On ternary cubic forms that determine central simple algebras of degree 3

Mélanie Raczek

November 13, 2008

Abstract

Fixing a field F of characteristic different from 2 and 3, we consider pairs (A, V) consisting of a degree 3 central simple F-algebra A and a 3-dimensional vector subspace V of the reduced trace zero elements of A which is totally isotropic for the trace quadratic form. Each such pair gives rise to a cubic form mapping an element of V to its cube; therefore we call it a cubic pair over F. Using the Okubo product in the case where F contains a primitive cube root of unity, and extending scalars otherwise, we give an explicit description of all isomorphism classes of such pairs over F. We deduce that a cubic form associated with an algebra in this manner determines the algebra up to (anti-)isomorphism.

Introduction

Consider a field F of characteristic different from 2. Let A be a quaternion algebra over F and let A^0 denote the subspace of reduced trace zero elements of A. Then for all $x \in A^0$ we have $x^2 \in F$. We thus obtain a quadratic form on A^0 mapping x to x^2. Up to the sign, this quadratic form is the norm form of the quaternion algebra restricted to A^0. By Theorem 2.5, p. 57, in [Lam, 2005], this quadratic form determines the quaternion algebra up to isomorphism.

In this paper we shall generalize this construction for algebras of degree 3. Consider a field F of characteristic different from 2 and 3 and let A be a degree 3 central simple algebra over F. Again let A^0 denote the subspace of reduced trace zero elements of A. Then the cube of an arbitrary element $x \in A^0$ need not be in F in general. In fact, it is in F if and only if the reduced trace of x^2 is equal to zero. Let $q\colon A^0 \to F$ be the trace quadratic form on A^0 (mapping x to the reduced trace of x^2). Then the Witt index of q is equal to 4 if F contains a primitive cube root of unity and is equal to 3 otherwise (see Lemma 0.1). In both cases there exist 3-dimensional subspaces of A^0 which are totally isotropic for the trace quadratic form. Each such vector subspace $V \subset A^0$ gives rise to a cubic form. In this paper we shall prove that this cubic form determines the algebra up to isomorphism or anti-isomorphism.

In the first two sections we shall give an explicit description of the pairs (A, V) where A and V are as above. In the first section we assume that the field F contains a primitive cube root of unity and we use the fact that we may write V in terms of the Okubo product. In the second section we assume that F does not contain a primitive cube root of unity, and we shall minimally extend the field F to use the results of the previous case. In the last section we use these descriptions to prove that a cubic form associated with a pair (A, V) determines A up to (anti-)isomorphism.

Throughout the paper, we denote by F a field of characteristic different from 2 and 3, by F_s a separable closure of F, and by Γ the absolute Galois group $\mathsf{Gal}(F_s/F)$. We fix $\omega \in F_s$ a primitive cube root of unity. We say that a pair (A, V) is a *cubic pair* over F if A is a degree 3 central simple F-algebra and V is a 3-dimensional subspace of A^0 (= the subspace of reduced trace zero elements of A) which is totally isotropic for the trace quadratic form. For a cubic pair (A, V) over F we define a cubic form

$$f_{A,V} \colon V \to F \colon x \mapsto x^3.$$

We say that $\Theta\colon (A, V) \to (B, W)$ is an *isomorphism of cubic pairs* over F if $\Theta\colon A \to B$ is an F-algebra isomorphism such that $\Theta(V) = W$. Note that if (A, V) and (B, W) are isomorphic then $f_{A,V}$ and $f_{B,W}$ are isometric (i.e. there exists an F-vector space isomorphism $\Theta\colon V \to W$ such that $f_{A,V} = f_{B,W} \circ \Theta$). For a field extension L over F we write A_L (resp. V_L) for $A \otimes_F L$ (resp. $V \otimes_F L$). Further we let Trd_A denote the reduced trace of A, and for an F-algebra K, we denote by Tr_K (resp. N_K) the trace (resp. the norm) of K.

Acknowledgement. Most results in this paper were already found in my PhD thesis (see [Raczek, 2007]). I would like to thank Jean-Pierre Tignol, my thesis supervisor, for his help during this work.

0. Some results on quadratic forms

Before we start the classification of cubic pairs, we need preliminary results on quadratic forms.

Let A be a degree 3 central simple algebra over F. First we shall compute the Witt index of the trace quadratic form of A.

Lemma 0.1 *Let q be the trace quadratic form on A^0. Then the Witt index of q is equal to 4 if F contains a primitive cube root of unity and is equal to 3 otherwise.*

Proof : There exists a splitting field L of A of odd degree over F. Indeed, we may choose $L := F$ if A is split and we choose a splitting field of degree 3 over F otherwise. Then straightforward computations show that the class of the form $q_L : A^0_L \to L$ is equal to the class of $2\langle 1, 3\rangle$ in the Witt ring of L. Hence, by Springer's Theorem about odd degree extensions (see Theorem 2.7, p. 194, in [Lam, 2005]), the Witt index of q is greater than or equal to 3 and it is 4 if and only if F contains a primitive cube root of unity. $\square$

Suppose that F contains a primitive cube root of unity.

Lemma 0.2 *Let V be a 3-dimensional totally isotropic subspace of A^0. Then there exist exactly two maximal totally isotropic subspaces W_1, W_2 of A^0 containing V; thus $V = W_1 \cap W_2$.*

Proof : A more general fact is proved in III.1.11 of [Chevalley, 1997]: in a quadratic space with a Witt index equal to n, all $(n-1)$-dimensional totally isotropic subspaces are contained in exactly two maximal totally isotropic subspaces. $\square$

1. Classification of cubic pairs over a field with a primitive cube root of unity

We assume that F contains a primitive cube root of unity.

1.1. Okubo product

Let A be a degree 3 central simple F-algebra. In [Knus *et al.*, 1998] the Okubo product over A^0 is defined as follows:

$$x \star y := \mu xy + (1-\mu)yx - \frac{1}{3}\mathsf{Trd}_A(xy) = \frac{1}{1-\omega}(yx - \omega xy) - \frac{1}{3}\mathsf{Trd}_A(xy)$$

where $\mu := \frac{1-\omega}{3}$. Let q denote the trace quadratic form on A^0. Because F contains a primitive cube root of unity, the Witt index of q is equal to 4. In [2008], Matzri interprets the results of van der Blij and Springer [1960] on triality, in the language of the Okubo product: he gives a description of the 4-dimensional subspaces of A which are totally isotropic for q, in terms of the Okubo product.

Theorem 1.1 (Matzri) *Let $u \in A^0 \setminus \{0\}$ be such that $\mathsf{Trd}_A(u^2) = 0$. Then $u \star A^0$ and $A^0 \star u$ are 4-dimensional totally isotropic subspaces of A^0. Moreover any 4-dimensional totally isotropic subspace is of this form.*

We may also write the 3-dimensional totally isotropic subspaces of A^0 in terms of the Okubo product. By Lemma 0.2, the 3-dimensional totally isotropic subspaces of A^0 are the intersections of two subspaces of the form $u \star A^0$ or $A^0 \star u$. We can be more precise using the following:

Theorem 1.2 (Matzri) *Let $u, v \in A^0 \setminus \{0\}$ be such that $\mathsf{Trd}_A(u^2) = 0$ and $\mathsf{Trd}_A(v^2) = 0$. Then*

1. *the dimension of $u \star A^0 \cap v \star A^0$ is even;*
2. *if $u \star v \neq 0$, then $\dim(u \star A^0 \cap A^0 \star v) = 1$;*
3. *if $u \star v = 0$, then $\dim(u \star A^0 \cap A^0 \star v) = 3$.*

Note that the Okubo product depends on the choice of the primitive cube root of unity. If we set, for a primitive cube root of unity ρ,

$$x \star_\rho y := \mu_\rho xy + (1 - \mu_\rho)yx - \frac{1}{3}\mathsf{Trd}_A(xy)$$

where $\mu_\rho := \frac{1-\rho}{3}$, then $x \star_\omega y = y \star_{\omega^2} x$. So by Theorem 1.2, the dimension of $A^0 \star u \cap A^0 \star v$ is also even.

Corollary 1.3 *Let (A, V) be a cubic pair over F. Then there exist nonzero $u, v \in A^0$ with $\mathsf{Trd}_A(u^2) = 0$, $\mathsf{Trd}_A(v^2) = 0$ and $u \star v = 0$ such that $V = u \star A^0 \cap A^0 \star v$.* □

The vectors u and v are in fact uniquely determined up to scalars. We shall prove this as a part of a more general statement:

Lemma 1.4 *If (B, W) is another cubic pair over F with $W = r \star B^0 \cap B^0 \star s$ as in Corollary 1.3, then an F-algebra isomorphism $\Theta\colon A \to B$ induces an isomorphism $\Theta\colon (A, V) \to (B, W)$ of cubic pairs if and only if $\Theta(u)F = rF$ and $\Theta(v)F = sF$.*

Proof: Assume that $\Theta\colon A \to B$ is an F-algebra isomorphism. If $\Theta(V) = W$, then $W = \Theta(u) \star B^0 \cap B^0 \star \Theta(v) = r \star B^0 \cap B^0 \star s$. By Lemma 0.2, we have

$$\{\Theta(u) \star B^0, B^0 \star \Theta(v)\} = \{r \star B^0, B^0 \star s\}.$$

If $r \star B^0 = B^0 \star \Theta(v)$, then $W = \Theta(u) \star B^0 \cap r \star B^0$ and by Theorem 1.2, the dimension of W is even. Hence $\Theta(u) \star B^0 = r \star B^0$ and $B^0 \star \Theta(v) = B^0 \star s$. By Theorem 2.10 in [Matzri, 2008], we then have $\Theta(u)F = rF$ and $\Theta(v)F = sF$. The converse is obvious. □

1.2. Classification

We shall describe a cubic pair (A, V) over F up to isomorphism. By Corollary 1.3, there exist nonzero $u, v \in A^0$ such that $\mathsf{Trd}_A(u^2) = 0$, $\mathsf{Trd}_A(v^2) = 0$, $u \star v = 0$ and $V = u \star A^0 \cap A^0 \star v$. Since $\mathsf{Trd}_A(v) = \mathsf{Trd}_A(v^2) = 0$ we have $v^3 \in F$, and similarly $u^3 \in F$. Note that $u \star v = 0$ implies that

$$vu = \frac{1-\omega}{3}\mathsf{Trd}_A(uv) + \omega uv.$$

Set $t := uv - \frac{1}{3}\mathsf{Trd}_A(uv)$. Then

$$tu = uvu - \frac{1}{3}\mathsf{Trd}_A(uv)u = \frac{1-\omega}{3}\mathsf{Trd}_A(uv)u + \omega u^2 v - \frac{1}{3}\mathsf{Trd}_A(uv)u = \omega ut$$

and similarly $vt = \omega tv$. We deduce that $t^3 \in F$. Indeed,

$$\begin{aligned} t^2 &= t\big(uv - \frac{1}{3}\mathsf{Trd}_A(uv)\big) \\ &= \omega utv - \frac{1}{3}\mathsf{Trd}_A(uv)t \\ &= \omega u^2v^2 + \frac{\omega^2}{3}\mathsf{Trd}_A(uv)uv + \frac{1}{9}\mathsf{Trd}_A(uv)^2 \end{aligned}$$

and thus

$$\begin{aligned} t^3 &= t^2\big(uv - \frac{1}{3}\mathsf{Trd}_A(uv)\big) \\ &= \omega^2 ut^2v - \frac{1}{3}\mathsf{Trd}_A(uv)t^2 \\ &= u^3v^3 - \frac{1}{27}\mathsf{Trd}_A(uv)^3 \in F. \end{aligned} \tag{1}$$

This implies in particular that $\mathsf{Trd}_A(t^2) = 0$, so $\mathsf{Trd}_A(u^2v^2) = \frac{1}{3}\mathsf{Trd}_A(uv)^2$.

We shall prove that $t^2, t^2u, t^2v \in V$. First we observe that

$$u \star A^0 = \{x \in A^0 \mid x \star u = 0\}.$$

Indeed, by Proposition (34.19) in [Knus *et al.*, 1998] we have $(u \star x) \star u = \frac{1}{6}\mathsf{Trd}_A(u^2)x = 0$ for all $x \in A^0$. Hence

$$u \star A^0 \subset \{x \in A^0 \mid x \star u = 0\} = \mathsf{ker}(R_u)$$

where $R_u \colon A^0 \to A^0 \colon x \mapsto x \star u$. But $\dim \mathsf{ker}(R_u) + \dim \mathsf{im}(R_u) = 8$, thus $\dim \mathsf{ker}(R_u) = \dim(u \star A^0) = 4$. Similarly,

$$A^0 \star v = \{x \in A^0 \mid v \star x = 0\}.$$

One can see that $t^2 \in V$ since $ut^2 = \omega t^2 u$ and $t^2 v = \omega v t^2$. Also $u(ut^2) = \omega(ut^2)u$ and $(vt^2)v = \omega v(vt^2)$ imply $(ut^2) \star u = 0$ and $v \star (vt^2) = 0$. Now

$$\begin{aligned} v \star (ut^2) &= \frac{1}{1-\omega}(ut^2 v - \omega v u t^2) - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\ &= \frac{1}{1-\omega}\Big(ut^2 v - \omega \frac{1-\omega}{3}\mathsf{Trd}_A(uv)t^2 - \omega^2 uvt^2\Big) - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\ &= ut^2 v - \frac{\omega}{3}\mathsf{Trd}_A(uv)t^2 - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\ &= \frac{\omega^2}{27}\mathsf{Trd}_A(uv)^3 - \frac{\omega^2}{9}\mathsf{Trd}_A(uv)\mathsf{Trd}_A(u^2 v^2) \\ &= 0. \end{aligned}$$

Since $(vt^2) \star u = \omega^2 v \star (ut^2) = 0$, we obtain that $ut^2, vt^2 \in V$. So $t^2, t^2 u, t^2 v \in V$ because $vt = \omega tv$ and $tu = \omega ut$.

To work out the classification of cubic pairs we shall distinguish different situations:

First case: We assume that $u, v, t \in A^\times$. Observe that $t = \frac{1}{1-\omega}(uv - vu) \neq 0$, hence u and v are linearly independent. Since $\mathsf{Trd}_A(u) = \mathsf{Trd}_A(v) = 0$, the vectors $1, u, v$ are also linearly independent. Therefore t^2, $t^2 u$, $t^2 v$ span V. Set $\xi := t^2$, $\eta := t^2 v$ and $\lambda := \frac{1}{3t^3}\mathsf{Trd}_A(uv)$. Because $t = uv - \frac{1}{3}\mathsf{Trd}_A(uv)$, we have $u = tv^{-1} + \frac{1}{3}\mathsf{Trd}_A(uv)v^{-1}$. One can check that

$$t^2 u = \frac{\omega}{v^3 t^3}\Big(\xi\eta^2 + \lambda\xi^2\eta^2\Big).$$

Finally A is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ and η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and V is the vector subspace spanned by ξ, η and $\xi\eta^2 + \lambda\xi^2\eta^2$ where $1 + \lambda^3 a \neq 0$ since $u^3 \neq 0$. In this basis of V, the cubic form $f_{A,V}$ takes the generalized Hesse normal form:

$$f_{A,V}\big(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2)\big) = ax^3 + by^3 + ab^2(1 + \lambda^3 a)z^3 - 3\omega^2 ab\lambda xyz.$$

The form $f_{A,V}$ is nonsingular.

Conversely, suppose that B is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the vector subspace spanned by ξ, η, $\xi\eta^2 + \lambda\xi^2\eta^2$, for some $a, b \in F^\times$ and $\lambda \in F$ such that $1 + \lambda^3 a \neq 0$. Then one can check that (B, W) is a cubic pair over F such that $f_{B,W}$ is nonsingular.

Second case: We suppose that $u, v \in A^\times$ and $t = 0$. Then $uv = \frac{1}{3}\mathsf{Trd}_A(uv) \in F^\times$. Thus we may assume that $u = v^2$. We need the following:

Lemma 1.5 *Let $\xi \in A^0$ be such that $\xi^3 \in F^\times$. Then there exists $\eta \in A$ such that $\eta^3 \in F^\times$ and $\xi\eta = \omega\eta\xi$.*

Proof: Assume that $\xi^3 \notin F^{\times 3}$, then $F(\xi)$ is a subfield of A. Let $\sigma\colon F(\xi) \to F(\xi)$ be the F-automorphism defined by $\sigma(\xi) = \omega^2\xi$. By the Skolem-Noether Theorem, there exists $\eta \in A^\times$ such that $\eta x \eta^{-1} = \sigma(x)$ for all $x \in F(\xi)$. In particular $\xi\eta = \omega\eta\xi$. Because $\eta = \omega\xi^{-1}\eta\xi$ and $\eta^2 = \omega^2\xi^{-1}\eta^2\xi$, we have $\mathsf{Trd}_A(\eta) = 0$, $\mathsf{Trd}_A(\eta^2) = 0$; so $\eta^3 \in F^\times$.

Now we suppose that $\xi^3 \in F^{\times 3}$. Then we may assume that $\xi^3 = 1$ and $A = \mathsf{M}_3(F)$. The minimal polynomial of ξ divides $t^3 - 1$, so ξ is diagonalizable and its eigen values are cube roots of unity. Hence we may assume that

$$\xi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_i \in \{1, \omega, \omega^2\}$. Since $\mathsf{tr}(\xi) = 0$ we have $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, \omega, \omega^2\}$. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

if necessary, we may assume that $\lambda_1 = 1$, $\lambda_2 = \omega$, $\lambda_3 = \omega^2$. Then

$$\eta := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is such that $\eta^3 \in F^\times$ and $\xi\eta = \omega\eta\xi$. $\square$

Let $w \in A$ be such that $w^3 \in F^\times$ and $vw = \omega wv$. Then the subspace of the elements x in A such that $vx = \omega^2 xv$ is spanned by w^2, vw^2, v^2w^2, and it is contained in V; therefore

$$V = \{x \in A \mid vx = \omega^2 xv\}.$$

Set $\xi := w^2$ and $\eta := vw^2$, then A is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ and η such that $\xi^3 = a$, $\eta^3 = b$ and $\xi\eta = \omega\eta\xi$, and V is the vector subspace spanned by ξ, η and $\xi^2\eta^2$. In this basis of V, the form $f_{A,V}$ takes the generalized Hesse normal form:

$$f_{A,V}(x\xi + y\eta + z\xi^2\eta^2) = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2 abxyz.$$

The form $f_{A,V}$ is singular; more precisely, it is *triangular*, i.e., there exist linearly independent forms $\varphi_1, \varphi_2, \varphi_3 \in (V \otimes_F F_s)^\star$ such that $f_{A,V} = \varphi_1\varphi_2\varphi_3$ as a cubic form over $V \otimes_F F_s$.

Conversely, suppose that B is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W is the vector subspace spanned by

ξ, η, $\xi^2\eta^2$, for some $a, b \in F^\times$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular.

Third case: We suppose that either $u \notin A^\times$, or $v \notin A^\times$, or $t \notin A^\times$ and $t \neq 0$. Then the algebra A is split, so we may assume that $A = \mathsf{M}_3(F)$. This case is less interesting and thus we shall not give an explicit description of the pair (A, V), but we shall only prove that the cubic form $f_{A,V}$ is singular and not triangular (it is possible to describe V by matrix computations distinguishing several cases; details can be found in [Raczek, 2007]).

To show that $f_{A,V}$ is not triangular, we first prove a more general Lemma on triangular forms.

Lemma 1.6 *Suppose that (B, W) is any cubic pair over F such that $f_{B,W}$ is triangular. Then $W = s^2 \star B^0 \cap B^0 \star s$ for some $s \in B^0$ such that s is invertible.*

Proof: We may assume that $F = F_s$ and $B = \mathsf{M}_3(F)$. Let $e_1, e_2, e_3 \in W$ be such that

$$f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_2x_3$$

for all $x_1, x_2, x_3 \in F$. Observe that $x^3 = \frac{1}{3}\mathsf{tr}(x^3)$ for all $x \in W$, hence $f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3)$ is equal to

$$\sum_{i=1}^{3} e_i^3 x_i^3 + \sum_{i \neq j} \mathsf{tr}(e_i^2 e_j) x_i^2 x_j + \mathsf{tr}(e_1e_2e_3 + e_2e_1e_3)x_1x_2x_3$$

for all $x_1, x_2, x_3 \in F$. We deduce that $\mathsf{tr}(e_i^2 e_j) = 0$ for all i, j. Set $e_2 = (x_{ij})$ and $e_3 = (y_{ij})$.
Suppose that $e_1^2 \neq 0$. Since $e_1^3 = 0$, we may assume that

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(e_2) = 0$, $\mathsf{tr}(e_1e_2) = 0$ and $\mathsf{tr}(e_1^2e_2) = 0$, we have

$$x_{33} = -x_{11} - x_{22}, \quad x_{32} = -x_{21}, \quad x_{31} = 0.$$

From $\mathsf{tr}(e_1e_2^2) = 0$ we deduce that $x_{21}(2x_{11}+x_{22}) = 0$. If $x_{21} = 0$ then $\mathsf{tr}(e_2^2) = 0$ and $e_2^3 = 0$ imply $x_{11} = x_{22} = 0$. Then $e_1e_2 + e_2e_1 = (x_{12} + x_{23})e_1^2$ and it contradicts the fact that $\mathsf{tr}(e_1^2e_3) = 0$ and $\mathsf{tr}(e_1e_2e_3+e_2e_1e_3) = 1$. If $x_{22} = -2x_{11}$ then

$$e_1e_2 + e_2e_1 = \begin{pmatrix} x_{21} & -x_{11} & x_{12} + x_{23} \\ 0 & 0 & -x_{11} \\ 0 & 0 & -x_{21} \end{pmatrix}.$$

By symmetry we know that

$$e_3 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & -2y_{11} & y_{23} \\ 0 & -y_{21} & y_{11} \end{pmatrix},$$

thus $\mathsf{tr}(e_1e_2e_3 + e_2e_1e_3) = 1$ is impossible.
Therefore $e_1^2 = 0$ (by symmetry we also have $e_2^2 = 0$, $e_3^2 = 0$) and we may assume that

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(e_2) = 0$ and $\mathsf{tr}(e_1e_2) = 0$, we have $x_{33} = -x_{11} - x_{22}$ and $x_{31} = 0$. Then $e_2^2 = 0$ implies $x_{21}x_{32} = 0$. Observe that

$$e_1e_2 + e_2e_1 = \begin{pmatrix} 0 & x_{32} & -x_{22} \\ 0 & 0 & x_{21} \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(e_1e_2e_3 + e_2e_1e_3) = 1$ we have either $x_{21} = 0 = y_{32}$ and $x_{32}, y_{21} \neq 0$ or $x_{21}, y_{23} \neq 0$ and $x_{32} = 0 = y_{21}$. Thus we may assume that $x_{21} = 0$ and $x_{32} = 1$. From $e_2^2 = 0$ we deduce that

$$e_2 = \begin{pmatrix} 0 & \alpha & -\alpha\beta \\ 0 & \beta & -\beta^2 \\ 0 & 1 & -\beta \end{pmatrix}$$

for some $\alpha, \beta \in F$. We may assume that $\alpha = \beta = 0$ since the invertible matrix

$$m = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $me_1m^{-1} = e_1$ and

$$me_2m^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly we see that

$$e_3 = \begin{pmatrix} \alpha & -\alpha^2 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\alpha \in F$. Again we may assume that $\alpha = 0$ conjugating by

$$\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if necessary.
Then one can check that $W = s^2 \star B^0 \cap B^0 \star s$ with

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

$\square$

In our case, the subspace V is equal to $u \star A^0 \cap A^0 \star v$ where either $u \notin A^\times$, or $v \notin A^\times$, or $t \notin A^\times$ and $t \neq 0$. Observe that if $u, v \in A^\times$, then $uF = v^2 F$ if and only if $t = 0$. Thus, by the previous Lemma, the form $f_{A,V}$ is not triangular.

To prove that $f_{A,V}$ is singular we shall distinguish different cases.

1. Suppose that $v \notin A^\times$.
If $\mathsf{tr}(uv) \neq 0$ then, by the relation (1), t is invertible; hence V is spanned by t^2, $t^2 u$ and $t^2 v$. Since $\mathsf{tr}\big(x(t^2v)^2\big) = 0$ for all $x \in V$, the point $t^2 v F_s$ of the projective plane $\mathbb{P}_V(F_s)$ is a singular zero of $f_{A,V}$.
If $v^2 \neq 0$ and $\mathsf{tr}(uv) = 0$, then $v^2 \in V$ and $v^2 F_s$ is a singular zero of $f_{A,V}$.
If $v^2 = 0$, then we may assume that

$$v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so} \quad u = \begin{pmatrix} \omega^2\alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \omega\alpha_1 & \alpha_4 \\ 0 & 0 & \alpha_1 \end{pmatrix}$$

for some $\alpha_i \in F$. If $\alpha_1 = 0$ then vF_s is a singular zero of $f_{A,V}$. If $\alpha_1 \neq 0$ then V is spanned by

$$\begin{pmatrix} \alpha_1 & 0 & -\alpha_3 \\ 0 & \omega\alpha_1 & 0 \\ 0 & 0 & \omega^2\alpha_1 \end{pmatrix}, \begin{pmatrix} 0 & (\omega-1)\alpha_1 & \alpha_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & (\omega-\omega^2)\alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the cubic curve associated with $f_{A,V}$ is a triple line.

2. Suppose that $u \notin A^\times$, then, by symmetry, we deduce that $f_{A,V}$ is also singular.

3. Suppose that $u, v \in A^\times$, $t \notin A^\times$ and $t \neq 0$.
If $t^2 \neq 0$ then $t^2 F_s$ is a singular zero of $f_{A,V}$.
If $t^2 = 0$, we shall prove that there exists a nonzero $s \in A$ such that $s^2 = 0$, $vs = \omega^2 sv$, $s(tv^{-1}) = (tv^{-1})s = 0$. Since $u = tv^{-1} + \frac{1}{3}\mathsf{tr}(uv)v^{-1}$, we then have $s \in V$ and so sF_s is a singular zero of $f_{A,V}$. Let $w \in A$ be such that $w^3 \in F^\times$ and $vw = \omega wv$. Since $v(tv^{-1}) = \omega(tv^{-1})v$ and $tv^{-1} \neq 0$, there exist $\alpha_i \in F$ not all zero such that $tv^{-1} = \alpha_0 w + \alpha_1 vw + \alpha_2 v^2 w$. But $(tv^{-1})^2 = \omega^2 t^2 v^{-2} = 0$, so $\alpha_0 \neq 0$, $\alpha_2 = \alpha_1^2\alpha_0^{-1}$ and $\alpha_0^3 = v^3\alpha_1^3$. Hence $v^3 \in F^{\times 3}$ and we may assume that

$v^3 = 1$. Replacing w by $\alpha_0^{-1}w$ if necessary, we may assume that $\alpha_0 = 1$. Then $\alpha_1 = 1$, ω or ω^2. Conjugating by w or w^{-1} if necessary, we may assume that $tv^{-1} = w + vw + v^2w$. Then we may choose $s = w^2 + vw^2 + v^2w^2$.

We summarize the above classification in the following Theorem.

Theorem 1.7 *Suppose that F contains a primitive cube root of unity. Let (A, V) be a cubic pair over F.*

1. *If $f_{A,V}$ is nonsingular, then*

$$(A, V) \cong \big((a, b)_{\omega,F}, \mathsf{span}_F\langle \xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2\rangle\big)$$

for some $a, b \in F^\times$, $\lambda \in F$ such that $1+\lambda^3a \neq 0$, where ξ, η are generators of the symbol algebra such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$. Conversely, let $a, b \in F^\times$, $\lambda \in F$ be such that $1 + \lambda^3a \neq 0$. Let B be the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the subspace spanned by ξ, η, $\xi\eta^2 + \lambda\xi^2\eta^2$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is nonsingular. In the basis $(\xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2)$, the form $f_{B,W}$ takes the generalized Hesse normal form:

$$\big(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2)\big)^3 = ax^3 + by^3 + ab^2(1 + \lambda^3a)z^3 - 3\omega^2ab\lambda xyz.$$

2. *If $f_{A,V}$ is triangular, then*

$$(A, V) \cong \big((a, b)_{\omega,F}, \mathsf{span}_F\langle \xi, \eta, \xi^2\eta^2\rangle\big)$$

for some $a, b \in F^\times$, where ξ, η are generators of the symbol algebra such that $\xi^3 = a$, $\eta^3 = b$ and $\xi\eta = \omega\eta\xi$. Conversely, let $a, b \in F^\times$, let B be the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the subspace spanned by ξ, η, $\xi^2\eta^2$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular. In the basis $(\xi, \eta, \xi^2\eta^2)$, the form $f_{B,W}$ takes the generalized Hesse normal form:

$$(x\xi + y\eta + z\xi^2\eta^2)^3 = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2abxyz.$$

3. *If $f_{A,V}$ is singular and not triangular, then A is split.*

2. Classification of cubic pairs over a field without primitive cube root of unity

Suppose that F does not contain a primitive cube root of unity. We shall give the classification of cubic pairs over F in the case where the associated cubic form is nonsingular or triangular. For the remaining cases we know by Theorem 1.7 that the algebra is split and it is just a matter of matrix computations to describe all the possible subspaces up to conjugacy (see [Raczek, 2007] for details). We shall extend the scalars to $F(\omega)$ to use the previous classification. To simplify notations, let T denote the reduced trace of $A_{F(\omega)}$. Throughout this section, we denote by σ the F-automorphism of $F(\omega)$ such that $\sigma(\omega) = \omega^2$.

2.1. Nonsingular form

Let (A, V) be a cubic pair over F such that $f_{A,V}$ is nonsingular. By Subsection 1.2, there exist nonzero $u, v \in A^0_{F(\omega)}$ such that $\mathsf{T}(u^2) = 0$, $\mathsf{T}(v^2) = 0$, $u \star v = 0$ and

$$V_{F(\omega)} = \Big(u \star A^0_{F(\omega)}\Big) \cap \Big(A^0_{F(\omega)} \star v\Big)$$

where $u, v, t := uv - \frac{1}{3}\mathsf{T}(uv) \in A^\times_{F(\omega)}$. Then $V_{F(\omega)}$ is spanned by t^2, t^2v and t^2u.

We extend σ to an F-automorphism of $A_{F(\omega)}$: for $x \in A_{F(\omega)}$ and $\lambda \in F(\omega)$, define $\sigma(x \otimes \lambda) = x \otimes \sigma(\lambda)$. Then A (resp. V) consists of the elements of $A_{F(\omega)}$ (resp. $V_{F(\omega)}$) which are fixed under σ. Note that $\sigma(x \star y) = \sigma(y) \star \sigma(x)$ for all $x, y \in A_{F(\omega)}$. Since $\sigma(V_{F(\omega)}) = V_{F(\omega)}$, we have

$$\Big(\sigma(v) \star A^0_{F(\omega)}\Big) \cap \Big(A^0_{F(\omega)} \star \sigma(u)\Big) = \Big(u \star A^0_{F(\omega)}\Big) \cap \Big(A^0_{F(\omega)} \star v\Big).$$

Hence there exists $\lambda \in F(\omega)^\times$ such that $\sigma(u) = \lambda v$. Replacing v by λv if necessary we may assume that $\sigma(u) = v$ and thus $\sigma(v) = u$. Recall that

$$vu = \frac{1-\omega}{3}\mathsf{T}(uv) + \omega uv,$$

hence we have

$$\begin{aligned} \sigma(t) &= \sigma(uv) - \frac{1}{3}\sigma\big(\mathsf{T}(uv)\big) \\ &= \sigma(u)\sigma(v) - \frac{1}{3}\mathsf{T}\big(\sigma(uv)\big) \\ &= vu - \frac{1}{3}\mathsf{T}(vu) \\ &= \omega t. \end{aligned}$$

Therefore $\omega^2 t \in A$ and $\omega t^2 \in V$. Set $e := \omega t^2$, then V is spanned by e, $e(v + u)$ and $e(\omega v + \omega^2 u)$.

We shall find a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L, ρ) such that $L \subset A$, the vector e and L generate A, and $ex = \rho(x)e$ for all $x \in L$. To do this we first construct an element $\eta \in A_{F(\omega)}$ such that $e\eta = \omega\eta e$, $\eta^3 \in F(\omega)^\times$ and $\sigma(\eta)\eta \in F^\times$. Recall that $t^3 = u^3v^3 - \frac{1}{27}\mathsf{T}(uv)^3$, $vt = \omega tv$, $tu = \omega ut$ and

$$t^2 = \omega u^2v^2 + \frac{\omega^2}{3}\mathsf{T}(uv)uv + \frac{1}{9}\mathsf{T}(uv)^2.$$

Set

$$\eta := \frac{\mathsf{T}(uv)}{3}v + \frac{1}{t^3}e^2v,$$

then $e\eta = \omega\eta e$, $\eta^3 = u^3v^6$ and

$$\begin{aligned}\sigma(\eta)\eta &= \frac{\mathsf{T}(uv)^2}{9}uv - \frac{\omega\mathsf{T}(uv)}{3}utv + \omega^2ut^2v \\ &= u^3v^3.\end{aligned}$$

We set $\lambda := u^3v^3$, so $\eta + \lambda\eta^{-1}$ is fixed under σ.
If $\eta^3 \notin F(\omega)^{\times 3}$, then set $L := F(\eta + \lambda\eta^{-1})$ and let $\rho\colon L \to L$ be the F-automorphism defined by $\rho(\eta + \lambda\eta^{-1}) = \omega\eta + \omega^2\lambda\eta^{-1}$. Then L is a cyclic extension of degree 3 over F which is contained in A and with a Galois group generated by ρ. Moreover $ex = \rho(x)e$ for all $x \in L$.
If $\eta^3 \in F(\omega)^{\times 3}$, then $\eta^3 = \nu^3$ for some $\nu \in F(\omega)^\times$. Replacing η by $\nu^{-1}\eta$ if necessary, we may assume that $\eta^3 = 1$ and $\sigma(\eta)\eta = 1$. Set

$$L := F \cdot 1 + F \cdot (\eta + \eta^{-1}) + F \cdot (\omega\eta + \omega^2\eta^{-1})$$

and define ρ as the F-automorphism of L such that $\rho(\eta + \eta^{-1}) = \omega\eta + \omega^2\eta^{-1}$ and $\rho(\omega\eta + \omega^2\eta^{-1}) = \omega^2\eta + \omega\eta^{-1}$. Since $(\eta + \eta^{-1})^2 = \eta + \eta^{-1} + 2$, the algebra L is isomorphic to $F \times F \times F$. Again (L, ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra such that $L \subset A$ and $ex = \rho(x)e$ for all $x \in L$.
In both cases we obtain that $A = \bigoplus_{i=0}^{2} Le^i$ where the multiplication in A is determined by $e^3 = a \in F^\times$ and $ex = \rho(x)e$ for all $x \in L$.

To finish the description of the pair (A, V), we shall write ev in function of e and η:

$$\begin{aligned}ev &= e\Big(\frac{\mathsf{T}(uv)}{3} + \frac{1}{t^3}e^2\Big)^{-1}\eta \\ &= \frac{1}{u^3v^3}(\alpha + a^{-1}\alpha^2e + e^2)\eta\end{aligned}$$

where $\alpha := -\mathsf{T}(uv)t^3/3$. Hence $V_{F(\omega)}$ is spanned by e, $(\alpha + a^{-1}\alpha^2e + e^2)\eta$ and $(\alpha + a^{-1}\alpha^2e + e^2)\lambda\eta^{-1}$. Note that $\alpha^3 \neq a^2$ since ev is invertible. We obtain that

$$V = \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2e + e^2)\theta, (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)\rangle$$

where $\theta = \eta + \lambda\eta^{-1} \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$.

We shall prove that the cubic form $f_{A,V}$ is isometric to the form

$$\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K \colon K \to F \colon x \mapsto \mathsf{Tr}_K(\beta x^3) - 3b\mathsf{N}_K(x)$$

for $K = F \times F(\omega)$ and, for some $\beta \in K^\times$ and $b \in F$. We set

$$v_1 := e, \;\; v_2 := (\alpha + a^{-1}\alpha^2e + e^2)\theta, \;\; v_3 := (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)$$

so that V is the vector space spanned by v_1, v_2, v_3. Then $f_{A,V}(xv_1 + yv_2 + zv_3)$ is equal to

$$\Big(xu_1 + \big(y + \omega z\big)u_2 + \big(y + \omega^2 z\big)u_3\Big)^3$$

where $u_1 := e$, $u_2 := (\alpha + a^{-1}\alpha^2 e + e^2)\eta$, $u_3 := (\alpha + a^{-1}\alpha^2 e + e^2)\lambda\eta^{-1}$. But $(xu_1 + yu_2 + zu_3)^3$ is equal to

$$ax^3 + \eta^3(a^{-1}\alpha^3 - a)^2 y^3 + \sigma(\eta^3)(a^{-1}\alpha^3 - a)^2 z^3 - 3\lambda\alpha(a^{-1}\alpha^3 - a)xyz.$$

Therefore $f_{A,V}$ is isometric to the form $\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K$ where $K := F \times F(\omega)$, $\beta := \big(a, \eta^3(a^{-1}\alpha^3 - a)^2\big)$ and $b := \lambda\alpha(a^{-1}\alpha^3 - a)$.

Observe that if (L, ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, then there exist $\lambda \in F^\times$ and $\phi \in L \otimes_F F(\omega)$ such that $\phi^3 \in F(\omega)^\times$, $\phi \notin F(\omega)$, $\sigma(\phi^3) = \lambda^3\phi^{-3}$ and

$$L = F \cdot 1 + F \cdot (\phi + \lambda\phi^{-1}) + F \cdot (\omega\phi + \omega^2\lambda\phi^{-1}).$$

Indeed, suppose that (L, ρ) is not split, then $L(\omega) \cong L \otimes_F F(\omega)$ is a Galois extension of F with a Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Let $\tilde{\sigma}, \tau$ be F-automorphisms of $L(\omega)$ such that L (resp. $F(\omega)$) is the subfield of $L(\omega)$ which is fixed under $\tilde{\sigma}$ (resp. τ). Let $\phi \in L(\omega)$ be such that $\phi^3 \in F(\omega)^\times$ and $\phi \notin F(\omega)$. Replacing ϕ by ϕ^{-1} if necessary, we may assume that $\tau(\phi) = \omega\phi$. Hence

$$\tau\tilde{\sigma}(\phi) = \tilde{\sigma}\tau(\phi) = \tilde{\sigma}(\omega\phi) = \omega^2\tilde{\sigma}(\phi).$$

Thus $\tilde{\sigma}(\phi) = \lambda\phi^{-1}$ for some $\lambda \in F(\omega)^\times$. But

$$\phi = \tilde{\sigma}(\lambda\phi^{-1}) = \tilde{\sigma}(\lambda)\lambda^{-1}\phi,$$

so $\tilde{\sigma}(\lambda) = \lambda$ and $\lambda \in F^\times$. Then we have $L = F(\phi + \lambda\phi^{-1})$ with

$$\sigma(\phi^3) = \big(\tilde{\sigma}(\phi)\big)^3 = (\lambda\phi^{-1})^3 = \lambda^3\phi^{-3}.$$

Suppose that $L = F \times F \times F$, then we may choose $\lambda = 1$ and $\phi = (1, \omega, \omega^2)$.

Now let (L, ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, $a \in F^\times$ and $\alpha \in F$ such that $\alpha^3 \neq a^2$. Set

$$(B, W) := \Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2 e + e^2)\theta, (\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)\rangle\Big),$$

where the multiplication in B is defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Then one can check that (B, W) is a cubic pair over F and $f_{B,W}$ is nonsingular.

We summarize this subsection by the following Theorem:

Theorem 2.1 *Suppose that F does not contain a primitive cube root of unity. Let (A, V) be a cubic pair over F such that $f_{A,V}$ is nonsingular. Then (A, V) is isomorphic to*

$$\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2 e + e^2)\theta, (\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)\rangle\Big)$$

for some Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L, ρ), $a \in F^\times$, $\alpha \in F$ such that $\alpha^3 \neq a^2$, where $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$ and $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Conversely, let (L, ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, $a \in F^\times$, $\alpha \in F$ such that $\alpha^3 \neq a^2$. Let $B = \bigoplus_{i=0}^{2} Le^i$ be the algebra with multiplication defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and let W be the subspace spanned by e, $(\alpha + a^{-1}\alpha^2 e + e^2)\theta$, $(\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)$, where $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is nonsingular. Let $\phi \in L \otimes_F F(\omega)$ and $\lambda \in F^\times$ be such that $\phi^3 \in F(\omega)^\times$, $\phi \notin F(\omega)$, $\sigma(\phi^3) = \lambda\phi^{-3}$ and $1, \phi + \lambda\phi^{-1}, \omega\phi + \omega^2\lambda\phi^{-1}$ span L, where σ is the nontrivial F-automorphism of $F(\omega)$. Then $f_{B,W}$ is isometric to $\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K$ where $K = F \times F(\omega)$, $\beta = \big(a, \phi^3(a^{-1}\alpha^3 - a)^2\big)$ and $b = \lambda\alpha(a^{-1}\alpha^3 - a)$.

2.2. Triangular form

Let (A, V) be a cubic pair over F such that $f_{A,V}$ is triangular. By Subsection 1.2, there exists $v \in A^0_{F(\omega)}$ such that $\mathsf{T}(v^2) = 0$, $v \in A^\times_{F(\omega)}$ and

$$V = \big(v^2 \star A^0_{F(\omega)}\big) \cap \Big(A^0_{F(\omega)} \star v\Big);$$

then $V = \{x \in A_{F(\omega)} \mid vx = \omega^2 xv\}$. Fix $e \in V$, then $ev = \omega ve$. We extend the F-automorphism σ of $F(\omega)$ to $A_{F(\omega)}$. Then $\sigma(v) = \lambda v^2$ for some $\lambda \in F(\omega)^\times$. Since

$$v = \sigma(\lambda v^2) = \sigma(\lambda)\lambda^2 v^4,$$

we deduce that $\sigma(\lambda)\lambda^2 v^3 = 1$. Hence

$$\sigma(\lambda v) = \sigma(\lambda)\sigma(v) = \lambda^{-2}v^{-3}\lambda v^2 = (\lambda v)^{-1};$$

so we may assume that $\sigma(v) = v^{-1}$. Set

$$L := F \cdot 1 + F \cdot (v + v^{-1}) + F \cdot (\omega v + \omega^2 v^{-1})$$

and define ρ as the F-automorphism of L such that $\rho(v + v^{-1}) = \omega v + \omega^2 v^{-1}$ and $\rho(\omega v + \omega^2 v^{-1}) = \omega^2 v + \omega v^{-1}$. Then (L, ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (note that (L, ρ) is split if and only if $v^3 \in F(\omega)^{\times 3}$). Moreover $L \subset A$, $ex = \rho(x)e$ for all $x \in L$, $A = \bigoplus_{i=0}^{2} Le^i$ and $V = eL$. It is easy to check that $f_{A,V}$ is isometric to $a\mathsf{N}_L$, where $a := e^3$.

Conversely, let (L, ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra and $a \in F^\times$. Set

$$(B, W) := \Big(\bigoplus_{i=0}^{2} Le^i, eL\Big)$$

where the multiplication in B is defined by $e^3 = a$ and $ex = \rho(x)e$ for all $x \in L$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular.

Thus we obtain:

Theorem 2.2 *Suppose that F does not contain a primitive cube root of unity. Let (A, V) be a cubic pair over F such that $f_{A,V}$ is triangular. Then (A, V) is isomorphic to*

$$\Big(\bigoplus_{i=0}^{2} Le^i, eL\Big)$$

for some Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L, ρ) and $a \in F^\times$, where $e^3 = a$ and $ex = \rho(x)e$ for all $x \in L$. Conversely, let (L, ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra and $a \in F^\times$. Let $B = \bigoplus_{i=0}^{2} Le^i$ be the algebra with multiplication defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and set $W := eL$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular. Moreover $f_{B,W}$ is isometric to $a\mathsf{N}_L$.

3. The form determines the algebra

Let (A, V) and (A', V') be cubic pairs over F and suppose that $f_{A,V}$ and $f_{A',V'}$ are isometric. In this section we shall prove that A and A' are either isomorphic or anti-isomorphic.

We may assume that F contains a primitive cube root of unity. Indeed, if $A \otimes_F F(\omega) \cong A' \otimes_F F(\omega)$, then $A \cong A'$ since A and A' are central simple algebras of degree 3 and $F(\omega)/F$ is an extension of degree at most 2. We may also assume that A is division, because there is nothing to prove if A and A' are split. Therefore, by Theorem 1.7, the cubic form $f_{A,V}$ is either nonsingular or triangular.

First case: Suppose that $f_{A,V}$ is nonsingular, then so is $f_{A',V'}$. By Theorem 1.7, there exist $a_i, a_i' \in F^\times$, $\lambda, \lambda' \in F$ such that $1 + \lambda^3 a_1, 1 + \lambda'^3 a_1' \neq 0$ and

$$A = (a_1, a_2)_{\omega,F}, \quad V = \mathsf{span}_F\langle \xi_1, \xi_2, \xi_1\xi_2^2 + \lambda\xi_1^2\xi_2^2\rangle$$
$$A' = (a_1', a_2')_{\omega,F}, \quad V' = \mathsf{span}_F\langle \xi_1', \xi_2', \xi_1'\xi_2'^2 + \lambda'\xi_1'^2\xi_2'^2\rangle$$

where A (resp. A') is generated by ξ_1, ξ_2 such that $\xi_i^3 = a_i$ and $\xi_1\xi_2 = \omega\xi_2\xi_1$ (resp. ξ_1', ξ_2' such that $\xi_i'^3 = a_i'$ and $\xi_1'\xi_2' = \omega\xi_2'\xi_1'$). Set

$$\xi_3 := \xi_1\xi_2^2 + \lambda\xi_1^2\xi_2^2, \;\; a_3 := \xi_3^3, \;\; \xi_3' := \xi_1'\xi_2'^2 + \lambda'\xi_1'^2\xi_2'^2, \;\; a_3' := \xi_3'^3.$$

We recall properties of nonsingular cubic forms and we refer to [Brieskorn and Knörrer, 1986] or [Hirschfeld, 1979] for more details. A nonsingular cubic form f on a 3-dimensional vector space V has 9 inflexion points in the projective plane $\mathbb{P}_V(F_s)$. There are four triangles (i.e. cubic curves associated with triangular cubic forms) in $\mathbb{P}_V(F_s)$ with the property that each inflexion point is incident with one and only one line of the triangle and each line of the triangle passes through exactly 3 inflexion points. These triangles are called *inflexional triangles* of f. For a triangular cubic form $g = \varphi_1\varphi_2\varphi_3$ over V, we denote by $g = 0$ the triangle formed by the zeros of the linear forms φ_i in $\mathbb{P}_V(F_s)$.

The map $V \to K := F \times F \times F$ which sends ξ_1, ξ_2, ξ_3 on the canonical basis of K is an F-vector space isomorphism. Under this isomorphism, $f_{A,V}$ is isometric to the form

$$\mathsf{Tr}_{K,\alpha} - 3b\mathsf{N}_K \colon K \to F \colon x \mapsto \mathsf{Tr}_K(\alpha x^3) - 3b\mathsf{N}_K(x)$$

where $\alpha = (a_1, a_2, a_3)$ and $b = \omega^2 a_1 a_2 \lambda$. The inflexional triangles of the form $\mathsf{Tr}_{K,\alpha} - 3b\mathsf{N}_K$ are $\mathsf{N}_K = 0$ and $\mathsf{Tr}_{K,\alpha} - 3\theta\mathsf{N}_K = 0$ for all $\theta \in F_s$ such that $\theta^3 = \mathsf{N}_K(\alpha)$. Let $(\varphi_1, \varphi_2, \varphi_3)$ denote the dual basis of (ξ_1, ξ_2, ξ_3), then under the previous isomorphism $V \to K$, the form $\varphi_1\varphi_2\varphi_3$ is isometric to N_K. Hence Γ acts trivially on the lines of the corresponding inflexional triangle. In fact, we have the following:

Lemma 3.1 *There exists a unique inflexional triangle of $f_{A,V}$ whose lines are defined over F.*

Proof: Suppose that Γ acts trivially on the lines of $\mathsf{Tr}_{K,\alpha} - 3\theta\mathsf{N}_K = 0$, for some $\theta \in F_s$ such that $\theta^3 = \mathsf{N}_K(\alpha)$. For $x = (x_1, x_2, x_3) \in K$, $\mathsf{Tr}_K(\alpha x^3) - 3\theta\mathsf{N}_K(x)$ is equal to

$$(\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3)(\theta_1 x_1 + \omega\theta_2 x_2 + \omega^2\theta_3 x_3)(\theta_1 x_1 + \omega^2\theta_2 x_2 + \omega\theta_3 x_3)$$

for some $\theta_i \in F_s$ such that $\theta_i^3 = a_i$ and $\theta_1\theta_2\theta_3 = \theta$. Since Γ acts trivially on the line $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 = 0$, there exists a nonzero $u \in F$ such that $\theta_2 = u\theta_1$. This implies that $a_2 = u^3 a_1$, which contradicts the assumption that A is division. $\square$

Let $(\varphi_1', \varphi_2', \varphi_3')$ be the dual basis of (ξ_1', ξ_2', ξ_3'). Then $\varphi_1'\varphi_2'\varphi_3' = 0$ is an inflexional triangle of $f_{A',V'}$ whose lines are defined over F. Let $\Theta \colon V \to V'$ be an F-vector space isomorphism such that $f_{A,V} = f_{A',V'} \circ \Theta$, then

$$\varphi_1\varphi_2\varphi_3 F = (\varphi_1'\varphi_2'\varphi_3' \circ \Theta)F.$$

Thus there exist $\lambda_i \in F^\times$ and a permutation π of $\{1, 2, 3\}$ such that

$$\varphi_i' \circ \Theta = \lambda_{\pi(i)}\varphi_{\pi(i)} \qquad \text{for all } i \in \{1, 2, 3\}.$$

For all $i, j \in \{1,2,3\}$, we have

$$\varphi'_i \circ \Theta(\xi_{\pi(j)}) = \lambda_{\pi(i)}\varphi_{\pi(i)}(\xi_{\pi(j)}) = \delta_{ij}\lambda_{\pi(i)},$$

hence $\Theta(\xi_{\pi(j)}) = \lambda_{\pi(j)}\xi'_j$. We obtain that $a_{\pi(j)} = \lambda^3_{\pi(j)}a'_j$ and $b = \lambda_1\lambda_2\lambda_3 b'$ where $b' = \omega^2 a'_1 a'_2 \lambda'$. But a_1 is the only scalar among the a_i's such that

$$\frac{a_1a_2a_3 - b^3}{a_i^2} \in F^{\times 3}.$$

Indeed $a_1a_2a_3 - b^3 = a_1^2a_2^3$, thus $a_1^{-2}(a_1a_2a_3 - b^3) \in F^{\times 3}$; if $a_2^{-2}(a_1a_2a_3 - b^3) \in F^{\times 3}$, then $a_1F^{\times 3} = a_2F^{\times 3}$ and it contradicts the assumption that A is division; similarly, $a_3^{-2}(a_1a_2a_3 - b^3) \notin F^{\times 3}$. On the other hand, we have $a_1'^{-2}(a'_1a'_2a'_3 - b'^3) \in F^{\times 3}$ and

$$\frac{a_1a_2a_3 - b^3}{a_{\pi(1)^2}} = \left(\frac{\lambda_1\lambda_2\lambda_3}{\lambda^2_{\pi(1)}}\right)^3 \frac{a'_1a'_2a'_3 - b'^3}{a_1'^2} \in F^{\times 3};$$

therefore $\pi(1) = 1$. If $\pi(2) = 2$, then

$$(\lambda_i\xi'_i)^3 = a_i \qquad \text{and} \qquad (\lambda_1\xi'_1)(\lambda_2\xi'_2) = \omega(\lambda_2\xi'_2)(\lambda_1\xi'_1),$$

thus $A' \cong A$. If $\pi(2) = 3$, then

$$(\lambda_1\xi'_1)^3 = a_1, \;\; (\lambda_2\xi'_3)^3 = a_2 \quad \text{and} \quad (\lambda_1\xi'_1)(\lambda_2\xi'_3) = \omega^2(\lambda_2\xi'_3)(\lambda_1\xi'_1),$$

thus $A' \cong A^{\mathsf{op}}$.

Second case: Suppose that $f_{A,V}$ is triangular. Then there exist $a_i, a'_i \in F^\times$ such that

$$A = (a_1, a_2)_{\omega,F}, \quad V = \mathsf{span}_F\langle \xi_1, \xi_2, \xi_1^2\xi_2^2\rangle$$
$$A' = (a'_1, a'_2)_{\omega,F}, \quad V' = \mathsf{span}_F\langle \xi'_1, \xi'_2, \xi_1'^2\xi_2'^2\rangle$$

where A (resp. A') is generated by ξ_1, ξ_2 such that $\xi_i^3 = a_i$ and $\xi_1\xi_2 = \omega\xi_2\xi_1$ (resp. ξ'_1, ξ'_2 such that $\xi_i'^3 = a'_i$ and $\xi'_1\xi'_2 = \omega\xi'_2\xi'_1$). Let $\theta \in F_s$ be a cube root of $a_1^{-1}a_2$. Since A is division, $\theta \notin F$. We have

$$f_{A,V}(x_1\xi_1 + x_2\xi_2 + x_3\xi_1^2\xi_2^2) = a_1\mathsf{N}_{F(\theta)}(x_1 + x_2\theta + \omega^2a_1x_3\theta^2).$$

Let $\theta' \in F_s$ be a cube root of $a_1'^{-1}a'_2$. Similarly, $f_{A',V'}(x_1\xi'_1 + x_2\xi'_2 + x_3\xi_1'^2\xi_2'^2)$ is equal to

$$a'_1(x_1 + x_2\theta' + \omega^2a'_1x_3\theta'^2)(x_1 + \omega x_2\theta' + \omega a'_1x_3\theta'^2)(x_1 + \omega^2x_2\theta' + a'_1x_3\theta'^2).$$

Since $f_{A,V} \cong f_{A',V'}$, we have $\theta' \notin F$ and, by Proposition 8 in [Raczek and Tignol, 2008], the fields $F(\theta)$ and $F(\theta')$ are isomorphic. We deduce that either

$\theta' F = \theta F$ or $\theta' F = \theta^2 F$. Identifying θ with $\xi_1^{-1}\xi_2$ (resp. θ' with $\xi_1'^{-1}\xi_2'$), we have

$$A = \bigoplus_{i=0}^{2} F(\theta)\xi_1^i \quad \text{and} \quad A' = \bigoplus_{i=0}^{2} F(\theta')\xi_1'^i$$

where $\xi_1\theta = \omega\theta\xi_1$ and $\xi_1'\theta' = \omega\theta'\xi_1'$. Because $f_{A,V}$ is isometric to $f_{A',V'}$, there exist $u_i \in F$ such that

$$a_1 \mathsf{N}_{F(\theta)}(u_1 + u_2\theta + u_3\theta^2) = a_1'.$$

Set $\eta_1 := \xi_1(u_1 + u_2\theta + u_3\theta^2)$, then

$$\eta_1^3 = a_1 \mathsf{N}_{F(\theta)}(u_1 + u_2\theta + u_3\theta^2) = a_1' \quad \text{and} \quad \eta_1\theta = \omega\theta\eta_1.$$

Hence $A = \bigoplus_{i=0}^{2} F(\theta)\eta_1^i$ with $\eta_1^3 = a_1'$ and $\eta_1\theta = \omega\theta\eta_1$. So

$$A \cong \begin{cases} A' & \text{if } \theta' F = \theta F, \\ A'^{\mathsf{op}} & \text{if } \theta' F = \theta^2 F. \end{cases}$$

We thus obtain the following:

Theorem 3.2 *Let (A, V) and (A', V') be cubic pairs over F. Suppose that $f_{A,V}$ and $f_{A',V'}$ are isometric, then the algebras A and A' are either isomorphic or anti-isomorphic.*

References

[1] [Egbert Brieskorn and Horst Knörrer, 1986] *Plane algebraic curves*, Birkhäuser Verlag, Basel.

[2] [Claude Chevalley, 1997] *The algebraic theory of spinors and Clifford algebras*, Springer-Verlag, Berlin.

[3] [James William Peter Hirschfeld, 1979] *Projective geometries over finite fields*, The Clarendon Press Oxford University Press, New York.

[4] [Tsit Yuen Lam, 2005], *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI.

[5] [Eli Matzri, 2008] Totally isotropic subspaces in degree 3 symbol algebras, preprint.

[6] [Mélanie Raczek, 2007] *Ternary cubic forms and central simple algebras of degree 3*, PhD thesis from Université Catholique de Louvain, supervised by Jean-Pierre Tignol.

[7] [Mélanie Raczek and Jean-Pierre Tignol, 2008] Ternary cubic forms and étale algebras, to appear in *Enseign. Math.*

[8] [Max-Albert Knus, Alexander Merkurjev, Markus Rost and Jean-Pierre Tignol, 1998] *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI.

[9] [Frederik van der Blij and Tonny Albert Springer, 1960] Octaves and triality, *Nieuw Arch. voor Wisk.* **8**, pp. 158–169.