THE IMAGE OF THE MAP FROM GROUP COHOMOLOGY TO GALOIS COHOMOLOGY

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Abstract. We study the image of the natural map from group cohomology to Galois cohomology by using motivic cohomology of classifying spaces.

1. Introduction

Let k be a field of $ch(k)=0$, which contains a p-th root of unity. Let G be a split affine algebraic group over k and W a faithful representation of G. Then G acts also on the function field $k(W)$. Let $k(W)^G$ be the invariant filed. Then we have the natural quotient map of groups $q: Gal(\bar{k}(\bar{W})/k(W)^G)\to G$. This induces the map of cohomologies

$$q^*: H^*(G;\mathbb{Z}/p)\to H^*(k(W)^G;\mathbb{Z}/p).$$

The purpose of this paper is to study the image $Im(q^*)$ by using the motivic cohomology defined by Suslin and Voevodsky [Vo1,3]. The image $Im(q^*)$ is called the *stable* cohomology in [Bo], [Bo-Pe-Ts]. The kernel $Ker(q^*)=Ng$ is called the (geometric) negligible ideal [Pe],[Sa].

Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology. Let $0\neq\tau\in H^{0,1}(Spec(k);\mathbb{Z}/p)\cong\mathbb{Z}/p$. Using affirmative solution of the Bloch-Kato conjecture by Voevodsky (and hence Beilinson-Lichtenbaum conjecture), the map q^* is decomposed as

$$q^*: H^{*,*}(BG;\mathbb{Z}/p)\to H^{*,*}(BG;\mathbb{Z}/p)/(\tau)\to H^{*,*}(Spec(k(W)^G);\mathbb{Z}/p).$$

where $H^{*,*}(BG;\mathbb{Z}/p)/(\tau)=H^{*,*}(BG;\mathbb{Z}/p)/(\tau H^{*,*-1}(BG;\mathbb{Z}/p))$ and BG is the classifying space of G defined by Totaro and Voevodsky ([To], [Vo1,4]).

By the Belinson-Lichtenbaum conjecture and the work of Bloch-Ogus [Bl-Og], we know

$$H^{*,*}(BG;\mathbb{Z}/p)/(\tau)\subset H^0_{Zar}(BG;H^*_{\mathbb{Z}/p})\subset H^*(k(W)^G;\mathbb{Z}/p).$$

1991 *Mathematics Subject Classification.* Primary 11E72, 12G05; Secondary 55R35.

Key words and phrases. cohomolgy invariant, classifying spaces, motivic cohomology.

Here $H^*_{\mathbb{Z}/p}$ is the Zarisky sheaf induced from the presheaf $H^*_{et}(V;\mathbb{Z}/p)$ for open subset V of BG. Therefore we see ([Or-Vi-Vo])

$$Im(q^*) = H^*(BG;\mathbb{Z}/p)/(Ng) \cong H^{*,*}(BG;\mathbb{Z}/p)/(\tau).$$

Note that the right hand side ring does not depend on the choice of W. We also note that the ideal (Ng) coincides the coniveau filtration $N^1H^*(BG;\mathbb{Z}/p)$ defined by Grothendieck.

In this paper we compute $Im(q^*)$ when $k = \mathbb{C}$ for abelian p-groups, symmetric group S_n, O_n, SO_n, $Spin_n$, PGL_p and exceptional groups. Extra special p-groups are also studied. For example, we see $H^*(BSpin_n;\mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2$ for all $n \geq 6$.

Recall that the cohomology invariant $Inv^*(G;\mathbb{Z}/p)$ is a ring of natural maps $H^1(F;G) \to H^*(F;\mathbb{Z}/p)$ for finitely generated fields F over k. This ring is very well studied for example see [Ga-Me-Se]. In particular, it is very useful to compute the essential dimension $ed(G)$ of G ([Re], [Br-Re-Vi]). Moreover, Totaro proved that

$$Inv^*(G;\mathbb{Z}/p) \cong H^0_{Zar}(BG;H^*_{\mathbb{Z}/p})$$

in a letter to Serre [Ga-Me-Se]. Hence $Im(q^*) \subset Inv^*(G;\mathbb{Z}/p)$. We use these results for some parts of this paper, however we also give new explanations of $Inv^*(G;\mathbb{Z}/p)$ for the case $k = \mathbb{C}$. For example,the image of (topological) Stiefel-Whitney class w_i of the map

$$H^*(BO_n;\mathbb{Z}/2) \to H^*(BO_n;\mathbb{Z}/2)/(Ng) \subset Inv^*(O_n;\mathbb{Z}/2)$$

is indeed the Stiefel-Whitney class w_i defined by Milnor and Serre as the natural function from quadratic forms to Milnor K-theories.

All examples stated above are detected by abelian p-subgroups A of G, i.e., the restriction map

$$Res : H^*(BG;\mathbb{Z}/p)/(Ng) \to \Pi_A H^*(BA;\mathbb{Z}/p)/(Ng)$$

is injective. (Indeed, most of the above cases are detected by only one elemenary abelian p-subgroup.)

Of course this detected property does not hold for general G. However to give examples is not so easy. Indeed, for a p-group G of exponent p, if $H^2(BG;\mathbb{Z}/p)/(Ng)$ is not detected by any $A \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then G is a counter example of the Noether's problem, namely, $k(W)^G$ is not purely transcendental over k. The examples of Saltman and Bogomolov are essentially of these types [Sa].

For each $n > 1$, we give an example G_n of a p-group $p \geq 3$, such that $H^{2n}(G_n;\mathbb{Z}/p)$ is not detected by abelian p-subgroups, while it does not implies a counter example of Noether's problem. Here the composition $Q_{2n-2}...Q_0$ of Milnor operations is used to see $x \notin Ng$ given $x \in H^{2n}(BG;\mathbb{Z}/p)$.

The second author learned theories of cohomology invariants from the discussions with Burt Totaro and Patrick Brosnan. Discussions with Alexander Vishik, Zinovy Reichstein, Kirill Zainoulline, Nikita Semenov, Masaharu Kaneda, Charles Vial and Dinesh Deshpande are very helpful to write this paper. The authors thank them very much.

2. MOTIVIC COHOMOLOGY

Let X be a smooth (quasi projective) variety. Let $H^{*,*'}(X;\mathbb{Z}/p)$ be the *mod*(p) motivic cohomology defined by Voevodsky and Suslin.

Recall that the (*mod* p) $B(n,p)$ condition holds if

$$H^{m,n}(X;\mathbb{Z}/p)\cong H^m_{et}(X;\mu_p^{\otimes n}) \qquad for\ all\ m\le n.$$

It is known that the $B(n,p)$ condition holds for $p=2$ or $n=2$ by Voevodsky([Vo1,2]), and Merkurjev-Suslin respectively. Quite recently M.Rost and V.Voevodsky ([Vo5],[Su-Jo],[Ro]) announced that $B(n,p)$ condition holds for each p and n. Hence the Bloch-Kato conjecture also holds. Therefore in this paper, we *always assume* the $B(n,p)$-condition and so the Bloch-Kato conjecture for all n,p.

Moreover we always assume that k contains a primitive p-th *root of unity*. For these cases, we see the isomorphism $H^m_{et}(X;\mu_p^{\otimes n})\cong H^m_{et}(X;\mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k);\mathbb{Z}/p)\cong\mathbb{Z}/p$. Hence

$$colim_i\tau^iH^{*,*'}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p).$$

Recall that $\mathbb{Z}/p(n)$ ([Vo1,2,3]) is the complex of sheaves in Zarisky topology such that $H^{m,n}(X;\mathbb{Z}/p)\cong H^m_{Zar}(X;\mathbb{Z}/p(n))$. Let α be the obvious map of sites from etale topology to Zarisky topology so that

$$H^m_{et}(X;\mathbb{Z}/p)\cong H^m_{et}(X;\mu_p^{\otimes n})\cong H^m_{Zar}(X,R\alpha_*\alpha^*\mathbb{Z}/p(n)).$$

For $k\le n$, let $\tau_{\le k}R\alpha_*\alpha^*\mathbb{Z}/p(n)$ be the canonical truncation of $R\alpha_*\alpha^*\mathbb{Z}/p(n)$ of level k. Then we have the short exact sequence of sheaves

$$\tau_{\le n-1}R\alpha_*\alpha^*\mathbb{Z}/p(n)\to\tau_{\le n}R\alpha_*\alpha^*\mathbb{Z}/p(n)\to H^n_{\mathbb{Z}/p}[-n]$$

where $H^n_{\mathbb{Z}/p}$ is the Zarisky sheaf induced from the presheaf $H^n_{et}(V;\mathbb{Z}/p)$ for open subset V of X. The Beilinson and Lichtenbaum conjecture (hence $B(n,p)$-condition) (see [Vo2,5]) implies

$$\mathbb{Z}/p(k)\cong\tau_{\le k}R\alpha_*\alpha^*\mathbb{Z}/p(n).$$

Hence we have ;

Lemma 2.1. *([Or-Vi-Vo], [Vo5]) There is the long exact sequence*

$$\to H^{m,n-1}(X;\mathbb{Z}/p)\xrightarrow{\times\tau}H^{m,n}(X;\mathbb{Z}/p)$$
$$\to H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})\to H^{m+1,n-1}(X;\mathbb{Z}/p)\to.$$

In particular, we have

Corollary 2.2. *The graded ring* $grH^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})$ *is isomorphic to*

$$H^{m,n}(X;\mathbb{Z}/p)/(\tau)\oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/p)$$

where $H^{m,n}(X;\mathbb{Z}/p)/(\tau)=H^{m,n}(X;\mathbb{Z}/p)/(\tau H^{m,n-1}(X;\mathbb{Z}/p))$.

Note that the above long exact sequence induces the τ-Bockstein spectral sequence

$$E(\tau)_1=H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p})\Longrightarrow colim_i\tau^iH^{*,*'}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p).$$

On the other hand, the filtration *coniveau* is given by

$$N^cH^m_{et}(X;\mathbb{Z}/p)=\cup_Z Ker\{H^m_{et}(X;\mathbb{Z}/p)\to H^m_{et}(X-Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim=c$. Grothendieck wrote down the E_1-term of the spectral sequence induced from the above coniveau filtration.

$$E(c)^{c,m-c}_1\cong\Pi_{x\in X^{(c)}}H^{m-c}_{et}(k(x);\mathbb{Z}/p)\Longrightarrow H^m_{et}(X;\mathbb{Z}/p)$$

where $X^{(c)}$ is the set of primes of codimension c and $k(x)$ is the residue field of x. We can regard $i_{x*}H^{m-c}_{et}(k(x);\mathbb{Z}/p)$ as a constant sheaf $H^{m-c}_{et}(k(x);\mathbb{Z}/p)$ on $\{\bar{x}\}$ and extend it by zero to X. Then the differentials of the spectral sequence give us a complex on sheaves on X

(2.9) $0\to H^q_{\mathbb{Z}/p}\to\Pi_{x\in X^{(0)}}i_{x*}H^q_{et}(k(x);\mathbb{Z}/p)\to\Pi_{x\in X^{(1)}}i_{x*}H^{q-1}_{et}(k(x);\mathbb{Z}/p)$
$$\to...\to\Pi_{x\in X^{(q)}}i_{x*}H^0_{et}(k(x);\mathbb{Z}/p)\to 0.$$

Bloch-Ogus [Bl-Og] proved that the above sequence of sheaves is exact and the E_2-term is given by

$$E(c)^{c,m-c}_2\cong H^c_{Zar}(X,H^{m-c}_{\mathbb{Z}/p}).$$

In particular, we have ;

Corollary 2.3.

$$H^0_{Zar}(X;H^m_{\mathbb{Z}/p})\cong Ker\{H^m_{et}(k(X);\mathbb{Z}/p)\to\Pi_{x\in X^{(1)}}H^{m-1}_{et}(k(x);\mathbb{Z}/p)\}.$$

By Deligne (foot note (1) in Remark 6.4 in [Bl-Og]) and Paranjape (Corollary 4.4 in [Pj]), it is proven that there is an isomorphism of the coniveau spectral sequence with the Leray spectral sequence for the map α. Hence we have ;

Theorem 2.4. *(Deligne, Parajape) There is the isomorphism* $E(c)^{c,m-c}_r\cong E(\tau)^{m,m-c}_{r-1}$ *of spectral sequences. Hence the filtrations are the same* $N^cH^m_{et}(X;\mathbb{Z}/p)=F^{m,m-c}_\tau$ *where*

$$F^{m,m-c}_\tau=Im(\times\tau^c:H^{m.m-c}(X;\mathbb{Z}/p)\to H^{m,m}(X;\mathbb{Z}/p)).$$

3. COHOMOLOGY OF GROUPS

Let G be a reductive algebraic group over k acting on an affine variety W. A point $x \in W$ is called stable if the orbit Gx is closed and the stabilizer group $Stab(x)$ is a finite group. Let us write by X^s the set of stable points in X. Then X^s is an open subset of X and there is the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & W//G \\ \uparrow{\scriptstyle open} & & \uparrow{\scriptstyle open} \\ W^s & \xrightarrow{\phi^s} & W^s/G, \end{array}$$

where $W//G = Spec(k[W]^G)$, the geometric quotient of X. We also note that the invariant field $k(W)^G$ is a quotient field of $k[W]^G$ when $X^s \neq \emptyset$.

Suppose $X^s \neq \emptyset$. Then $H^{*,*'}(W//G;\mathbb{Z}/p) = H^{*,*'}(Spec(k[W]^G);\mathbb{Z}/p)$ and $k(W^s/G) = k(W//G) = k(W)^G$. Hence we have the diagram

$$\begin{array}{ccc} H^{*,*'}(W//G;\mathbb{Z}/p) & \longrightarrow & H^{*,*'}(Spec(k(W)^G);\mathbb{Z}/p) \\ \downarrow & & {\scriptstyle =}\downarrow \\ H^{*,*'}(W^s/G;\mathbb{Z}/p) & \xrightarrow{\psi} & H^{*,*'}(Spec(k(W^s/G);\mathbb{Z}/p) \end{array}$$

Restrict W as an affine space $W = \oplus k$ and let $\rho : G \to W = \oplus k$ a faithful representation. Let $U_n = W - S$ be an open set of W such that G act freely U where $codim_W S = n$. (Of course U is an open subset of W^s.) Then the classifying space of G is defined as $colim_{n\to\infty}(U_n/G)$. Then the $mod(p)$ motivic cohomology (for degree $* < 2n$) of BG is given by ([Vo4],[To])

$$H^{*,*'}(BG;\mathbb{Z}/p) \cong lim_{n\to\infty} H^{*,*'}(U_n/G;\mathbb{Z}/p).$$

In particular, by $BL(p,*)$ condition, we have

$$H^{*,*}(BG;\mathbb{Z}/p) \cong H^*_{et}(BG;\mathbb{Z}/p) = H^*(G;\mathbb{Z}/p) \otimes H^*(k;\mathbb{Z}/p)$$

where the last group is the cohomology group of the Galois group G (when G is finite). Thus from the above diagram, we have the map

$$\psi : H^{*,*'}(BG;\mathbb{Z}/p) \to H^{*,*'}(Spec(k(W)^G);\mathbb{Z}/p).$$

This map $\psi^{*,*} = \psi^*_{et}$ is explains also as follows. Let Γ is the absolute Galois group $\Gamma = Gal(k(\bar{W})/k(W)^G)$. Then the group $G = Gal(k(W)/k(W)^G)$ is a quotient group of the absolute Galois group Γ.

Then the map ψ_{et} is the induced map from the quotient $q : \Gamma \to G$, i.e.,

$$\psi_{et}^* = q^* : H^*(G;\mathbb{Z}/p) \to H^*(\Gamma;\mathbb{Z}/p) = H^*(k(W)^G;\mathbb{Z}/p).$$

Lemma 3.1.

$$Im(\psi^*) \cong H^{*,*}(BG;\mathbb{Z}/p)/(\tau).$$

Proof. For each field F, by the Bloch-Kato conjecture, $H^*(F;\mathbb{Z}/p)$ is generated by elements in $H^1_{et}(F;\mathbb{Z}/p) \cong H^{1,1}(Spec(F);\mathbb{Z}/p)$. So

$$\psi^{*,*-1} : H^{*,*-1}(BG;\mathbb{Z}/p) \to H^{*,*-1}(Spec(k(W)^G);\mathbb{Z}/p) = 0.$$

Hence the map ψ^* is expressed as a composition

$$H^{*,*}(BG;\mathbb{Z}/p) \to H^{*,*}(BG;\mathbb{Z}/p)/(\tau) \to H^*_{et}(k(W)^G;\mathbb{Z}/p).$$

The first map is of course surjective and we only need the injectivity of the second map. Indeed, from Corollary 2.2 and 2.3, we see

$$H^{*,*}(BG;\mathbb{Z}/p)/(\tau) \subset H^0_{Zar}(BG;H^*_{\mathbb{Z}/p}) \subset H^*(k(W)^G;\mathbb{Z}/p).$$

□

Recall the coniveau filtration given in §2

$$N^cH^m_{et}(X;\mathbb{Z}/p) = \cup_Z Ker\{H^m_{et}(X;\mathbb{Z}/p) \to H^m_{et}(X - Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim = c$. From Theorem 2.4, we see

Corollary 3.2. $Im(q^*) \cong H^*_{et}(BG;\mathbb{Z}/p)/(N^1H^*_{et}(BG;\mathbb{Z}/p))$.

According to Saltman (and Serre), we say an element $x \in H^*(G;\mathbb{Z}/p)$ is geometrically negligible if $\psi^*(x) = 0$. Let us write $Ng = Ng(G) = Ker(\psi^*)$. From the above lemma, it is immediate

$$Ng(G) = N^1H^*_{et}(BG;\mathbb{Z}/p)$$

$$= Im(\times\tau|H^{*,*-1}(BG;\mathbb{Z}/p) \to H^{*,*}(BG;\mathbb{Z}/p))$$

and we have

$$Im(\psi^*) = H^*(BG;\mathbb{Z}/p)/(Ng) \cong H^*(BG;\mathbb{Z}/p)/(N^1) \cong H^{*,*}(BG;\mathbb{Z}/p)/(\tau).$$

Lemma 3.3. *For each element x in $H^*_{et}(BG;\mathbb{Z}/p)$, the images of cohomology (Bockstein, reduced) operations $\beta(x)$, $P^i(x)$ for $i > 0$ are in $Ng(G)$ (hence $x^p \in Ng(G)$). The image of Gysin map $g_*(x)$ are also in $Ng(G)$.*

Proof. The cohomology operations act as ([Vo2,4])

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+1,*}(X;\mathbb{Z}/p)$$

$$P^i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2i(p-1),*+i(p-1)}(X;\mathbb{Z}/p).$$

For an element $x \in H^{m,n}(X;\mathbb{Z}/p)$, define the difference degree $d(x) = m-n$. Then if $d(x)=0$, then $d(\beta(x))>0$ and $d(P^i(x))>0$. Hence these elements are in $Im(\tau)$ as a subset of $H^{*,*}(X;\mathbb{Z}/p)$.

For the embedding $X \subset Y$ of codimension c, the Gysin map is defined on

$$g_* : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2c,*+c}(Y;\mathbb{Z}/p).$$

By the same reason as the cases of cohomology operations, we get lemma. □

Here we give a sufficient condition for $x \notin Ng(G)$. Voevodsky define the Milnor operation Q_i also in the mod p motivic cohomology

$$Q_i : H^{*,*'}(-;\mathbb{Z}/p) \to H^{*+2p^n-1,*'+p^n-1}(-;\mathbb{Z}/p).$$

Define the weight $w(x) = 2*'-*$ for element (or operation) $x \in H^{*,*'}(X;\mathbb{Z}/p)$, e.g., $w(\tau)=2$, $w(Q_i)=-1$ and $w(P^i)=0$.

Lemma 3.4. *Let $x \in H^n_{et}(BG;\mathbb{Z}/p)$ and $Q_{n-2}...Q_0(x) \neq 0$. Then $0 \neq x \in H^n_{et}(BG;\mathbb{Z}/p)/(Ng)$.*

Proof. Identify x as an element in $H^{n,n}(BG;\mathbb{Z}/2)$. Suppose that $x = \tau\bar{x}$. So $w(\bar{x}) = n-2$ since $w(\tau)=2$. Then $t_{\mathbb{C}}(\bar{x}) = t_{\mathbb{C}}(x) = x$ where $t_{\mathbb{C}} : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*,*'}(X(\mathbb{C});\mathbb{Z}/p)$ is the realization map.

The operation Q_i descends the weight one. Let $\bar{\psi} = Q_{n-2}...Q_0(\bar{x})$. Then $w(\bar{\psi}) = -1$ but $t_{\mathbb{C}}(\bar{\psi}) = Q_{n-2}...Q_0(x) \neq 0$. This is a contradiction since $w(y) \geq 0$ for each nonzero element $y \in H^{*,*'}(Y;\mathbb{Z}/p)$ and for smooth Y. □

We have the Kunneth formula for the etale cohomology of coefficient $\mathbb{Z}/p$. Since Ng is an ideal, we have the surjection

$$H^*_{et}(BG_1;\mathbb{Z}/p)/(Ng) \otimes_{H^*(k;\mathbb{Z}/p)} H^*_{et}(BG_2;\mathbb{Z}/p)/(Ng)$$

$$\to H^*_{et}(B(G_1\times G_2);\mathbb{Z}/p)/(Ng).$$

However it does not need isomorphic, because there is the possibility that $x_1 \otimes x_2 \in Ng(G_1\times G_2)$ but $x_1 \notin Ng(G_1),\ x_2 \notin Ng(G_2)$.

4. COHOMOLOGY INVARIANT

Recall that $H^1(k;G)$ is the first non abelian Galois cohomology set of G, which represents the set of G-torsors over k. It is very important to study $H^1(k;G)$, for example $H^1(k;O_n)$ is isomorphic to the set of isomorphism classes of non generate quadratic forms over k of rank n. (For details, see the excellent book [Ga-Me-Se].)

The cohomology invariant is defined by

$$Inv^i(G,\mathbb{Z}/p) = Func(H^1(F;G) \to H^i(F;\mathbb{Z}/p))$$

where $Func$ means natural functions for each fields F over k. The cohomology invariant is studied by many authors. The cohomology invariants $Inv^*(G;\mathbb{Z}/p)$ are computed (for example in [Ga-Me-Se]) for groups elementary abelian 2-groups, O_n,SO_n, G_2,... It is also stated in [Ga-Me-Se] that for many G (but not all) $Inv^*(G;\mathbb{Z}/p)$ are detected by $Inv^*(H;\mathbb{Z}/p)$ for elementary abelian p-subgroups H.

Let $x \in H^0(BG;H^i_{\mathbb{Z}/p})$. Given a G-torsor E over F, we can construct $x(E) \in H^i_{et}(F;\mathbb{Z}/p)$. Roughly speaking, we can identify E as the pullback of some map $f : Spec(F) \to BG$. So we can take $x(E) = f^*(x) \in H^0(Spec(F);H^i_{\mathbb{Z}/p}) = H^i_{et}(F;\mathbb{Z}/p)$. Indeed, Totaro proved [Ga-Me-Se] the following theorem in a letter to Serre.

Theorem 4.1. $Inv^*(G;\mathbb{Z}/p) \cong H^0(BG;H^*_{\mathbb{Z}/p})$.

Therefore we see

Corollary 4.2. $Im(\psi^*) \cong H^*(BG;\mathbb{Z}/p)/(Ng) \subset Inv^*(G;\mathbb{Z}/p)$.

5. ABELIAN p-GROUPS

Let us write $H^{*,*'} = H^{*,*'}(Spec(k);\mathbb{Z}/p)$ and $H^* = H^{*,*} = K^*_M(k)/p$ so that $H^{*,*'} \cong H^*[\tau]$. First consider the case $G = \mathbb{Z}/p^r$. The $mod(p)$ motivic cohomology is computed (as the case $\mathbb{Z}/p$ in [Vo])

$$H^{*,*'}(B\mathbb{Z}/p^r;\mathbb{Z}/p) \cong H^{*,*'}[y(r)] \otimes \Lambda(x(r)) \quad |y(r)| = 2,\ |x(r)| = 1.$$

(When $p = 2$ and $r = 1$, we see by Voevodsky ([Vo2,4])

$$x(1)^2 = \tau y(1) + \rho x(1) \quad with\ \rho = (-1) \in H^1 = k^*/(k^*)^2.)$$

For the inclusion $i : \mathbb{Z}/p^r \subset \mathbb{Z}/p^s$ and quotient map $q : \mathbb{Z}/p^s \to \mathbb{Z}/p^r$, for $s \geq r$, we have

$$i^*(y(s)) = y(r),\ i^*(x(s)) = 0, \quad q^*(y(r)) = 0,\ q^*(x(r)) = x(s).$$

Moreover we still know $x(r) \in H^{1,1}(BG;\mathbb{Z}/p)$ and $y(r) \in H^{2,1}(BG;\mathbb{Z}/p)$. Thus we see $((Ker(\tau)|H^{*,*'}(B\mathbb{Z}/p^r;\mathbb{Z}/p) = 0)$.

$$Inv^*(\mathbb{Z}/p^r;\mathbb{Z}/p) \cong H^{*,*}(\mathbb{Z}/p^r;\mathbb{Z}/p)/(\tau) = H^*\{1, x(r)\}.$$

Next consider their product $G = \mathbb{Z}/p^{r_1} \times ... \times \mathbb{Z}/p^{r_s}$. The cohomology $H^{*,*'}(B\mathbb{Z}/p^r;\mathbb{Z}/p)$ has the Kunneth formula. Hence the motivic cohomology is given

$$H^{*,*'}(BG;\mathbb{Z}/p) \cong H^{*,*'}[y(r_1),...,y(r_s)] \otimes \Lambda(x(r_1),..,x(r_s))$$

where $x(r_i) \in H^{1,1}(BG;\mathbb{Z}/p)$ and $y(r_i) \in H^{2,1}(BG;\mathbb{Z}/p)$.

Recall that $H^*(BG;\mathbb{Z}/p)/(Ng) \cong H^{*,*}(BG;\mathbb{Z}/p)/(\tau)$ (Lemma 3.1). Then we get

Lemma 5.1. *Let G be an abelian p-group, i.e., $G = \oplus_i \mathbb{Z}/(p^{r_i})$. Then*

$$Inv^*(G;\mathbb{Z}/p) \cong H^*(G;\mathbb{Z}/p)/(Ng) \cong H^* \otimes \Lambda(x(r_1),...,x(r_s))$$

$$when\ p = 2\ r_i = 1, \quad x(r_i)^2 = \rho x(r_i).$$

The elementary 2-groups cases are stated in Theorem 16.4 in [Ga-Me-Se].

The Q_i-operation acts on $H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p)$ by $Q_i(x) = y^{p^i}$ (while $Q_i(x(j)) = 0$ for all $j > 1$). We consider Q_i action on

$$H^{*,*'}(B(\mathbb{Z}/p)^s;\mathbb{Z}/p) \cong H^{*,*'}[y_1,...,y_s] \otimes \Lambda(x_1,...,x_s).$$

Each Q_i is a derivation, and hence

$$Q_0...Q_{s-1}(x_1...x_s) = \sum sgn(j_1,...,j_s) y_1^{p^{j_1}} y_2^{p^{j_2}} ... y_s^{p^{j_s}} \neq 0$$

where $(j_1,...,j_s)$ are permutations of $(0,...,s-1)$. Thus we see that this case satisfies the sufficient condition of Lemma 3.3 while the other cases does not), in fact $x_1...x_s \notin Ng(G)$.

Let us say that an element $x \in H^*(BG;\mathbb{Z}/p)/(Ng)$ is *detected* by an elemntary abelian p-subgroup A if $Res(x) \neq 0$ for

$$Res : H^*(BG;\mathbb{Z}/p)/(Ng) \to H^*(BA;\mathbb{Z}/p)/(Ng).$$

The following lemma is immediate from the above arguments.

Lemma 5.2. *If $x \in H^n(BG;\mathbb{Z}/p)/(Ng)$ is detected by elementary abelian p-subgroups, then $Q_{n-1}...Q_0(x) \neq 0$ in $H^*(BG;\mathbb{Z}/p)$.*

6. Cases $G = O_n$ and SO_n

Hereafter, in this paper (except for §11), we assume that $k = \mathbb{C}$ otherwise stated.

lt is well known that

$$H^*(BO_n;\mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^{\times n};\mathbb{Z}/p)^{S_n} \cong \mathbb{Z}/2[w_1,...,w_n]$$

where S_n is the n-th symmetric group, and w_i is the Stiefel-Whitney class representing the i-the elementary symmetric function. we easily see

$Q_{i-1}...Q_0(w_i) = y_1^{p^{i-1}} y_2^{p^{p-2}} ...y_i + ... \neq 0 \in \mathbb{Z}/2[y_1, ..., y_n] \subset H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2)$

and hence $w_i \notin Ng(G)$. Recall the Wu formula

$$Sq^i w_k = \sum_j^i \begin{pmatrix} k-j-1 \\ i-j \end{pmatrix} w_{k+i-j} w_j \quad (0 \leq i \leq k).$$

Many cases of product of $w_i w_j$ are in $Ng(G)$, e.g.,$w_i^2 \in Ng(G)$. More precisely, the motivic cohomology of BO_n is computed for $k = \mathbb{C}$ (Theorem 8.1 in [Ya3])

$$H^{*,*'}(BO_n; \mathbb{Z}/2) \cong H^{*,*'}((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/p)^{S_n}.$$

Given $x \in H^*(BG; \mathbb{Z}/p)$, let us define the weight $w(x)$ as the smallest weight $w(x')$ such that $t_{\mathbb{C}}(x') = x$ with $x' \in H^{*,*'}(BG; \mathbb{Z}/p)$. Indeed, the weight of the symmetric polynomial

$$t = \sum x_1^{2i_1+1} ... x_k^{2k+1} x_{k+1}^{2j_1} ... x_{k+q}^{2j_q} \quad in\ H^*((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/2)$$

(with $0 \leq i_1 \leq ... \leq i_k$, $0 \leq j_1 \leq ... \leq j_q$) is given by $w(t) = k$. Hence if $t \notin Ng$, then $w(t) = deg(t)$ and this implies $t = x_1 ... x_i = w_i$.

Theorem 6.1.

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_n\}.$$

In fact, $Inv(O_n; \mathbb{Z}/2)$ is well known (Theorem 17.3 in [Ga-Me-Se]) for general k ;

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*\{1, w_1, ..., w_n\}.$$

We consider the multiplicative structure of $Inv^*(O_n; \mathbb{Z}/2)$. From the Wu formula, we see

$$Sq^1(w_{2i}) = w_{2i+1} + w_{2i} w_1 \in Ng(O_n).$$

Hence $w_{2i+1} = w_{2i} w_1$ in $Inv^*(O_n; \mathbb{Z}/2)$. By Rost and Kahn [Ka], the divided power operation can be defined in $K_*^M(F)/p$ compatible with fields F over k (and hence $Inv^*(G; \mathbb{Z}/p)$) if $\sqrt{-1} \in k$. Vial showed [Via] that the divided power operations are only compatible maps (natural maps) with field extensions over k. Moreover Becher [Be] showed that $\gamma_n(w_2) = w_{2n}$. (See also Milnor p133 in [Mi].)

Theorem 6.2. *(Becher [Be], [Via]) Let $\sqrt{-1} \in k$. Then $Inv^*(O_n; \mathbb{Z}/2)$ is generated by w_1 and w_2 as an H^*-ring with divided powers by*

$$\gamma_i(w_2) = w_{2i}, \quad w_{2i+1} = w_{2i} w_1.$$

Next consider the case $G = SO_n$. It is well known that

$$H^*(BSO_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(w_1) \cong \mathbb{Z}/2[w_2, ..., w_n].$$

Let $n = 2m+1 = odd$. For this case, there is the isomorphism

$$O_{2m+1} \cong SO_{2m+1} \times \mathbb{Z}/2.$$

Let $p : O_n \to SO_n$ is the projection and $i : SO_n \to O_n$ the inclusion. We consider the induced map p^* and i^* on the mod 2 motivic cohomology of their classifying spaces. Since $p^*(w_1) = 0$, we see $w_{2i+1} \in Ng(G)$ from the above theorem (in fact, $Sq^1(w_{2i}) = w_{2i+1}$ in $H^*(BSO_n; \mathbb{Z}/2)$). Moreover $i^* w_{2i} = w_{2i} \ mod(Ng(G))$. Thus we have

Theorem 6.3. *For $G = SO_{2m+1}$, we have*

$$Inv^*(G; \mathbb{Z}/2) \cong H^*(BG; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, ..., w_{2m}\}.$$

Moreover $Inv^*(SO_{2m+1}; \mathbb{Z}/2)$ is still computed (Theorem 19.1 in [Ga-Me-Se]) for general k ;

$$Inv^*(SO_{2m+1}; \mathbb{Z}/2) \cong H^*\{1, w_2, ..., w_{2m}\}.$$

Now consider the case $n = 2m = even$. This case the mod 2 motivic cohomology is not computed even $k = \mathbb{C}$ for $n > 4$. However we compute $H^{*,*}(BSO_n; \mathbb{Z}/2)/(\tau)$ easily. Consider the inclusion

$$SO_{2m-1} \stackrel{i_1}{\to} SO_{2m} \stackrel{i_2}{\to} SO_{2m+1}.$$

Since the restriction map for $i < m$

$$i_1^*(w_{2i}) \neq 0 \in H^{*,*}(BSO_{2m-1}; \mathbb{Z}/2),$$

we see $w_{2i} \notin Ng(SO_{2m})$. Moreover we know that $w(w_n) = n-2$ in Lemma 9.2 in [Ya3]. On the other hand, the monomial w_I of not w_{2i} are all in $Ng(SO_{2m+1})$ and so $i_2^*(w_I) \in Ng(SO_{2m})$.

Thus we have

Theorem 6.4. $H^*(BSO_{2m}; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, ..., w_{2m-2}\}$.

Next we study $Inv^*(SO_{2m}; \mathbb{Z}/2)$. There is an element (Lemma 9.3 in [Ya3]) in the motivic cohomology

$$u_{m-1} \in H^{n,n-2}(BSO_n; \mathbb{Z}/2) \quad with \ \tau u_{m-1} = 0.$$

So there is the nonzero element

$$u \in Ker(\tau | H^{n,n-2}(BG; \mathbb{Z}/2)) \subset H^0(BG; H^{n-1}_{\mathbb{Z}/2}).$$

On the other hand in [Ga-Me-Su], it is proved (Theorem 20.6) for general k

$$Inv^*(SO_{2m}; \mathbb{Z}/2) \cong H^*\{1, w_2, ..., w_{n-2}\} \oplus (Im(I_\delta)).$$

Here when $k=\mathbb{C}$, $Im(I_\delta)\cong \mathbb{Z}/2\{u\}$ with $deg(u)=n-1$ from Proposition 20.1 in [Ga-Me-Se]. Thus we see

Theorem 6.5. *Let $G=SO_{2m}$ and $m\ge 2$. Then for $deg(u)=2m-1$,*

$$Inv^*(G;\mathbb{Z}/2)\cong H^*(BG;\mathbb{Z}/2)/(Ng)\oplus \mathbb{Z}/2\{u\}.$$

From 22.10 in [Ga-Me-Se], it is known that

$$Res: Inv^*(G;\mathbb{Z}/p)\to Inv^*(H;\mathbb{Z}/p)$$

is injective for $p=2$, $G=SO_n$ and $H=(\mathbb{Z}/2)^{n-1}$. Hence for

$$Res: H^0(BSO_n;H^*_{\mathbb{Z}/2})\to H^0(B(\mathbb{Z}/2)^{n-1};H^*_{\mathbb{Z}/2})$$

we have $Res(u)=x_1...x_{n-1}$. Of course $u\notin H^{*,*}(BSO_n;\mathbb{Z}/2)/(\tau)$ but $x_1...x_{n-1}\in H^{*,*}(B(\mathbb{Z}/2)^{n-1};\mathbb{Z}/2)/(\tau)$.

Recall that in Corollary 2.2 $grH^0(BG;H^*_{\mathbb{Z}/p})$ is defined by the filtration $H^{*,*}(BG;\mathbb{Z}/p)/(\tau)\subset H^0(BG;H^*_{\mathbb{Z}/p})$. So note that

$$Res: grH^0(BSO_n;H^*_{\mathbb{Z}/2})\to grH^0(B(\mathbb{Z}/2)^{n-1};H^*_{\mathbb{Z}/2})$$

is not injective (in Corollary 2.2).

7. $Spin_n$ and exceptional groups

The $mod(2)$ cohomology of $BSpin_n$ is computed by Quillen

$$H^*(BSpin_n;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_{2^h}(\Delta)]\otimes \mathbb{Z}/2[w_2,...,w_n]/(Q_iw_2|0\le i\le h)$$

where $w_i(\Delta)$ (resp. w_i) is the Stiefel-Whitney class of a spin representation Δ (resp. usual representation $Spin_n\to SO_n$), and 2^h is the the Radon-Hurwitz number (See [Qu] p.210). By the result of Becher (Theorem 6.2), we have

Theorem 7.1. *$H^*(BSpin_n;\mathbb{Z}/2)/(Ng)=\mathbb{Z}/2$ for $n>4$.*

Proof. Let us write representations $j: Spin_n\to SO_n$ and $\Delta: Spin_n\to SO_N$. We consider the induced map in Galois cohomolgy

$$j^*: H^*_{et}(k(W)^{SO_n};\mathbb{Z}/2)\to H^*_{et}(k(W)^{Spin_n};\mathbb{Z}/2).$$

By the Quillen's result, we see $j^*(w_2)=0$. By Rost and Kahn [Ka], the divided powers naturally act on $K_*^M(F)/p$ for field F over k. Hence from Becher theorem (Theorem 6.2),we get

$$j^*(w_{2i})=j^*(\gamma_i(w_2))=\gamma_i(j^*(w_2))=0.$$

Similarly $w_2(\Delta)=0$ implies $w_{2^h}(\Delta)=0$ if $n>4$. □

Corollary 7.2. *For $*>0$, there is the isomorphism*

$$Inv^*(Spin_n;\mathbb{Z}/2)\cong Ker(\tau)|H^{*+1,*-1}(BSpin_n;\mathbb{Z}/2).$$

The $mod(2)$ motivic cohomology of $BSpin_7$ is computed in (Theorem 9.6 in [Ya3]). We can easily see the above theorem also from the concrete computation. Moreover there are τ-torsion elements

$$y_2 \in H^{4,2}(Spin_7;\mathbb{Z}/2), \quad y_2' \in H^{5,3}(Bspin_7;\mathbb{Z}/2).$$

Therefore we can take $u \in H^0(BSpin_7, H^3_{\mathbb{Z}/2}), v \in H^0(BSpin_7, H^4_{\mathbb{Z}/2})$.

Theorem 7.3.

$$Inv^*(BSpin_7;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1,u,v\} \quad |u|=3,\ |v|=4.$$

We consider exceptional Lie group types G_2, F_4 and split E_6. In [Ga-Me-Su], it is proved (Teorem 18.1, Theorem 22.5)

$$Inv^*(G_2;\mathbb{Z}/2) \cong Inv^*(E_6;\mathbb{Z}/2) \cong H^*\{1,u\},$$
$$Inv^*(F_4;\mathbb{Z}/2) \cong H^*\{1,u,f_5\}.$$

(Unfortunately, we can not reexplain f_5 by using $H^{*,*'}(BF_4;\mathbb{Z}/2)$, which is not computed yet.)

Moreover restriction image for elementary abelin 2-subgroup of rank 3 (resp. rank 5) is injective for G_2, E_6 (resp. for F_4), see 22.10 in [Ga-Me-Se].

Theorem 7.4. *Let $G = G_2, F_4$. Then $H^*(BG;\mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.*

Proof. It is known that the inclusion $i : G_2 \to Spin_7$ induces the epimorphism $i^* : H^*(BSpin_7;\mathbb{Z}/2) \to H^*(BG_2;\mathbb{Z}/2)$. Hence the result for $G = G_2$ follows from $H^*(BSpin_7;\mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.

It is also known that the inclusion $i' : Spin_9 \to F_4$ induces the injection $i'^* : H^*(BF_4;\mathbb{Z}/2) \to H^*(BSpin_9;\mathbb{Z}/2)$. The groups $Spin_9$ and F_4 has the same maximal abelin 2-group H (of rank 5). We consider the restriction to this H. The fact $Res(H^*(BSpin_9;\mathbb{Z}/2)/(Ng)) = \mathbb{Z}/2$ implies the results for F_4, because the restriction is injective from [Ga-Me-Se]. □

Here we give an example. The $mod(2)$ cohomology is well known

$$H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7].$$

The motivic cohomology is given in Theorem 7.5 in [Ya3]. In particular, c_4, c_6, c_7 are the Chern classes (in $H^{2*,*}(BG_2;\mathbb{Z}/2)$) so that $\tau^i c_i = w_i^2$. Let us write simply by $Q(n)$ the exterior algebra $\Lambda(Q_0, ..., Q_n)$.

Theorem 7.5. *The motivic cohomology $H^{*,*'}(BG_2;\mathbb{Z}/2)$ is isomorphic to*

$$\mathbb{Z}/2[c_6,c_4]\otimes(\mathbb{Z}/2\{y_2\}\oplus\mathbb{Z}/2[\tau]\otimes(\mathbb{Z}/2\{1\}\oplus(\mathbb{Z}/2[c_7]Q(2)-\mathbb{Z}/2\{1\})\{a\})$$

where $deg(y) = (4,2)$, and a is a virtual element with $deg(a) = (3,3)$ so that $c_7 a = w_6 w_7 w_4$, $Q_0 a = w_4$, $Q_1 a = w_6$ and $Q_2 a = w_6 w_4$.

Let us write $Q_{i_1}...Q_{i_s}(a)$ by $Q_{i_1...i_s}$ and $w_{i_1}...w_{i_s}$ by $w_{i_1...i_s}$. Then $(\mathbb{Z}/2[c_7]Q(2)-\mathbb{Z}/2\{1\})\{a\}$ is written as

$$\mathbb{Z}/2[c_7]\{c_7a=w_{467}, Q_0=w_4, Q_1=w_6, Q_{01}=w_7,$$
$$Q_2=w_{46}, Q_{02}=w_{47}, Q_{12}=w_{67}, Q_{012}=c_7\}.$$

Note that all $w_{i_1...i_s}$ (which are generators of the $\mathbb{Z}/2[c_4,c_6,c_7]$-module $H^*(G_2;\mathbb{Z}/2)$) appeared indeed in the above. Moreover it is immediate that all elements except for 1 and a are in $Ng(G_2)$, and $y\in Ker(\tau)|H^{4,2}(BG_2;\mathbb{Z}/2)$.

Next we consider odd prime case. In 22.10 in [Ga-Me-Se] it is stated that the restriction image of $Inv^*(G;\mathbb{Z}/p)$ to some (maximal) elementary abelian p-group H is injective for $(G,p)=(F_4,3),(E_6,3),(E_7,3)$ or $(E_8,5)$.

In these cases, each exceptional Lie group has two conjugacy classes of maximal elementary abelian p-groups. One is the subgroup of a maximal torus and the other is a nontoral A. Let us write $i_A: A\to G$ and $i_T: T\to B$ be the inclusions. Tezuka and Kameko proves ([Ka], [Ka-Ya]) that the following map is injective

$$i_A^*\times i_T^*: H^*(BG;\mathbb{Z}/p)\to H^*(BA;\mathbb{Z}/p)\times H^*(BT;\mathbb{Z}/p),$$

namely, $H^*(BG;\mathbb{Z}/p)$ is detected by A and T. Since $H^*(BT;\mathbb{Z}/p)$ is non-nilpotent, the above group H must be A.

Theorem 7.6. *Let $(G,p)=(F_4,3),(E_6,3),(E_7,3)$ or $(E_8,5)$. Then $H^*(BG;\mathbb{Z}/p)/(Ng)=\mathbb{Z}/p$.*

Proof. The restriciton image to A is studied in [Ka-Ya]. Images are generated as a ring by Chern classes and

$$Q_I(x_1..x_s)\in H^*(BA;\mathbb{Z}/p)\cong \mathbb{Z}/p[y_1,...,y_s]\otimes\Lambda(x_1,...,x_s)$$

where $Q_I=Q_{i_1}...Q_{is}$ for some $I\neq\emptyset$. (Note $x_1...x_s\notin Im(i_A^*)$.) Hence $i_A^*(H^+(BG;\mathbb{Z}/p))\in Ng(A)$. □

Here we give an example. The mod 3 cohomology of BF_4 is completely determined by Toda.

Theorem 7.7. *([Toda]) The cohomology $H^*(BF_4;\mathbb{Z}/3)$ is isomorphic to*

$$\mathbb{Z}/3[x_{36},x_{48}]\otimes(\mathbb{Z}/3[x_4,x_8]\otimes\{1,x_{20},x_{20}^2\}+\mathbb{Z}/3[x_{26}]\otimes\Lambda(x_9)\otimes\{1,x_{20},x_{21},x_{25}\})$$

where the above two terms have the intersection $\{1,x_{20}\}$.

Indeed, we see that $x_{26}|A=Q_0Q_1Q_2(u_3)$, $x_{36}|A=c_{3,1}$, $x_{48}|A=c_{3,2}$, $x_4|A=Q_0(u_3)$, $x_8|A=Q_1(u_3)$, $x_{20}|A=Q_2(u_3)$, $x_9|A=Q_0Q_1(u_3)$, $x_{21}|A=Q_0Q_2(u_3)$, $x_{25}|A=Q_1Q_2(u_3)$. Here $u_3=x_1x_2x_3$ and x_{36} and x_{48} are represented by Chern classes.

8. $GL_n(\mathbb{F}_\ell)$ AND PGL_p

Of course, there is the isomorphism

$$H^{*,*'}(BGL_n;\mathbb{Z}/p)\cong H^{*,*'}[c_1,...,c_n]$$

where c_i is the Chern class. Hence $Inv^*(GL_n;\mathbb{Z}/p)\cong\mathbb{Z}/p$ (for $k=\mathbb{C}$). Let G be a finite group such that

$$(8.1)\quad H^*(BG;\mathbb{Z}/p)\cong\mathbb{Z}/p[c_{i_1},...,c_{i_n}]\otimes\Lambda(e_{i_1},...,e_{i_n})$$

where $\beta_{j_s}(e_{i_s})=c_{i_s}$ where β_{j_s} is the higher Bockstein operation, and c_{i_s} is a Chern class of some representation of G, e.g., $G=GL_n(\mathbb{F}_\ell)$ where ℓ is prime to p.

Theorem 8.1. *Let G be a finite group given as (8.1) so that $i_1\geq 2$. Then $H^*(BG;\mathbb{Z}/p)/(Ng)\cong\mathbb{Z}/p$.*

Proof. First note that $H^*(BG;\mathbb{Z}/p)/(Ng)$ is a quotient $\Lambda(e_{i_1},...,e_{i_n})$. Since the motivic cohomology has the transfer map, we know that each element x in $H^{*,*'}(BG;\mathbb{Z})$ has the exponent dividing $|G|$. Hence there is $e_s'\in H^{*,*-1}(BG;\mathbb{Z}/p)$ such that

$$\beta_{s'}(e_s')=\tau^{i_s-2}c_{i_s}.$$

From Lemma 2.1, we know that $\tau: H^{*,*-1}(X;\mathbb{Z}/p)\to H^{*,*}(X;\mathbb{Z}/p)$ is injective. Hence $\tau e_s'\neq 0$. This means $\beta_{s''}(\tau e_s')=\tau^{i_s-1}c_{i_s}$ for $s''\leq s'$. So $s''=i_s$ and we can take $\tau e_s'=e_{i_s}\ mod(Ideal(e_{i_1},...,e_{i_{s-1}}))$. By induction on s, we can prove all $e_{i_s}\in Ng$. □

Let p be an odd prime and denote by PGL_p the projective group which is the quotient of the general linear group GL_p by the center $\mathbb{G}_m$. Its ordinary $mod(p)$ cohomology and the Chow ring are known by Vistoli [Vi] and Kameko-Yagita [Ka-Ya].

To state the cohomology $H^*(BPGL_p;\mathbb{Z}/p)$, we recall the Dickson algebra. Let $A\cong(\mathbb{Z}/p)^n$ be an elementary abelian p-group of rank n, and $H^*(BA)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes\Lambda(x_1,...,x_n)$. The Dickson algebra is

$$D_n=\mathbb{Z}/p[y_1,...,y_n]^{GL_n(\mathbb{F}_p)}\cong\mathbb{Z}/p[c_{n,0},...,c_{n,n-1}]$$

with $|c_{n,i}|=2(p^n-p^i)$. The invariant ring under $SL_n(\mathbb{F}_p)$ is also given

$$SD_n=\mathbb{Z}/p[y_1,...,y_n]^{SL_n(\mathbb{F}_p)}\cong D_n\{1,e_n,...,e_n^{p-2}\}\quad with\ e_n^{p-1}=c_{n,0}.$$

We also recall the Mui's result by using Q_i according to Kameko and Mimura [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)}\cong SD_n/(e_n)\oplus SD_n\otimes Q(n-1)\{u_n\}$$

where $u_n=x_1...x_n$ and $e_n=Q_0...Q_{n-1}u_n$. (Here note

$$SD_n/(e_n)\cong D_n/(c_{n,0})\cong\mathbb{Z}/p[c_{n,1},...,c_{n,n-1}].)$$

Theorem 8.2. *(*[Vi],[Ka-Ya]*) There is the isomorphism*

$$H^*(BPGL_p;\mathbb{Z}/p)\cong M\oplus N$$

where $M\cong \mathbb{Z}/p[x_4,x_6,\cdots,x_{2p}]$ *as modules (but not rings) and*

$$N\cong SD_2\otimes Q(1)\{u_2\}\cong \mathbb{Z}/p[e_2,c_{2,1}]\otimes Q(1)\{u_2\}.$$

Theorem 8.3. *(*[Vi],[Ka-Ya]*) There is the additive isomorphism*

$$CH^*(BPGL_p)/p\cong M\oplus SD_2\{Q_0Q_1u_2\}.$$

It is also proved that $Ker(\tau)|H^{*,*'}(BPGL_p;\mathbb{Z}/p)=0$ in Theorem 10.4 in [Ya3].

Theorem 8.4.

$$Inv^*(PGL_p;\mathbb{Z}/p)\cong H^*(G;\mathbb{Z}/p)/(Ng)\cong \mathbb{Z}/2\{1,u\}\quad |u|=2.$$

This fact is also shown in [Ga-Me-Se].

9. Symmetric group S_n

Let S_n be the Symmetric group generated by permutations of n-letters. The permutations induce the natural representation $S_n\to O_n$. Let us write by w_i its Stiefel-Whitney class. Then it is proved for example in [Ga-Me-Se] that for general k

Theorem 9.1. $Inv^*(S_n;\mathbb{Z}/2)\cong H^*\{1,w_1,...,w_{[n/2]}\}$.

Let A be a subgroup of S_n generated by the transpositions $(2i-1,2j)$ for $1\le j\le [n/2]$ so that $A\cong \oplus^{[n/2]}\mathbb{Z}/2$. Then $Inv^*(S_n;\mathbb{Z}/2)$ is detected by the group A.

In this section, by using the cohomology $H^*(BS_n;\mathbb{Z}/2)$, we will re-explain above facts (for $H^*(BS_n;\mathbb{Z}/2)$) but when $k=\mathbb{C}$ and $n=2^m$. (We assume $k=\mathbb{C}$.)

V.Voevodsky showed (for the definition of the reduced power operation)

$$H^{*,*'}(BS_p;\mathbb{Z}/p)\cong H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p)^{S_p}\cong \mathbb{Z}/p[Y]\otimes\Lambda(X)$$

where $Y=y^{p-1}$ and $X=y^{p-2}x$ in $H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p)\cong \mathbb{Z}/p[\tau,y]\otimes\Lambda(x)$. Hence $H^*(BS_p;\mathbb{Z}/p)/(Ng)=\mathbb{Z}/p$ for $p\ne 2$ but $H^*(BS_2;\mathbb{Z}/2)/(Ng)\cong \Lambda(x)$.

Now we restrict $p=2$ and let $n=2^m$. Consider the natural embedding

$$g_m: V_m=\oplus^m\mathbb{Z}/2\to \mathbb{Z}/2\wr ...\wr \mathbb{Z}/2\to S_{2^m}$$

where $-\wr-$ is the wreath product. Then it is known (Theorem 3.23 in [Ma-Mi]) that $H^*(BS_{2^m};\mathbb{Z}/2)$ is detected by $S_{2^{m-1}}\times S_{2^{m-1}}$ and V_m.

Let $f_m : V_m \stackrel{g_m}{\to} S_n \to O_n$. The the total Whitney class is written as

$$w(f_m) = 1 + w(m-1) + w(m-2) + ... + w(0)$$

where $w(i) = w_{2^m - 2^i}(f_m)$. Moreover $Sq^{2^i}(w(i+1)) = w(i)$. In fact the image of g_m^* is just the Dickson algebra

$$Im(g_m^*) = \mathbb{Z}/2[x_1, ..., x_m]^{GL_2(\mathbb{F}_2)} = \mathbb{Z}/2[w(m-1), ..., w(0)].$$

(Here $w(i)^2$ corresponds to $c_{m,i}$ for odd prime cases stated in the preceding section.)

By induction on m, we assume

$$H^*(BS_{2^{m-1}}; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-2}}\}.$$

Then considering the restriction

$$H^*(BS_n; \mathbb{Z}/2)/(Ng) \to H^*(BS_{n-1}; \mathbb{Z}/2)/(Ng) \otimes H^*(BS_{n-1}; \mathbb{Z}/2)/(Ng),$$

we see

$$H^*(BS_n; \mathbb{Z}/2)/(Ng) \supset \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-2}}, ..., w_{2^{m-1}}\}.$$

Note by construction of maps g_m, f_m, we see $g_m^* w_{2^{m-1}} = w(m-1)$. Hence

$$w(j)' = Sq^{2^j} ... Sq^{2^{m-2}} w_{2^{m-1}} \in Ng(S_n)$$

with $g_m^*(w(j)') = w(j)$. Thus we have

Proposition 9.2. *$H^*(BS_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, ..., w_{2^{m-1}}\}$.*

In [Ga-Me-Se], it is also shown $Inv^*(S_n; \mathbb{Z}/p) = \mathbb{Z}/p$ for odd prime p. Let $n = p^m$. The symmetric group S_n has a subgroup isomorphic to

$$S(m) = \mathbb{Z}/p \wr S_{p^{m-1}} \cong (S_{p^{m-1}})^p \rhd \mathbb{Z}/p$$

of the index prime to p. Hence $H^*(BS_n; \mathbb{Z}/p) \subset H^*(BS(m); \mathbb{Z}/p)$.

We consider the Hochshild-Serre spectral sequence

$$E_2^{*,*'} = H^*(\mathbb{Z}/p; H^*((BS_{p^{m-1}})^p; \mathbb{Z}/p)) \Longrightarrow H^*(BS(m); \mathbb{Z}/p).$$

Let us write by σ the generator of the cyclic group $\mathbb{Z}/p$. Let $T = (1-\sigma)$ and $N = (1+\sigma+...+\sigma^{p-1})$. For a $\mathbb{Z}/p$-module M, the homology is written

$$H^*(\mathbb{Z}/p; M) = \begin{cases} Ker(T) = M^{\mathbb{Z}/p} & * = 0 \\ Ker(T)/Im(N) & * = even > 0 \\ Ker(N)/Im(T) & * = odd. \end{cases}$$

Let $\{x_i\}$ be a $\mathbb{Z}/p$ basis of $H^*(BS_{p^{m-1}}; \mathbb{Z}/p)$. Then the basis of

$$S = H^*((BS_{p^{m-1}})^p; \mathbb{Z}/p) \cong \mathbb{Z}/p\{x_{i_1} \otimes x_{i_2} \otimes ....x_{i_p}\}.$$

decomposes as $I \cup F$ with $I = \{x_i \otimes ... \otimes x_i\}$ and

$$F = \{x_{i_1} \otimes ... \otimes x_{i_s} | i_k \neq i_\ell \ for \ \ some \ k \neq \ell\}.$$

The generator σ acts on F freely but invariants on I. Then cohomology is computed

$$H^*(\mathbb{Z}/p; \mathbb{Z}/p\{I\}) \cong \mathbb{Z}/p[y] \otimes \Lambda(x) \otimes \mathbb{Z}/p\{I\} \quad |x| = 1, \ |y| = 2$$

$$H^*(\mathbb{Z}/p; \mathbb{Z}/p\{F\}) \cong \mathbb{Z}/p\{F\}^{\mathbb{Z}/p}.$$

Since $S^{\mathbb{Z}/p} = \mathbb{Z}/p\{F\}^{\mathbb{Z}/p} \oplus \mathbb{Z}/p\{I\}$, we have the isomorphism

$$E_2^{*,*'} \cong S^{\mathbb{Z}/p} \oplus \mathbb{Z}/p\{I\} \otimes \mathbb{Z}/p[y]\{x, y\}.$$

It is well known that $S^{\mathbb{Z}/p} \subset H^*(BG; \mathbb{Z}/p)$ by Nakaoka (also Totaro [To])). Hence this spectral sequence collapses from the E_2-term. Thus we have the well known result ;

Theorem 9.3. *(Nakaoka)*

$$H^*(BS(m); \mathbb{Z}/p) \cong E_2^{*,*'} \cong S^{\mathbb{Z}/p} \oplus H^*(BS_{p^{m-1}}; \mathbb{Z}/p)^{[p]} \otimes \mathbb{Z}/p[y]\{x, y\}$$

where $A^{[p]}$ is the graded algebra whose degree is given by p-times of degree of A.

By induction on m, we may assume $H^*(BS_{p^{m-1}}; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$. Hence $\mathbb{Z}/p\{I\}/(Ng) = \mathbb{Z}/p$. Therefore there is the surjection

$$\mathbb{Z}/p\{1, x\} \to H^*(BS; \mathbb{Z}/p)/(Ng).$$

But $H^2(BS_n; \mathbb{Z}/p) = 0$. Hence $H^+(BS_n; \mathbb{Z}/p)/(Ng) = 0$.

Proposition 9.4. *$H^*(BS_{p^m}; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$.*

We note here about the restriction image for $g_m : V_m = \mathbb{Z}/p \to S_n$. The restriction image is contained in the Dickson algebra as stated in the preceding section

$$grH^*(BA)^{GL_n(\mathbb{F}_p)} \cong D_n/(c_{n,0}) \oplus D_n \otimes Q(n-1)\{e_n^{p-2}u_n\}$$

where $u_n = x_1...x_n$ and $e_n = Q_0...Q_{n-1}u_n$. (Here note $D_n/(c_{n,0}) = \mathbb{Z}/p[c_{n,1}, ..., c_{n,n-1}]$.) We know D_n is in the image of g_m^*. However it is known that $e_n^{p-2}u_n \notin g_m^*$ for $n \geq 3$ (p.196 in [Ad-Mi]).

10. EXTRASPECIAL p-GROUPS

We assume that p is an odd prime. The extraspecial p-group $E_n = p_+^{1+2n}$ is the group such that exponent is p, its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \longrightarrow C \xrightarrow{i} E_n \xrightarrow{\pi} V \longrightarrow 0$$

with $V = \oplus^{2n}\mathbb{Z}/p$. (For details of the cohomology of E_n see [Te-Ya].) We can take generators $a_1, ..., a_{2n}, c \in E_n$ such that $\pi(a_1), .., \pi(a_{2n})$ (resp. c) make a base of V (resp. C) such that

$$[a_{2i-1}, a_{2i}] = c \quad and \quad [a_{2i-1}, a_j] = 1 \ \ if \ j \neq 2i.$$

We note that E_n is also the central product of the n-copies of E_1

$$E_n \cong E_1 \cdots E_1 = E_1 \times_{\langle c\rangle} E_1 ... \times_{\langle c\rangle} E_1.$$

Take cohomologies

$$H^*(BC; \mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(z), \quad \beta z = u,$$

$$H^*(BV; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ..., y_{2n}] \otimes \Lambda(x_1, ...x_{2n}), \quad \beta x_i = y_i,$$

identifying the dual of a_i (resp.c) with x_i (resp. z). That means

$$H^1(E_n; \mathbb{Z}/p) \cong Hom(E_n; \mathbb{Z}/p) \ni x_i : a_j \mapsto \delta_{ij}.$$

The central extension is expressed by

$$f = \sum_{i=1}^{n} x_{2i-1}x_{2i} \in H^2(BV; \mathbb{Z}/p).$$

Hence $\pi^* f = 0$ in $H^2(BE_n; \mathbb{Z}/p)$. We consider the Hochshild-Serre spectral sequence

$$E_2^{*,*'} \cong H^*(BV; \mathbb{Z}/p) \otimes H^*(BC; \mathbb{Z}/p) \Longrightarrow H^*(BE_n; \mathbb{Z}/p).$$

Hence the first nonzero differential is $d_2(z) = f$ and the next differential is

$$d_3(u) = d_3(Q_0(z)) = Q_0(f) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1}.$$

In particular

$$E_4^{0,*} \cong \mathbb{Z}/p[y_1, ..., y_{2n}] \otimes \Lambda(x_1, ...x_{2n})/(f, Q_0(f)).$$

Lemma 10.1. *We have the inclusion*

$$\Lambda(x_1, ..., x_{2n})/(f) \subset H^*(BE_n; \mathbb{Z}/p).$$

Proof. We consider similar group E_n' such that its center is $C \cong \mathbb{Z}/p$ and there is the extension

$$0 \longrightarrow C \xrightarrow{i} E_n' \xrightarrow{\pi} V' \longrightarrow 0$$

but $V' = \oplus^{2n}\mathbb{Z}_p$ such that there is the quotient map $q : E_n' \to E_n$. We also consider the spectral sequence

$$E_2^{*,*'} \cong H^*(BV';\mathbb{Z}/p) \otimes H^*(BC;\mathbb{Z}/p) \Longrightarrow H^*(BE_n';\mathbb{Z}/p).$$

Here $H^*(BV';\mathbb{Z}/p) \cong \Lambda(x_1,...x_{2n})$. The first nonzero differential is $d_2(z) = f$ but the second differential is

$$d_3(u) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1} = 0.$$

Hence we have

$$H^*(BE_n';\mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(x_1,...,x_{2n})/(f).$$

From the map $q^* : H^*(BE_n;\mathbb{Z}/p) \to H^*(BE_n';\mathbb{Z}/p)$, we get the result. □

However $H^*(BE_n;\mathbb{Z}/p)/(Ng) \not\cong \Lambda(x_1,...,x_{2n})/(f)$, infact, when $n = 1$, from Theorem 3.3 in [Ya3] we see

Proposition 10.2.

$$H^*(Bp_+^{1+2};\mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{1, x_1, x_2, a_1', a_2'\} \quad deg(a_i') = 2.$$

When $p = 2$, the situation becomes well. The extraspecial 2-group $D(n) = 2_+^{1+2n}$ in the n-th central extension of the dihedral group D_8 of order 8. It has the central extension

$$0 \to \mathbb{Z}/2 \to D(n) \to V \to 0$$

with $V = \oplus^{2n}\mathbb{Z}/2$. Hence $H^*(BV;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_{2n}]$. Then using the Hochschild-Serre spectral sequence, Quillen proved [Qu]

$$H^*(BD(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_{2n}]/(f, Q_0(f),...,Q_{n-2}(f))\otimes\mathbb{Z}[w_{2^n}(\Delta)].$$

(In fact when n is the real case (i.e., $n = -1, 0, 1\ mod(8)$), the cohomology $H^*(BSpin_n;\mathbb{Z}/2)$ injects into $H^*(BD(n);\mathbb{Z}/2)$.) Here w_i (resp. $w_{2^n}(\Delta)$) is the Stiefel-Whitney class of usual representation from the above extension (resp. 2^n-dimensional representation which restrict nonzero on the center). Moreover Quillen proves following two theorems (Theorem 5.10-11 in [Qu])

Theorem 10.3. *([Qu]) $H^*(BD(n);\mathbb{Z}/2)$ is detected by the product of cohomoloogy of maximal elementary abeian groups.*

Theorem 10.4. *([Qu]) The nonzero Stiefel-Whitney $w_i(\Delta)$ are those of degrees 2^n and $2^n - 2^i$ for $0 \leq i < n$.*

In fact $w_i(\Delta)$ generates the Dickson algebra in the cohomology of the maximal elementary abelian 2-group as stated in the preceding section.

Proposition 10.5. *When $n > 2$, there is the surjection*

$$\Lambda(x_1, ..., x_{2n})/(f) \to H^*(BD(n); \mathbb{Z}/2)/(Ng).$$

Proof. By the same arguments with $p = odd$, we see

$$\Lambda = \Lambda(x_1, ..., x_{2n})/(f) \subset H^*(BD(n); \mathbb{Z}/2).$$

The fact $w_2(\Delta) = 0$ follows from the above second Quillen's theorem. Hence we have $w_{2^n}(\Delta) \in Ng$ from Becher's theorem (Theorem 6.2). Thus we get the theorem. □

Let $0 \neq x \in \Lambda$. Then by the Quillen's theorem, $i_A^*(x) \neq 0$ for

$$i_A^* : H^*(BD(n); \mathbb{Z}/2) \to H^*(BA; \mathbb{Z}/2)$$

with $A \cong \mathbb{Z}/2 \oplus ... \oplus \mathbb{Z}/2$. Let $H^*(BA; \mathbb{Z}/2)/(Ng) \cong \Lambda(x_1', ..., x_m')$. Then $i_A^*(x_i) \in \mathbb{Z}/2\{x_1', ..., x_m'\}$ and hence the map is restricted

$$i_A^* : \Lambda(x_1, ..., x_{2n})/(f) \to \Lambda(x_1', ..., x_m').$$

However this map is not need injective. In fact there is a possibility of $i_A^*(x) \in Ng(A)$, e.g., $i_A^*(x_1x_2) = (x_1')^2 \in Ng(A)$.

11. UNRAMIFIED THEORY

In this section, we assume that k is an *algebraic closed field* of $ch(k) = 0$.

Let K be a function field of k, that is finitely generated as a field over k. Here we recall the definition of unramifed cohomology of $H^*(K; \mathbb{Z}/p)$ according to Saltman, Peyre and Colliot-Thelene. We denote by $P(K/k)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and that the fraction field $Fr(A)$ of A is K. If A belongs to $P(K/k)$, then for the residue field κ_A, we can define the residue map $\partial_A : H^*(K; \mathbb{Z}/p) \to H^{*-1}(\kappa_A; \mathbb{Z}/p)$ as follows.

Let $\hat{K}_A$ be the completion, $\hat{K}_A^{nr}$ the maximal unramified extension of K_A and $\bar{K}_A$ is an algebraic closure of K_A. Put $I_A = Gal(\bar{K}_A/\hat{K}_A^{nr})$ and $G_A = Gal(\bar{K}_A/\hat{K}_A)$.

$$\bar{K}_A \overset{I_A}{--} \hat{K}_A \overset{G_A/I_A}{--} \hat{K}_A -- K.$$

Then ∂_A is defined as the composition of maps

$$\partial : H^*(K; \mathbb{Z}/p) \to H^*(\hat{K}_A; \mathbb{Z}/p)$$

$$\overset{proj.}{\to} H^{*-1}(G_A/I_A) \otimes H^1(I_A; \mathbb{Z}/p) \cong H^{*-1}(\kappa_A; \mathbb{Z}/p).$$

Here we used that $\hat{K}_A \cong I_A \oplus (G_A/I_A)$ ([Sa]) and $I_A \cong \hat{\mathbb{Z}}$. Moreover $H^*(\hat{\mathbb{Z}};\mathbb{Z}/p) \cong \mathbb{Z}/p$ if $* = 0,1$ and $\cong 0$ otherwise.

Then we can define the unramified cohomology

$$H^*_{nr}(K;\mathbb{Z}/p) = \cap_{A\in P(K/k)} Ker(H^*(K;\mathbb{Z}/p) \stackrel{\partial_A}{\to} H^{*-1}(\kappa_A;\mathbb{Z}/p)).$$

Namely, when X is complete, the residue map is the same as the differential d_1 of the coniveau spectral sequence given in §2, and hence the unramified cohomology is just the E_2-term $H^0_{Zar}(X;H^m_{\mathbb{Z}/p})$ of the coniveau spectral sequence.

Corollary 11.1. *When X is complete, there is the isomorphism*

$$H^*_{nr}(k(X);\mathbb{Z}/p) \cong H^{*,*}(X;\mathbb{Z}/p)/(\tau) \oplus Ker(\tau)|H^{*+1,*-1}(X;\mathbb{Z}/p).$$

Now we consider the case $X = BG$; non complete cases. Let $W//G$ be the scheme which has the coordinate ring $k[W]^G$. Then it is known that $W//G$ contains U_n/G as an open set. Here U_n is the open set given in §3, where G acts freely. Hence we have for $* < n$

$$H^0_{Zar}(W//G;H^*_{\mathbb{Z}/p}) \subset H^0_{Zar}(U_n/G;H^*_{\mathbb{Z}/p}).$$

Since G is reductive, it is also known that the fraction field of $k[W]^G$ is $k(W)^G$, that is $k(W//G) = k(W)^G$. Thus we get

Lemma 11.2. *For $* < n$, we have*

$$H^*_{nr}(K(W)^G;\mathbb{Z}/p) = H^0_{Zar}(W//G;H^*_{\mathbb{Z}/p})$$
$$\subset H^0(U_n/G;H^*_{\mathbb{Z}/p}) = H^0(BG;H^*_{\mathbb{Z}/p)}).$$

According to Peyre [Pe], we define a subring $H^*_{nr}(G;\mathbb{Z}/p)$ of $H^*_{et}(BG;\mathbb{Z}/p)$ as follows. Let $P(G)$ be the set of elements $g \in G$ such that $I = \langle g\rangle \cong \mathbb{Z}/p^s$ for some $s \geq 1$ but $g \neq h^p$ for any $h \in G$. Then the centralizer is written as

$$Z_G(I) \cong I \oplus H \qquad with\ H = Z_G(I)/I.$$

Let us write by ∂_g the composition map

$$\partial_g : H^*(BG;\mathbb{Z}/p) \to H^*(BZ_G(I);\mathbb{Z}/p)$$
$$\cong H^{*-*'}(BH;\mathbb{Z}/p) \otimes H^{*'}(BI;\mathbb{Z}/p) \stackrel{proj.}{\to} H^{*-1}(BH;\mathbb{Z}/p)$$

using $H^1(BI;\mathbb{Z}/p) \cong \mathbb{Z}/p$. Then define the unramified cohomology by

$$H^*_{nr}(G;\mathbb{Z}/p) = \bigcap_{g\in P(G)} Ker(H^*(BG;\mathbb{Z}/p) \stackrel{\partial_g}{\to} H^{*-1}(BH;\mathbb{Z}/p)).$$

Remark. The restriction map $H^1(B\langle g\rangle;\mathbb{Z}/p) \to H^1(B\langle g^p\rangle;\mathbb{Z}/p)$ is always zero. Hence we need not consider the case $I = \langle g^p\rangle$.

Theorem 11.3. *(Peyre [Pe]) Let W be a faithful representaion of G. Let q is the quotient map $q : Gal(k(\bar{W})/k(W)^G) \to Gal(k(W)/k(W)^G) = G$. Then*

$$q^*(H^*_{nr}(G;\mathbb{Z}/p)) \subset H^*_{nr}(k(W)^G;\mathbb{Z}/p).$$

Proof. Arguments of pages 203 to 206 in the proof of Proposition 3 in [Pe] work exchanging $H^3(-;\mathbb{Q}/\mathbb{Z}(-))$ to $H^*(-;\mathbb{Z}/p)$. Indeed we have the commutative diagram ((12) in [Pe])

$$\begin{array}{ccc} H^*(G;\mathbb{Z}/p) & \xrightarrow{\partial_{D,g}} & H^{*-1}(D;\mathbb{Z}/p) \\ \downarrow & & \downarrow \\ H^*(k(W)^G;\mathbb{Z}/p) & \xrightarrow{\partial_A} & H^{*-1}(\kappa_A;\mathbb{Z}/p) \end{array}$$

where D is the decomposition subgroup of G. If $x \in H^*_{nr}(G;\mathbb{Z}/p)$, then $\partial_{D,g}(x) = 0$ and hence $\partial_A(\rho^*(x)) = 0$. □

It is well known (Theorem 4.1.5 in [Co-Te]) that if K is purely transcendental over k (i.e., $K \cong k(x_1, ..., x_n)$ for indeterminate x_i), then

$$H^*(k;\mathbb{Z}/p) \cong H^*_{nr}(K;\mathbb{Z}/p) \quad for \ * > 0.$$

When k is algebraic closed field, it is immediate $H^+(k;\mathbb{Z}/p) = 0$.

Corollary 11.4. *Suppose that*

$$0 \neq x \in H^*(BG;\mathbb{Z}/p)/(Ng) \quad and \quad x \in H^*_{nr}(G;\mathbb{Z}/p).$$

Then $q^*(x) \neq 0$ in $H^*_{nr}(k(W)^G;\mathbb{Z}/p)$. Hence $k(W)^G/k$ is not purely trascendental.

Proof. From the preceding theorem, we have

$$q^*(x) \in H^*_{nr}(k(W)^G;\mathbb{Z}/p) \subset H^*(k(W)^G;\mathbb{Z}/p).$$

It is nonzero since

$$H^*(BG;\mathbb{Z}/p)/(Ng) \subset H^0_{Zar}(BG;H^*_{\mathbb{Z}/p}) \subset H^*(k(W)^G;\mathbb{Z}/p).$$

□

Corollary 11.5. *Let G be a $p-$group of exponent p. If $H^2(BG;\mathbb{Z}/p)/(Ng)$ is not detected by $\mathbb{Z}/p \times \mathbb{Z}/p$, then $k(W)^G$ is not purely transcendental.*

Proof. Suppose that $x \notin H^2_{un}(G;\mathbb{Z}/p)$. Then we can take $x = \sum x_1x_2$ such that $x_1|H^1(BI;\mathbb{Z}/p) \neq 0$ with $I \cong \mathbb{Z}/p$ and for $H = Z_G(I)/I$, $x_2|H^1(BH;\mathbb{Z}/p) \neq 0$. Here

$$H^1(BH;\mathbb{Z}/p)/(Ng) = H^1(BH;\mathbb{Z}/p) \cong Hom(H,\mathbb{Z}/p).$$

Hence x_2 defines subgroup $J \cong \mathbb{Z}/p$ of H such that $x_2|H^1(BJ;\mathbb{Z}/p) \neq 0$ (since H is exponent p). Thus the element x is detected by the subgroup $I \oplus J \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. □

12. Saltman's example

Let G be the group defined by

$$0 \to \langle c_3, c_4\rangle \to G \to E_2 = p_+^{1+4} \to 0$$

$$with \quad [a_1, a_3] = c_3, \quad and \quad [a_1, a_4] = c_4.$$

Saltman and Bogomolov showed that $H^2_{nr}(k(W)^G; \mathbb{Q}/\mathbb{Z}) \neq 0$ for this group. We will see the $\mathbb{Z}/p$-coefficient case.

Lemma 12.1. *The* 3*-dimensional cohomology* $H^3(BG; \mathbb{Z}/p)$ *contains the* $\mathbb{Z}/p$*-module*

$$A = \mathbb{Z}/p\{y_i x_j | 1 \leq i, j \leq 4\}/(Q_0(f), Q_0(x_1x_3), Q_0(x_1x_4)).$$

Proof. Consider the central extension

$$0 \to \langle c, c_3, c_4\rangle \to G \to V = \oplus^4\mathbb{Z}/p \to 0$$

and induced spectral sequence

$$E_2^{*,*'} \cong \mathbb{Z}/p[y_1, ..., y_4, u, u_3, u_4] \otimes \Lambda(x_1, ..., x_4, z, z_3, z_4)$$

converging to $H^*(BG; \mathbb{Z}/p)$. The first differential is

$$d_2(z) = f, \quad d_2(z_3) = x_1x_3, \quad d_2(z_4) = x_1x_4.$$

The second nonzero differential is

$$d_3(u) = Q_0(f), \quad d_3(u_3) = Q_0(x_1x_3), \quad d_3(u_4) = Q_0(x_1x_4).$$

Thus we see

$$A \cong E_4^{3,0} \cong E_\infty^{3,0} \subset H^3(BG; \mathbb{Z}/p).$$

□

Theorem 12.2. *(Saltman [Sa]) We have*

$$0 \neq x_1x_2 \in H^2_{nr}(G; \mathbb{Z}/p) \cap H^{2,2}(BG; \mathbb{Z}/p)/(\tau).$$

Hence $k(W)^G$ *is not purely transcendental.*

Proof. From the preceding lemma, we see $Q_0(x_1x_2) \neq 0$ in $H^3(BG; \mathbb{Z}/p)$. Hence we see $x_1x_2 \notin Ng(G)$ from Lemma 3.4.

Next we will show $x_1x_2 \in H_{nr}(G; \mathbb{Z}/p)$. Suppose this is not the case. By the definition, this means that there is an element g and $h \in Z_G(\langle g\rangle)$ such that

$$x_i|\langle g\rangle \neq 0, \quad x_j|\langle h\rangle \neq 0 \quad for\ \{i, j\} = \{1, 2\}.$$

Let us write

$$a^\Lambda = a_1^{\lambda_1}...a_4^{\lambda_4} \quad for\ \Lambda = (\lambda_1, ..., \lambda_4),$$

$$a^M = a_1^{\mu}...a_4^{\mu_4} \quad for\ M = (\mu_1, ..., \mu_4).$$

Note that $x_i|\langle a^\Lambda\rangle = \lambda_i$. The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4} \quad where \ \ d = \lambda_1\mu_2 - \lambda_2\mu_1 + \lambda_3\mu_4 - \lambda_4\mu_3,$$

$$d_3 = \lambda_1\mu_3 - \lambda_1\mu_3, \qquad d_4 = \lambda_1\mu_4 - \lambda_4\mu_1.$$

Take $g = a^\Lambda$. Exchanging $x_1 \mapsto x_1 + \lambda x_2$ or $x_1 \mapsto x_2$ (if necessary), we can take

$$\lambda_1 = 1, \lambda_2 = 0 \quad \Lambda = (1, 0, \lambda_3, \lambda_4).$$

(Hence $x_1\langle g\rangle = 1$ and $x_2|\langle g\rangle = 0$.) Then we can take $h = a^M$ so that $x_2|\langle h\rangle = 1$ and $h \notin \langle g\rangle$, that means

$$\mu_1 = 0, \ \mu_2 = 1 \quad M = (0, 1, \mu_3, \mu_4).$$

From $d_3 = 0$ and $d_4 = 0$, we see

$$\mu_3 = \lambda_3\mu_1 = 0 \quad and \quad \mu_4 = \lambda_4\mu_1 = 0$$

(and hence $M = (0, 1, 0, 0)$). Therefore $d = 1 \times 1 - 0 + 0 - 0 = 1$. This is a contradiction. Hence we have proved $x_1x_2 \in H_{nr}(G; \mathbb{Z}/p)$. $\square$

13. NON DETECTED EXAMPLES

We consider group G'_m and element $\xi_m \in H^*(BG'_m; \mathbb{Z}/p)/(Ng)$ which is not detected by elementary abelina p-groups, while it is not in $H_{nr}(G'_m; \mathbb{Z}/p)$.

Let G' be the group defined by

$$0 \to \langle c_3, c_4\rangle \to G' \to E_2 = p_+^{1+4} \to 0$$

$$with \quad [a_1, a_3] = c_3, \quad and \quad [a_2, a_4] = c_4.$$

Difference of the definitions of G and G' is just $[a_2, a_4] = c_4$. However note G' has the rank 2 elementary abeilan p-subgroup

$$\langle a_1a_3^{-1}, a_2a_4^{-1}\rangle \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

while G does not.

Let G'_m be the central product of $m-1$-th copies of G' and one G

$$G'_m = G \cdot G' \cdot ... \cdot G' = G \times_{\langle c\rangle} G' \times_{\langle c\rangle} ... \times_{\langle c\rangle} G'$$

which is generated by $a_1, ..., a_{4m}$, c, $c_3, c_4, c_7, ..., c_{4m-1}, c_{4m}$. The comutators are given for $1 \le i \le m$

$$[a_{4i-3}, a_{4i-2}] = [a_{4i-1}, a_{4i}] = c,$$

$$[a_{4i-3}, a_{4i-1}] = c_{4i-1}, \quad and \quad [a_{4i-2}, a_{4i}] = c_{4i} \ for \ i \neq 1,$$

$$but \quad [a_1, a_4] = c_4.$$

Let us write $a^\Lambda = a_1^{\lambda_1}...a_{4m}^{\lambda_{4m}}$ and $\Lambda = (\lambda_1, ..., \lambda_{4m})$. Note also $x_i|\langle a^\Lambda\rangle = \lambda_i$. The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4} .... c_{4m}^{d_{4m}}$$

$$where\ \ d=\sum_i \lambda_{4i-3}\mu_{4i-2}-\lambda_{4i-2}\mu_{4i-3}+\lambda_{4i-1}\mu_{4i}-\lambda_{4i}\mu_{4i-1},$$

$$d_{4i-1}=\lambda_{4i-3}\mu_{4i-1}-\lambda_{4i-1}\mu_{4i-3},\qquad d_{4i}=\lambda_{4i-2}\mu_{4i}-\lambda_{4i}\mu_{4i-2}\ (i\neq 1),$$

$$and\qquad d_4=\lambda_1\mu_4-\lambda_4\mu_1.$$

Lemma 13.1. *Let* $\xi_m = x_1x_2x_5x_6....x_{4m-3}x_{4m-2} \in H^*(BG'_m;\mathbb{Z}/p)$. *Then* $\xi_m \in H^*(BG'_n;\mathbb{Z}/p)/(Ng)$ *is not detected by elementary abelian* $p-groups$.

Proof. Suppose that ξ_m is detected by

$$\langle g_1,g_2,...,g_{2m}\rangle \cong \mathbb{Z}/p\oplus ...\oplus\mathbb{Z}/p$$

such that $\xi_m|\langle g_1,...,g_{2m}\rangle \neq 0$, and $x_i|\langle g_i\rangle\neq 0$ for some i. Take generators adequately, let $g_i=a^{\Lambda_i}$ such that

$$\Lambda_1=(1,0,*,*|0,0,*,*|....|0,0,*,*),$$

$$\Lambda_2=(0,1,*,*|0,0,*,*|....|0,0,*,*),$$

$$\Lambda_5=(0,0,*,*|1,0,*,*|....|0,0,*,*),$$

$$\Lambda_6=(0,0,*,*|0,1,*,*|....|0,0,*,*),$$

....

$$\Lambda_{4m-3}=(0,0,*,*|0,0,*,*|....|1,0,*,*),$$

$$\Lambda_{4m-2}=(0,0,*,*|0,0,*,*|....|0,1,*,*).$$

Let $i>1$. Consider the commutativity of $L=\Lambda_{4i-3}$ and $M=\Lambda_1$. Since $d_{4i-1}=\lambda_{4i-3}\mu_{4i-1}-\lambda_{4i-1}\mu_{4i-3}=0$, we see

$$\mu_{4i-1}=0\quad from\ \mu_{4i-3}=0\ and\ \lambda_{4i-3}=1.$$

Similarly we have $\mu_{4i}=0$ from the commutativity of Λ_{4i-2} and Λ_1. Thus we see

$$\Lambda_1=(1,0,*,*|0,0,0,0|....|0,0,0,0).$$

We also have $\Lambda_2=(0,1,*,*|0,0,0,0|....|0,0,0,0)$. Let $\Lambda=\Lambda_1$ and $M=\Lambda_2$. By the commutativity and facts $d_3=d_4=0$, we see $\mu_3=\mu_4=0$, that is,

$$\Lambda_2=(0,1,0,0|0,0,0,0|....|0,0,0,0).$$

However this is a contradiction, indeed,

$$d=\sum_i \lambda_{4i-3}\mu_{4i-2}-\lambda_{4i-2}\mu_{4i-3}+\lambda_{4i-1}\mu_{4i}-\lambda_{4i}\mu_{4i-1}=1\neq 0.$$

□

Lemma 13.2. $Q_0...Q_{2m-2}(\xi_m)\neq 0$ *in* $H^*(BG_m;\mathbb{Z}/p)$.

Proof. We recall that G'_m is the central product of G and copies of G', that means

$$0 \to \oplus^m \mathbb{Z}/p \to G \times G' \times ... \times G' \stackrel{pr.}{\to} G'_m \to 0.$$

The map induces the map of cohomologies

$$pr.^* : H^*(BG'_m;\mathbb{Z}/p) \to H^*(BG;\mathbb{Z}/p) \otimes H^*(BG';\mathbb{Z}/p)^{\otimes n}.$$

The operation Q_i is derivative, we have

$$Q_0...Q_{2m-2}(\xi_m) =$$

$$Q_0(x_1x_2)Q_1Q_2(x_5x_6)...Q_{2m-3}Q_{2m-2}(x_{4m-3}x_{4m-2}) + ...$$

Here from Lemma 12.1,

$$Q_1(x_1x_2) = y_1x_2 - y_2x_1 \neq 0 \quad in\ H^*(BG;\mathbb{Z}/p).$$

For $i > 1$, we see

$$Q_{2i-3}Q_{2i-2}(x_{4i-3}x_{4i-2}) = y_{4i-3}^{p^{2i-2}} y_{4i-2}^{p^{2i-3}} - y_{4i-3}^{p^{2i-3}} y_{4i-2}^{p^{2i-2}}.$$

This is nonzero in $H^*(BG';\mathbb{Z}/p)$ because it is nonzero in the restriction image

$$H^*(BG';\mathbb{Z}/p) \to H^*(B\langle a_1a_3^{-1}, a_2a_4^{-1}\rangle;\mathbb{Z}/p) \cong \mathbb{Z}/p[y'_1, y'_2] \otimes \Lambda(x'_1, x'_2)$$

where $y_{4i-3}, y_{4i-1} \mapsto y'_1$ and $y_{4i-2}, -y_{4i} \mapsto y'_2$. □

Corollary 13.3. *Elements in $H^{2m}(BG'_m;\mathbb{Z}/p)/(Ng)$ is not detected by abelian subgroups, namely, the restriction map*

$$Res : H^{2m}(BG'_m;\mathbb{Z}/p)/(Ng) \to \Pi_{A;abelian} H^{2m}(BA;\mathbb{Z}/p)/(Ng)$$

is not injective.

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