JACOBSON'S THEOREM FOR BILINEAR FORMS IN CHARACTERISTIC 2

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Abstract

The aim of this note is to extend to bilinear forms in characteristic 2 a result of Jacobson which states that over any field, two Albert quadratic forms are similar if and only if they have the same Clifford invariant.

1. *Introduction*

Throughout this note F denotes a field. To a biquaternion algebras $Q_1 \otimes_F Q_2$, we attach a 6-dimesnional quadratic form φ given by $N_{Q_1} \perp -N_{Q_2} \simeq \mathbf{H} \perp \varphi$, where N_{Q_i} is the norm form of the quaternion algebra Q_i, and $\mathbf{H}$ is the hyperbolic plane ($\perp$ and $\cong$ mean the orthogonal sum and isometry, respectively). The form φ has trivial signed discriminant (*resp.* trivial Arf invariant) if the characteristic is $\neq 2$ (*resp.* if the characteristic is 2). We call such a form an Albert quadratic form. A well-known result of Jacobson states that two biquaternion algebras are isomorphic if and only if their corresponding Albert quadratic forms are similar [**3**]. In other words, this result says that two Albert quadratic forms are similar if and only if they have the same Clifford invariant. Using a method based on quadratic forms theory, Mammone and Shapiro recovered Jacobson's result [**7**], and also completed it in characteristic 2 (see [**7**, comments in the middle of page 529]). The Clifford invariant of a quadratic form (nonsingular quadratic form if the characteristic is 2) is defined in the 2-torsion ${}_2\mathrm{Br}(F)$ of the Brauer group of F. It is well-known that ${}_2\mathrm{Br}(F)$ is isomorphic to I^2F/I^3F (*resp.* I_q^2F/I_q^3F) if the characteristic is not 2 (*resp.* if the characteristic is 2), where $I^kF = (IF)^k$, $I_q^kF = I^{k-1}F \otimes W_q(F)$ and IF denotes the ideal of the Witt ring of even dimensional quadratic forms or bilinear forms according as the characteristic is different or equal to 2, and $W_q(F)$ denotes the Witt group of nonsingualr quadratic forms [**1**]. These isomorphisms are due to Merkurjev [**8**] and Sah [**10**], respectively. Since in characteristic 2 we should distinguish between quadratic and bilinear forms, it is natural to ask whether an analogue of Jacobson's result holds for bilinear forms in characteristic 2. As for quadratic forms, an Albert bilinear form means a 6-dimensional form with trivial determinant. Of course there is no notion of Clifford invariant for bilinear forms in characteristic 2, but we have a result of Kato which gives an analogue of Merkurjev's and Sah's results cited before (see Theorem 2.1), and where the group of finite sums of logarithmic differential forms (see below) is used as for ${}_2\mathrm{Br}(F)$ in the case of quadratic forms. We will see that this ingredient suffices to get a result similar to that of Jacobson.

Recall that for $n \geq 1$, one denotes by $\Omega_F^n = \bigwedge^n \Omega_F^1$ the vector space of n-differential forms over F, where Ω_F^1 is the F-vector space generated by symbols dx, $x \in F$, subject to the relations: $d(x+y) = dx + dy$ and $d(xy) = xdy + ydx$, for $x, y \in F$. An element $\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ is called an n-logarithmic symbol. A sum of n-logarithmic symbols has length k if it is a sum of k n-logarithmic symbols but not a sum of less that k n-logarithmic symbols.

For $a_1, \cdots, a_n \in F^* := F\backslash\{0\}$, let $\langle a_1, \cdots, a_n\rangle_b$ denote the bilinear form given by $\sum_{i=1}^n a_i x_i y_i$. It is clear that any Albert bilinear form γ is isometric to $\alpha\langle r, s, rs, u, v, uv\rangle_b$

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for suitable scalars $\alpha, r, s, u, v \in F^*$. We associate to such a form γ the following sum of 2-logarithmic symbols:

$$e^2(\gamma) := \frac{dr}{r} \wedge \frac{ds}{s} + \frac{du}{u} \wedge \frac{dv}{v} \in \Omega_F^2.$$

This is an invariant of γ modulo I^3F (see below). Our main result in this note is the following theorem:

Theorem 1.1. *Let F be a field of characteristic 2, and let γ_1, γ_2 be two Albert bilinear forms. Then we have the following statements:*
(1) The length of $e^2(\gamma_1)$ is 2, 1 or 0 according as the Witt index of γ_1 is 0, 1 or 3.
(2) γ_1 is similar to γ_2 if and only if $e^2(\gamma_1) = e^2(\gamma_2)$.

By using Theorem 2.1 of Kato, we reduce the proof of Theorem 1.1 to the use of methods from quadratic and bilinear forms theory as was used in [**7**]. However, some of our results differ from those given in [**7**], and their proofs require more details. The reason is that for bilinear forms in characteristic 2 some classical results, like the Witt cancellation and the representation criterion, fail. Moreover, we will be based on the connection between totally singular quadratic forms and bilinear forms, and the notion of norm degree introduced in [**2**, Section 8] will play a crucial rôle.

The rest of this note is organized as follows. We finish this section by giving backgrounds on quadratic and bilinear forms in characteristic 2. The next section in devoted to some results needed for the proof of Theorem 1.1, and more specifically it will concern the similarity of 4-dimensional bilinear forms in characteristic 2, and then in the third section we prove the theorem.

Form now on, we assume that F is of characteristic 2. The expression "bilinear form" means "regular symmetric bilinear form of finite dimension". To keep this note self-contained we briefly recall some notions. More details can be found in [**1**], [**2**].

For a quadratic (or bilinear) form φ, we denote by $\dim \varphi$ its dimension. Two quadratic (or bilinear) forms φ and ψ are called similar if $\varphi \cong \alpha\psi$ for some scalar $\alpha \in F^*$. A quadratic (or bilinear) form φ is called a subform of another form ψ if there exists a form φ' such that $\psi \cong \varphi \perp \varphi'$.

To any bilinear form B with underlying vector space V, we associate a unique quadratic form $\widetilde{B}$ given by: $\widetilde{B}(v) = B(v,v)$ for $v \in V$. A quadratic form φ is called totally singular if $\varphi \cong \widetilde{B}$ for some bilinear form B. If $B \cong \langle a_1, \cdots, a_n\rangle_b$, then we denote $\widetilde{B}$ by $\langle a_1, \cdots, a_n\rangle$.

For a field extension K/F and a quadratic (or bilinear) form φ, the form $\varphi \otimes K$ is denoted by φ_K.

For a quadratic form φ, we denote by $F(\varphi)$ its function field, i.e., the function field of the affine quadric given by $\varphi = 0$. The function field of a bilinear form B is by definition the field $F(\widetilde{B})$.

A quadratic form φ with underlying vector space V is isotropic if there exists $v \in V\backslash\{0\}$ such that $\varphi(v) = 0$. A bilinear form B is isotropic if $\widetilde{B}$ is isotropic too. A form (quadratic or bilinear) is anisotropic if it is not isotropic.

Any bilinear form B decomposes as $B \cong B_{\text{an}} \perp M_1 \perp \cdots \perp M_n$, where B_{an} is anisotropic and M_i is given by the matrix $\begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$ for some $a_i \in F^*$ $(1 \le i \le n)$. The form B_{an} is unique up to isometry, we call it the anisotropic part of B. The interger n is called the Witt index of B and it is denoted by $i_W(B)$. The form B is called metabolic if $2i_W(B) = \dim B$. Two bilinear forms B and C are called equivalent, denoted by $B \sim C$, if $B \perp M \cong C \perp M'$ for some metabolic forms M and M'. By the uniqueness of the anisotropic part the condition $B \sim C$ implies $B_{\text{an}} \cong C_{\text{an}}$. Recall that the isotropy of a bilinear form B is equivalent to say that $i_W(B) \ge 1$.

The form $B = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b$ is called an n-fold bilinear Pfister form, and is denoted by $\langle\langle a_1, \cdots, a_n\rangle\rangle_b$. In this case, we denote $\widetilde{B}$ by $\langle\langle a_1, \cdots, a_n\rangle\rangle$ and we call it an n-fold quasi-Pfister form. Recall that a bilinear Pfister form is isotropic if and only if it is metabolic, and for any integer $n \geq 1$, the ideal I^nF is additively generated by n-fold bilinear Pfister forms.

We say that a totally singular form φ is a quasi-Pfister neighbor if there exists a quasi-Pfister form π such that φ is similar to a subform of π and $2\dim\varphi > \dim\pi$. In this case, π is unique up to isometry, and for any field extension K/F, the forms φ_K and π_K are simultaneously isotropic or anisotropic.

The norm field of a nonzero totally singular form φ, denoted by $N_F(\varphi)$, is the field $F^2(\alpha\beta \mid \alpha, \beta \in D_F(\varphi))$, where $D_F(\varphi)$ is the set of scalars in F^* represented by φ. We denote by $\mathrm{ndeg}_F(\varphi)$ the integer $[N_F(\varphi) : F^2]$ and we call it the norm degree of φ. It is clear that $N_F(\varphi) = N_F(\alpha\varphi)$ for any scalar $\alpha \in F^*$. If φ is anisotropic and $2^n < \dim\varphi \leq 2^{n+1}$, then $\mathrm{ndeg}_F(\varphi) \geq 2^{n+1}$, and $\mathrm{ndeg}_F(\varphi) = 2^{n+1}$ if and only if φ is a quasi-Pfister neighbor. If ψ is a quadratic form such that $\varphi_{F(\psi)}$ is isotropic, then ψ is totally singular and $N_F(\psi) \subset N_F(\varphi)$. Moreover, there is a bijection between anisotropic n-fold quasi-Pfister forms and purely inseparable extensions of F^2 of degree 2^n inside F, it is given by $F^2(a_1, \cdots, a_n) \leftrightarrow \langle\langle a_1, \cdots, a_n\rangle\rangle$. We refer to [**2**, Section 8] for more details on norm field and some of its applications.

2. *Preliminaries*

It is clear that the map $d : F \longrightarrow \Omega_F^1$: $x \mapsto dx$, extends to a map $d : \Omega_F^n \longrightarrow \Omega_F^{n+1}$ defined by:

$$d\left(x dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\right) = dx \wedge dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

There is a well-defined homomorphism $\wp_n : \Omega_F^n \longrightarrow \Omega_F^n/d\Omega_F^{n-1}$ given on generators by:

$$\wp_n\left(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right) = \overline{(x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}}.$$

We write $\nu_F(n)$ the kernel of this map. A crucial result that we will use is the following theorem due to Kato which gives a link between this kernel and bilinear forms. Kato also established a relation between the cokernel of $\wp_n$ and quadratic forms, but we don't need it here.

Theorem 2.1 ([**4**])**.** *For any integer $n \geq 1$, there is an isomorphism $e^n : I^nF/I^{n+1}F \xrightarrow{\sim} \nu_F(n)$, given by:*

$$e^n\left(\sum \langle\langle a_1, \cdots, a_n\rangle\rangle_b + I^{n+1}F\right) = \sum \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

We will use this theorem in the case $n = 2$. Another result that we need is the following theorem which gives information on the dimensions of bilinear forms in I^nF:

Theorem 2.2. *Let $B \in I^nF$ be anisotropic ($n \geq 1$). Then:*
*(1) ([**6**]) $\dim B = 0$ or $\dim B \geq 2^n$.*
(2) If $\dim B > 2^n$, then $\dim B \geq 2^n + 2^{n-1}$.

Statement (2) can be deduced from a result of Vishik [**11**] by the same argument used for the proof of [**6**, Prop. 5.7].

Lemma 2.3. *Let B be an anisotropic bilinear form and $d \in F^*\backslash F^{*2}$. Then, B becomes metabolic over $F(\sqrt{d})$ if and only if $B \cong \alpha_1\langle 1, d + x_1^2\rangle_b \perp \cdots \perp \alpha_n\langle 1, d + x_n^2\rangle_b$ for some $\alpha_i, x_i \in F$ with $\alpha_i \neq 0$, $1 \leq i \leq n$.*

Proof. Use [**5**, Lem. 3.4] and the fact that for $a \neq 0, b \in F$, the bilinear form given by the matrix $\begin{pmatrix} a & b \\ b & ad \end{pmatrix}$ is isometric to $\alpha \left\langle 1, d + x^2 \right\rangle_b$ for some $\alpha \neq 0, x \in F$. ☐

Some results on bilinear forms of dimension 4 will be needed, more particularly properties on the similarity between such forms. Before we give our contribution in this direction (Proposition 2.5 and Corollary 2.6), we start by clarifying the situation whether a 4-dimensional bilinear form becomes isotropic over the inseparable quadratic extension given by its determinant. Recall that an anisotropic quadratic form of dimension 4 (nonsingular if the characteristic is 2) stays anisotropic over the quadratic extension given by its signed discriminant (or the separable quadratic extension given by its Arf invariant). For bilinear forms in characteristic 2 the situation is different as shows the following proposition:

Proposition 2.4. *Let B be a bilinear form of dimension 4 whose determinant is not trivial. Then, B becomes isotropic over the quadratic inseparable extension given by its determinant if and only if* $\mathrm{ndeg}_F(\widetilde{B}) \leq 4$.

Proof. Write $B = \alpha \langle r, s, rs, d \rangle_b$ for suitable scalars $\alpha, r, s \in F^*$. One has $N_F(\widetilde{B}) = F^2(r, s, d)$ and thus $\mathrm{ndeg}_F(\widetilde{B}) \leq 8$. We may suppose that B is anisotropic, in particular $\langle r, s, rs \rangle_b$ is anisotropic too, and thus $[F^2(r, s) : F^2] = 4$. Hence, $\mathrm{ndeg}_F(\widetilde{B}) \in \{4, 8\}$.

Suppose $\mathrm{ndeg}_F(\widetilde{B}) = 4$. Hence, $d \in F^2(r, s)$. Since $d \notin F^2$, one can write $F^2(r, s) = F^2(d, k)$ for some $k \in F^*$. In particular, $\langle\langle r, s \rangle\rangle \cong \langle\langle d, k \rangle\rangle$. Now it is clear that $\langle\langle r, s \rangle\rangle_{F(\sqrt{d})}$ is isotropic. Then, $\langle r, s, rs, d \rangle_{F(\sqrt{d})}$ is also isotropic, i.e., $B_{F(\sqrt{d})}$ is isotropic.

Conversely, if $B_{F(\sqrt{d})}$ is isotropic, then $(\langle\langle r, s \rangle\rangle_b)_{F(\sqrt{d})}$ is metabolic. By Lemma 2.3 it is clear $\langle\langle r, s \rangle\rangle_b \cong \left\langle\left\langle d + x^2, y \right\rangle\right\rangle$ for suitable scalars $x, y \in F$. Hence, $d \in F^2(r, s)$ and then $\mathrm{ndeg}_F(\widetilde{B}) = 4$. ☐

The following proposition is in the spirit of a result due to Wadsworth [**12**, Theorem 7]. In our case, the notion of norm degree plays an essential rôle:

Proposition 2.5. *Let $B = \langle r, s, rs, d \rangle_b$ and $C = \langle u, v, uv, d \rangle_b$ be two anisotropic bilinear forms of dimension 4 having the same determinant d. Suppose that $\mathrm{ndeg}_F(\widetilde{B}) = 8$ and $\langle r, s, rs, u, v, uv \rangle$ is isotropic. Then, B and C are similar over $F(\sqrt{d})$ if and only if there exists $x \in F^*$ such that $\left\langle r, s, rs, d + x^2 \right\rangle$ is similar to $\left\langle u, v, uv, d + x^2 \right\rangle$.*

Proof. It is clear that $N_F(\widetilde{B}) = F^2(r, s, d)$. Suppose that $\left\langle r, s, rs, d + x^2 \right\rangle_b$ is similar to $\left\langle u, v, uv, d + x^2 \right\rangle_b$ for some $x \in F$. Since $(\left\langle d + x^2 \right\rangle_b)_{F(\sqrt{d})} \cong (\langle 1 \rangle_b)_{F(\sqrt{d})} \cong (\langle d \rangle_b)_{F(\sqrt{d})}$, it is clear that the forms $\langle r, s, rs, d \rangle_b$ and $\langle u, v, uv, d \rangle_b$ are similar over $F(\sqrt{d})$.

Conversely, suppose that $B_{F(\sqrt{d})}$ is similar to $C_{F(\sqrt{d})}$. Then, by using the multiplicativity of bilinear Pfister forms, we get that $(\langle\langle r, s \rangle\rangle_b \perp \langle\langle u, v \rangle\rangle_b)_{F(\sqrt{d})}$ is metabolic. The isotropy of $\langle r, s, rs, u, v, uv \rangle_b$ and Lemma 2.3 imply that

$$\langle\langle r, s \rangle\rangle_b \perp \langle\langle u, v \rangle\rangle_b \sim \alpha \left\langle 1, d + x^2 \right\rangle_b \perp \beta \left\langle 1, d + y^2 \right\rangle_b$$

for suitable $\alpha, \beta, x, y \in F$ with $\alpha, \beta \neq 0$. But by comparing determinants in the last relation, we may suppose that $x = y$. Moreover, by the subform theorem for bilinear forms [**5**, Prop. 1.1], we may suppose that α is represented by $\langle r, s, rs \rangle_b$, and thus $\langle\langle r, s \rangle\rangle_b \cong \alpha \langle\langle r, s \rangle\rangle_b$. Hence we get

$$\alpha(\langle\langle r, s \rangle\rangle_b \perp \left\langle 1, d + x^2 \right\rangle_b) \sim \langle\langle u, v \rangle\rangle_b \perp \beta \left\langle 1, d + x^2 \right\rangle_b.$$

The anisotropic part of $\langle\langle r, s \rangle\rangle_b \perp \left\langle 1, d + x^2 \right\rangle_b$ has dimension 4, and thus it is isometric to $\left\langle r, s, rs, d + x^2 \right\rangle_b$, otherwise the form $\langle r, s, rs \rangle_b$ would represent $d + x^2$, and thus $d \in F^2(r, s)$, a contradiction with $\mathrm{ndeg}_F(\widetilde{B}) = 8$. By the uniqueness of the anisotropic part, the form $\langle\langle u, v \rangle\rangle_b \perp \beta \left\langle 1, d + x^2 \right\rangle_b$ also has an anisotropic part of dimension 4. It follows from the multiplicativity of bilinear Pfister forms that $\alpha\left\langle r,s,rs,d+x^2\right\rangle_b \cong \lambda\left\langle u,v,uv,d+x^2\right\rangle_b$ for some scalar $\lambda \in F^*$. Hence, the claim. $\square$

We don't know if Proposition 2.5 remains true in the case of norm degree 4. As a corollary of this proposition we get the following:

Corollary 2.6. *Let B and C be anisotropic bilinear forms of dimension 4 such that $B \perp C \in I^3F$. Then, B is similar to C.*

Proof. The forms B and C have the same determinant since $B \perp C \in I^3F$. Set $B = \alpha\langle r,s,rs,l\rangle_b$ and $C = \beta\langle u,v,uv,l\rangle_b$. We may suppose that l is not a square, otherwise we get the similarity by Theorem 2.2 and the multiplicativity of bilinear Pfister forms.

Since $B \perp C \in I^3F$, it follows from Theorem 2.1 that $e^2(B \perp C) = \frac{du}{u} \wedge \frac{dv}{v} + \frac{dr}{r} \wedge \frac{ds}{s} + \frac{d(\alpha\beta)}{\alpha\beta} \wedge \frac{dl}{l} = 0$. Hence, by statement (1) of Theorem 1.1 the form $\langle r,s,rs,u,v,uv\rangle_b$ is isotropic (note that we can use statement (1) of Theorem 1.1 since its proof is independent of this corollary). Moreover, Theorem 2.2(1) implies that the forms $B_{F(C)}$ and $C_{F(B)}$ are isotropic, and thus $N_F(\widetilde{B}) = N_F(\widetilde{C}) = F^2(r,s,l)$. Since $B \perp C \in I^3F$, we get $(\langle\langle r,s\rangle\rangle_b \perp \langle\langle u,v\rangle\rangle_b)_{F(\sqrt{l})} \in I^3F(\sqrt{l})$, and again by Theorem 2.2(1) we deduce the isometry $(\langle\langle r,s\rangle\rangle_b)_{F(\sqrt{l})} \cong (\langle\langle u,v\rangle\rangle_b)_{F(\sqrt{l})}$. Now it is clear that $B_{F(\sqrt{l})}$ similar to $C_{F(\sqrt{l})}$. We discuss two cases:

(1) Suppose that $\operatorname{ndeg}_F(\widetilde{B}) = 8$: In this case we conclude by Proposition 2.5 that $\left\langle r,s,rs,l+x^2\right\rangle_b \cong k\left\langle u,v,uv,l+x^2\right\rangle_b$ for some scalars $x, k \neq 0 \in F$. In particular,

$$\langle\langle r,s\rangle\rangle_b \sim \left\langle 1,l+x^2\right\rangle_b \perp k\,\langle\langle u,v\rangle\rangle_b \perp k\left\langle 1,l+x^2\right\rangle_b.$$

If we combine this relation with $B \perp C \in I^3F$, it is clear that modulo I^3F we get

$$\alpha k\left\langle 1,l+x^2\right\rangle_b \perp \alpha\left\langle l,l+x^2\right\rangle_b \perp \beta\langle 1,l\rangle_b \in I^3F.$$

Hence

$$\alpha k\left\langle 1,l+x^2\right\rangle_b \perp \alpha\left\langle l,l+x^2\right\rangle_b \sim \beta\langle 1,l\rangle_b.$$

Since $\langle r,s,rs,l\rangle_b \sim k\,\langle\langle u,v\rangle\rangle_b \perp k\left\langle 1,l+x^2\right\rangle_b \perp \left\langle l,l+x^2\right\rangle_b$, it follows that $\langle r,s,rs,l\rangle_b \sim k\,\langle\langle u,v\rangle\rangle_b \perp \alpha\beta\langle 1,l\rangle_b$. By the uniqueness of the anisotropic part and the multiplicativity of bilinear Pfister forms, we conclude that $B \cong mC$ for some scalar $m \in F^*$.

(2) Suppose that $\operatorname{ndeg}_F(\widetilde{B}) = 4$: Since $\widetilde{\langle r,s,rs\rangle_b}$ is anisotropic, the form $\langle\langle r,s\rangle\rangle_b$ is anisotropic too, and then $[F^2(r,s) : F^2] = 4$. Hence, $N_F(\widetilde{B}) = N_F(\widetilde{C}) = F^2(r,s) = F^2(u,v)$ and $l \in F^2(r,s)$.

(i) If $B \perp C$ is isotropic, then it is metabolic and thus $B \cong C$.

(ii) If $B \perp C$ is anisotropic. By Theorem 2.2(1), $B \perp C$ becomes metabolic over its function field. It follows from [**5**, Cor. 5.5] that $B \perp C$ is similar to 3-fold bilinear Pfister form. Hence, $\operatorname{ndeg}_F(\widetilde{B \perp C}) = 8 = \operatorname{ndeg}_F(\widetilde{\alpha(B \perp C)})$. By a simple computation with the fact $l \in F^2(r,s) = F^2(u,v)$, one has $N_F(\alpha(B \perp C)) = F^2(r,s,\alpha\beta)$. Moreover, $\langle\langle r,s\rangle\rangle \cong \langle\langle u,v\rangle\rangle$ since $F^2(r,s) = F^2(u,v)$. Hence, $\langle u,v,uv\rangle$ is isotropic over $K := F(\langle\langle r,s\rangle\rangle)$ since it is a quasi-Pfister neighbor of $\langle\langle u,v\rangle\rangle$. Consequently, $(\langle u,v,uv\rangle_b)_K \sim (\langle 1\rangle_b)_K$ and similarly $(\langle r,s,rs\rangle_b)_K \sim (\langle 1\rangle_b)_K$. Hence, after extending the relation $\alpha(B \perp C) \in I^3F$ to the field K, we conclude that $(\langle\langle l,\alpha\beta\rangle\rangle_b)_K$ is metabolic. In particular, $\langle\langle l,\alpha\beta\rangle\rangle \cong \langle\langle e,f\rangle\rangle$ for some $e,f \in F^2(r,s)$ [**5**, Th. 1.2]. This implies that $\alpha\beta \in F^2(r,s)$, and thus $[F^2(r,s,\alpha\beta) : F^2] = 4$, a contradiction. $\square$

3. *Proof of Theorem 1.1*

(1) (i) If $e^2(\gamma_1)$ has length 2, then it is clear that $i_W(\gamma_1) = 0$.

(ii) If $e^2(\gamma_1)$ has length 1: Let $k,l \in F^*$ be such that $e^2(\gamma_1) = \frac{dk}{k} \wedge \frac{dl}{l}$. Let K be the function field of $\tau = \langle\langle k,l\rangle\rangle_b$. Since τ_K is metabolic, one has $e^2(\gamma_1)_K = 0$. It follows from Theorem

2.1 that $(\gamma_1)_K \in I^3K$, and by Theorem 2.2 $(\gamma_1)_K$ is metabolic. Since $\mathrm{ndeg}_F(\widetilde{\tau}) = 4$ because τ is anisotropic, it follows from [**5**, Th. 1.2] that γ_1 is isotropic. Moreover, the form γ_1 is not metabolic by reason of lenght, hence $i_W(\gamma_1) = 1$.

(iii) If $e^2(\gamma_1)$ has length 0, then it follows from Theorems 2.1 and 2.2 that $i_W(\gamma_1) = 3$.

(2) Let γ_1, γ_2 be two Albert bilinear forms. We have to show that γ_1 is similar to γ_2 if and only if $e^2(\gamma_1) = e^2(\gamma_2)$.

Suppose that γ_1 is similar to γ_2. Then, $\gamma_1 \perp \gamma_2 \in I^3F$. Il follows from Theorem 2.1 that $e^2(\gamma_1) = e^2(\gamma_2)$.

Conversely, suppose that $e^2(\gamma_1) = e^2(\gamma_2)$. Then, again by Theorem 2.1 $\gamma_1 \perp \gamma_2 \in I^3F$. After multiplying, if necessary, γ_1 and γ_2 by suitable scalars, we may suppose that $\gamma_1 \perp \gamma_2$ is isotropic. By Theorem 2.2(2) the Witt index of $\gamma_1 \perp \gamma_2$ is at least 2. Let $k, l \in F^*$ be such that $\gamma_1 \cong k\langle 1, l\rangle \perp B$ and $\gamma_2 = k\langle 1, l\rangle \perp C$ for some 4-dimensional bilinear forms B and C which have the same determinant l. Write $B = \alpha\langle r, s, rs, l\rangle$ and $C = \beta\langle u, v, uv, l\rangle$. An easy computation of $e^2(\gamma_1) = e^2(\gamma_2)$ gives that

$$\frac{dr}{r} \wedge \frac{ds}{s} + \frac{dl}{l} \wedge \frac{d(k\alpha)}{k\alpha} = \frac{du}{u} \wedge \frac{dv}{v} + \frac{dl}{l} \wedge \frac{d(k\beta)}{k\beta}.$$

In particular,

$$\frac{dr}{r} \wedge \frac{ds}{s} + \frac{du}{u} \wedge \frac{dv}{v} = \frac{dl}{l} \wedge \frac{d(k^2\alpha\beta)}{k^2\alpha\beta}.$$

We conclude by statement (1) that $\langle r, s, rs, u, v, uv\rangle_b$ is isotropic. Since $B \perp C \in I^3F$, it follows from Corollary 2.6 that $B \cong mC$ for some scalar $m \in F^*$. Since $\gamma_1 \perp \gamma_2 \in I^3F$, one deduces that $m\gamma_1 \perp \gamma_2 \in I^3F$, i.e., $mk\langle 1, l\rangle \perp k\langle 1, l\rangle \in I^3F$. Hence, $mk\langle 1, l\rangle \cong k\langle 1, l\rangle$, and $\gamma_1 \cong m\gamma_2$.

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