QUADRATIC FORMS OF DIMENSION 8 WITH TRIVIAL DISCRIMINANT AND CLIFFORD ALGEBRA OF INDEX 4.

ALEXANDRE MASQUELEIN, ANNE QUÉGUINER-MATHIEU, AND JEAN-PIERRE TIGNOL

Abstract. Izhboldin and Karpenko proved in [IK00, Thm 16.10] that any quadratic form of dimension 8 with trivial discriminant and Clifford algebra of index 4 is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form. We give a new proof of this result, based on a theorem of decomposability for degree 8 and index 4 algebras with orthogonal involution.

Let WF denote the Witt ring of a field F of characteristic different from 2. As explained in [Lam05, X.5 and XII.2], one would like to describe those quadratic forms whose Witt class belongs to the nth power I^nF of the fundamental ideal IF of WF. By the Arason-Pfister Hauptsatz, such a form is hyperbolic if it has dimension $< 2^n$ and similar to a Pfister form if it has dimension 2^n. More generally, Vishik's Gap Theorem gives the possible dimensions of anisotropic forms in I^nF.

In addition, one may describe explicitly, for some small values of n, low dimensional anisotropic quadratic forms in I^nF. This is the case, in particular, for $n = 2$, that is for even-dimensional quadratic forms with trivial discriminant. In dimension 6, it is well known that such a form is similar to an Albert form, and uniquely determined up to similarity by its Clifford invariant. In dimension 8, if the index of the Clifford algebra is ≤ 4, Izhboldin and Karpenko proved in [IK00, Thm 16.10] that it is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form.

The purpose of this paper is to give a new proof of Izhboldin and Karpenko's result. Our proof is in the framework of algebras with involution, and does not use Rost's description of 14-dimensional forms in I^3F (see [IK00, Rmk 16.11.2]). More precisely, we use triality [KMRT98, (42.3)] to translate the question into a question on algebras of degree 8 and index 4 with orthogonal involution. Our main tool then is a decomposability theorem (Thm. 1.1), proven in § 3. We also use a refinement of a statement of Arason [Ara75, 4.18] describing the even part of the Clifford algebra of a transfer (see Prop. 2.1 below).

1. Notations and statement of the theorem

Throughout the paper, we work over a base field F of characteristic different from 2. We refer the reader to [KMRT98] and [Lam05] for background information on algebras with involution and on quadratic forms. However, we depart from the notation in [Lam05] by using $\langle\langle a_1, \dots, a_n \rangle\rangle$ to denote the n-fold Pfister form $\otimes_{i=1}^{n} \langle 1, -a_i \rangle$. For any quadratic space (V, ϕ) over F, we let Ad_ϕ be the algebra

Date: February 6, 2009.
The third author is supported in part by the F.R.S.–FNRS (Belgium).

with involution $(\mathrm{End}_F(V), \mathrm{ad}_\phi)$, where ad_ϕ is the adjoint involution with respect to ϕ, denoted by σ_ϕ in [KMRT98].

For any field extension L/F, we denote by $GP_n(L)$ the set of quadratic forms that are similar to n-fold Pfister forms. This notation extends to the quadratic étale extension $F \times F$ by $GP_n(F \times F) = GP_n(F) \times GP_n(F)$. For any quadratic form ψ over L, let $\mathcal{C}(\psi)$ be its full Clifford algebra, with even part $\mathcal{C}_0(\psi)$. Both $\mathcal{C}(\psi)$ and $\mathcal{C}_0(\psi)$ are endowed with a canonical involution, which is the identity on the underlying vector space, denoted by γ (see [KMRT98, p.89]). If ψ has even dimension and trivial discriminant, then its even Clifford algebra splits as a direct product $\mathcal{C}_+(\psi) \times \mathcal{C}_-(\psi)$, for some isomorphic central simple algebras $\mathcal{C}_+(\psi)$ and $\mathcal{C}_-(\psi)$ over F (see [Lam05, V, Thm 2.5]). Those algebras are Brauer-equivalent to the full Clifford algebra of ψ and their Brauer class is the Clifford invariant of ψ. Assume moreover that $\dim(\psi) \equiv 0 \bmod 4$. As explained in [KMRT98, (8.4)], the involution γ then induces an involution on each factor of $\mathcal{C}_0(\psi)$, and one may easily check that the isomorphism between the two factors described in the proof of [Lam05, V, Thm 2.5] preserves the involution, so that we actually get a decomposition $(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-)$, with $(\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-)$.

Let L/F be a quadratic field extension. For any quadratic form ψ over L, we let $\mathrm{tr}_\star(\psi)$ be the transfer of ψ, associated to the trace map $\mathrm{tr} : L \to F$, as defined in [Lam05, VII.1.2]. This definition extends to the split étale case $L = F \times F$ and leads to $\mathrm{tr}_\star(\psi, \psi') = \psi + \psi'$. On the other hand, for any algebra A over L, we let $N_{L/F}(A)$ be its norm, as defined in [KMRT98, §3.B]. Recall that the Brauer class of $N_{L/F}(A)$ is the corestriction of the Brauer class of A. Moreover, if A is endowed with an involution of the first kind σ, then the tensor product $\sigma \otimes \sigma$ restricts to an involution $N_{L/F}(\sigma)$ on $N_{L/F}(A)$. We use the following notation: $N_{L/F}(A, \sigma) = (N_{L/F}(A), N_{L/F}(\sigma))$. In the split étale case, we get $N_{F\times F/F}((A, \sigma), (A', \sigma')) = (A, \sigma) \otimes (A', \sigma')$ (see [KMRT98, §15.B]).

Let (A, σ) be a degree 8 algebra with orthogonal involution. We assume that (A, σ) is *totally decomposable*, that is, isomorphic to a tensor product of three quaternion algebras with involution,

$$(A, \sigma) = \otimes_{i=1}^3 (Q_i, \sigma_i).$$

If A is split (resp. has index 2), then (A, σ) admits a decomposition as above in which each quaternion algebra (resp. each but one) is split (see [Bec08]). Our main result is the following theorem:

Theorem 1.1. *Let (A, σ) be a degree 8 totally decomposable algebra with orthogonal involution. If the index of A is ≤ 4, then there exists $\lambda \in F^\times$ and a biquaternion algebra with orthogonal involution (D, θ) such that*

$$(A, \sigma) \simeq (D, \theta) \otimes \mathrm{Ad}_{\langle\langle \lambda \rangle\rangle}.$$

The theorem readily follows from Becher's results mentioned above if A has index 1 or 2; it is proven in § 3 for algebras of index 4. For algebras of index ≤ 2, we may even assume that (D, θ) decomposes as a tensor product of two quaternion algebras with involution; this is not the case anymore if A has index 4, as was shown by Sivatski [Siv05, Prop. 5].

Using triality, we easily deduce the following from Theorem 1.1:

Theorem 1.2 (Izhboldin-Karpenko)**.** *Let ϕ be an 8-dimensional quadratic form over F. The following are equivalent:*

(i) ϕ has trivial discriminant and Clifford invariant of index ≤ 4;

(ii) there exists a quadratic étale extension L/F and a form $\psi \in GP_2(L)$ such that $\phi = \mathrm{tr}_\star(\psi)$.

If $\phi = \mathrm{tr}_\star(\psi)$ for some $\psi \in GP_2(L)$, it follows from some direct computation made in [IK00, §16] that ϕ has trivial discriminant and Clifford invariant of index ≤ 4.

Assume conversely that ϕ has trivial discriminant. By the Arason-Pfister Hauptsatz, ϕ is in $GP_3(F)$ if and only if it has trivial Clifford invariant. More generally, it is well-known that ϕ decomposes as $\phi = \langle\langle a\rangle\rangle q$ for some $a \in F^\times$ and some 4-dimensional quadratic form q over F if and only if its Clifford invariant has index ≤ 2 (see for instance [Kne77, Ex 9.12]). Hence, in both cases, ϕ decomposes as a sum $\phi = \pi_1 + \pi_2$ of two forms $\pi_1, \pi_2 \in GP_2(F)$. This proves that condition (ii) holds with $L = F \times F$.

In section 4 below, we finish this proof by treating the index 4 case. This part of the proof differs from the argument given in [IK00]. In particular, we do not use Rost's description of 14-dimensional forms in I^3F.

2. Clifford algebra of the transfer of a quadratic form

Let L/F be a quadratic field extension. By Arason [Ara75, 4.18], for any quadratic form $\psi \in GP_2(L)$, the Clifford invariant of the transfer $\mathrm{tr}_\star(\psi)$ coincides with the corestriction of the Clifford invariant of ψ. In this section, we extend this result, taking into account the algebras with involution rather than just the Brauer classes. More precisely, we prove:

Proposition 2.1. *Let $L = F[X]/(X^2 - d)$ be a quadratic étale extension of F. Consider a quadratic form ψ over L with $\dim(\psi) \equiv 0 \bmod 4$ and $d_\pm(\psi) = 1$, so that its even Clifford algebra decomposes as*

$$(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-), \text{ with } (\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-).$$

For any $\lambda \in L^\times$ represented by ψ, the two components of the even Clifford algebra of the transfer of ψ are both isomorphic to

$$(\mathcal{C}_+(\mathrm{tr}_\star(\psi)), \gamma_+) \simeq \mathrm{Ad}_{\langle\langle -dN_{L/F}(\lambda)\rangle\rangle} \otimes N_{L/F}(\mathcal{C}_+(\psi), \gamma_+).$$

Proof. In the split étale case $L = F \times F$, the quadratic form ψ is a couple (ϕ, ϕ') of two quadratic forms over F with

$$\dim(\phi) = \dim(\phi') \equiv 0 \bmod 4 \quad \text{and} \quad d_\pm(\phi) = d_\pm(\phi') = 1 \in F^\star/F^{\star 2}.$$

Pick λ and λ' in F respectively represented by ϕ and ϕ'; the norm $N_{F\times F/F}(\lambda, \lambda')$ is $\lambda\lambda'$. So the following lemma proves the proposition in that case:

Lemma 2.2. *Let ϕ and ϕ' be two quadratic forms over F of the same dimension $n \equiv 0 \bmod 4$ and trivial discriminant. For any λ and $\lambda' \in F^\times$, respectively represented by ϕ and ϕ', the components of the even Clifford algebra of the orthogonal sum $\phi + \phi'$ are isomorphic to*

$$(\mathcal{C}_+(\phi + \phi'), \gamma_+) \simeq \mathrm{Ad}_{\langle\langle -\lambda\lambda'\rangle\rangle} \otimes (\mathcal{C}_+(\phi), \gamma_+) \otimes (\mathcal{C}_+(\phi'), \gamma_+).$$

Proof of Lemma 2.2. Denote by V and V' the underlying quadratic spaces. The natural embeddings $V \hookrightarrow V \oplus V'$ and $V' \hookrightarrow V \oplus V'$ induce F-algebra homomorphisms

$$\mathcal{C}(\phi) \to \mathcal{C}(\phi + \phi') \text{ and } \mathcal{C}(\phi') \to \mathcal{C}(\phi + \phi').$$

One may easily check that the images of the even parts centralize each other, so that we get an F-algebra homomorphism

$$(\mathcal{C}_0(\phi),\gamma)\otimes(\mathcal{C}_0(\phi'),\gamma)\to(\mathcal{C}_0(\phi+\phi'),\gamma).$$

Pick orthogonal bases $(e_1,\dots,e_n)$ of (V,ϕ) and $(e'_1,\dots,e'_n)$ of (V',ϕ'). The basis of $\mathcal{C}_0(\phi+\phi')$ consisting of products of an even number of vectors of the set $\{e_1,\dots,e_n,e'_1,\dots,e'_n\}$ as described in [Lam05, V, cor 1.9] clearly contains the image of a basis of $\mathcal{C}_0(\phi)\otimes\mathcal{C}_0(\phi')$, so that the homomorphism above is injective. In the sequel, we will identify $\mathcal{C}_0(\phi)$ and $\mathcal{C}_0(\phi')$ with their images in $\mathcal{C}_0(\phi+\phi')$.

Consider the element $z=e_1\dots e_n\in\mathcal{C}_0(\phi)$. As explained in [Lam05, V, Thm2.2], for any $v\in V$, one has $vz=-zv\in\mathcal{C}(\phi)$ and z generates the center of $\mathcal{C}_0(\phi)$. Since ϕ has dimension 0 mod 4 and trivial discriminant, this element z is γ-symmetric, and multiplying e_1 by a scalar if necessary, we may assume $z^2=1$. The two components of $\mathcal{C}_0(\phi)$ are $\mathcal{C}_+(\phi)=\mathcal{C}_0(\phi)\frac{1+z}{2}$ and $\mathcal{C}_-(\phi)=\mathcal{C}_0(\phi)\frac{1-z}{2}$. Consider similarly $z'=e'_1\dots e'_n$, with $\gamma(z')=z'$ and assume $z'^2=1$. The product zz' also has square 1 and generates the center of $\mathcal{C}_0(\phi+\phi')$. We denote by ε the idempotent $\varepsilon=\frac{1+zz'}{2}$, so that $\mathcal{C}_+(\phi+\phi')=\mathcal{C}_0(\phi+\phi')\varepsilon$ and $\mathcal{C}_-(\phi+\phi')=\mathcal{C}_0(\phi+\phi')(1-\varepsilon)$.

Let us now fix two vectors $v\in V$ and $v'\in V'$ such that $\phi(v)=\lambda$ and $\phi'(v')=\lambda'$. Since $\frac{1+z}{2}v^{-1}=v^{-1}\frac{1-z}{2}$, we have $vxv^{-1}\in\mathcal{C}_-(\phi)$ for any $x\in\mathcal{C}_+(\phi)$. Using this identification between the two components, we may diagonally embed $\mathcal{C}_+(\phi)$ in $\mathcal{C}_0(\phi)$ by considering $x\in\mathcal{C}_+(\phi)\mapsto x+vxv^{-1}\in\mathcal{C}_0(\phi)$. Similarly, we may embed $\mathcal{C}_+(\phi')$ in $\mathcal{C}_0(\phi')$ by $x'\in\mathcal{C}_+(\phi')\mapsto x'+v'x'v'^{-1}\in\mathcal{C}_0(\phi')$. Combining those two maps with the morphism

$$\mathcal{C}_0(\phi)\otimes\mathcal{C}_0(\phi')\to\mathcal{C}_0(\phi+\phi'),$$

and the projection

$$y\in\mathcal{C}_0(\phi+\phi')\mapsto y\varepsilon\in\mathcal{C}_+(\phi+\phi'),$$

we get an algebra homomorphism

$$\begin{array}{ccc}\mathcal{C}_+(\phi)\otimes\mathcal{C}_+(\phi') & \to & \mathcal{C}_+(\phi+\phi'),\\ x\otimes x' & \mapsto & (x+vxv^{-1})(x'+v'x'v'^{-1})\varepsilon.\end{array}$$

One may easily check on generators that this map is not trivial; hence it is injective. To conclude the proof, it only remains to identify the centralizer of the image, which by dimension count has degree 2. It clearly contains $\frac{z+z'}{2}\varepsilon$ and $vv'\varepsilon$. Moreover, these two elements anticommute, have square ε and $-\lambda\lambda'\varepsilon$, and are respectively symmetric and skew-symmetric under γ. Hence they generate a split quaternion algebra, with orthogonal involution of discriminant $-\lambda\lambda'$, which is isomorphic to $\mathrm{Ad}_{\langle\langle-\lambda\lambda'\rangle\rangle}$. □

This concludes the proof in the split étale case. Until the end of this section, we assume L is a quadratic field extension of F, with non-trivial F-automorphism denoted by ι. To prove the proposition in this case, we will use the following description of the transfer of a quadratic form and its Clifford algebra.

Let ψ be any quadratic form over L, defined on the vector space V. We consider its conjugate ${}^{\iota}V=\{{}^{\iota}v,\ v\in V\}$ with the following operations ${}^{\iota}v_1+{}^{\iota}v_2={}^{\iota}(v_1+v_2)$ and $\lambda\cdot{}^{\iota}v={}^{\iota}(\iota(\lambda)\cdot v)$, for any v_1,v_2 and v in V and $\lambda\in L$. Clearly, ${}^{\iota}\psi({}^{\iota}v)=\iota(\psi(v))$ is a quadratic form on ${}^{\iota}V$. One may easily check from the definition given in [Lam05, VII §1] that the quadratic form $\mathrm{tr}_\star(\psi)$ is nothing but the restriction of $\psi+{}^{\iota}\psi$ to the F-vector space of fixed points $(V \oplus {}^\iota V)^s$, where s is the switch semi-linear automorphism defined on the direct sum $V \oplus {}^\iota V$ by $s(v_1 + {}^\iota v_2) = v_2 + {}^\iota v_1$.

Moreover, s induces a semi-linear automorphism of order 2 of the tensor algebra $T(V \oplus {}^\iota V)$ which preserves the ideal generated by the elements

$$(v_1 + {}^\iota v_2) \otimes (v_1 + {}^\iota v_2) - \big(\psi(v_1) + {}^\iota\psi({}^\iota v_2)\big).$$

Hence, we get a semi-linear automorphism s of order 2 on the Clifford algebra $\mathcal{C}(\psi + {}^\iota\psi)$, which commutes with the canonical involution. The set of fixed points $\big(\mathcal{C}(\psi + {}^\iota\psi)\big)^s$ is an F-algebra; the involution γ restricts to an F-linear involution which we denote by γ_s. We then have:

Lemma 2.3. *The natural embedding* $(V \oplus {}^\iota V) \hookrightarrow \mathcal{C}(\psi + {}^\iota\psi)$*, restricted to* $(V + {}^\iota V)^s$*, induces an isomorphism of graded algebras*

$$\big(\mathcal{C}(\mathrm{tr}_\star(\psi)), \gamma\big) \tilde{\to} \big((\mathcal{C}(\psi + {}^\iota\psi))^s, \gamma_s\big).$$

Proof of Lemma 2.3. The natural embedding $(V \oplus {}^\iota V) \hookrightarrow \mathcal{C}(\psi + {}^\iota\psi)$ restricts to an injective map $i : (V + {}^\iota V)^s \hookrightarrow \mathcal{C}(\psi + {}^\iota\psi)^s$, which clearly satisfies

$$i(w)^2 = (\psi + {}^\iota\psi)(w) \quad \text{for any } w \in (V \oplus {}^\iota V)^s.$$

By the universal property of Clifford algebras, it extends to a non-trivial algebra homomorphism $\mathcal{C}(\mathrm{tr}_\star(\psi)) \mapsto \mathcal{C}(\psi + {}^\iota\psi)^s$, which clearly preserves the grading. Since $\mathcal{C}(\mathrm{tr}_\star(\psi))$ is simple, and both algebras have the same dimension, it is an isomorphism. Clearly, γ coincides with γ_s under this isomorphism. □

Hence, we want to describe one component of $\mathcal{C}_0(\mathrm{tr}_\star(\psi)) \simeq (\mathcal{C}_0(\psi + {}^\iota\psi))^s$. We proceed as in the split étale case. Fix an orthogonal basis $e_1, \dots e_n$ of V over L such that $\psi(e_n) = \lambda$. The elements ${}^\iota e_1, \dots, {}^\iota e_n$ are an orthogonal basis of ${}^\iota V$ and ${}^\iota\psi({}^\iota e_n) = \iota(\lambda)$. We may moreover assume that $z = e_1 \dots e_n$ and ${}^\iota z = {}^\iota e_1 \dots {}^\iota e_n$ have square 1. Since the idempotent $\varepsilon = \frac{1 + z^\iota z}{2} \in C_0(\psi + {}^\iota\psi)$ satisfies $s(\varepsilon) = \varepsilon$, the semilinear automorphism s preserves each factor $\mathcal{C}_+(\psi + {}^\iota\psi)$ and $\mathcal{C}_-(\psi + {}^\iota\psi)$. Hence, the components of $\mathcal{C}_0(\mathrm{tr}_\star(\psi))$ are

$$\mathcal{C}_0(\mathrm{tr}_\star(\psi)) = (\mathcal{C}_+(\psi + {}^\iota\psi))^s \times (\mathcal{C}_-(\psi + {}^\iota\psi))^s.$$

Moreover, by Lemma 2.2, we have

$$\mathcal{C}_+(\psi + {}^\iota\psi) \simeq \mathrm{Ad}_{\langle\langle -\lambda\iota(\lambda)\rangle\rangle} \otimes (\mathcal{C}_+(\psi), \gamma) \otimes (\mathcal{C}_+({}^\iota\psi), \gamma),$$

and it remains to understand the action of the switch automorphism on this tensor product. First, one may identify $\mathcal{C}_+({}^\iota\psi)$ with the algebra ${}^\iota\mathcal{C}_+(\psi)$ defined by

$${}^\iota\mathcal{C}_+(\psi) = \{{}^\iota x,\ x \in \mathcal{C}_+(\psi)\},$$

with the operations

$${}^\iota x + {}^\iota y = {}^\iota(x + y),\quad {}^\iota x {}^\iota y = {}^\iota(xy) \text{ and } {}^\iota(\lambda x) = \iota(\lambda){}^\iota x,$$

for all $x, y \in \mathcal{C}_+(\psi)$ and $\lambda \in L$. Clearly, the switch automorphism acts on the tensor product

$$\mathcal{C}_+(\psi) \otimes \mathcal{C}_+({}^\iota\psi) \simeq \mathcal{C}_+(\psi) \otimes {}^\iota\mathcal{C}_+(\psi),$$

by

$$s(x \otimes {}^\iota y) = y \otimes {}^\iota x,$$

and by definition of the corestriction (see [KMRT98, 3.B]), the F-subalgebra of fixed points is

$$\big((\mathcal{C}_+(\psi), \gamma) \otimes ({}^\iota\mathcal{C}_+(\psi), \gamma)\big)^s = N_{L/F}(\mathcal{C}_+(\psi), \gamma).$$

It remains to understand the action of the switch on the centralizer, which is the split quaternion algebra over L generated by $x = \frac{z+{}^\iota z}{2}\varepsilon$ and $y = e_n{}^\iota e_n\varepsilon$. The element x clearly is s-symmetric, while y satisfies $s(y) = -y$. Let δ be a generator of the quadratic extension L/F, so that $\iota(\delta) = -\delta$ and $\delta^2 = d$. Since the switch map s is L/F semi-linear, we may replace y by δy which now satisfies $s(\delta y) = \delta y$. Hence, the set of fixed points under s is the split quaternion algebra over F generated by x and δy. Since $(\delta y)^2 = -dN_{L/F}(\lambda)$, it is isomorphic to $\mathrm{Ad}_{\langle\langle -dN_{L/F}(\lambda)\rangle\rangle}$. □

3. Proof of the decomposability theorem

In this section, we finish the proof of Theorem 1.1. Let $(A,\sigma) = \otimes_{i=1}^3(Q_i,\sigma_i)$ be a product of three quaternion algebras with orthogonal involution. We assume that A has index 4, so that it is Brauer-equivalent to a biquaternion division algebra D. We have to prove that (A,σ) is isomorphic to $(D,\theta)\otimes \mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}$ for a well chosen involution θ on D and some $\lambda \in F^\times$.

The algebra D is endowed with an orthogonal involution τ, and we may represent

$$(A,\sigma) = (\mathrm{End}_D(M), \mathrm{ad}_h),$$

for some 2-dimensional hermitian module (M,h) over (D,τ). Let us consider a diagonalisation $\langle a_1, a_2\rangle$ of h, and define

$$\theta = \mathrm{Int}(a_1^{-1})\circ\tau.$$

With respect to this new involution, we get another representation

$$(A,\sigma) = (\mathrm{End}_D(M), \mathrm{ad}_{h'}),$$

where h' is a hermitian form over (D,θ) which diagonalises as $h' = \langle 1, -a\rangle$ for some θ-symmetric element $a \in D^\times$. The theorem now follows from the following lemma:

Lemma 3.1. *The involutions θ and $\theta' = \mathrm{Int}(a^{-1})\circ\theta$ of the biquaternion algebra D are conjugate.*

Indeed, assume there exists $u \in A^\times$ such that $\theta = \mathrm{Int}(u)\circ\theta'\circ\mathrm{Int}(u^{-1})$. We then have $\theta = \mathrm{Int}(ua^{-1})\circ\theta\circ\mathrm{Int}(u^{-1}) = \theta\circ\mathrm{Int}(\theta(u)^{-1}au^{-1})$. Hence, there exists $\lambda \in F^\times$ such that $\theta(u)^{-1}au^{-1} = \lambda$, that is $a = \lambda\theta(u)u$. This implies that the hermitian form $h' = \langle 1, -a\rangle$ over (D,θ) is isometric to $\langle 1, -\lambda\rangle$. Since $\lambda \in F^\times$, we get $(A,\sigma) = (\mathrm{End}_D(M), \mathrm{ad}_{\langle 1,-\lambda\rangle}) = (D,\theta)\otimes\mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}$, and it only remains to prove the lemma.

Proof of Lemma 3.1. We want to compare the orthogonal involutions θ and θ' of the biquaternion algebra D. By [LT99, Prop. 2], they are conjugate if and only if their Clifford algebras $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic as F-algebras. This can be proven as follows.

Since (A,σ) is a product of three quaternion algebras with involution, we know from [KMRT98, (42.11)] that the discriminant of σ is 1 and its Clifford algebra has one split component.

On the other hand, the representation $(A,\sigma) = (\mathrm{End}_D(M), \mathrm{ad}_{\langle 1,-a\rangle})$ tells us that (A,σ) is an orthogonal sum, as in [Dej95], of (D,θ) and (D,θ'). Hence its invariants can be computed in terms of those of (D,θ) and (D,θ'). By [Dej95, Prop. 2.3.3], the discriminant of σ is the product of the discriminants of θ and θ'. So θ and θ' have the same discriminant, and we may identify the centers Z and Z' of their Clifford algebras in two different ways. We are in the situation described in [LT99, p. 265], where the Clifford algebra of such an orthogonal sum is computed. In particular, since one component of the Clifford algebra of (A,σ) is split, it follows from [LT99, Lem 1] that

$$\mathcal{C}\simeq\mathcal{C}' \qquad \text{or} \qquad \mathcal{C}\simeq{}^{\iota}\mathcal{C}',$$

depending on the chosen identification between Z and Z'. In both cases, $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic as F-algebras, and this concludes the proof. □

4. A new proof of Izhboldin and Karpenko's theorem

Let ϕ be an 8-dimensional quadratic form over F with trivial discriminant and Clifford invariant of index 4. We denote by (A,σ) one component of its even Clifford algebra, so that

$$(\mathcal{C}_0(\phi),\gamma)\simeq(A,\sigma)\times(A,\sigma),$$

where A is an index 4 central simple algebra over F, with orthogonal involution σ.

By triality [KMRT98, (42.3)], the involution σ has trivial discriminant and its Clifford algebra is

$$\mathcal{C}(A,\sigma)=\mathrm{Ad}_\phi\times(A,\sigma).$$

In particular, it has a split component, so that the algebra with involution (A,σ) is isomorphic to a tensor product of three quaternion algebras with involution (see [KMRT98, (42.11)]). Hence we can apply our decomposability theorem 1.1, and write $(A,\sigma)=(D,\theta)\otimes\mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}$ for some biquaternion division algebra with orthogonal involution (D,θ) and some $\lambda\in F^\times$.

Let us denote by d the discriminant of θ, and let $L=F[X]/(X^2-d)$ be the corresponding quadratic étale extension. Consider the image δ of X in L. By Tao's computation of the Clifford algebra of a tensor product [Tao95, Thm. 4.12], the components of $\mathcal{C}(A,\sigma)$ are Brauer-equivalent to the quaternion algebra (d,λ) over F and the tensor product $(d,\lambda)\otimes A$. Since A has index 4, the split component has to be (d,λ), so that λ is a norm of L/F, say $\lambda=N_{L/F}(\mu)$.

Consider now the Clifford algebra of (D,θ). It is a quaternion algebra Q over L, endowed with its canonical (symplectic) involution γ. Denote by n_Q the norm form of Q, that is $n_Q=\langle\langle\alpha,\beta\rangle\rangle$ if $Q=(\alpha,\beta)_L$. It is a 2-fold Pfister form and for any $\ell\in L^\star$, $(\mathcal{C}_+(\langle\ell\rangle n_Q),\gamma_+)\simeq(Q,\gamma)$. Moreover, by the equivalence of categories $A_1^2\equiv D_2$ described in [KMRT98, (15.7)], the algebra with involution (D,θ) is canonically isomorphic to $N_{L/F}(Q,\gamma)$.

Hence we get that $(A,\sigma)=N_{L/F}(Q,\gamma)\otimes\mathrm{Ad}_{\langle\langle -dN_{L/F}(\delta\mu)\rangle\rangle}$. By Proposition 2.1, this implies that

$$(A,\sigma)\times(A,\sigma)\simeq(\mathcal{C}_0(\mathrm{tr}_\star(\psi)),\gamma),$$

where $\psi=\langle\delta\mu\rangle n_Q$. Applying again triality [KMRT98, (42.3)], we get that the split component Ad_ϕ of the Clifford algebra of (A,σ) also is isomorphic to $\mathrm{Ad}_{\mathrm{tr}_\star(\psi)}$, so that the quadratic forms ϕ and $\mathrm{tr}_\star(\psi)$ are similar. This concludes the proof since ψ belongs to $GP_2(L)$.

Remark. Let ϕ and (A,σ) be as above, and let $L=F[X]/(X^2-d)$ be a fixed quadratic étale extension of F. It follows from the proof that the quadratic form ϕ is isometric to the transfer of a form $\psi\in GP_2(L)$ if and only if (A,σ) admits a decomposition $(A,\sigma)=\mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}\otimes(D,\theta)$, with $d_\pm(\theta)=d$. In particular, the quadratic form ϕ is a sum of two forms similar to 2-fold Pfister forms exactly when the algebra with involution (A,σ) admits a decomposition as $(D,\theta)\otimes\mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}$ with θ of discriminant 1, that is when it decomposes as a tensor product of three quaternion algebras with involution, with one split factor.

Such a decomposition does not always exist, as was shown by Sivatski [Siv05, Prop 5]. This reflects the fact that 8-dimensional quadratic forms ϕ with trivial discriminant and Clifford algebra of index ≤ 4 do not always decompose as a sum of two forms similar to two-fold Pfister forms (see [IK00, §16] and [HT98] for explicit examples).

References

[Ara75] J. K. Arason – "Cohomologische Invarianten quadratischer Formen", *J. Alg.* **36** (1975), p. 448–491.

[Bec08] K. J. Becher – "A proof of the Pfister factor conjecture", *Invent. Math.* **173** (2008), no. 1, p. 1–6.

[Dej95] I. Dejaiffe – "Somme orthogonale d'algèbres à involution et algèbre de Clifford", *Comm. Algebra* **26(5)** (1995), p. 1589–1612.

[HT98] D. W. Hoffmann et J.-P. Tignol – "On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property", *Doc. Math.* **3** (1998), p. 189–214 (electronic).

[IK00] O. T. Izhboldin et N. A. Karpenko – "Some new examples in the theory of quadratic forms", *Math. Z.* **234** (2000), no. 4, p. 647–695.

[KMRT98] M.-A. Knus, S. Merkurjev, M. Rost et J.-P. Tignol – *The book of involutions*, Colloquium Publ., vol. 44, Amer. Math. Soc., Providence, RI, 1998.

[Kne77] M. Knebusch – "Generic splitting of quadratic forms. II", *Proc. London Math. Soc. (3)* **34** (1977), no. 1, p. 1–31.

[Lam05] T.-Y. Lam – *Introduction to quadratic forms over fields*, Grad. Studies in Math., vol. 67, Amer. Math. Soc., 2005.

[LT99] D. W. Lewis et J.-P. Tignol – "Classification theorems for central simple algebras with involution", *Manuscripta Math.* **100** (1999), no. 3, p. 259–276, With an appendix by R. Parimala.

[Siv05] A. S. Sivatski – "Applications of Clifford algebras to involutions and quadratic forms", *Comm. Algebra* **33** (2005), no. 3, p. 937–951.

[Tao95] D. Tao – "The generalized even Clifford algebra", *J. Algebra* **172** (1995), no. 1, p. 184–204.

Département de Mathématiques, Université catholique de Louvain, Chemin du cyclotron, 2, B1348 Louvain-la-Neuve, Belgique

E-mail address: Alexandre.Masquelein@uclouvain.be

Université Paris 13 (LAGA), CNRS (UMR 7539), Université Paris 12 (IUFM), 93430 Villetaneuse, France

E-mail address: queguin@math.univ-paris13.fr

URL: http://www-math.univ-paris13.fr/~queguin/

Département de Mathématiques, Université catholique de Louvain, Chemin du cyclotron, 2, B1348 Louvain-la-Neuve, Belgique

E-mail address: jean-pierre.tignol@uclouvain.be

URL: http://wwww.math.ucl.ac.be/membres/tignol