3.3)

$$H^3_{\text{ét}}(R, \mu_l^{\otimes 3}) \to H^3(K, \mu_l^{\otimes 3}) \xrightarrow{\partial} H^2(k, \mu_l^{\otimes 2}) \to H^4_{\text{ét}}(R, \mu_l^{\otimes 3})$$

Since R is complete, $H^n_{\text{ét}}(R, \mu_l^{\otimes 3}) \simeq H^n(k, \mu_l^{\otimes 3})$ ([Mi], p.224 Corollary 2.7). Hence $H^n_{\text{ét}}(R, \mu_l^{\otimes 3}) = 0$ for $n \geq 3$, by the hypothesis. In particular $\partial : H^3(K, \mu_l^{\otimes 3}) \to H^2(k, \mu_l^{\otimes 2})$ is an isomorphism and $H^3_{\text{nr}}(K, \mu_l^{\otimes 3}) = 0$. Let $u, v \in R$ be units and $\pi \in R$ a parameter. Then we have $\partial((u)\cdot(v)\cdot(\pi)) = (\overline{u})\cdot(\overline{v})$. Let $\zeta \in H^3(K, \mu_l^{\otimes 3})$. Since every element in $H^2(k, \mu_l^{\otimes 2})$ is a symbol, we have $\partial(\zeta) = (\overline{u}) \cdot (\overline{v})$ for some units $u, v \in R$. Since ∂ is an isomorphism, we have $\zeta = (u) \cdot (v) \cdot (\pi)$. Thus every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol. $\square$

Corollary 1.2 Let k be a p-adic field and K the function field of an integral curve over k. Let l be a prime not equal to p. Let K_v be the completion of K at a discrete valuation of K. Then $H^3_{nr}(K_v, \mu_l^{\otimes 3}) = 0$. Suppose further that K contains a primitive l^{th} root of unity. Then every element in $H^3(K_v, \mu_l^{\otimes 3})$ is a symbol.

Proof. Let v be a discrete valuation of K and K_v the completion of K at v. The residue field $\kappa(v)$ at v is either a p-adic field or a function field of a curve over a finite field of characteristic p. In either case, the cohomological dimension of $\kappa(v)$ is 2 and hence $H^n(\kappa(v), \mu_l^{\otimes 3}) = 0$ for $n \geq 3$. By (1.1), $H^3_{nr}(K_v, \mu_l^{\otimes 3}) = 0$.

If $\kappa(v)$ is a local field, by local class field theory, every finite dimension central division algebra over $\kappa(v)$ is split by an unramified (cyclic) extension. If $\kappa(v)$ is a function field of a curve over finite field, then by a classical theorem of Hasse-Brauer-Noether-Albert, every finite dimensional central division algebra over $\kappa(v)$ is split by a cyclic extension. Since $\kappa(v)$ contains a primitive l^{th} of unity, every element in $H^2(\kappa(v), \mathbf{Z}/l\mathbf{Z})$ is a symbol. By (1.1), every element in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is a symbol. $\square$

Let $\mathcal{X}$ be a regular integral scheme of dimension d, with field of fractions F. Let $\mathcal{X}^1$ be the set of points of $\mathcal{X}$ of codimension 1. A point $x \in \mathcal{X}^1$ gives rise to a discrete valuation v_x on F. The residue field of this discrete valuation ring is denoted by $\kappa(x)$ or $\kappa(v_x)$. The corresponding residue homomorphism is denoted by ∂_x. We say that an element $\zeta \in H^n(F, \mu_l^{\otimes m})$ is *unramified* at x if $\partial_x(\zeta) = 0$; otherwise it is said to be *ramified* at x. We define the ramification divisor $\operatorname{ram}_{\mathcal{X}}(\zeta) = \sum x$ as x runs over $\mathcal{X}^1$ where ζ is ramified. The unramified cohomology on $\mathcal{X}$, denoted by $H^n_{nr}(F/\mathcal{X}, \mu_l^{\otimes m})$, is defined as the intersection of kernels of the residue homomorphisms $\partial_x : H^n(F, \mu_l^{\otimes m}) \to H^{n-1}(\kappa(x), \mu_l^{\otimes(m-1)})$, x running over $\mathcal{X}^1$. We say that $\zeta \in H^n(F, \mu_l^{\otimes m})$ is *unramified on* $\mathcal{X}$ if $\zeta \in H^n_{nr}(F/\mathcal{X}, \mu_l^{\otimes m})$. If $\mathcal{X} = \operatorname{Spec}(R)$, then we also say that ζ is unramified on R if it is unramified on $\mathcal{X}$. Suppose C is an irreducible subscheme of $\mathcal{X}$ of codimension 1. Then the generic point x of C belongs to $\mathcal{X}^1$ and we set $\partial_x = \partial_C$. If $\alpha \in H^n(F, \mu_l^{\otimes m})$ is unramified at x, then we say that α is *unramified* at C.

Let k be a p-adic field and K the function field of a smooth, projective, geometrically integral curve X over k. By the resolution of singularities for surfaces (cf. [Li1] and [Li2]), there exists a regular, projective model $\mathcal{X}$ of X over the ring of integers $\mathcal{O}_k$ of k. We call such an $\mathcal{X}$ a *regular projective model of* K. Since the generic fibre X of $\mathcal{X}$ is geometrically integral, it follows that the special fibre $\overline{\mathcal{X}}$ is connected. Further if D is a divisor on $\mathcal{X}$, there exists a proper birational morphism $\mathcal{X}' \to \mathcal{X}$ such that the total transform of D on $\mathcal{X}'$ is a divisor with normal crossings (cf. [Sh], Theorem, p.38 and Remark 2, p. 43). We use this result throughout this paper without further reference.

Let k be a p-adic field and K the function field of a smooth, projective, geometrically integral curve over k. Let l be a prime not equal to p. Assume that k contains a primitive l^{th} of unity. Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular projective model of K such that the ramification locus $\mathrm{ram}_{\mathcal{X}}(\alpha)$ is a union of regular curves with normal crossings. Let P be a closed point in the intersection of two regular curves C and E in $\mathrm{ram}_{\mathcal{X}}(\alpha)$. Suppose that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbf{Z}/l\mathbf{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbf{Z}/l\mathbf{Z})$ are unramified at P. Let $u(P), v(P) \in H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$ be the specialisations at P of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ respectively. Following Saltman ([S3], §2), we say that P is a *cool point* if $u(P), v(P)$ are trivial and a *chilly point* if $u(P)$ and $v(P)$ generate the same subgroup of $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$ and neither of them is trivial. If $u(P)$ and $v(P)$ do not generate the same subgroup of $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$, then P is said to be a *hot point*. Let $\mathcal{O}_{\mathcal{X},P}$ be the regular local ring at P and π, δ prime elements in $\mathcal{O}_{\mathcal{X},P}$ which define C and E respectively at P. The condition that $\partial_C(\alpha) \in H^1(\kappa(C), \mathbf{Z}/l\mathbf{Z})$ and $\partial_E(\alpha) \in H^1(\kappa(E), \mathbf{Z}/l\mathbf{Z})$ are unramified at P is equivalent to the condition $\alpha = \alpha' + (u, \pi) + (v, \delta)$ for some units $u, v \in \mathcal{O}_{\mathcal{X},P}$ and α' unramified on $\mathcal{O}_{\mathcal{X},P}$ ([S2], §2). The specialisations of $\partial_C(\alpha)$ and $\partial_E(\alpha)$ in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z}) \simeq \kappa(P)^*/\kappa(P)^{*^l}$ are given by the images of u and v in $\kappa(P)$.

Proposition 1.3 ([S3], 2.5) If the index of α is l, then there are no hot points for α.

Suppose P is a chilly point. Then $v(P) = u(P)^s$ for some s with $1 \leq s \leq l-1$ and s is called the *coefficient* of P with respect to π. To get some compatibility for these coefficients, Saltman associates to α and $\mathcal{X}$

the following graph: The set of vertices is the set of irreducible curves in $\mathrm{ram}_{\mathcal{X}}(\alpha)$ and there is an edge between two vertices if there is a chilly point in the intersection of the two irreducible curves corresponding to the vertices. A loop in this graph is called a *chilly loop*.

Proposition 1.4 ([S3], 2.6, 2.9) There exists a projective model $\mathcal{X}$ of K such that there are no chilly loops and no cool points on $\mathcal{X}$ for α.

Let F be a field of characteristic not equal to 2. The u-invariant of F, denoted by $u(F)$, is defined as follows:

$$u(F) = \sup\{\mathrm{rk}(q) \mid q \text{ an anisotropic quadratic form over } F\}.$$

For $a_1, \cdots, a_n \in F^*$, we denote the diagonal quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$ by $<a_1, \cdots, a_n>$. Let $W(F)$ be the Witt ring of quadratic forms over F and $I(F)$ be the ideal of $W(F)$ consisting of even dimension forms. Let $I^n(F)$ be the n^{th} power of the ideal $I(F)$. For $a_1, \cdots, a_n \in F^*$, let $<<a_1, \cdots, a_n>>$ denote the n-fold Pfister form $<1, a_1> \otimes \cdots \otimes <1, a_n>$. The abelian group $I^n(F)$ is generated by n-fold Pfister forms. The dimension modulo 2 gives an isomorphism $e_0 : W(F)/I(F) \to H^0(F, \mathbf{Z}/2\mathbf{Z})$. The discriminant gives an isomorphism $e_1 : I(F)/I^2(F) \to H^1(F, \mathbf{Z}/2\mathbf{Z})$. The classical result of Merkurjev ([M]), asserts that the Clifford invariant gives an isomorphism $e_2 : I^2(F)/I^3(F) \to H^2(F, \mathbf{Z}/2\mathbf{Z})$.

Let $P_n(F)$ be the set of isometry classes of n-fold Pfister forms over F. There is a well-defined map ([A])

$$e_n : P_n(F) \to H^n(F, \mathbf{Z}/2\mathbf{Z})$$

given by $e_n(<1, a_1> \otimes \cdots <1, a_n>) = (-a_1) \cdots (-a_n) \in H^n(F, \mathbf{Z}/2\mathbf{Z})$.

A quadratic form version of the Milnor conjecture asserts that e_n induces a surjective homomorphism $I^n(F) \to H^n(F, \mathbf{Z}/2\mathbf{Z})$ with kernel $I^{n+1}(F)$. This conjecture was proved by Voevodsky, Orlov and Vishik. In this paper we are interested in fields of 2-cohomological dimension at most 3. For such fields, the above Milnor's conjecture is already proved by Arason, Elman and Jacob ([AEJ], Corollary 4 and Theorem 2), using the theorem of Markurjev ([M]).

Let q_1 and q_2 be two quadratic forms over F. We write $q_1 = q_2$ if they represent the same class in the Witt group $W(F)$. We write $q_1 \simeq q_2$, if q_1 and q_2 are isometric quadratic forms. We note that if the dimensions of q_1 and q_2 are equal and $q_1 = q_2$, then $q_1 \simeq q_2$.

2. Divisors on Arithmetic Surfaces

In this section we recall a few results from a paper of Saltman ([S3]) on divisors on arithmetic surfaces.

Let $\mathcal{Z}$ be a connected, reduced scheme of finite type over a Noetherian ring. Let $\mathcal{O}^*_{\mathcal{Z}}$ be the sheaf of units in the structure sheaf $\mathcal{O}_{\mathcal{Z}}$. Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{Z}$. For each $P \in \mathcal{P}$, let $\kappa(P)$ be the residue field at P and $\iota_P : \mathrm{Spec}(\kappa(P)) \to \mathcal{Z}$ be the natural morphism. Consider the sheaf $\mathcal{P}^* = \oplus_{P\in\mathcal{P}} \iota_P^* \kappa(P)^*$, where $\kappa(P)^*$ denotes the group of units in $\kappa(P)$. Then there is a surjective morphism of sheaves $\mathcal{O}^*_{\mathcal{Z}} \to \mathcal{P}^*$ given by the evaluation at each $P \in \mathcal{P}$. Let $\mathcal{O}^{*(1)}_{\mathcal{Z},\mathcal{P}}$ be its kernel. When there is no ambiguity we denote $\mathcal{O}^{*(1)}_{\mathcal{Z},\mathcal{P}}$ by $\mathcal{O}^{*(1)}_{\mathcal{P}}$. Let $\mathcal{K}$ be the sheaf of total quotient rings on $\mathcal{Z}$ and $\mathcal{K}^*$ be the sheaf of groups given by units in $\mathcal{K}$. Every element $\gamma \in H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ can be represented by a family $\{U_i, f_i\}$, where U_i are open sets in $\mathcal{Z}$, $f_i \in \mathcal{K}^*(U_i)$ and $f_i f_j^{-1} \in \mathcal{O}^*(U_i \cap U_j)$. We say that an element $\gamma = \{U_i, f_i\}$ of $H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ *avoids* $\mathcal{P}$ if each f_i is a unit at P for all $P \in U_i \cap \mathcal{P}$. Let $H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ be the subgroup of $H^0(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*)$ consisting of those γ which avoid $\mathcal{P}$. Let $K^* = H^0(\mathcal{Z}, \mathcal{K}^*)$ and $K^*_{\mathcal{P}}$ be the subgroup of K^* consisting of those functions which are units at all $P \in \mathcal{P}$. We have a natural inclusion $K^*_{\mathcal{P}} \to H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P\in\mathcal{P}} \kappa(P)^*)$.

We have the following

Proposition 2.1 ([S3], 1.6) Let $\mathcal{Z}$ be a connected, reduced scheme of finite type over a Noetherian ring. Then

$$H^1(\mathcal{Z}, \mathcal{O}^{*(1)}_{\mathcal{P}}) \simeq \frac{H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P\in\mathcal{P}} \kappa(P)^*)}{K^*_{\mathcal{P}}}.$$

Let k be a p-adic field and $\mathcal{O}_k$ the ring of integers of k. Let $\mathcal{X}$ be a connected regular surface with a projective morphism $\eta : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k)$. Let $\overline{\mathcal{X}}$ be the reduced special fibre of η. Assume that $\overline{\mathcal{X}}$ is connected. Note that $\overline{\mathcal{X}}$ is connected if the generic fibre is geometrically integral. Let $\mathcal{P}$ be a finite set of closed points in $\mathcal{X}$. Since every closed point of $\mathcal{X}$ is in $\overline{\mathcal{X}}$, $\mathcal{P}$ is also a subset of closed points of $\overline{\mathcal{X}}$. Let m be an integer coprime with p.

Proposition 2.2 ([S3], 1.7) The canonical map $H^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}}) \to H^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})$ induces an isomorphism

$$\frac{H^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}})}{mH^1(\mathcal{X}, \mathcal{O}^{*(1)}_{\mathcal{X},\mathcal{P}})} \simeq \frac{H^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})}{mH^1(\overline{\mathcal{X}}, \mathcal{O}^{*(1)}_{\overline{\mathcal{X}},\mathcal{P}})}.$$

Let $\mathcal{X}$ be as above. Suppose that $\overline{\mathcal{X}}$ is a union of regular curves $F_1, \cdots, F_m$ on $\mathcal{X}$ with only normal crossings. Let $\mathcal{P}$ be a finite set of closed points of $\mathcal{X}$ including all the points of $F_i \cap F_j$, $i \neq j$ and at least one point from each F_i. Let E be a divisor on $\mathcal{X}$ whose support does not pass through any point of $\mathcal{P}$. In particular, no F_i is in the support of E. Hence there are only finitely many closed points $Q_1, \cdots, Q_n$ on the support of E. For each closed point Q_i on the support of E, let D_i be a regular geometric curve (i.e. a curve not contained in the special fibre) on $\mathcal{X}$ such that Q_i is the multiplicity one intersection of D_i and $\overline{\mathcal{X}}$. Such a curve exists by ([S3], 1.1). We note that any closed point on $\mathcal{X}$ is a point of codimension 2 and there is a unique closed point on any geometric curve on $\mathcal{X}$ (cf. §1).

The following is extracted from ([S3], §5).

Proposition 2.3 Let $\mathcal{X}, \mathcal{P}, E, Q_i, D_i$ be as above. For each closed point Q_i, let m_i be the intersection multiplicity of the support of E and the special fibre $\overline{\mathcal{X}}$ at Q_i. Let l be a prime not equal to p. Then there exist $\nu \in K^*$ and a divisor E' on $\mathcal{X}$ such that

$$(\nu) = -E + \sum_{i=1}^{n} m_i D_i + lE'$$

and $\nu(P) \in \kappa(P)^{*^l}$ for each $P \in \mathcal{P}$.

Proof. Let F be the divisor on $\mathcal{X}$ given by $\sum F_i$. Let $\gamma \in Pic(\mathcal{X})$ be the line bundle equivalent to the class of the divisor $-E$ and $\overline{\gamma} \in Pic(\overline{\mathcal{X}})$ its image. Since the support of E does not pass through the points of $\mathcal{P}$ and $\mathcal{P}$ contains all the points of intersection of distinct F_i, E and F intersect only at smooth points of $\overline{\mathcal{X}}$. In particular, we have $\overline{\gamma} = -\sum m_i Q_i$. Let $\gamma' \in H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{P}})$ be the element which, under the isomorphism of (2.1), corresponds to the class of the element $(-E + \sum m_i D_i, 1)$ in $H^0_{\mathcal{P}}(\mathcal{X}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P \in \mathcal{P}} \kappa(P)^*)$. Since the m_i's are intersection multiplicities of E and $\overline{\mathcal{X}}$ at Q_i and the image of $\sum m_i D_i$ in $H^0_{\mathcal{P}}(\overline{\mathcal{X}}, \mathcal{K}^*/\mathcal{O}^*)$ is $\sum m_i Q_i$, the image $\overline{\gamma'}$ of γ' in $H^1(\overline{\mathcal{X}}, \mathcal{O}^*_{\mathcal{P}})$ is zero. By (2.2), we have $\gamma' \in lH^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{P}})$. Using (2.1), there exists $(E', (\lambda_P)) \in H^0_{\mathcal{P}}(\mathcal{Z}, \mathcal{K}^*/\mathcal{O}^*) \oplus (\oplus_{P \in \mathcal{P}} \kappa(P)^*)$ such that $(-E + \sum m_i D_i, 1) = l(E', (\lambda_P)) = (lE', (\lambda_P^l))$ modulo $K^*_{\mathcal{P}}$. Thus there exists $\nu \in K^*_{\mathcal{P}} \subset K^*$ such that $(\nu) = (-E + \sum m_i D_i, 1) - (lE', (\lambda^l_P))$. i.e. $(\nu) = -E + \sum m_i D_i - lE'$ and $\nu(P) = \lambda^l_P$ for each $P \in \mathcal{P}$. $\square$

3. A local-global principle

Let k be a p-adic field, $\mathcal{O}_k$ be its ring of integers and K the function field of a smooth, projective, geometrically integral curve over k. Let l be a prime not equal to p. Throughout this section, except in 3.6, we assume that k contains a primitive l^{th} root of unity. We fix a generator ρ for μ_l and identify μ_l with $\mathbf{Z}/l\mathbf{Z}$.

Lemma 3.1 Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular projective model of K. Assume that the ramification locus $\text{ram}_{\mathcal{X}}(\alpha)$ is a union of regular curves $\{C_1, \cdots, C_r\}$ with only normal crossings. Let T be a finite set of closed points of $\mathcal{X}$ including the points of $C_i \cap C_j$, for all $i \neq j$. Let D be an irreducible curve on $\mathcal{X}$ which is not in the ramification locus of α and does not pass through any point in T. Then D intersects C_i at points P where $\partial_{C_i}(\alpha)$ is unramified. Suppose further that at such points P, $\partial_{C_i}(\alpha)$ specializes to 0 in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$. Then α is unramified at D and specializes to 0 in $H^2(\kappa(D), \mu_l)$.

Proof. Since k contains a primitive l^{th} of unity, we fix a generator ρ for μ_l and identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$. Let P be a point in the intersection of D and the support of $\text{ram}_{\mathcal{X}}(\alpha)$. Since D does not pass through the points of T and T contains all the points of intersection of distinct C_j, the point P belongs to a unique curve C_i in the support of $\text{ram}_{\mathcal{X}}(\alpha)$. We have ([S1], 1.2) $\alpha = \alpha' + (u, \pi)$, where α' is unramified on $\mathcal{O}_{\mathcal{X},P}$, $u \in \mathcal{O}_{\mathcal{X},P}$ is a unit and $\pi \in \mathcal{O}_{\mathcal{X},P}$ is a prime defining the curve C_i at P. Therefore $\partial_{C_i}(\alpha) = (\overline{u}) \in H^1(\kappa(C_i), \mathbf{Z}/l\mathbf{Z})$ is unramified at P.

Suppose that $\partial_{C_i}(\alpha)$ specialises to zero in $H^1(\kappa(P), \mathbf{Z}/l\mathbf{Z})$. Since D is not in the ramification locus of α, α is unramified at D. Let $\overline{\alpha}$ be the specialization of α in $H^2(\kappa(D), \mathbf{Z}/l\mathbf{Z})$. Since $\kappa(D)$ is either a p-adic field or a function field of a curve over a finite field, to show that $\overline{\alpha}$ is zero, by class field theory it is enough to show that $\overline{\alpha}$ is unramified at every discrete valuation of $\kappa(D)$.

Let v be a discrete valuation of $\kappa(D)$ and R the corresponding discrete valuation ring. Then there exists a closed point P of D such that R is a localization of the integral closure of the one dimensional local ring $\mathcal{O}_{D,P}$ of P on D. The local ring $\mathcal{O}_{D,P}$ is a quotient of the local ring $\mathcal{O}_{\mathcal{X},P}$.

Suppose P is not on the ramification locus of α. Then α is unramified on $\mathcal{O}_{\mathcal{X},P}$ and hence $\overline{\alpha}$ on $\overline{\mathcal{O}}_{D,P}$. In particular $\overline{\alpha}$ is unramified at R.

Suppose P is on the ramification locus of α. As before, we have $\alpha = \alpha' + (u, \pi)$, where α' is unramified on $\mathcal{O}_{\mathcal{X},P}$, $u \in \mathcal{O}_{\mathcal{X},P}$ is a unit and $\pi \in \mathcal{O}_{\mathcal{X},P}$ is a prime defining the curve C_i at P. Therefore $\partial_{C_i}(\alpha) = \overline{u}$ in $\kappa(C_i)^*/\kappa(C_i)^{*^l}$. Since, by the assumption, $\partial_{C_i}(\alpha)$ specializes to 0 at P, $u(P) \in \kappa(P)^{*^l}$. We have $\overline{\alpha} = \overline{\alpha'} + (\overline{u}, \overline{\pi}) \in H^2(\kappa(D), \mathbf{Z}/l\mathbf{Z})$. Since α' is unramified at P, the residue of $\overline{\alpha}$ at R is $(u(P))^{\nu(\overline{\pi})}$. Since $\kappa(P)$ is contained in the residue field of the discrete valuation ring R and $u(P)$ is an l^{th} power in $\kappa(P)$, it follows that $\overline{\alpha}$ is unramified at R. □

Proposition 3.2 Let K and l be as above. Let $\alpha \in H^2(K, \mu_l)$ with index l. Let $\mathcal{X}$ be a regular projective model of K such that the ramification locus $\mathrm{ram}_{\mathcal{X}}(\alpha)$ and the special fibre of $\mathcal{X}$ are a union of regular curves with only normal crossings and α has no cool points and no chilly loops on $\mathcal{X}$ (cf. 1.4). Let s_i be the corresponding coefficients (cf. §1). Let $F_1, \cdots, F_r$ be irreducible regular curves on $\mathcal{X}$ which are not in $\mathrm{ram}_{\mathcal{X}}(\alpha) = \{C_1, \cdots, C_n\}$ and such that $\{F_1, \cdots, F_r\} \cup \mathrm{ram}_{\mathcal{X}}(\alpha)$ have only normal crossings. Let $m_1, \cdots, m_r$ be integers. Then there exists $f \in K^*$ such that

$$\mathrm{div}_{\mathcal{X}}(f) = \sum s_i C_i + \sum m_s F_s + \sum n_j D_j + lE',$$

where $D_1, \cdots, D_t$ are irreducible curves which are not equal to C_i and F_s for all i and s and α specializes to zero at D_j for all j and $(n_j, l) = 1$.

Proof. Let T be a finite set of closed points of $\mathcal{X}$ containing all the points of intersection of distinct C_i and F_s and at least one point from each C_i and F_s. By a semilocal argument, we choose $g \in K^*$ such that $\mathrm{div}_{\mathcal{X}}(g) =$

$\sum s_iC_i + \sum m_sF_s + G$ where G is a divisor on $\mathcal{X}$ whose support does not contain any of C_i or F_s and does not intersect T.

Since α has no cool points and no chilly loops on $\mathcal{X}$, by ([S3], Prop. 4.5), there exists $u \in K^*$ such that $\mathrm{div}_{\mathcal{X}}(ug) = \sum s_iC_i + \sum m_sF_s + E$, where E is a divisor on $\mathcal{X}$ whose support does not contain any C_i or F_s, does not pass through the points in T and either E intersect C_i at a point P where the specialization of $\partial_{C_i}(\alpha)$ is 0 or the intersection multiplicity $(E \cdot C_i)_P$ is a multiple of l.

Suppose C_i for some i is a geometric curve on $\mathcal{X}$. Since every closed point of $\mathcal{X}$ is on the special fibre $\overline{\mathcal{X}}$, the closed point of C_i is in T. Since the support of E avoids all the points in T, the support of E does not intersect C_i. Thus the support of E intersects only those C_i which are in the special fibre $\overline{\mathcal{X}}$. Let $Q_1, \cdots, Q_t$ be the points of intersection of the support of the divisor E and the special fibre with intersection multiplicity n_j at Q_j coprime with l, for $1 \leq j \leq r$. In particular we have $\partial_{C_j}(\alpha) = 0$ for $1 \leq j \leq r$. For each Q_j, let D_j be a regular geometric curve on $\mathcal{X}$ such that Q_j is the multiplicity one intersection of D_j and $\overline{\mathcal{X}}$ (cf. paragraph after 2.2). Then by (2.3) there exists $\nu \in K^*$ such that $\mathrm{div}_{\mathcal{X}}(\nu) = -E + \sum n_jD_j + lE'$ and $\nu(P) \in \kappa(P)^{*^l}$ for all $P \in T$. Let $f = ug\nu \in K^*$. Then we have

$$\mathrm{div}_{\mathcal{X}}(f) = \sum s_iC_i + \sum m_sF_s + \sum n_jD_j + lE'.$$

Since each Q_j is the only closed point on D_j and $\partial_{C_i}(\alpha)$ specialises to zero at Q_j, by (3.1), the α specializes to 0 at D_j. Thus f has all the required properties. $\square$

Lemma 3.3 Let $\alpha \in H^2(K, \mu_l)$ and let v be a discrete valuation of K. Let $u \in K^*$ be a unit at v such that $\overline{u} \in \kappa(v)^* \setminus \kappa(v)^{*^l}$. Suppose further that if α is ramified at v, $\partial_v(\alpha) = [L] \in H^1(\kappa(v), \mathbf{Z}/l\mathbf{Z})$, where $L = K(u^{\frac{1}{l}})$. Then, for any $g \in L^*$, the image of $(N_{L/K}(g)) \cdot \alpha \in H^3(K_v, \mu_l^{\otimes 2})$ is zero.

Proof. We identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$ as before. Since u is a unit at v and $\overline{u} \notin \kappa(v)^{*^l}$, there is a unique discrete valuation $\tilde{v}$ of L extending the valuation v of K, which is unramified with residual degree l. In particular $v(N_{L/K}(g))$ is a multiple of l. Thus if $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified at v, then $(N_{L/K}(g)) \cdot \alpha' \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified. Since $H^3_{nr}(K_v, \mathbf{Z}/l\mathbf{Z}) = 0$

(cf. 1.2), we have $(N_{L/K}(g)) \cdot \alpha' = 0$ for any $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ which is unramified at v. In particular, if α is unramified at v, then $\alpha \cdot (N_{L/K}(g)) = 0$.

Suppose that α is ramified at v. Then by the choice of u, we have $\alpha = \alpha' + (u) \cdot (\pi_v)$, where π_v is a parameter at v and $\alpha' \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ is unramified at v. Thus we have $(N_{L/K}(g)) \cdot \alpha = (N_{L/K}(g)) \cdot \alpha' + (N_{L/K}(g)) \cdot (u) \cdot (\pi_v) = (N_{L/K}(g)) \cdot (u) \cdot (\pi_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. Since $L_v = K_v(u^{\frac{1}{l}})$, we have $((N_{L/K}(g)) \cdot (u) = 0 \in H^2(K_v, \mathbf{Z}/l\mathbf{Z})$ and $(N_{L/K}(g)) \cdot \alpha = 0$ in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. $\square$

Theorem 3.4 Let K and l be as above. Let $\alpha \in H^2(K, \mu_l)$ and $\zeta \in H^3(K, \mu_l^{\otimes 2})$. Assume that the index of α is l. Let $\mathcal{X}$ be a regular projective model of K. Suppose that for each $x \in \mathcal{X}^1$, there exists $f_x \in K_x^*$ such that $\zeta = \alpha \cup (f_x) \in H^3(K_x, \mu_l^{\otimes 2})$, where K_x is the completion of K at the discrete valuation given by x. Then there exists $f \in K^*$ such that $\zeta = \alpha \cup (f) \in H^3(K, \mu_l^{\otimes 2})$.

Proof. We identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$ as before. By weak approximation, we may find $f \in K^*$ such that $(f) = (f_v) \in H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ for all the discrete valuations corresponding to the irreducible curves in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta)$. Let

$$\mathrm{div}_{\mathcal{X}}(f) = C' + \sum m_i F_i + lE,$$

where C' is a divisor with support contained in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta)$, F_i's are distinct irreducible curves which are not in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta)$, m_i coprime with l and E some divisor on $\mathcal{X}$.

For any $C_j \in ram_{\mathcal{X}}(\zeta) \setminus ram_{\mathcal{X}}(\alpha)$, let $\lambda_j \in \kappa(C_j)^* \setminus \kappa(C_j)^{*^l}$. By weak approximation, we choose $u \in K^*$ with $\overline{u} = \partial_{C_i}(\alpha) \in H^1(\kappa(C_i), \mathbf{Z}/l\mathbf{Z})$ for all $C_i \in ram_{\mathcal{X}}(\alpha)$, $\nu_{F_i}(u) = m_i$, where ν_{F_i} is the discrete valuation at F_i and $\overline{u} = \lambda_j$ for any $C_j \in ram_{\mathcal{X}}(\zeta) \setminus ram_{\mathcal{X}}(\alpha)$. In particular u is a unit at the generic point of C_j and $\overline{u} \notin \kappa(C_j)^{*^l}$ for any $C_j \in ram_{\mathcal{X}}(\zeta) \setminus ram_{\mathcal{X}}(\alpha)$.

Let $L = K(u^{\frac{1}{l}})$. Let $\eta : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ in L. Since $\nu_{F_i}(u) = m_i$ and m_i is coprime with l, $\eta : \mathcal{Y} \to \mathcal{X}$ is ramified at F_i. In particular there is a unique irreducible curve $\tilde{F}_i$ in $\mathcal{Y}$ such that $\eta(\tilde{F}_i) = F_i$ and $\kappa(F_i) = \kappa(\tilde{F}_i)$.

Let $\pi : \tilde{\mathcal{Y}} \to \mathcal{Y}$ be a proper birational morphism such that the ramification locus $ram_{\tilde{\mathcal{Y}}}(\alpha_L)$ of α_L on $\tilde{\mathcal{Y}}$ and the strict transform of the curves $\tilde{F}_i$ on $\tilde{\mathcal{Y}}$

is a union of regular curves with only normal crossings and there are no cool points and no chilly loops for α_L on $\tilde{\mathcal{Y}}$ (cf 1.4). We denote the strict transforms of $\tilde{F}_i$ by $\tilde{F}_i$ again. By (3.2), there exists $g \in L^*$ such that

$$div_{\tilde{\mathcal{Y}}}(g) = C + \sum -m_i\tilde{F}_i + \sum n_j D_j + lD,$$

where the support of C is contained in $ram_{\tilde{\mathcal{Y}}}(\alpha_L)$ and D_j's are irreducible curves which are not in $ram_{\tilde{\mathcal{Y}}}(\alpha_L)$ and α_L specializes to zero at all D_j's.

We now claim that $\zeta = \alpha \cup (fN_{L/K}(g))$. Since the group $H^3_{nr}(K/\mathcal{X}, \mathbf{Z}/l\mathbf{Z}) = 0$ ([K], 5.2), it is enough to show that $\zeta - \alpha \cup (fN_{L/K}(g))$ is unramified on $\mathcal{X}$. Let S be an irreducible curve on $\mathcal{X}$. Since the residue map ∂_S factors through the completion K_S, it suffices to show that $\zeta - \alpha \cup (fN_{L/K}(g)) = 0$ over K_S.

Suppose S is not in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta) \cup Supp(fN_{L/K}(g))$. Then each of ζ and $\alpha \cdot (fN_{L/K}(g))$ is unramified at S.

Suppose that S is in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta)$. Then by the choice of f we have $(f) = (f_v) \in H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ where v is the discrete valuation associated to S. Hence $\zeta = \alpha \cup (f)$ over the completion K_S of K at the discrete valuation given by S. It follows from (3.3) that $(N_{L/K}(g)) \cup \alpha = 0$ over K_S and $\zeta = \alpha \cup (fN_{L/K}(g))$ over K_S.

Suppose that S is in the support of $div_{\mathcal{X}}(fN_{L/K}(g))$ and not in $ram_{\mathcal{X}}(\alpha) \cup ram_{\mathcal{X}}(\zeta)$. Then α is unramified at S. We show that in this case $\alpha \cup (fN_{L/K}(g)) = 0$ over K_S. We have

$$\begin{aligned} div_{\mathcal{X}}(fN_{L/K}(g)) &= div_{\mathcal{X}}(f) + div_{\mathcal{X}}(N_{L/K}(g)) \\ &= C' + \textstyle\sum m_iF_i + lE + \eta_*\pi_*(C + \sum -m_i\tilde{F}_i + \sum n_jD_j + lD) \\ &= C' + \eta_*\pi_*(C) + \textstyle\sum n_j\eta_*\pi_*(D_j) + l\eta_*\pi_*(E). \end{aligned}$$

We note that if D_j maps to a point, then $\eta_*\pi_*(D_j) = 0$. Since the support of C is contained in $ram_{\tilde{\mathcal{Y}}}(\alpha_L)$, the support of $\eta_*\pi_*(C)$ is contained in $ram_{\mathcal{X}}(\alpha)$. Thus S is in the support of $\eta_*\pi_*(D_j)$ for some j or S is in the support of $l\eta_*\pi_*(E)$. In the later case, clearly $\alpha \cdot (fN_{L/K}(g))$ is unramified at S and hence $\alpha \cdot (fN_{L/K}(g)) = 0$ over K_S. Suppose S is in the support of $\eta_*\pi_*(D_j)$ for some j. In this case, if D_j lies over an inert curve, then $\eta_*\pi_*(D_j)$ is a multiple of l and we are done. Suppose that D_j lies over a split curve. Since α_L specializes to zero at D_j, it follows that α specializes to zero at $\eta_*\pi_*(D_j)$ and we are done. $\square$

Theorem 3.5. Let k be a p-adic field and K a function field of a curve over k. Let l be a prime not equal to p. Suppose that all the l^{th} roots of unity are in K. Then every element in $H^3(K, \mu_l^{\otimes 3})$ is a symbol.

Proof. We again identify the Galois modules $\mu_l^{\otimes j}$ with $\mathbf{Z}/l\mathbf{Z}$.

Let v be a discrete valuation of K and K_v the completion of K at v. By (1.2), every element in $H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ is a symbol.

Let $\zeta \in H^3(K, \mathbf{Z}/l\mathbf{Z})$ and $\mathcal{X}$ be a regular projective model of K. Let v be a discrete valuation of K corresponding to an irreducible curve in $\mathrm{ram}_{\mathcal{X}}(\zeta)$. Then we have $\zeta = (f_v) \cdot (g_v) \cdot (h_v)$ for some $f_v, g_v, h_v \in K_v^*$. By weak approximation, we can find $f, g \in K^*$ such that $(f) = (f_v)$ and $(g) = (g_v)$ in $H^1(K_v, \mathbf{Z}/l\mathbf{Z})$ for all discrete valuations v corresponding to the irreducible curves in $\mathrm{ram}_{\mathcal{X}}(\zeta)$. Let v be a discrete valuation of K corresponding to an irreducible curve C in $\mathcal{X}$. If C is in $\mathrm{ram}_{\mathcal{X}}(\zeta)$, then by the choice of f and g we have $\zeta = (f) \cdot (g) \cdot (h_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. If C is not in the $\mathrm{ram}_{\mathcal{X}}(\zeta)$, then $\zeta \in H^3_{\mathrm{nr}}(K_v, \mathbf{Z}/l\mathbf{Z}) \simeq H^3(\kappa(v), \mathbf{Z}/l\mathbf{Z}) = 0$. In particular we have $\zeta = (f) \cdot (g) \cdot (1) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$. Let $\alpha = (f) \cdot (g) \in H^2(K, \mathbf{Z}/l\mathbf{Z})$. Then we have $\zeta = \alpha \cdot (h'_v) \in H^3(K_v, \mathbf{Z}/l\mathbf{Z})$ for some $h'_v \in K_v^*$ for each discrete valuation v of K associated to any point of $\mathcal{X}^1$. By (3.4), there exists $h \in K^*$ such that $\zeta = \alpha \cdot (h) = (f) \cdot (g) \cdot (h) \in H^3(K, \mathbf{Z}/l\mathbf{Z})$. □

Remark 3.6. We remark that all the results of this section can be extended to the situation where k does not necessarily contain a primitive l^{th} root of unity. This can be achieved by going to the extension k' of k obtained by adjoining a primitive l^{th} of unity to k and noting that the extension k'/k is unramified of degree $l - 1$. We do not use this remark in the sequel.

4. The u-invariant

In (4.1) and (4.2) below, we give some necessary conditions for a field k to have the u-invariant less than or equal to 8. If K is the function field of a curve over a p-adic field and K_v is the completion of K at a discrete valuation v of K, then the residue field $\kappa(v)$ of K_v, which is either a global field of positive characteristic or a p-adic field, has u-invariant 4. By a theorem of Springer, $u(K_v) = 8$ and we use (4.1) and (4.2) for K_v.

Proposition 4.1 Let K be a field of characteristic not equal to 2. Suppose that $u(K) \leq 8$. Then $I^4(K) = 0$ and every element in $I^3(K)$ is a 3-fold Pfister form. Further if ϕ is a 3-fold Pfister form and q_2 a rank 2 quadratic form over K, then there exists $f, g, h \in K^*$ such that f is a value of q_2 and $\phi = <1, f><1, g><1, h>$.

Proof. Suppose that $u(K) = 8$. Then every 4-fold Pfister form is isotropic and hence hyperbolic; in particular, $I^4(K) = 0$. Let $\phi_1 = <1, f_1><1, g_1><1, h_1>$ and $\phi_2 = <1, f_2><1, g_2><1, h_2>$ be two anisotropic 3-fold Pfister forms. Since $u(K) \leq 8$, the Witt index of $\phi_1 - \phi_2$ is at least 4. In particular $\phi_1 - \phi_2$ is isotropic, i.e. ϕ_1 and ϕ_2 represent a common value $a \in K$. Since ϕ_1 and ϕ_2 are anisotropic, $a \neq 0$. We have $\phi_1 = <a> \perp \phi'_1$ and $\phi_2 = <a> \perp \phi'_2$ for some quadratic form ϕ_1 and ϕ_2 over K ([Sc], p.7, Lemma 3.4). Since the Witt index of $\phi_1 - \phi_2$ is at least 4, the Witt index of $\phi'_1 - \phi'_2$ is at least 3. Repeating this process, we see that there exists a quadratic form $< a, \; b, \; c, \; d >$ over K which is a subform of both ϕ_1 and ϕ_2 . Let C be a conic given by the quadratic form $abc < a, b, c > = < bc, ac, ab >$. Since $abc <a, b, c>$ is isotropic over the function field $K(C)$ of C (cf. [Sc], p.154, Remark 5.2(iv)) and $abc <a, b, c>$ is a subform of $abc\phi_1$ and $abc\phi_2$, ϕ_1 and ϕ_2 are isotropic over $K(C)$. Since ϕ_1 and ϕ_2 are Pfister forms, they are hyperbolic over $k(C)$ ([Sc], p.144, Cor. 1.5). Let $\psi = <1, bc><1, ac>$ and $k(\psi)$ be the function field of the quadratic form ψ. Then $k(C)$ and $k(\psi)$ are birational (cf. [Sc], p.154) and hence ϕ_1 and ϕ_2 are hyperbolic over $k(\psi)$. Therefore $\phi_1 \simeq <1, bc><1, ac><a_1, b_1>$ and $\phi_2 \simeq <1, bc><1, ac><a_2, b_2>$ for some $a_1, a_2, b_1, b_2 \in K^*$ (cf. [Sc], p.155, Th.5.4). Since $I^4(K) = 0$, we have $\phi_1 \simeq <1, bc><1, ac><1, a_1b_1>$ and $\phi_2 \simeq <1, bc><1, ac><1, a_2b_2>$. We have

$$\begin{aligned}
\phi_1 + \phi_2 &= \phi_1 - \phi_2 \\
&= <1, bc><1, ac><1, a_1b_1, -1, -a_2b_2> \\
&= <1, bc><1, ac><a_1b_1, -a_2b_2> \\
&= a_1b_1 <1, bc><1, ac><1, -a_1a_2b_1b_2> \\
&= <1, bc><1, ac><1, -a_1a_2b_1b_2> .
\end{aligned}$$

Thus the sum of any two 3-fold Pfister forms is a Pfister form in $I^3(K)$ and every element in $I^3(K)$ is the class of a 3-fold Pfister form.

Let $\phi = <1, a><1, b><1, c>$ be a 3-fold Pfister form and ϕ' be its pure subform. Let q_2 be a quadratic form over K of dimension 2. Since $\dim(\phi') = 7$

and $u(K) \leq 8$, the quadratic form $\phi' - q_2$ is isotropic. Therefore there exists $f \in K^*$ which is a value of q_2 and $\phi' \simeq <f> + \phi''$ for some quadratic form ϕ'' over K. Hence by ([Sc], p.143), $\phi = <1, f><1, b'><1, c'>$ for some $b', c' \in K^*$. □

Proposition 4.2. Let K be a field of characteristic not equal to 2. Suppose that $u(K) \leq 8$. Let $\phi = <1, f><1, a><1, b>$ be a 3-fold Pfister form over K and q_3 a quadratic form over K of dimension 3. Then there exist $g, h \in K^*$ such that g is a value of q_3 and $\phi = <1, f><1, g><1, h>$.

Proof. Let $\psi = <1, f><a, b, ab>$. Since $u(K) \leq 8$, the quadratic form $\psi - q_3$ is isotropic. Hence there exists $g \in K^*$ which is a common value of q_3 and ψ. Thus, $\psi \simeq <g> + \psi_1$ for some quadratic form ψ_1 over K. Since ψ is hyperbolic over $K(\sqrt{-f})$, $\psi_1 \simeq <1, f><a_1, b_1> + <g_1>$ for some $a_1, b_1, g_1 \in K^*$. By comparing the determinants, we get $g_1 = gf$ modulo squares. Hence $\psi = <1, f><g, a_1, b_1>$ and $\phi = <1, f> + \psi = <1, f><1, g, a_1, b_1>$. The form ϕ is isotropic and hence hyperbolic over the function field of the conic given by $<f, g, fg>$. Hence, as in 4.1, $\phi = \lambda <1, f><1, g><1, h>$ for some $\lambda, h \in K^*$. Since $I^4(K) = 0$, $\phi = <1, f><1, g><1, h>$ with g a value of q_3. □

Proposition 4.3. Let K be a field of characteristic not equal to 2. Assume the following:

1. Every element in $H^2(K, \mu_2)$ is a sum of at most 2 symbols.
2. Every element in $I^3(K)$ is equal to a 3-fold Pfister form.
3. If ϕ is a 3-fold Pfister form and q_2 is a quadratic form over K of dimension 2, then $\phi = <1, f><1, g><1, h>$ for some $f, g, h \in K^*$ with f a value of q_2.
4. If $\phi = <1, f><1, a><1, b>$ is a 3-fold Pfister form and q_3 a quadratic form over K of dimension 3, then $\phi = <1, f><1, g><1, h>$ for some $g, h \in K^*$ with g a value of q_3.
5. $I^4(K) = 0$.

Then $u(K) \leq 8$.

Proof. Let q be a quadratic form over K of dimension 9. Since every element in $H^2(K,\mu_2)$ is a sum of at most 2 symbols, as in ([PS], proof of 4.5), we find a quadratic form $q_5 = \lambda < 1, a_1, a_2, a_3, a_4 >$ over K such that $\phi = q + q_5 \in I^3(K)$. By the assumptions 2, 3 and 4, there exist $f, g, h \in K^*$ such that $\phi =< 1, f >< 1, g >< 1, h >$ and f is a value of $< a_1, a_2 >$ and g is a value of $< fa_1a_2, a_3, a_4 >$. We have $< a_1, a_2 >\simeq< f, fa_1a_2 >$ and $< fa_1a_2, a_2, a_3 >\simeq< g, g_1, g_2 >$ for some $g_1, g_2 \in K^*$. Since $I^4(K) = 0$, we have $\lambda\phi = \phi$ and

$$\begin{aligned}
\lambda q &= \lambda q + \lambda q_5 - \lambda q_5 \\
&= \lambda\phi - \lambda q_5 \\
&= \phi - \lambda q_5 \\
&= <1, f><1, g><1, h> - <1, a_1, a_2, a_3, a_4> \\
&= <1, f><1, g><1, h> - <1, f, g, g_1, g_2> \\
&= <gf> + <1, f><h, gh> - <g_1, g_2> .
\end{aligned}$$

Since the dimension of λq is 9 and the dimension of $<gf>\perp<1, f><h, gh> - <g_1, g_2>$ is 7, it follows that λq and hence q is isotropic over K. □

Proposition 4.4. Let k be a p-adic field, $p \neq 2$ and K a function field of a curve over k. Let ϕ be a 3-fold Pfister form over K and q_2 a quadratic form over K of dimension 2. Then there exist $f, a, b \in K^*$ such that f is a value of q_2 and $\phi =<1, f><1, a><1, b>$.

Proof. Let $\zeta = e_3(\phi) \in H^3(K, \mu_2)$. Let $\mathcal{X}$ be a projective regular model of K. Let C be an irreducible curve on $\mathcal{X}$ and v the discrete valuation given by C. Let K_v be the completion of K at v. Since the residue field $\kappa(v) = \kappa(C)$ is either a p-adic field or a function field of a curve over a finite field, $u(\kappa(v)) = 4$ and $u(K_v) = 8$ ([Sc], p.209). By (4.1), there exist $f_v, a_v, b_v \in K_v^*$ such that f_v is a value of q_2 over K_v and $\phi =< 1, f_v >< 1, a_v >< 1, b_v >$ over K_v. By weak approximation, we can find $f, a \in K^*$ such that f is a value of q_2 over K and $f = f_v, a = a_v$ modulo $K_v^{*^2}$ for all discrete valuations v corresponding to the irreducible curves C in the support of $\mathrm{ram}_{\mathcal{X}}(\zeta)$. Let C be any irreducible curve on $\mathcal{X}$ and v the discrete valuation of K given by C. If C is in the support of $\mathrm{ram}_{\mathcal{X}}(\zeta)$, then by the choice of f and a, we have $\zeta = e_3(\phi) = (-f) \cdot (-a) \cdot (-b_v)$ over K_v. If C is not in the support of $\mathrm{ram}_{\mathcal{X}}(\zeta)$, then $\zeta \in H^3_{\mathrm{nr}}(K_v, \mu_2) \simeq H^3(\kappa(v), \mu_2) = 0$. In particular we have $\zeta = (-f)\cdot(-a)\cdot(1)$ over K_v. Let $\alpha = (-f)\cdot(-a) \in H^2(K,\mu_2)$. By (3.4), there exists $b \in K^*$ such that $\zeta = \alpha \cdot (-b) \in H^3(K,\mu_2)$. Since $e_3 : I^3(K) \to H^3(K,\mu_2)$ is an isomorphism, we have $\phi = <1,f><1,a><1,b>$ as required. $\square$.

There is a different proof of the Proposition 4.4 in ([PS], 4.4)!

Proposition 4.5 Let k be a p-adic field, $p \neq 2$ and K a function field of a curve over k. Let $\phi = <1,f><1,a><1,b>$ be a 3-fold Pfister form over K and q_3 a quadratic form over K of dimension 3. Then there exist $g,h \in K^*$ such that g is a value of q_3 and $\phi = <1,f><1,g><1,h>$.

Proof. Let $\zeta = e_3(\phi) = (-f)\cdot(-a)\cdot(-b) \in H^3(K,\mu_2)$. Let $\mathcal{X}$ be a projective regular model of K. Let C be an irreducible curve on $\mathcal{X}$ and v the discrete valuation of K given by C. Let K_v be the completion of K at v. Then as in the proof of (4.4), we have $u(K_v) = 8$. Thus by (4.2), there exist $g_v, h_v \in K_v^*$ such that g_v is a value of the quadratic form q_3 and $\phi = <1,f><1,g_v><1,h_v>$ over K_v. By weak approximation, we can find $g \in K^*$ such that g is a value of q_3 over K and $g = g_v$ modulo $K_v^{*^2}$ for all discrete valuations v of K given by the irreducible curves C in $\mathrm{ram}_{\mathcal{X}}(\zeta)$. Let C be an irreducible curve on $\mathcal{X}$ and v the discrete valuation of K given by C. By the choice of g it is clear that $\zeta = e_3(\phi) = (-f)\cdot(-g)\cdot(-h_v)$ for all the discrete valuations v of K given by the irreducible curves C in the support of $\mathrm{ram}_{\mathcal{X}}(\zeta)$. If C is not in the support of $\mathrm{ram}_{\mathcal{X}}(\zeta)$, then as is the proof of (4.4), we have $\zeta = (-f)\cdot(-g)\cdot(1)$ over K_v. Let $\alpha = (-f)\cdot(-g) \in H^2(K,\mu_2)$. By (3.4), there exists $h \in K^*$ such that $\zeta = \alpha \cup (-h) = (-f)\cdot(-g)\cdot(-h)$. Since $e_3 : I^3(K) \to H^3(K,\mu_2)$ is an isomorphism, $\phi = <1,f><1,g><1,h>$.

Theorem 4.6 Let K be a function field of a curve over a p-adic field k. If $p \neq 2$, then $u(K) = 8$.

Proof. Let K be a function field of a curve over a p-adic field k. Assume that $p \neq 2$. By a theorem of Saltman ([S1], 3.4, cf. [S2]), every element in $H^2(K,\mu_2)$ is a sum of at most 2 symbols. Since the cohomological dimension of K is 3, we also have $I^4(K) \simeq H^4(K,\mu_2) = 0$ ([AEJ]). Now the theorem follows from (4.4), (4.5) and (4.3). $\square$.

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