The structure of the group $\mathbf{G}(k[t])$: Variations on a theme of Soulé

Benedictus Margaux

Abstract. Following Soulé's ideas [14] we give a presentation of the abstract group $\mathbf{G}(k[t])$ for any semisimple (connected) simply connected absolutely almost simple k–group $\mathbf{G}$. As an application, we give a description of $\mathbf{G}(k[t])$ in terms of direct limits, and show that the Whitehead group and the naïve group of connected components of $\mathbf{G}$ coincide.

Keywords: Bruhat-Tits buildings, arithmetic groups, simplicial fundamental domain.
MSC: 20G10, 22

1 Introduction

Let k be a field and let $\mathbf{G}$ be a semisimple simply connected absolutely almost simple k–group. For $\mathbf{G}$ split, Soulé [14] has given a presentation of the group $\mathbf{G}(k[t])$, thus extending a theorem of Nagao [8] for $\mathbf{SL}_2$ (see also [11, II.1.6]). The goal of this note is to provide a presentation of $\mathbf{G}(k[t])$ in the general case.

We will follow Soulé's original ideas and study the action of $\mathbf{G}(k[t])$ on the Bruhat-Tits building [4] of $\mathbf{G}$ corresponding to the field $K = k((\frac{1}{t}))$, where K is viewed as the completion of $k(t)$ with respect to the valuation at ∞. As an application, we show that the Whitehead group of $\mathbf{G}$ coincides with the naïve group of connected components of $\mathbf{G}$.

2 Structure of the group $\mathbf{G}(k[t])$

Throughout k and $\mathbf{G}$ will be as above. For convenience the group $\mathbf{G}(k[t])$ will be denoted by Γ.

2.1 Notation and statement of the main Theorem

Let $\mathbf{S}$ be a maximal k–split torus of $\mathbf{G}$, and $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing $\mathbf{S}$. Recall that $\mathbf{S}_K$ is a maximal K-split torus of $\mathbf{G}_K$. Let $\widetilde{k}/k$ a finite Galois extension which splits $\mathbf{T}$ (hence also $\mathbf{G}$). Set $\mathcal{G} = \mathrm{Gal}(\widetilde{k}/k)$ and $\widetilde{\mathbf{T}} = \mathbf{T} \underset{k}{\times} \widetilde{k}$.

Let $\widetilde{\mathbf{G}} = \mathbf{G} \underset{k}{\times} \widetilde{k}$ and $\widetilde{\mathbf{S}} = \mathbf{S} \underset{k}{\times} \widetilde{k}$. We choose compatible orderings on the root systems $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ and $\widetilde{\Phi} = \Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ (see [1]). We then have a set Δ of relative simple roots and a set $\widetilde{\Delta}$ of absolute simple roots.

It will be convenient to essentially maintain the same notation than in Soulé's paper, namely:

- $A = k[t]$, $K = k((\frac{1}{t}))$, $G = \mathbf{G}(K)$;
- ω the valuation defined on K the valuation on K at ∞, that is, the valuation on K having $\mathcal{O} = k[[\frac{1}{t}]]$ as its ring of integers.

We also have the analogous to the above objects for $\widetilde{k}$, namely

- $\widetilde{A} = \widetilde{k}[t]$, $\widetilde{K} = \widetilde{k}((\frac{1}{t}))$, $\widetilde{\Gamma} = \mathbf{G}(\widetilde{A})$, and $\widetilde{\mathcal{O}} = \widetilde{k}[[\frac{1}{t}]]$.

At the level of buildings we set.

- $\mathcal{T}$ the (affine) Bruhat-Tits building of the K–group $\mathbf{G}_K := \mathbf{G} \times_k K$ and $\widetilde{\mathcal{T}}$ the Bruhat-Tits building of the $\tilde{K}$–group $\mathbf{G}_{\tilde{K}} := \mathbf{G} \times_k \tilde{K}$ [4, §4.2].[1] We recall that both $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ have a natural simplicial complex structure [4, §4.2.23].

Recall that $\mathcal{T}$ is equipped with an action of $\mathbf{G}(K)$ and that $\widetilde{\mathcal{T}}$ is equipped with an action of $\mathbf{G}(\widetilde{K}) \rtimes \mathcal{G}$. We have an isometric embedding $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ which identifies $\mathcal{T}$ with $\widetilde{\mathcal{T}}^{\mathcal{G}}$. The hyperspecial group $\mathbf{G}(\widetilde{O})$ of $\mathbf{G}(\widetilde{K})$ fixes a unique point $\widetilde{\phi}$ of $\widetilde{\mathcal{T}}$ [3, §9.1.9.c]. This point descends to a point ϕ of $\mathcal{T}$.

We denote by $\mathcal{A}$ the standard apartment of $\mathcal{T}$ associated to $\mathbf{S}$ (this is a real affine space) and similarly by $\widetilde{\mathcal{A}}$ the standard apartment associated to $\widetilde{\mathbf{T}}$. The point $\widetilde{\phi}$ belongs to $\widetilde{\mathcal{A}}$ (*ibid.*). Since $\mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \Big(\mathrm{Hom}_{\widetilde{k}-gr}(\mathbf{G}_{m,\widetilde{k}}, \widetilde{\mathbf{T}}) \otimes_{\mathbb{Z}} \mathbb{R}\Big)^{\mathcal{G}}$ [4, §4.2], we have $j(\mathcal{A}) = \widetilde{\mathcal{A}}^{\mathcal{G}}$ so ϕ belongs to $\mathcal{A}$ and

$$\mathcal{A} = \phi \, + \, \mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By means of the canonical pairing $\langle \, , \, \rangle : \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \times \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \to \mathbb{Z}$ we can then define the *sector* (quartier)

$$\mathcal{Q} := \phi + D \;\text{ where }\; D := \Big\{ v \in \mathrm{Hom}_{k-gr}(\mathbf{S}, \mathbf{G}_m) \otimes_{\mathbb{Z}} \mathbb{R} \;\mid\; \langle b, \lambda \rangle \geq 0 \;\; \forall \;\; b \in \Delta \Big\}.$$

[1] Since $\mathbf{G} \times_K \widetilde{K}$ is split, the assumptions of [4, §5.1.1.1] are satisfied. This allows us to do away with the "standard" assumption that the base field k be perfect.

The following result generalizes Soulé's theorem [14].

Theorem 2.1. *The set $\mathcal{Q}$ is a simplicial fundamental domain for the action of* $\mathbf{G}(k[t])$ *on $\mathcal{T}$. In other words, any simplex of $\mathcal{T}$ is equivalent under the action of* $\mathbf{G}(k[t])$ *to a unique simplex of $\mathcal{Q}$.*

2.2 Buildings and valuations

Let $\mathbf{P}$ be the minimal parabolic k–subgroup of $\mathbf{G}$ defined by $\mathbf{S}$ and Δ. We denote by $\mathbf{U} = R_u(\mathbf{P})$ the unipotent radical of $\mathbf{P}$.

We denote by $\widetilde{\mathbf{U}}_{\widetilde{a}}$ the split unipotent subgroup associated to a root $\widetilde{a} \in \widetilde{\Phi}$, and by $\widetilde{a}^\vee : \mathbf{SL}_2 \to \mathbf{G}$ the corresponding standard homomorphism (see [12, §2.2]).

The set of positive (resp. negative) roots with respect to the basis Δ of Φ will be denoted by Φ^+ (resp. Φ^-). Given $b \in \Phi$, the subset of absolute roots

$$\widetilde{\Phi}^b := \left\{ \widetilde{a} \in \widetilde{\Phi} \ \mid \ \widetilde{a}_{|S \times_k \widetilde{k}} = b \ \text{ or } \ 2b \right\}$$

is positively closed in $\widetilde{\Phi}$. It defines then a split $\widetilde{k}$-unipotent subgroup $\widetilde{\mathbf{U}}_b$ of $\widetilde{\mathbf{G}}$ which descends to a split k–unipotent subgroup $\mathbf{U}_b$ of $\mathbf{G}$. As is [3], we make the convention that $\mathbf{U}_{2b} = 1$ if $2b \notin \Phi$.

For $I \subset \Delta$, we define along standard lines

$$\mathbf{S}_I = \Big(\bigcap_{b \in I} \ker(b) \Big)^0 \subset \mathbf{S}, \ \mathbf{L}_I = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}_I) \text{ and } \mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I.$$

Thus $\mathbf{P}_I$ is the standard parabolic subgroup of $\mathbf{G}$ of type I and $\mathbf{L}_I$ its standard Levi subgroup (see [1, §21.11]). Recall that the root system $\Phi(\mathbf{L}_I, \mathbf{S}) = [I]$ is the subroot system of Φ consisting of roots which are linear combinations of I; the split unipotent k-group $\mathbf{U}_I$ is the subgroup of $\mathbf{U}$ generated by the $\mathbf{U}_b$ with b running over $\Phi^+ \setminus [I]$.

Given $\widetilde{a} \in \widetilde{\Phi}$, the group $\widetilde{U}_{\widetilde{a}} := \widetilde{\mathbf{U}}_{\widetilde{a}}(\widetilde{K}) = \widetilde{K}$ is equipped with the valuation ω, which we denote by $\widetilde{\varphi}_a : \widetilde{U}_a \to \mathbb{R} \cup \{\infty\}$. This defines the Chevalley-Steinberg "donnée radicielle valuée" $\Big(T(\widetilde{K}), (\widetilde{U}_{\widetilde{a}}, M_{\widetilde{a}})_{\widetilde{a} \in \widetilde{\Phi}} \Big)$ where

$$M_{\widetilde{a}} = T(\widetilde{K}) \ \widetilde{a}^\vee \Big(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Big)$$

[3, example 6.2.3.b], and also a filtration $(\widetilde{U}_{\widetilde{a},m})_{m \in \mathbb{Z}}$ of $\widetilde{U}_{\widetilde{a}}$ where

$$\widetilde{U}_{\widetilde{a},m} := \widetilde{\varphi}^{-1}([m, +\infty[).$$

Note that $\widetilde{U}_{\widetilde{a},0} = \widetilde{\mathbf{U}}_{\widetilde{a}}(\widetilde{\mathcal{O}})$.

A crucial point of Bruhat-Tits theory is the descent of this data to $G = \mathbf{G}(K)$ [4, §5.1]. Given $b \in \Phi$, the commutative group $U_b := \mathbf{U}_b(K)$ is equipped with the descended valuation $\varphi_b : U_b \to \mathbb{R} \cup \{\infty\}$. The definition of φ_b is delicate, and is given as follows [4, §5.1.16]. Define

$$\widetilde{U}_{b,m} := \prod_{\widetilde{a} \in \widetilde{\Phi}^b,\, \widetilde{a}_{|S \times_k \widetilde{k}} = b} \widetilde{U}_{\widetilde{a},m} \ . \prod_{\widetilde{a} \in \widetilde{\Phi}^b,\, \widetilde{a}_{|S \times_k \widetilde{k}} = 2b} \widetilde{U}_{\widetilde{a},2m} \qquad (m \in \mathbb{R}).$$

Then U_b is a subgroup of $\mathbf{U}_b(\widetilde{K}) = \widetilde{U}_b = \bigcup_{m \in \mathbb{R}} \widetilde{U}_{b,m}$ and the descended valuation is defined by

$$\varphi_b(u) := \mathrm{Sup}\Big\{ m \in \mathbb{R} \mid u \in \widetilde{U}_{b,m} \Big\}.$$

Note that[2] $\Theta_b := \varphi_b\big(U_b \setminus \{e\}\big)$ is either $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$. As above, it gives rises to a filtration $(U_{b,m})_{m \in \Theta_b}$ of U_b such that $U_{b,0} = \mathbf{U}_b(\mathcal{O})$.

Again me make the convention that $U_{2b} = 1$ if $2b \notin \Phi$.

2.3 Description of the isotropy group of a vertex

Given $\Omega \subset \mathcal{Q}$, we denote by Γ_Ω the corresponding isotropy subgroup, namely the elements of Γ that fix all elements of Ω. We introduce an analogous definition and notation for $j(\Omega) \in \widetilde{\mathcal{A}}$. By Galois descent we have

$$\Gamma_\Omega = \big(\widetilde{\Gamma}_{j(\Omega)}\big)^{\mathcal{G}}. \tag{2.1}$$

In particular, since $\widetilde{\Gamma}_{\widetilde{\phi}} = \mathbf{G}(\widetilde{\mathcal{O}}) \cap \widetilde{\Gamma} = \mathbf{G}(\widetilde{k})$ [13, §1.1], we have $\Gamma_\phi = \big(\widetilde{\Gamma}_{\widetilde{\phi}}\big)^{\mathcal{G}} = \mathbf{G}(\widetilde{k})^{\mathcal{G}} = \mathbf{G}(k)$.

If $x \in \mathcal{Q} \setminus \{\phi\}$ and if $[x[$ is the half-line of origin x and direction $\overrightarrow{\phi x}$, we claim that

$$\Gamma_x = \Gamma_{[x[}.$$

If $\mathbf{G}$ is split, this is proven in Soulé's paper by reduction to the case of $\mathbf{SL}_n$. By applying the identity (2.1) to x and $[x[$, our claim now readily follows from the absolute case.

The isotropy of $[x[$ in $G = \mathbf{G}(K)$ is the Bruhat-Tits abstract parahoric group $P_{[x[}$ [3, § 7.1]. We have

$$P_{[x[} = U_{[x[}.H$$

[2] We use the notation Θ_b, rather than the more standard Γ_b found in [3], to avoid any possible confusion with the notation used in Soulé's paper.

where $H = \mathrm{Fix}_G(\mathcal{A})$. By [4, §5.2.2], we have

$$H = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})(\mathcal{O}).$$

The group $U_{[x[}$ is defined by means of the function [3, §6.4.2]

$$f_{[x[} : \Phi \to \mathbb{R} \cup \{\infty\}, \;\; b \mapsto \inf\Big\{ s \in \mathbb{R} \;\mid\; b(y) + s \geq 0 \text{ for all } \;\; y \in [x[\Big\}.$$

Hence

$$f_{[x[}(b) = \begin{cases} 0 & \text{if } b(x) = 0, \\ -b(x) & \text{if } b(x) > 0, \\ \infty & \text{if } b(x) < 0. \end{cases}$$

The group $U_{[x[} \subset G$ is then the subgroup of G generated by the $U_{b,m}$ for $b \in \Phi^+$ and $m \geq -b(x)$ ($m \in \Theta_b$), together with the $\mathbf{U}_b(\mathcal{O})$ for $b \in \Phi^-$ such that $b(x) = 0$. In other words, by distinguishing positive roots which vanish at x, we see that $U_{[x[}$ is the subgroup of G generated by subgroups of the following three "shapes":

(I) $U_{b,m}$ for $b \in \Phi^+$ such that $b(x) > 0$ and $m \in \Theta_b$ such that $m \geq -b(x)$;

(II) $\mathbf{U}_b(\mathcal{O})$ for $b \in \Phi^+$ such that $b(x) = 0$;

(III) $\mathbf{U}_b(\mathcal{O})$ for $b \in \Phi^-$ such that $b(x) = 0$.

Define $U^{\pm}_{[x[} := U_{[x[} \cap \mathbf{U}^{\pm}(K)$ as in [3, §6.4.2]. By definition $U^+_{[x[}$ and $U^-_{[x[}$ generate $U_{[x[}$. On the other hand, $U^+_{[x[}$ (resp. $U^-_{[x[}$) is the subgroup of $U_{[x[}$ generated by the subgroups of type (I) and (II) (resp.(III)) [3, prop. 6.4.9]. Define the subset of roots

$$I_x = \big\{\, b \in \Delta \;\mid\; b(x) = 0 \,\big\}.$$

This definition makes sense if x is an element of $\mathcal{A}$, and we then have $I_\phi = \Delta$.

Lemma 2.2. *We have*

$$[I_x] \cap \Phi^+ = \big\{\, b \in \Phi^+ \;\mid\; b(x) = 0 \,\big\}, \tag{2.2}$$

$$\Phi^+ \setminus [I_x] = \big\{\, b \in \Phi^+ \;\mid\; b(x) > 0 \big\} \text{ and} \tag{2.3}$$

$$[I_x] \cap \Phi^- = \big\{\, b \in \Phi^- \;\mid\; b(x) = 0 \,\big\}. \tag{2.4}$$

Proof. Observe that if $b \in [I_x]$, b is a linear combination of elements of I_x, hence $b(x) = 0$. This implies that $[I_x] \cap \Phi^+ \subset \big\{\, b \in \Phi^+ \;\mid\; b(x) = 0 \,\big\}$. Conversely, let b be a positive root such that $b(x) = 0$. Then $b = \sum_{c \in \Delta} n_c\, c$ where the n_c's are non-negative integers. Hence $\sum_{c \in \Delta} n_c\, c(x) = 0$. Since $x \in \mathcal{Q}$, we have $c(x) \geq 0$. Therefore $n_c\, c(x) = 0$ and b is a linear combination of elements of I_x. This shows (2.2). Since

$$\big\{\, b \in \Phi^+ \;\mid\; b(x) \neq 0 \big\} = \big\{\, b \in \Phi^+ \;\mid\; b(x) > 0 \big\},$$

we get also (2.3). Similar considerations apply to (2.4). □

It follows from (2.2) and (2.4) respectively that the subgroups of shape (II) and (III) are subgroups of $\mathbf{L}_{I_x}(\mathcal{O})$. Furthermore, (2.3) shows that the subgroups of shape (I) are subgroups of $\mathbf{U}_{I_x}(K)$. Hence we get the following inclusion

$$U_{[x[} \subset (U_{[x[} \cap \mathbf{U}_{I_x}(K)) \rtimes \mathbf{L}_{I_x}(\mathcal{O}) \subset \mathbf{P}_{I_x}(K). \tag{2.5}$$

Lemma 2.3. *1.* $\mathbf{L}_{I_x}(\mathcal{O}) \subset P_{[x[} \subset \mathbf{U}_{I_x}(K) \rtimes \mathbf{L}_{I_x}(\mathcal{O}) \subset \mathbf{P}_{I_x}(K)$*;*

2. $\mathbf{U}_{I_x}(K) \cap P_{[x[} \subset U^+_{[x[}$*;*

3. $\bigcup_{z \geq 1} \big(U^+_{[zx[} \cap \mathbf{U}_{I_x}(K)\big) = \mathbf{U}_{I_x}(K)$.

Proof. Let $I = I_x$.
(1) Since $U_{[x[} \subset \mathbf{U}_I(K) \rtimes \mathbf{L}_I(\mathcal{O})$ and $\mathcal{Z}_{\mathbf{G}}(\mathbf{S}) \subset \mathbf{L}_I$ it follows that $P_{[x[} = U_{[x[}.H = U_{[x[}.\mathcal{Z}_{\mathbf{G}}(\mathbf{S})(\mathcal{O})$ is a subgroup of $\mathbf{U}_I(K) \rtimes \mathbf{L}_I(\mathcal{O})$.

Let us show that $\mathbf{L}_I(\mathcal{O}) \subset P_{[x[}$. Let $\mathbf{V}_I$ be the unipotent radical of the minimal standard parabolic subgroup of $\mathbf{L}_I$, namely the k–subgroup of $\mathbf{U}$ generated by the $\mathbf{U}_b$ such that $b \in \Phi^+$ and $b(x) = 0$. We have [10, th. XXVI.5.1]

$$\bigcup_{g \in \mathbf{V}_I(k)} g\Omega = \mathbf{L}_I$$

where Ω stands for the big cell $\mathbf{V}_I^- \times_k \mathcal{Z}_{\mathbf{G}}(S) \times_k \mathbf{V}_I$ of $\mathbf{L}_I$. Since $\mathcal{O}$ is local, it follows that

$$\mathbf{L}_I(\mathcal{O}) = \mathbf{V}_I(k).\Omega(\mathcal{O}) = \mathbf{V}_I(k).\mathbf{V}_I^-(\mathcal{O})\,.\,H\,.\,\mathbf{V}_I(\mathcal{O}).$$

We conclude that $\mathbf{L}_I(\mathcal{O}) \subset P_{[x[}$.
(2) We claim that $\mathbf{U}(K) \cap P_{[x[} = U^+_{[x[}$. Note that this establishes (2) since $\mathbf{U}_I(K) \subset \mathbf{U}(K)$. To prove the claim it we need to show that $\mathbf{U}(K) \cap P_{[x[} \subset U^+_{[x[}$ (the reversed inclusion is obvious). With the notations of [3, §7], we have $\mathbf{U}(K) = U^+_D$ where D is the direction of the sector $\mathcal{Q}$. By *loc. cit.* 7.1.4, we have

$$P_{[x[} \cap \mathbf{U}(K) = U_{[x[+D}$$

where $U_{[x[+D}$ is the subgroup of $\mathbf{G}(K)$ attached to the subset $[x[+D = x + D$ of $\mathcal{A}$. This group is defined by means of the function [3, §6.4.2]

$$f_{x+D} : \Phi \to \mathbb{R} \cup \{\infty\}, \;\; b \mapsto \inf\Big\{ s \in \mathbb{R} \;\mid\; b(y) + s \geq 0 \text{ for all } \;\; y \in x + D \Big\}.$$

Hence

$$f_{x+D}(b) = \begin{cases} -b(x) & \text{if } b > 0, \\ \infty & \text{if } b < 0, \end{cases}$$

so $U_{x+D} = U^+_{[x[}$ as desired.
(3) If $b \in \Phi^+$ satisfies $b(x) > 0$, then the number

$$\mathrm{Inf}\Big\{ m \in \Theta_b \mid m + b(zx) \geq 0 \Big\}$$

tends to $-\infty$ as z tends to ∞. This readily yields that $\bigcup_{z \geq 1} \big(U^+_{[zx[} \cap \mathbf{U}_I(K) \big) = \mathbf{U}_I(K)$. $\square$

Remark 2.4. Geometrically speaking, the K–parabolic $\mathbf{P}_{I_x} \times_k K$ is attached to the extremity of the half line $[x[$ in the spherical building at infinity [6, § 16.9]. Since $P_{[x[}$ is the isotropy group of the half line $[x[$, it fixes its extremity. This point of view yields another way to prove the inclusion $P_{[x[} \subset \mathbf{P}_{I_x}(K)$ which is part of Lemma 2.3(1).

Given $b \in \Phi$ we set

$$m_x(b) := \mathrm{Inf}\Big\{ m \in \Theta_b \mid m + b(x) \geq 0 \Big\}.$$

Since $\Gamma_x = P_{[x[} \cap \Gamma$, we have the inclusion

$$\Big\langle (U_{b, m_x(b)}.U_{2b, m_x(2b)}) \cap \Gamma,\ b \in \Phi,\ b(x) \geq 0 \Big\rangle \quad \subset \quad \Gamma_x. \tag{2.6}$$

Proposition 2.5. *1.* $\Gamma_x = \big(\Gamma_x \cap \mathbf{U}_{I_x}(K) \big) \rtimes \mathbf{L}_{I_x}(k)$*;*

2. $\Gamma_x = \Big\langle (U_{b, m_x(b)}.U_{2b, m_x(2b)}) \cap \Gamma, b(x) > 0 \Big\rangle \rtimes \mathbf{L}_{I_x}(k)$*;*

3. $\bigcup_{z \geq 1} \Gamma_{zx} = \mathbf{U}_{I_x}(k[t]) \rtimes \mathbf{L}_{I_x}(k)$*.*

Proof. To cut down on the notation we set $I = I_x$.
(1) According to Lemma 2.3.(1) $\mathbf{L}_I(k) = \Gamma \cap \mathbf{L}_I(\mathcal{O})$ fixes the point x. Hence the inclusion

$$\big(\Gamma_x \cap \mathbf{U}_I(K) \big) \rtimes \mathbf{L}_I(k) \subset \Gamma_x.$$

To prove the reverse inclusion we make use of the projection $\mathbf{P}_I(K) \to \mathbf{L}_I(K)$. The image of Γ_x inside $\mathbf{L}_I(K)$ is a subgroup of $\mathbf{L}_I(A)$. On the other hand, by Lemma 2.3.(1), the image of P_x inside $\mathbf{L}_I(K)$ is the subgroup $\mathbf{L}_I(\mathcal{O})$. Hence the image of Γ_x inside $\mathbf{L}_I(K)$ is a subgroup of $\mathbf{L}_I(A) \cap \mathbf{L}_I(\mathcal{O}) = \mathbf{L}_I(k)$. We thus have an exact sequence

$$1 \to \big(\Gamma_x \cap \mathbf{U}_I(K) \big) \to \Gamma_x \to \mathbf{L}_I(k)$$

which is a split surjection.
(2) Put $V := \left\langle (U_{b,m_x(b)}.U_{2b,m_x(2b)}) \cap \Gamma,\, b \in \Phi,\, b(x) > 0 \right\rangle$. This is a subgroup of Γ_x (2.6) and of $U_I(K)$ (2.5). So $V \subset \Gamma_x \cap U_I(K)$. For showing the reverse inclusion, it suffices to show that

$$\Gamma_x \cap \mathbf{U}_I(K) \quad \subset \quad \left\langle (U_{b,m_x(b)}.U_{2b,m_x(2b)}) \cap \Gamma, b(x) \geq 0 \right\rangle. \tag{2.7}$$

From Lemma 2.3.(3) we have

$$\Gamma_x \cap \mathbf{U}_I(K) \subset \Gamma \cap U^+_{[x[}.$$

Accordingly, it will suffice to show that $\Gamma_x \cap U^+_{[x[}$ is a subgroup of the right handside of (2.7). Let $\Phi^+_{red} = \{b_1, ..., b_N\}$ be the subset of reduced positive roots (with an arbitrary order). The product induces a isomorphism of k–varieties $\prod_{j=1}^{N} \mathbf{U}_{b_j} \xrightarrow{\sim} \mathbf{U}$ [1, prop. 21.9]. In particular, we have compatible bijections

$$\begin{array}{ccc} \prod_{j=1}^{N} \mathbf{U}_{b_j}(K) & \xrightarrow{\sim} & \mathbf{U}(K) \\ \cup & & \cup \\ \prod_{j=1}^{N} \mathbf{U}_{b_j}(A) & \xrightarrow{\sim} & \mathbf{U}(A). \end{array} \tag{2.8}$$

By comparing these with the bijection [3, §6.4.9]

$$\prod_{j=1}^{N} U_{b_j,m_x(b_j)}.U_{2b_j,m_x(2b_j)} \quad \xrightarrow{\sim} \quad U^+_{[x[},$$

we see that $\Gamma_x \cap \mathbf{U}_I(K) \subset U^+_{[x[} \cap \mathbf{U}(A)$ consists of products of elements $(U_{b_j,m_x(b_j)}.U_{2b_j,m_x(2b_j)}) \cap \Gamma$ with $b_j(x) \geq 0$.
(3) This follows from (1) and Lemma 2.3.(3). □

2.4 Action on the star of certain points

We will now make use of the spherical building $\mathcal{B}(\mathbf{G})$ of $\mathbf{G}$ [16, §5]. Recall that $\mathcal{B}(\mathbf{G})$ is a simplicial complex whose simplex are the k-parabolic subgroups of $\mathbf{G}$. If $\mathbf{Q}$ is such a parabolic subgroup, the faces of its associated simplex are the simplexes associated to the maximal proper k-parabolic subgroups of $\mathbf{Q}$. The standard apartment $\mathfrak{A}$ of $\mathcal{B}(\mathbf{G})$ is the subcomplex of k-parabolic subgroups containing $\mathbf{S}$ and the standard chamber $\mathfrak{C}$ is the simplex associated to the minimal k-parabolic subgroup $\mathbf{P}$. We denote by $\mathbf{W} = \mathbf{N_G}(\mathbf{S})/\mathcal{Z}_\mathbf{G}(\mathbf{S})$ the relative Weyl group of $\mathbf{G}$.

If $x \in \mathcal{T}$, we denote by $\mathcal{L}_x$ the *star* of x (étoile in french) of x,[3] i.e. the subspace of $\mathcal{T}$ consisting of facets F such that $x \in \overline{F}$ [4, §4.6.33].

We denote by $\mathbf{S}_* = \mathrm{Hom}_{k-gr}(\mathbf{G}_m, \mathbf{S})$ the group of cocharacters of $\mathbf{S}$. Inside the apartment $\mathcal{A} = \phi + \mathbf{S}_* \otimes_\mathbb{Z} \mathbb{R}$, this corresponds to the lattice of points having type 0, i.e. the type of ϕ. The action of $\mathbf{S}(K)$ on $\mathcal{T}$ preserves $\mathcal{A}$. More precisely, the element $s \in S(K)$ acts on $\mathcal{A}$ as the translation by the vector v_s defined by the property [4, §5.1.22]

$$\langle v_s, b\rangle = -\omega(b(s)) \quad \forall b \in \Phi. \tag{2.9}$$

We denote by $\mathcal{C} \subset S_* \otimes_\mathbb{Z} \mathbb{R}$ the vector chamber such that $\phi + \mathcal{C}$ is the unique chamber of the sector $\mathcal{Q}$ which contains the special point ϕ in its adherence [3, §1.3.11].

Lemma 2.6. *Let x be a point of $\mathbf{S}_* \cap \mathcal{Q}$. Then the chambers of $\mathcal{L}_x \cap \mathcal{Q}$ are the $x + w\mathcal{C}$ for $w \in \mathbf{W}(k)$ satisfying $I_x \subset w.\Phi^+$.*

Proof. Set $I = I_x$. The chambers of $\mathcal{L}_x$ are the $x + w\mathcal{C}$ with $w \in \mathbf{W}(k)$. Let $y \in \mathcal{C}$. If $x + w\mathcal{C} \subset \mathcal{Q}$, then

$$b(x + w.y) = b(x) + (w^{-1}.b)(y) \geq 0 \ \ \forall b \in \Delta.$$

It follows that if $b \in I$, i.e. $b(x) = 0$, then $(w^{-1}.b)(y) \geq 0$, and therefore $b \in w(\Phi_+)$. Conversely, if $w \in \mathbf{W}(k)$ satisfies $I \subset w(\Phi_+)$, then the above inequality holds for ϵy for all $b \in \Delta$ for $\epsilon > 0$ small enough. Thus $x + w\,.\,(\epsilon y) \in \mathcal{Q}$ and $x + w\mathcal{C} \subset \mathcal{Q}$. $\square$

Lemma 2.7. *Let I be a subset of Δ, and set $\mathbf{W}_I := \mathbf{N}_{\mathbf{L}_I}(\mathbf{S})/\mathcal{Z}_\mathbf{G}(\mathbf{S})$. Let $\mathfrak{A}_I$ be the union of the $w\overline{\mathfrak{C}}$ for w running over the elements of $\mathbf{W}(k)$ satisfying $I \subset w.\Phi^+$.*

1. *$\mathbf{W}_I(k)\,.\,\mathfrak{A}_I = \mathfrak{A}$.*
2. *$\mathbf{P}_I(k)\,.\,\mathfrak{A}_I = \mathcal{B}(\mathbf{G})$.*

Proof. (1) We reason by induction on the cardinality of I. If $I = \emptyset$, then $\mathfrak{A}_I = \mathfrak{A}$ and there is nothing to prove. Assume that $I = I' \cup \{b\}$. We are given a chamber $w\mathfrak{C}$ of $\mathfrak{A}$ with $w \in \mathbf{W}(k)$. We want to show that $w\mathfrak{C}$ is equivalent under $\mathbf{W}_I(k)$ to a

[3] The terminology *link* is also used in the literature.

chamber of $\mathfrak{A}_I$. Since $\mathbf{W}_{I'}(k) \subset \mathbf{W}_I(k)$, we can assume by the induction hypothesis that $I' \subset w.\Phi^+$. If $b \in w.\Phi^+$, we have $I \subset w.\Phi^+$. The other case is when $-b \in w.\Phi^+$. Let $s_b \in \mathbf{W}_I(k)$ the reflexion associated to b. Then $s_b(b) = -b$, hence $b \subset s_b w.\Phi^+$. For $b' \in I'$, we have $s_b(b') = b' + mb$ where m is non-negative. Therefore

$$b' = s_b^2(b') = s_b(b' + mb) = s_b(b') - mb \in \; s_b w.\Phi^+.$$

We conclude that $I \subset s_b w.\Phi^+$ and $s_b.(w\mathfrak{C}) \subset \mathfrak{A}_I$.
(2) Again it suffices to prove that any chamber of $\mathcal{B}(\mathbf{G})$ is equivalent under $\mathbf{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. Let $\mathfrak{C}'$ be a chamber of $\mathcal{B}(\mathbf{G})$. Let $\mathbf{P}'$ be the underlying minimal k–parabolic subgroup. By [2, prop. 4.4.b] $\mathbf{P}_I \cap \mathbf{P}'$ contains a maximal k-split torus of $\mathbf{P}_I$. Since maximal k–split tori of $\mathbf{P}_I$ are conjugate under $\mathbf{U}_I(k)$, it follows that there exists $u \in \mathbf{U}_I(k)$ such that $u\mathbf{S}u^{-1} \subset \mathbf{P}_I \cap \mathbf{P}'$, hence $\mathbf{S} \subset u^{-1}\mathbf{P}'u$. So we can assume that $\mathbf{S} \subset \mathbf{P}'$, i.e. that $\mathfrak{C}' \subset \mathfrak{A}$. Then $\mathfrak{C}' = w\mathfrak{C}$ for some $w \in \mathbf{W}(k)$. By 1), $\mathfrak{C}'$ is then equivalent under $\mathbf{W}_I(k)$ to a chamber of $\mathfrak{A}_I$. Since $\mathbf{N}_{\mathbf{L}_I}(\mathbf{S})(k)$ maps onto $\mathbf{W}_I(k)$, we conclude that $\mathfrak{C}'$ is then equivalent under $\mathbf{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. $\square$

We come now to the following important step in Soulé's proof.

Lemma 2.8. *Let $x \in \mathbf{S}_* \cap \mathcal{Q}$. Then $\Gamma_x.(\mathcal{L}_x \cap \mathcal{Q}) = \mathcal{L}_x$.*

Proof. We will make use of the canonical smooth model $\mathfrak{P}_x/\mathcal{O}$ of the parahoric subgroup associated to x [4, §5.2]. As an $\mathcal{O}$–group scheme $\mathfrak{P}_x$ is isomorphic to $\mathbf{G} \times_k \mathcal{O}$, and we have an identification $\mathfrak{P}_x(\mathcal{O}) = P_x$. The star $\mathcal{L}_x$ is the spherical building of $\mathfrak{P}_x \times_{\mathcal{O}} k \cong \mathbf{G}$ [4, §5.1.32]. Set for convenience $I = I_x$. By Lemma 2.6, $\mathcal{L}_x \cap \mathcal{Q}$ is identified with $\mathfrak{A}_I$ in the spherical building $\mathcal{B}(\mathbf{G})$. Furthermore, the chamber $x + \mathcal{C}$ identifies with $\mathfrak{C}$.

The inclusion $\Gamma_x.(\mathcal{L}_x \cap \mathcal{Q}) \subset \mathcal{L}_x$ is clear. Let us prove the reverse inclusion. By definition, there exists $\lambda \in \mathbf{S}_* \cap \mathcal{Q}$ such that $x = \lambda$. Define $g_\lambda = \lambda(\frac{1}{t})^{-1} = \lambda(t) \in \mathbf{S}(K)$. Since $x = g_\lambda.\phi$ by (2.9) above, we have

$$P_x = g_\lambda \, P_\phi \, g_\lambda^{-1}. \tag{2.10}$$

Thus

$$\mathfrak{P}_x(\mathcal{O}) \cong P_x = g_\lambda \, \mathbf{G}(\mathcal{O}) \, g_\lambda^{-1} \subset \mathbf{G}(K).$$

In view of Lemma 2.7.2, it will suffice to establish the following.

Claim 2.9. *The image of the composite map*

$$\Gamma_x \quad \subset \quad P_x \longrightarrow \big(\mathfrak{P}_x \times_{\mathcal{O}} k\big)(k) \cong \mathbf{G}(k)$$

contains $\mathbf{P}_I(k)$.

The group $\mathbf{L}_I(k)$ commutes with g_λ inside $\mathbf{G}(k(t))$, and it is therefore included in the image in question (as we have already observed in Proposition 2.5). So it is enough to check that $g_\lambda \mathbf{U}(k) g_\lambda^{-1} \subset \Gamma_x$, or equivalently that $g_\lambda \mathbf{U}(k) g_\lambda^{-1} \subset \Gamma$. This can be verified by working over the field $\widetilde{k}$ and checking the inclusion for the subgroups $\mathbf{U}_b(\widetilde{k})$ of $\mathbf{U}(\widetilde{k})$ for $b \in \Phi^+$. To verify this we use that the product map induces a decomposition (with the notation of §2.2)

$$\prod_{\widetilde{a} \in \widetilde{\Phi}^b,\, \widetilde{a}_{|S \times_k \widetilde{k}} = b} \widetilde{\mathbf{U}}_{\widetilde{a}}(\widetilde{k}) \ . \prod_{\widetilde{a} \in \widetilde{\Phi}^b,\, \widetilde{a}_{|S \times_k \widetilde{k}} = 2b} \widetilde{\mathbf{U}}_{\widetilde{a}}(\widetilde{k}) \ \xrightarrow{\sim} \ \mathbf{U}_b(\widetilde{k}).$$

For $\widetilde{a} \in \widetilde{\Phi}^b$ and $s \in \widetilde{k}$, we have

$$g_\lambda \, \mathbf{U}_{\widetilde{a}}(s) \, g_\lambda^{-1} = \begin{cases} \widetilde{\mathbf{U}}_{\widetilde{a}}(t^{\langle b, \lambda \rangle} \, s) & \text{if } \widetilde{a}_{|S \times_k \widetilde{k}} = b, \\ \widetilde{\mathbf{U}}_{\widetilde{a}}(t^{2 \langle b, \lambda \rangle} \, s) & \text{if } \widetilde{a}_{|S \times_k \widetilde{k}} = 2b. \end{cases}$$

Hence $g_\lambda \, \mathbf{U}_{\widetilde{a}}(s) \, g_\lambda^{-1} \subset \widetilde{\Gamma}$. This establishes Claim 2.9. The proof of Lemma 2.8 is now complete □

2.5 End of the proof of Theorem 2.1

We now finish the proof of the main Theorem.

Two distinct points of $\mathcal{Q}$ are not equivalent under Γ. Since two different points of $\widetilde{\mathcal{Q}}$ are not equivalent under $\widetilde{\Gamma}$ [14, 1.3], it follows that two distinct points in $\mathcal{Q}$ are not equivalent under Γ.

A point of $\mathcal{T}$ of type 0 *is equivalent to a point of* $\mathcal{Q}$. We denote by $M \subset \mathbf{S}(K) = \mathbf{S}_* \otimes K^\times$ the subgroup generated by the $\lambda(t)$ for λ running over $\mathbf{S}_*$. We denote by $M_+ \subset M$ the semigroup generated by the $\lambda(t)$ for λ satisfying $\langle b, \lambda \rangle \geq 0$ for all $b \in \Delta$. By a result of Raghunathan [9, th. 3.4],[4] we have the decomposition

$$\mathbf{G}(K) = \Gamma \, . \, M \, . \, \mathbf{G}(\mathcal{O}).$$

Again, since $\mathbf{N}_{\mathbf{G}}(\mathbf{S})(k)$ maps onto $\mathbf{W}(k)$ and $\mathbf{W}(k).M_+ = M$, we have actually a decomposition

$$\mathbf{G}(K) = \Gamma \, . \, M_+ \, . \, \mathbf{G}(\mathcal{O}).$$

Since $\mathbf{G}(K)/\mathbf{G}(\mathcal{O})$ is the set of points of type 0 of $\mathcal{T}$, this shows that every such point of $\mathcal{T}$ is Γ–conjugated to a point of $M.\phi$. But $M_+.\phi \subset \mathcal{Q}$, so we conclude that every such point of $\mathcal{T}$ is Γ–conjugated to a point of $\mathcal{Q}$.

[4] This reference presupposes that the base field k is infinite, but this assumption is not necessary (see [7, III.3.4.2] for details).

Every point of $\mathcal{T}$ is equivalent to a point of $\mathcal{Q}$. Let y be a point of $\mathcal{T}$. Let F be a chamber of $\mathcal{T}$ containing y. Then $\overline{F}$ contains a (unique) point x whose type is that of ϕ. By the preceding step, we can assume that $x \in \mathcal{Q}$. Then y belongs to $\mathcal{L}_x$ and Lemma 2.8 shows that y is equivalent under Γ to a point of $\mathcal{Q}$.

¿From the above it follows that $\mathcal{T} = \Gamma.\mathcal{Q}$, as it is stated in the Theorem. $\square$

3 Applications

We give two applications of the main Theorem. The notation and assumptions are as in the previous section. We begin by recalling some basic facts about direct limits of groups.

3.1 Direct limits of groups

Direct limits of groups occur in geometric group theory [11]. In what follows we will repeatedly encounter the following situation: We are given a family of subgroups $(H_\lambda)_{\lambda\in\Lambda}$ of a group H (indexed by some set Λ) and we wish to consider the group which is the direct limit of the groups $(H_\lambda, H_\lambda \cap H_\mu)_{\lambda,\mu\in\Lambda}$ where the only transition maps are the inclusions $H_\lambda \cap H_\mu \subset H_\lambda$ and $H_\lambda \cap H_\mu \subset H_\mu$. We call the resulting group *the direct limit of the family $(H_\lambda)_{\lambda\in\Lambda}$ with respect to their intersections.*[5]

Let T be an abstract simplicial complex, E the set of its vertices, and Φ the set of its simplexes. Denote by X the geometric realization of T. Let H be a group which acts in a simplicial way on T, and for which there exists a simplicial fundamental domain T'. Recall that T' is a subcomplex of T such that if E' (resp. Φ') denotes the set of vertices (resp. simplexes) of T', then for every $s \in \Phi$, there exists a unique $s' \in \Phi'$ such that $s \in H.s'$.

The isotropy subgroup of H corresponding to an element z (respectively a subset M) of either T or X will be denoted by H_z (respectively H_M).

Theorem 3.1. *(Soulé, [13]) Let T, X, H, T' be as above. Assume that X is connected and simply connected and that the geometric realization X' of T' is connected. Then the group H is the direct limit of the family of isotropy subgroups $(H_M)_{M\in E'}$ with respect to their intersections.*

Higher dimensional generalizations of this result have been established by Chebotarev [5]. As pointed out by one of the referees, when X has additional structures there are other presentations which are useful in practice.

[5]Another terminology, which is a slight abuse of language, is that H is the sum of the H_M amalgamated over their intersections [11, II.1.7].

Proposition 3.2. *Under the hypothesis of Theorem 3.1, assume that X is equipped with a distance d such that*

i) for any two points x, y, there is a unique geodesic linking x and y;

ii) for any $x \in X$, there exists an open neighbourhood D_x of x such that for any simplex F of X, $D_x \cap F \neq \emptyset \Longrightarrow x \in \overline{F}$;

iii) H acts isometrically on X.

Furthermore, we assume that

iv) for each simplex F of X the stabilizer of F (as a set) coincides with the isotropy group (pointwise stabilizer) of $\overline{F}$.

Then

1. *The group H is the direct limit of the family $(H_M \cap H_N)_{M,N\in E'}$ with transition maps $H_M \cap H_N \to H_M$ and $H_M \cap H_N \to H_N$ for M, N belonging to an edge of X'.*[6]

2. *The group H is the direct limit of the family of isotropy subgroups $(H_x)_{x\in X'}$ with respect to their intersections.*

Note that when X is a tree the first statement of the Proposition allows us to recover a classical result [11, §4.5, th. 10].

Remark 3.3. Note that the first statement of the Proposition is different than that of Theorem 3.1. The point is that two vertices of X' do not necessarily belong to a common edge. In other words, the presentation of H given by Proposition 3.2.(1) has less relations than the one given by Theorem 3.1.

Proof. We prove both statements at the same time. We denote by $H^\dagger$ the first limit and by $H^\sharp$ the second one. We have an obvious surjective map $H^\dagger \to H$, while the inclusion $E' \subset X$ gives rise to a map $H \to H^\sharp$. We denote by $\xi : H^\dagger \to H \to H^\sharp$ the composition of these two maps. It is enough then to show that $H \to H^\sharp$ is surjective, and to produce a section $\theta : H^\sharp \to H^\dagger$ of ξ.

If $x \in X$, we denote by $F_x \subset X$ the (open) simplex attached to x. Since every F_x contains in its closure a vertex M, our hypothesis on stabilizers implies that $H_x \subset H_M$. It follows that $H \to H^\sharp$ is surjective.

To define the splitting $\theta : H^\sharp \to H^\dagger$ we proceed as follows. We are given $x \in X$, and $M \in E'$ such that $M \in \overline{F}_x$. Since the action is simplicial, we have $H_x = H_{F_x}$. By our hypothesis on the stabilizers, we have then the inclusion $H_x \subset H_M \subset H$.

[6] By taking $M = N$ in E' we see that the groups H_M are part of our family. Observe that if M, N are vertices of a common edge F, then $H_N \cap H_M$ is nothing but the isotropy group of $\overline{F}$.

Step 1 : The composite map

$$\theta_{x,M} : H_x \to H_M \to H^\dagger$$

does not depend of the choice of M:

We note that two distinct choices M, N of vertices of $\overline{F}_x$ define an edge of X', so that the maps $H_x \to H_M \to H^\dagger$ and $H_x \to H_M \to H^\dagger$ agree since they agree on $H_M \cap H_N$. This establishes Step 1, and defines a map $\theta_x : H_x \to H^\sharp$.

Step 2: If $y \in \overline{F}_x$, then θ_x and θ_y agree on the subgroup H_x of H_y :

Since $\overline{F}_y \subset \overline{F}_x$, we can pick a vertice $M \in \overline{F}_y$. By definition $\theta_{x,M}$ and $\theta_{y,M}$ agree on H_y. Hence θ_x and θ_y agree on H_y by the first step.

Step 3: Connexity argument. We are given $x, y \in X$ and we want to show that θ_x and θ_y agree on $H_x \cap H_y$. Since $H_x \cap H_y$ acts trivially on the geodesic $[x, y]$, we have $H_x \cap H_y \subset H_z$ for all $z \in [x, y]$. We consider then the restrictions $\Theta_z : H_x \cap H_y \subset H_z \to H^\dagger$ of θ_z to $H_x \cap H_y$ for z running over $[x, y]$.

Recall that D_z stands for the open neighbourhood of $z \in X$ given by the hypothesis *ii*).

Step 4: If $z \in [x, y]$, then $\Theta_z = \Theta_{z'}$ for all $z' \in D_z \cap [x, y]$. Since $z' \in F_{z'} \cap D_z$, assumption *ii*) implies that $z \in \overline{F}_{z'}$. Step 2 shows that θ_z and $\theta_{z'}$ agree on $H_{z'} \subset H_z$, hence $\Theta_z = \Theta_{z'}$.

We now finish the proof of the Proposition. Since the $D_z \cap [x, y]$ define an open covering of the connected space $[x, y]$, Step 3 implies that Θ_z does not depend on z. In particular θ_x and θ_y agree on $H_x \cap H_y$. By the universal property defining $H^\sharp$, we obtain a map $\theta : H^\sharp \to H^\dagger$. By construction $\theta \circ \xi = id_{H^\dagger}$. □

For future use we record the following.

Lemma 3.4. *Let H be a group which is the direct limit of a family of subgroups $(H_\alpha)_{\alpha\in\Lambda}$ of H with respect to their intersections.*

1. *Let $\Lambda' \subset \Lambda$ be a directed subset, i.e. for all $\alpha, \beta \in \Lambda'$, there exists $\gamma \in \Lambda'$ such that $H_\alpha \subset H_\gamma$, $H_\beta \subset H_\gamma$. Then the direct limit of the family $(H_\alpha)_{\alpha\in\Lambda'}$ with respect to their intersections is canonically isomorphic to the subgroup $\bigcup_{\alpha\in\Lambda'} H_\alpha$ of H.*

2. *Let $\Lambda = \sqcup_{j\in J}\Lambda_j$ be a partition of Λ in directed subsets. For $j \in J$, denote by*

$$H_j := \bigcup_{\alpha\in\Lambda_j} H_\alpha$$

the subgroup of H associated to Λ_j. Then H is the direct limit of the family of subgroups $(H_j)_{j\in J}$ of H with respect to their intersections.

Proof. (1) Note that $\bigcup_{\alpha \in \Lambda'} H_\alpha$ is a subgroup of H since Λ' is directed. For any group M we have

$$\mathrm{Hom}_{gr}(H', M) = \varprojlim_{\alpha \in \Lambda'} \mathrm{Hom}_{gr}(H_\alpha, M),$$

whence the statement.

(2) Denote by $\widetilde{H}$ the direct limit of the family of subgroups $(H_j)_{j \in J}$ of H with respect to their intersections. The inclusion maps $H_j \subset H$ agree over their intersections, hence give rise to a natural map $\xi : \widetilde{H} \to H$. For defining the reverse map, denote by $\alpha \mapsto j(\alpha)$ the map $\Lambda \to J$ which maps α to the unique index j such that $\alpha \in \Lambda_j$. We then get maps

$$H_\alpha \hookrightarrow H_{j(\alpha)} \to \widetilde{H} \qquad (\alpha \in \Lambda).$$

Since these maps agree over their intersections they yield a map $\eta : H \to \widetilde{H}$. Given that the images of the H_α generate H (resp. $\widetilde{H}$), we get that $\eta \circ \xi = id_{\widetilde{H}}$ and $\xi \circ \eta = id_H$. □

3.2 The group $\mathbf{G}(k[t])$ as a direct limit

Theorem 3.1 yields.

Corollary 3.5. *Let V be the set of vertices of $\mathcal{Q}$. The group $\Gamma = \mathbf{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x \in V}$ with respect to their intersections.* □

¿From the Corollary we see that Γ is generated by the Γ_x. By Proposition 2.5.(1) Γ_x consists of products of elements of $\mathbf{G}(k)$ and elements of $\mathbf{U}(k[t])$, where $\mathbf{U}$ stands for the unipotent radical of the minimal parabolic subgroup attached to $\mathbf{S}$ and Δ. From this we obtain.

Corollary 3.6. $\mathbf{G}(k[t]) = \Big\langle \mathbf{G}(k), \mathbf{U}(k[t]) \Big\rangle$. □

Another presentation of Γ is given by means of Proposition 3.2.(2).

Corollary 3.7. *The group $\Gamma = \mathbf{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x \in \mathcal{Q}}$ with respect to their intersections.*

Proof. We have to check that hypothesis (i) through (iv) of Proposition 3.2 are satisfied for the action of Γ on the Bruhat-Tits building $\mathcal{T}$, which is a metric space.

(i) Two points of $\mathcal{T}$ are linked by a unique geodesic [3, §2.5].

(ii) Lemma 2.5.11 of *loc. cit.* guarantees that for any $x \in X$, there exists an open ball D_x of center x such that for any simplex F of X, $D_x \cap F \neq \emptyset \Longrightarrow x \in \overline{F}$.

(iii) The group $\mathbf{G}(K)$ acts isometrically on $\mathcal{T}$ (*ibid*).

(iv) Since $\mathbf{G}$ is simply connected, the stabilizer of a simplex F of $\mathcal{T}$ (or facet with the terminology of Bruhat-Tits) under $\Gamma \subset \mathbf{G}(K)$ is also its pointwise stabilizer [4, prop. 4.6.32] and also of $\overline{F}$ [3, prop. 2.4.13].

The Corollary now follows from Proposition 3.2. $\square$

We shall now give a nicer presentation of Γ. Given a subset $I \subset \Delta$, define

$$\mathcal{Q}_I := \big\{x \in \mathcal{Q} \mid I_x = I\big\}.$$

It is a subcone of $\mathcal{Q}$, i.e. $z\mathcal{Q}_I \subset \mathcal{Q}_I$ for all $z > 0$. Define the subgroup $\Gamma_I = \mathbf{U}_I(k[t]) \rtimes \mathbf{L}_I(k)$.

Lemma 3.8. *1. The $(\Gamma_x)_{x\in\mathcal{Q}_I}$ form a directed family of subgroups of Γ.*

2. Γ_I is the direct limit of the Γ_x for $x \in \mathcal{Q}_I$.

Proof. (1) The sector $\mathcal{Q}$ is equipped with the partial order $x \leq y$ if $y - x \in \mathcal{Q}$. By restriction, we get a partial order on $\mathcal{Q}_I$ which is directed. Indeed, given $x, y \in \mathcal{Q}_I$, we have $x + y \in \mathcal{Q}_I$ and $x + y \geq x$ and $x + y \geq y$.

Let x, y be elements of $\mathcal{Q}_I$ such that $x \leq y$. Then $b(y) \geq b(x)$ for all $b \in [I]^+$, hence $m_y(b) \leq m_x(b)$ for all $b \in [I]^+$. It follows that for $b \in [I]^+$ we have

$$U_{b,m_x(b)}.U_{2b,m_x(2b)} \quad\subset\quad U_{b,m_y(b)}.U_{2b,m_y(2b)}.$$

Now Proposition 2.5.(2) shows that $\Gamma_x \subset \Gamma_y$. Since $\mathcal{Q}_I$ is a directed subset of $\mathcal{Q}$, we conclude that the $(\Gamma_x)_{x\in\mathcal{Q}_I}$ form a directed family of subgroups of Γ.
(2) By Lemma 3.4.(1), it is enough to show that

$$\bigcup_{x\in\mathcal{Q}_I} \Gamma_x \quad=\quad \Gamma_I. \tag{3.1}$$

Proposition 2.5.1 shows that the inclusion $\subset$ holds. Conversely, suppose that we are given an element $g \in \Gamma_I$. Let $x \in \mathcal{Q}_I$. By Proposition 2.5.(3) there exists a real number $z \geq 1$ such that $g \in \Gamma_{zx}$. Since $zx \in \mathcal{Q}_I$, g belongs to the left handside of 3.1. $\square$

Theorem 3.9. *The group $\Gamma = \mathbf{G}(k[t])$ is the direct limit of the family of subgroups $(\Gamma_I)_{I\subset\Delta}$ with respect to their intersections*

Proof. Lemma 3.8.(2) shows that Γ_I is the limit of the directed family of subgroups $(\Gamma_x)_{x\in\mathcal{Q}_I}$. To finish the proof we apply Lemma 3.4.(2) to the decomposition

$$\mathcal{Q} = \sqcup_{I\subset\Delta}\ \mathcal{Q}_I$$

of $\mathcal{Q}$ into directed subsets. $\square$

3.3 Application to Whitehead groups

Let $\mathbf{G}(k)^+$ be the (normal) subgroup of $\mathbf{G}(k)$ generated by the $(R_u\mathbf{P})(k)$ for $\mathbf{P}$ running over all parabolic k–subgroups of $\mathbf{G}$. If $\mathrm{card}(k) \geq 4$, Tits has shown that every proper normal subgroup of $\mathbf{G}(k)^+$ is central [15]. The quotient $W(k, \mathbf{G}) = \mathbf{G}(k)/\mathbf{G}(k)^+$ is the Whitehead group of $\mathbf{G}$ [17]. By Tits' result this group detects whether $\mathbf{G}(k)$ is projectively simple.

It turns out that the Whitehead group admits another characterization. Denote by $H\mathbf{G}(k)$ the (normal) subgroup of $\mathbf{G}(k)$ which consists in elements $g \in \mathbf{G}(k)$ for which there exists an element $h \in \Gamma = \mathbf{G}(k[t])$ such that $h(0) = e$ and $h(1) = g$. We denote by $\pi_0(k, \mathbf{G}) = \mathbf{G}(k)/H\mathbf{G}(k)$ this naïve group of connected components of $\mathbf{G}$.

Theorem 3.10. *There is a canonical isomorphism* $W(k, \mathbf{G}) \xrightarrow{\sim} \pi_0(k, \mathbf{G})$.

Proof. The unipotent radical $\mathbf{V}$ of a k–parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ is a split unipotent group, so it satisfies $H(\mathbf{V})(k) = \mathbf{V}(k)$. Hence we have $\mathbf{G}(k)^+ \subset H\mathbf{G}(k)$ and a surjection $\mathbf{G}(k)/\mathbf{G}(k)^+ \to \pi_0(k, \mathbf{G}) = \mathbf{G}(k)/H\mathbf{G}(k)$. It remains to show that $H\mathbf{G}(k) \subset \mathbf{G}(k)^+$. Let $g \in H\mathbf{G}(k)$, and choose $h \in \mathbf{G}(k[t])$ satisfying $h(0) = e$ and $h(1) = g$. According to Corollary 3.6, the element h can be written in the form

$$h = g_1 u_1\, g_2\, u_2 \cdots g_n u_n$$

with $g_i \in \mathbf{G}(k)$ and $u_i \in \mathbf{U}(k[t])$ where $\mathbf{U}$ is the unipotent radical of a minimal parabolic k–subgroup of $\mathbf{G}$. We can assume that $u_i(0) = e$, so the condition $h(0) = e$ reads $g_1 \cdots g_n = e$. It follows that

$$h = g_1'\, u_1\, {g'}_1^{-1} \cdots g_n'\, u_n\, {g'}_n^{-1}$$

with $g_1' = g_1$, $g_2' = g_1 g_2, \ldots, g_n' = g_1 \cdots g_n = e \in \mathbf{G}(k)$. Hence $g = h(1) = g_1'\, u_1(1)\, {g'}_1^{-1} \cdots {g'}_n\, u_n(1)\, {g'}_n^{-1} \in \mathbf{G}(k)^+$ as desired. □

Acknowledgements: The author wishes to express his sincere gratitude to Gopal Prasad and the referees for their useful comments.

References

[1] A. Borel, *Linear Algebraic Groups (Second enlarged edition)*, Graduate text in Mathematics **126** (1991), Springer.

[2] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[3] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251.

[4] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Etudes Sci. Publ. Math. **60** (1984), 197–376.

[5] N. G. Chebotarëv, The decomposition of groups acting on complexes with a fundamental domain. (Russian) Uspekhi Mat. Nauk **37** (1982), no. 4(226), 191–192, english translation in Russian Math Surveys **37** (1982), no. 4, 103-105.

[6] P. Garrett, *Buildings and classical groups*, Chapman and Hall, London, 1997.

[7] P. Gille, *Torseurs sur la droite affine et R-équivalence*, thèse de doctorat (Orsay, 1994), home page of the author.

[8] H. Nagao, *On* GL(2, $K[x]$), J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 117–121.

[9] M. S. Raghunathan, *Principal bundles admitting a rational section*, Invent. math. **116** (1994), 409-423.

[10] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[11] J.-P. Serre, *Arbres, amalgames,* SL_2, rédigé avec la collaboration de Hyman Bass, Astérisque **46** (1977).

[12] T. A. Springer, *Reductive groups*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 3–27, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[13] C. Soulé, *Groupes opérant sur un complexe simplicial avec domaine fondamental*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), 607–609.

[14] C. Soulé, *Chevalley groups over polynomial rings*, Homological group theory (Proc. Sympos., Durham, 1977), 359–367, London Math. Soc. Lecture Note Ser. **36** (1979), Cambridge Univ. Press.

[15] J. Tits, *Algebraic and abstract simple groups*, Ann. of Math. **80** (1964), 313–329.

[16] J. Tits, *Buidings of spherical type and finite BN-pairs*, Lect. Notes in Math. **396** (1973), Springer.

[17] J. Tits, Groupes de Whitehead de groupes algébriques simples sur un corps (d'après V. P. Platonov et al.), Séminaire Bourbaki, 29e année (1976/77), Exp. No. 505, pp. 218–236, Lecture Notes in Math. **677** (1978).

Laboratoire de Recherche "Princess Stephanie"
Monte Carlo 51840, Monaco.
benedictus.margaux@gmail.com