

SPECIALIZATION OF FORMS IN THE PRESENCE OF CHARACTERISTIC 2: FIRST STEPS

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1. INTRODUCTION

We outline a specialization theory of quadratic and (symmetric) bilinear forms with respect to a place $\lambda : K \rightarrow L \cup \infty$. Here K, L denote fields of any characteristic. We have to make a distinction between bilinear forms and quadratic forms and study them both over fields and valuation rings.

For bilinear forms this turns out to be essentially as easy as in the case $\text{char } L \neq 2$, albeit no general cancellation law holds for nondegenerate bilinear forms over a valuation domain \mathcal{O} , in which 2 is not a unit. For quadratic forms things are more difficult mainly for two reasons. 1) Forms cannot be diagonalized. 2) The quasilinear part of an anisotropic form over \mathcal{O} may become isotropic over the residue class field of \mathcal{O} .

Nevertheless a somewhat restricted specialization theory for quadratic forms is possible which is good enough to establish a fully fledged generic splitting theory. On the other hand it seems, that for bilinear forms no generic splitting is possible. (Most probably there does not exist a “generic zero field” for a bilinear form over a field of characteristic 2.) But specialization of bilinear forms is nevertheless important for generic splitting of quadratic forms, since a bilinear form and a quadratic form can be multiplied via tensor product to give another quadratic form.

All this is explicated in a recent book by the author [Spez]. The book contains more material than outlined here. In particular its last chapter IV gives a specialization theory of forms under “quadratic places”, much more tricky than the theory for ordinary places. Miraculously this leads to a generic splitting theory with respect to quadratic places which is as satisfactory as for ordinary places.

If φ is a quadratic form over a field K which has “good reduction” with respect to a place $\lambda : K \rightarrow L \cup \infty$ then our specialization theory gives a quadratic form $\lambda_*(\varphi)$ over L . We also develop a theory of “weak specialization”, which associates to φ only a Witt class $\lambda_W(\varphi)$ of forms over L , but under a more general condition on φ than just having good reduction. In the present article weak specialization plays only an auxiliary role in order to define specializations $\lambda_*(\varphi)$. But weak specialization is a key notion in establishing the specialization theory for quadratic places (not described here, cf. [Spez, Chap. IV]).

The book [Spez] is in German. It is now in the process of translation into English by Thomas Unger. A preprint of the first two chapters is available [Spez’].

Everything said in §2 – §6 of the present article can be found with proofs and/or references in these two chapters. I have freely borrowed from passages in Unger’s translation. I also give almost no references here to the work of others, referring to the references in the book instead.

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2. SPECIALIZATION OF SYMMETRIC BILINEAR FORMS

We are given a place $\lambda : K \rightarrow L \cup \infty$ and a symmetric bilinear form φ , i.e., a polynomial

$$\varphi(x, y) = \sum_{i,j=1}^n a_{ij}x_iy_j \quad (1)$$

over K in two sets of variables $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, with coefficients $a_{ij} = a_{ji} \in K$. Under suitable conditions (“good reduction”, see below) we want to associate to φ a symmetric bilinear form $\lambda_*(\varphi)$ over L in a reasonable way.

We assume that φ is nondegenerate, i.e., $\det(a_{ij}) \in K^*$, and we want that $\lambda_*(\varphi)$ is again a nondegenerate form, of the same dimension $n = \dim \varphi$ as φ .

For the rest of this section a form always means a *nondegenerate symmetric bilinear form*. We denote the form φ above by the symmetric matrix (a_{ij}) . Nondegeneracy of φ means that $\det(a_{ij}) \neq 0$.

We call two forms $\varphi = (a_{ij}), \psi = (b_{ij})$ *isometric* (= isomorphic), and write $\varphi \cong \psi$, if $\dim \varphi = \dim \psi$ and ψ is obtained from φ by a linear change of coordinates, in matrix notation

$$(b_{ij}) = {}^tU(a_{ij})U \quad (2)$$

with some $U \in GL(n, K)$.

Let \mathcal{O}_λ denote the valuation ring of $\lambda, \mathcal{O}_\lambda = \{x \in K \mid \lambda(x) \neq \infty\}$.

Definition 2.1. We say that the form $\varphi = (a_{ij})$ has *good reduction* with respect to the place $\lambda : K \rightarrow L \cup \infty$, if there exists a symmetric matrix (b_{ij}) with coefficients in \mathcal{O}_λ and $\det(b_{ij})$ a unit of \mathcal{O}_λ , such that φ is isometric to the form (b_{ij}) over K . Alternatively we then say that φ is λ -*unimodular*, and we call an isometry $\varphi \cong (b_{ij})$ a λ -*unimodular representation* of φ .

In this situation we are tempted to define

$$\lambda_*(\varphi) := (\lambda(b_{ij})), \quad (3)$$

hoping that - up to isometry - the form $(\lambda(b_{ij}))$ does not depend on the choice of the λ -unimodular representation of φ .

(N.B.: We do not care to identify a form with an isometric form, thus abusively speaking of “forms” instead of isometry classes of forms.)

In this hope justified? The answer will be

“Yes”, if $\text{char } L \neq 2$, and “Nearly”, if $\text{char } L = 2$.

Our approach to the question will be via Witt rings. We briefly recall the definition of the Witt ring $W(K)$. We call two forms φ and ψ over K *stably isometric*, if there exists a form χ over K such that $\varphi \perp \chi \cong \psi \perp \chi$. We then write

$\varphi \approx \psi$. If $\text{char } K \neq 2$ then $\varphi \approx \psi$ implies $\varphi \cong \psi$ by Witt's cancellation theorem. For $\text{char } K = 2$, this is false.

Definition 2.2. We say that two forms φ and ψ over K are *Witt equivalent*, and then write $\varphi \sim \psi$, if there exist numbers $r, s \in \mathbb{N}_0$ such that

$$\varphi \perp r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \approx \psi \perp s \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Witt (equivalence) class of a form φ will be denoted by $\{\varphi\}$.

Witt classes can be added and multiplied as follows:

$$\{\varphi\} + \{\psi\} := \{\varphi \perp \psi\},$$

$$\{\varphi\} \cdot \{\psi\} := \{\varphi \otimes \psi\},$$

where \perp and \otimes denote the usual orthogonal sum and tensor product of symmetric bilinear forms. In this way the set of the Witt classes over K becomes a well defined commutative ring with 1, the *Witt ring* $W(K)$. The zero element is given by the class $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ (or by the zero-dimensional form $\varphi = 0$, which we admit), and the unit element by the class $\{(1)\}$ of the one-dimensional form (1). For any form φ , we have $\{\varphi\} + \{-\varphi\} = 0$.

A good insight into Witt equivalence is given by the following Proposition 2.3. First a bit of notation. A form φ of dimension n is called *isotropic*, if there exists some $x \in K^n, x \neq 0$, with $\varphi(x, x) = 0$, and *anisotropic* otherwise. φ is called *metabolic* if

$$\varphi \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \dots \perp \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}$$

for some $i > 0$ and $a_1, \dots, a_r \in K$.

Every form φ has a decomposition

$$\varphi \cong \varphi_0 \perp \varphi_1$$

with φ_0 anisotropic and φ_1 metabolic, called a *Witt decomposition* of φ .

Proposition 2.3. *Let $\varphi \cong \varphi_0 \perp \varphi_1$ and $\psi \cong \psi_0 \perp \psi_1$ be Witt decompositions of two forms φ and ψ . Then $\varphi \sim \psi$ iff $\varphi_0 \cong \psi_0$.* \square

In particular, the anisotropic part φ_0 of φ is uniquely determined by φ up to isometry. We call φ_0 the *kernel form* of φ and write $\varphi_0 = \ker(\varphi)$.

(Alternatively we may call φ_0 the *anisotropic part* of φ and write $\varphi_0 = \varphi_{an}$.)

As a consequence of Proposition 2.3 we state

Corollary 2.4. *$\varphi \approx \psi$ iff $\varphi \sim \psi$ and $\dim \varphi = \dim \psi$.* \square

Given elements $a_1, \dots, a_n \in K^*$, we denote the diagonal form

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

more succinctly by $\langle a_1, \dots, a_n \rangle$. We have the rules

$$\begin{aligned} \langle a_1, \dots, a_n \rangle \perp \langle b_1, \dots, b_m \rangle &\cong \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle, \\ \langle a_1, \dots, a_n \rangle \otimes \langle b_1, \dots, b_m \rangle &\cong \langle a_1 b_1, \dots, a_1 b_m, a_2 b_1, \dots, a_n b_m \rangle, \\ \langle a, -a \rangle &\cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

If for a form φ there exists at least one vector x with $\varphi(x, x) \neq 0$, then φ has an orthogonal basis, i.e. φ can be diagonalized, $\varphi \cong \langle a_1, \dots, a_n \rangle$ for some $a_i \in K^*$. Otherwise φ is an orthogonal sum $m \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of copies of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, hence $\varphi \sim 0$. Thus the Witt ring $W(K)$ is additively generated by the classes $\{\langle a \rangle\}$ with a running through K^* .

As a very special case of Proposition 1.1 we observe that two classes $\{\langle a \rangle\}, \{\langle b \rangle\}$ are equal iff $\langle a \rangle \cong \langle b \rangle$ iff the square classes aK^* and bK^{*2} are equal. In the following we identify the set of these Witt classes, and also the set of isometry classes of one-dimensional forms over K , with the group $Q(K) = K^*/K^{*2}$ of square classes of K .

We have $\langle a \rangle \otimes \langle b \rangle = \langle ab \rangle$, and thus may - and will - regard $Q(K)$ as a subgroup of the group $W(K)^*$ of units of the Witt ring $W(K)$.

We return to the place $\lambda : K \rightarrow L \cup \infty$ with valuation ring $\mathcal{O} := \mathcal{O}_\lambda$. Our specialization theory of bilinear forms is based on the following theorem.

Theorem 2.5. *There exists a well defined additive map $\lambda_W : W(K) \rightarrow W(L)$, which can be characterized as follows. If a is a unit of \mathcal{O} , then $\lambda_W(\langle a \rangle) = \langle \lambda(a) \rangle$. If a square class $\langle a \rangle = aK^{*2}$ does not contain a unit of \mathcal{O} , then $\lambda_W(\langle a \rangle) = 0$.¹*

This can be proved by using a description of the additive group of $W(K)$ by generators and relations. We have an additive map Λ_W from the group ring $\mathbb{Z}[Q(K)]$ to $W(L)$, which maps a group element $\langle a \rangle \in Q(K)$ to $\langle \lambda(a) \rangle$ if $a \in \mathcal{O}^*$, and to 0 if aK^{*2} does not contain a unit of \mathcal{O} .

The obvious surjection $\mathbb{Z}[Q(K)] \rightarrow W(K)$ has a kernel \mathfrak{a} which can be described explicitly (cf. [Spez, §2]).

One then verifies that $\Lambda_W(\mathfrak{a}) = 0$. Thus Λ_W factors through an additive map $\lambda_W : W(K) \rightarrow W(L)$ with the properties stated in the theorem.

Proposition 2.6. *Assume that the form φ has good reduction under λ , and $\varphi \cong (b_{ij})$ is a λ -unimodular representation of φ . Then*

$$\lambda_W(\{\varphi\}) = \{\langle \lambda(b_{ij}) \rangle\} \tag{4}$$

□

This is obvious from Theorem 1.3 if (b_{ij}) is a diagonal matrix. In the general case one has to argue that the symmetric matrix can be “diagonalized over \mathcal{O} ”, i.e.,

¹The letter W in the notation λ_W refers to “Witt” or “weak” (cf. also §5).

there exists an equation

$$(b_{ij}) = {}^t U \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} U \quad (5)$$

with $c_i \in \mathcal{O}^*$ and $U \in GL(n, \mathcal{O})$.

This is not always true, but becomes true if we replace (b_{ij}) , say, by $(b_{ij}) \perp \langle 1, -1 \rangle$. The proof is best understood in the geometric setting to be developed in §3.

Corollary 2.7. *Assume that φ and ψ are forms over K with good reduction and that $\varphi \cong (a_{ij}), \psi \cong (b_{ij})$ are λ -unimodular representations. If $\varphi \approx \psi$, then $(\lambda(b_{ij})) \approx (\lambda(c_{ij}))$.*

Proof. We conclude by Theorem 2.5 that the forms $(\lambda(b_{ij}))$ and $(\lambda(c_{ij}))$ are Witt equivalent, and then by Proposition 2.4 that they are stably isometric, since they have the same dimension. \square

In particular, if $\varphi \cong (a_{ij})$ and $\varphi \cong (b_{ij})$ are two λ -unimodular representations of a form φ over K then the forms $(\lambda(a_{ij}))$ and $(\lambda(b_{ij}))$ over L are stably isometric. Abusively we call $(\lambda(a_{ij}))$ “the” specialization of φ under λ , and denote this form by $\lambda_*(\varphi)$, although $\lambda_*(\varphi)$ is uniquely determined by φ and λ only up to stable isometry.

3. BILINEAR MODULES

We now switch to the “geometric language” for bilinear and - later (§4) - quadratic forms. Everything said in this section is very well known.

We first fix the basic notation valid for the rest of the paper. \mathcal{O} always denotes a valuation domain, \mathfrak{m} its maximal ideal, $k = \mathcal{O}/\mathfrak{m}$ its residue class field and $K = \text{Quot}(\mathcal{O})$ its quotient field, \mathcal{O}^* denotes the group of units of \mathcal{O} , hence $\mathcal{O}^* = \mathcal{O} \setminus \mathfrak{m}$.

The case $\mathfrak{m} = \{0\}$, i.e., $\mathcal{O} = K$, is by no means excluded.

A bilinear module $M = (M, B)$ over \mathcal{O} consists of an \mathcal{O} -module M and a symmetric bilinear form $B : M \times M \rightarrow \mathcal{O}$. If nothing else is said, we tacitly assume that the \mathcal{O} -module M is free of finite rank n . We write $n = \dim M$. If e_1, \dots, e_n is a basis of M , then B is given by the symmetric $n \times n$ -matrix (a_{ij}) with $a_{ij} = B(e_i, e_j)$.

Abusively we denote $M = (M, B)$, or better, its isometry class by this matrix (a_{ij}) . If e_1, \dots, e_n is an orthogonal basis, $a_{ij} = a_i \delta_{ij}$, we denote the bilinear module M also by $\langle a_1, \dots, a_n \rangle$.

We call the bilinear module M (or the form B) non degenerate if B gives an isomorphism of \mathcal{O} -modules $x \mapsto B(x, -)$ from M to its dual module $\tilde{M} = \text{Hom}_{\mathcal{O}}(M, \mathcal{O})$. This means that $\det(a_{ij}) \in \mathcal{O}^*$. We then also say that M is a bilinear space over \mathcal{O} .

It is well known that if M is a bilinear space containing a vector x with $B(x, x) \in \mathcal{O}^*$, then M has an orthogonal basis, hence $M \cong \langle a_1, \dots, a_n \rangle$ with $a_i \in \mathcal{O}^*$. This fills the gap in our sketch of proof of §2, Proposition 2.6.

We call a submodule N of a bilinear space M a subspace of M , if N is a direct summand of the module M . It will be helpful to remember that every finitely

generated torsion free \mathcal{O} -module is free. Thus a submodule N of M is a subspace iff M/N is torsion free.

For any subset S of a bilinear module M the module

$$S^\perp = \{x \in M \mid B(x, S) = 0\}$$

is a direct summand of M , since M/S^\perp is clearly torsion free and finitely generated.

We call a bilinear space M *isotropic*, if M contains a subspace $U \neq 0$, which is “totally isotropic”, i.e., $B(U, U) = 0$, in other terms, $U \subset U^\perp$. Otherwise we call M *anisotropic*.

Since \mathcal{O} has no zero divisors, and every finitely generated ideal of \mathcal{O} is principal, it is easily seen that M is isotropic iff there exists a vector $x \neq 0$ in M with $B(x, x) = 0$.

Indeed, we may always write $x = cz$ with $c \in \mathcal{O}$ and z a *primitive vector* of M , i.e., a vector z , such that $\mathcal{O}z$ is a direct summand of the module M . { N.B.: If e_1, \dots, e_n is a basis of M and $z = a_1e_1 + \dots + a_n e_n$, then z is primitive iff $a_1\mathcal{O} + \dots + a_n\mathcal{O} = \mathcal{O}$. }

We call M *metabolic* if M contains a subspace $U = U^\perp$. Equivalently we can say, that M is metabolic iff M contains a totally isotropic subspace U with $2 \dim U = \dim M$. Every metabolic space M has an orthogonal decomposition

$$M \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \dots \perp \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}$$

with some $a_i \in \mathcal{O}$. Notice that in the case $\mathcal{O} = K$ our present terminology is in complete harmony with §2, identifying isometry classes of spaces and isometry classes of forms in the obvious way.

Every bilinear space M is an orthogonal sum of an anisotropic space M_0 and a metabolic space M_1 . But now, in contrast to the case $\mathcal{O} = K$ (cf. Prop. 2.3), the isometry class of M_0 usually is not uniquely determined by M , if $2 \notin \mathcal{O}^*$.

For the rest of this section “space” always means “bilinear space”. Exactly as in §2 we define stable isometry (\approx) and Witt equivalence (\sim) of forms over \mathcal{O} , and then proceed as there to the Witt ring $W(\mathcal{O})$ consisting of the Witt (equivalence) classes of spaces.

We denote the Witt class of a space M by $\{M\}$. It turns out that $\{M\} = 0$, i.e., $M \sim 0$, iff M is metabolic. Also, for every space $M = (M, B)$ the space $(M, B) \perp (M, -B)$ is metabolic. Thus, abbreviating the space $(M, -B)$ by $-M$, we have $\{-M\} = -\{M\}$ in $W(\mathcal{O})$.

The bilinear form B on M extends in a unique way to a K -bilinear form B' on the K -vector space $E := K \otimes_{\mathcal{O}} M$ obeying the formula

$$B'(c \otimes x, d \otimes y) = cd B(x, y) \tag{6}$$

for $x, y \in M$ and $c, d \in K$. Identifying an element x of M with $1 \otimes x \in E$, we regard the free module M as an \mathcal{O} -submodule of E . We then have $B'|_{M \times M} = B$. A basis e_1, \dots, e_n of M over \mathcal{O} is also a basis of E over K , and the spaces M and E have the same symmetric matrix (a_{ij}) with respect to e_1, \dots, e_n . We often write

B instead of B' .

If U is a subspace of E , then $U \cap M$ is a subspace of M , and $K \cdot (U \cap M) = U$. In this way the subspaces of E correspond uniquely to the subspaces of M . Clearly U is totally isotropic iff $U \cap M$ is totally isotropic. Thus the following proposition is pretty obvious.

Proposition 3.1. *Let M be a space over \mathcal{O} and $E := K \otimes_{\mathcal{O}} M$.*

- a) *E is isotropic iff M is isotropic.*
- b) *E is metabolic iff M is metabolic.*

It follows that the natural map $\{M\} \mapsto \{K \otimes_{\mathcal{O}} M\}$ from $W(\mathcal{O})$ to $W(K)$, which is a ring homomorphism, is injective. We will often regard $W(\mathcal{O})$ as a subring of $W(K)$.

The square class group $Q(\mathcal{O}) := \mathcal{O}^*/\mathcal{O}^{*2}$ of \mathcal{O} injects into $Q(K) = K^*/K^{*2}$ since clearly every unit of \mathcal{O} which is a square in K is a square in \mathcal{O} . As previously in the case of fields we identify a square class $a\mathcal{O}^{*2}$, $a \in \mathcal{O}^*$, with the one-dimensional space $\langle a \rangle$ over \mathcal{O} (more precisely, with its isometry class), and then observe that the natural map $Q(\mathcal{O}) \rightarrow W(\mathcal{O})$ is injective, due to a natural commuting square

$$\begin{array}{ccc} Q(\mathcal{O}) & \longrightarrow & W(\mathcal{O}) \\ \downarrow & & \downarrow \\ Q(K) & \hookrightarrow & W(K) \end{array} \quad (7)$$

Thus $Q(\mathcal{O})$ can – and will – be also viewed as a subgroup of $W(\mathcal{O})^*$. In other terms, if $a, b \in \mathcal{O}^*$ then $\langle a \rangle \sim \langle b \rangle$ iff $\langle a \rangle \cong \langle b \rangle$.

Without invoking the commutative square (7) this can be also proved by use of the signed determinant

$$d(M) := \langle (-1)^{\frac{n(n-1)}{2}} \det(a_{ij}) \rangle \quad (8)$$

of an n -dimensional space $M \cong (a_{ij})$ over \mathcal{O} .

We switch to a place $\lambda : K \rightarrow L \cup \infty$ with valuation domain $\mathcal{O} = \mathcal{O}_\lambda$. The notation from §2 (λ_W , good reduction, λ_* etc.) will be freely used for spaces instead of forms.

Our place λ restricts to a ring homomorphism $\lambda|_{\mathcal{O}}$ from \mathcal{O} to L , and $\lambda|_{\mathcal{O}}$ factors through a field embedding $\bar{\lambda} : k \hookrightarrow L$. The definition of good reduction (Def. 2.1) and specialization under λ now reads as follows.

Scholium 3.2. *A bilinear space E over K has good reduction under λ iff $E \cong K \otimes_{\mathcal{O}} M$ for some bilinear space M over \mathcal{O} . In this case*

$$\lambda_*(E) \approx L \otimes_{\lambda} M = L \otimes_{\bar{\lambda}} \bar{M}. \quad (9)$$

Here $L \otimes_\lambda M$ denotes the scalar extension of the bilinear module M to L via $\lambda | \mathcal{O}$,² and \overline{M} is the bilinear space $M/\mathfrak{m}M$ over k obtained from M by reduction modulo \mathfrak{m} .

□

Example 3.3. Every metabolic space over K has good reduction. This follows easily from the fact that, for any $a, c \in K$ we have $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} ac^2 & 1 \\ 1 & 0 \end{pmatrix}$. □

Corollary 2.7 tells us that, if E and F are spaces with good reduction and $E \approx F$, then $\lambda_*(E) \approx \lambda_*(F)$. { In particular $\lambda_*(E)$ is well defined up to stable isometry. } This can now be proved in another, more transparent way as follows.

Let $E \cong K \otimes_{\mathcal{O}} M$, $F \cong K \otimes_{\mathcal{O}} N$ with spaces M, N over \mathcal{O} . Then $K \otimes_{\mathcal{O}} (M \perp -N) \cong E \perp -F$ is metabolic, hence $M \perp -N$ is metabolic by Proposition 3.1, and this implies that

$$L \otimes_\lambda (M \perp -N) \cong L \otimes_\lambda M \perp (-L \otimes_\lambda N)$$

is metabolic. Thus $L \otimes_\lambda M$ and $L \otimes_\lambda N$ are Witt equivalent. Since these spaces have the same dimension, they are stably isomorphic.

We also want to describe the map λ_W from §2 in geometric language.

Preparing for this we add more notation, which will be important also for later sections.

We choose a surjective valuation $v : K \rightarrow \Gamma \cup \infty$, essentially unique, associated with our valuation domain \mathcal{O} . So $\Gamma \cong K^*/\mathcal{O}^*$. { We use additive notation for Γ , so $v(xy) = v(x) + v(y)$. } We regard $Q(\mathcal{O})$ as a subgroup of $Q(K)$, and we choose a complement Σ of $Q(\mathcal{O})$ in $Q(K)$, i.e., a subgroup Σ of $Q(K)$ with $Q(K) = Q(\mathcal{O}) \times \Sigma$.

This is possible, since the group $Q(K)$ is elementary abelian of exponent 2. Further, we choose, for every square class $\sigma \in \Sigma$ an element $s \in \mathcal{O}$ with $\sigma = \langle s \rangle$. For $\sigma = 1$ we choose the representative $s = 1$. Let S be the set of these elements s . For every $a \in K^*$, there exists exactly one $s \in S$ and elements $\varepsilon \in \mathcal{O}^*, b \in K^*$ with $a = \varepsilon s b^2$. Since $K^*/\mathcal{O}^* \cong \Gamma$, it is clear that S (resp. Σ) is a system of representatives of $\Gamma/2\Gamma$ in K^* (resp. $Q(K)$) for the homomorphism from K^* (resp. $Q(K)$) onto $\Gamma/2\Gamma$ determined by $v : K^* \rightarrow \Gamma$.

Definition 3.4. A λ -modular decomposition of a bilinear space E over K is an orthogonal decomposition

$$E \cong \bigsqcup_{s \in S} \langle s \rangle \otimes (K \otimes_{\mathcal{O}} M_s)$$

with every M_s a space over \mathcal{O} and only finitely many $M_s \neq 0$. Here the unadorned \otimes means tensor product over K . Instead of “ λ -modular” we also use the word “ \mathcal{O} -modular”, since not the place λ but only the valuation domain \mathcal{O} is involved.

□

Every space E over K has a λ -modular decomposition. Indeed, we may decompose E orthogonally in one-dimensional spaces and metabolic planes, usually in

²The bilinear form of $L \otimes_\lambda M$ is defined by a formula analogous to (6) above.

many ways. One-dimensional spaces are products $\langle s \rangle \otimes \langle \varepsilon \rangle$ with $s \in S, \varepsilon \in \mathcal{O}^*$, and metabolic spaces are orthogonal sums of forms $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ with $a \in \mathcal{O}$. One then simply gathers summands belonging to the same $s \in S$.

The following is now obvious from §2.

Scholium 3.5. *Assume that*

$$E \cong \perp_{s \in S} \langle s \rangle \otimes (K \otimes_{\mathcal{O}} M_s)$$

is a λ -modular decomposition of a space E over K . Then

$$\lambda_W(\{E\}) = \{L \otimes_{\lambda} M_1\}. \quad (10)$$

□

In particular the space $L \otimes_{\mathcal{O}} M_1$ over L is uniquely determined by E up to Witt equivalence. In contrast to Scholium 3.2 we do not have a proof of this fact in simple geometric terms. Thus we cannot assert that the present “geometric language” supersedes the “algebraic language” of §2. We call the space $L \otimes_{\mathcal{O}} M_1$ a *weak specialization of E* with respect to λ .

We add an important result about good reduction. Starting from now we often abbreviate “good reduction” by “GR”.

First notice the trivial fact, that, if E and F are spaces over K with GR under λ , then $E \perp F$ has again GR under λ , and

$$\lambda_*(E \perp F) \approx \lambda_*(E) \perp \lambda_*(F). \quad (11)$$

Theorem 3.6. *Let E and F bilinear spaces over K . Assume that F and $E \perp F$ have GR under λ . Then E has GR under λ .*

Proof. Adding $-F$ to the space F we retreat to the case that F is metabolic. Let $E \perp F \cong K \otimes_{\mathcal{O}} N$ with N a space over \mathcal{O} . We choose decomposition $E \cong E_0 \perp E_1$ and $N \cong N_0 \perp N_1$ with E_0 and N_0 anisotropic, E_1 and N_1 metabolic.

Then

$$E_0 \perp E_1 \perp F \cong K \otimes_{\mathcal{O}} N_0 \perp K \otimes_{\mathcal{O}} N_1.$$

The spaces E_0 and $K \otimes_{\mathcal{O}} N_0$ are anisotropic, and the spaces $E_1 \perp F$ and $K \otimes_{\mathcal{O}} N_1$ are metabolic. We conclude by Proposition 2.3 that $E_0 \cong K \otimes_{\mathcal{O}} N_0$. Thus E_0 has GR. The space E_1 is metabolic, hence also has GR. Thus $E \cong E_0 \perp E_1$ has GR. □

4. QUADRATIC MODULES

We retain the notation and conventions of §3. In particular, \mathcal{O} denotes a valuation domain, and modules over \mathcal{O} will be free of finite rank, if nothing else is said.

A *quadratic module* $M = (M, q)$ over \mathcal{O} is an \mathcal{O} -module M equipped with a quadratic form q . This is a function $q : M \rightarrow \mathcal{O}$ such that $q(cx) = c^2q(x)$ for $c \in \mathcal{O}, x \in M$, and the map $B_q : M \times M \rightarrow \mathcal{O}$ given by

$$B_q(x, y) = q(x + y) - q(x) - q(y) \quad (12)$$

is \mathcal{O} -bilinear. If e_1, \dots, e_n is a basis of M then q is determined by the values $a_i = q(e_i)$, $a_{ij} = B(e_i, e_j)$ for $i \neq j$. More precisely,

$$q\left(\sum_1^n x_i e_i\right) = \sum_{i=1}^n a_i x_i^2 + \sum_{i < j} a_{ij} x_i x_j \quad (13)$$

for $x_1, \dots, x_n \in \mathcal{O}$.

We have an obvious notion of isometry (= isomorphism) between quadratic module over \mathcal{O} , and most often will be only interested in the isometry class of a quadratic module (M, q) . Slightly abusively we abbreviate a quadratic module M with quadratic form (13) by the symmetric matrix $[a_{ij}]$ in square brackets, with a_{ij} as above and $a_{ii} := a_i$. The *associated bilinear module* $\widetilde{M} := (M, B_q)$ is described by the matrix (b_{ij}) with $b_{ii} = 2a_i$, and $b_{ij} = a_{ij}$ for $i \neq j$.

The *orthogonal sum* of two quadratic modules (M_1, q_1) and (M_2, q_2) over \mathcal{O} is defined by

$$(M_1, q_1) \perp (M_2, q_2) := (M_1 \oplus M_2, q_1 \perp q_2)$$

with

$$(q_1 \perp q_2)(x_1 + x_2) := q_1(x_1) + q_2(x_2)$$

($x_1 \in M_1, x_2 \in M_2$). Notice that the associated bilinear module of $(M_1, q_1) \perp (M_2, q_2)$ is the orthogonal sum $\widetilde{M}_1 \perp \widetilde{M}_2$.

Orthogonality in a quadratic module $M = (M, q)$ refers to the bilinear form B_q . In particular, if N_1 and N_2 are submodules of M , then $M = N_1 \perp N_2$ means that $M = N_1 \oplus N_2$ as an \mathcal{O} -module and $B_q(N_1, N_2) = 0$. The following fact will be used frequently.

Lemma 4.1. *Let $M = (M, q)$ be a quadratic module. Assume that N is a submodule of M and the bilinear form $B_q|_{N \times N}$ is non-degenerate. Then*

$$M = N \oplus N^\perp.$$

□

Often we will denote a quadratic module by one letter, say M , without specifying the quadratic form on M . We then usually denote this form by q and the associated bilinear form B_q by B .

If $2 \neq 0$ in \mathcal{O} , then a quadratic module M may be viewed as a bilinear module with $B(x, x) \in 2\mathcal{O}$ for every $x \in M$ via the formula $B(x, x) = 2q(x)$, and if 2 is a unit of \mathcal{O} we may identify in this way quadratic and bilinear modules over \mathcal{O} . But, if $2 = 0$ in \mathcal{O} , bilinear and quadratic modules over \mathcal{O} are rather different objects.

Definition 4.2.

- a) A quadratic module $N = (N, q)$ is called *quasilinear* if $B_q = 0$.
- b) If M is any quadratic module over \mathcal{O} , then the quadratic module

$$M^\perp := \{x \in M \mid B(x, M) = 0\}$$

with the form $q|_{M^\perp}$ is called the *quasilinear part* of M . We denote it by $QL(M)$.

M^\perp is a direct summand of the \mathcal{O} -module M . Choosing any submodule N of M with $M = N \oplus M^\perp$ we have

$$M \cong N \perp QL(M)$$

as quadratic module. Moreover, $QL(M)$ is an orthogonal sum of quadratic modules of dimension (= rank) 1,

$$QL(M) = [a_1] \perp \dots \perp [a_n]$$

with $a_i \in \mathcal{O}$.

The following definition will be central for our theory of good reduction and specialization of a quadratic form over a field under a place.

Definition 4.3. We call a quadratic module (M, q) over \mathcal{O} *nondegenerate*, if it satisfies the following conditions:

- (Q0) M is free of finite rank.
- (Q1) The bilinear form \overline{B}_q , induced by B_q on M/M^\perp in the obvious way, is nondegenerate.
- (Q2) $q(x) \in \mathcal{O}^*$ for every vector x in M^\perp , which is primitive in M^\perp (and hence in M).

If instead of (Q2), the following condition is satisfied

$$(Q2') \quad QL(M) = 0 \text{ or } QL(M) \cong [\varepsilon] \text{ with } \varepsilon \in \mathcal{O}^*,$$

then we call (M, q) *regular*.

In the special case $M^\perp = 0$, we call (M, q) *strictly regular*.

Comment on conditions (Q2) and (Q2').

If $2 \neq 0$ in \mathcal{O} , then $q|_{M^\perp} = 0$ and the requirement (Q2) implies $M^\perp = 0$, hence implies - in conjunction with (Q0) and (Q1) - strict regularity. If $2 \in \mathcal{O}^*$, then the nondegenerate quadratic \mathcal{O} -modules are the same objects as the nondegenerate bilinear \mathcal{O} -modules, as defined in §3.

Suppose now that $2 = 0$ in \mathcal{O} . The condition $M^\perp = 0$, in other words, strict regularity, is very natural but too limited for applications. Indeed, if $M^\perp = 0$, then the bilinear module (M, B_q) is nondegenerate and we have $B_q(x, x) = 2q(x) = 0$ for every $x \in M$. This implies that M has even dimension, as is well-known. (To prove this, consider the bilinear space $K \otimes_{\mathcal{O}} \widetilde{M}$.) So, if we insist on using strict regularity, we can only deal with quadratic forms of even dimension.

On the other hand, property Q2 has an annoying effect: Q2 is not always preserved under a base extension. If $\mathcal{O}' \supset \mathcal{O}$ is another valuation domain, whose maximal ideal \mathfrak{m}' lies over \mathfrak{m} , i.e., $\mathfrak{m}' \cap \mathcal{O} = \mathfrak{m}$, and if M is non degenerate, then $\mathcal{O}' \otimes_{\mathcal{O}} M$ can be degenerate. However, if M satisfies (Q2'), this clearly cannot happen. \square

In the case $\mathfrak{m} = 0$, i.e., $\mathcal{O} = K$, we call a non degenerate quadratic \mathcal{O} -module a *quadratic space* over K .

We gather some facts about nondegenerate quadratic modules, all to be found in [Spez, §6]. In the following $M = (M, q)$ is a quadratic module over \mathcal{O} .

Fact 4.4. *If M is nondegenerate then M is an orthogonal sum of quadratic modules $\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$, with $\alpha, \beta \in \mathcal{O}, 1 - 4\alpha\beta \in \mathcal{O}^*$, and modules $[\varepsilon]$ with $\varepsilon \in \mathcal{O}^*$. \square*

Fact 4.5. *If M is regular and $\dim M$ is even, then M is strictly regular and equal to an orthogonal sum of modules $\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$, with $\alpha, \beta \in \mathcal{O}, 1 - 4\alpha\beta \in \mathcal{O}^*$. \square*

Fact 4.6. *Assume that M is nondegenerate. Then every primitive vector $e \in M$ with $q(e) = 0$ can be completed to a hyperbolic vector pair, i.e., a pair e, f with $q(f) = 0$ and $B(e, f) = 1$. \square*

As an illustration, how our conditions Q1 and Q2 can be put to work, we give the proof of 4.6. We choose a decomposition $M = N \perp M^\perp$ and write $e = x + y$ with $x \in N, y \in M^\perp$. Suppose for the sake of contradiction that the vector x is not primitive in N , hence not primitive in M . Then y is primitive in M^\perp , and thus $q(y) \in \mathcal{O}^*$ by condition Q2. Hence also $q(x) = -q(y) \in \mathcal{O}^*$ and x has to be primitive, a contradiction.

Thus x is primitive in N . Since B_q is nondegenerate on N , there exists some $z \in N$ with $B_q(x, z) = 1$. We also have $B_q(e, z) = 1$. Clearly $f := z - q(z)e$ completes the vector e to a hyperbolic pair. \square

Fact 4.7. *(Cancellation theorem) If M and N are quadratic \mathcal{O} -modules and G is a strictly regular quadratic \mathcal{O} -module with $M \perp G \cong N \perp G$, then $M \cong N$. \square*

We call a quadratic \mathcal{O} -module M *isotropic*, if M contains a vector $x \neq 0$ with $q(x) = 0$, *anisotropic* otherwise. We call M *hyperbolic* if M is isometric to an orthogonal sum $r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of r copies of the ‘‘hyperbolic plane’’ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for some $r \in \mathbb{N}$. As in the bilinear case one proves easily:

Fact 4.8. *M is isotropic iff the quadratic module $K \otimes_{\mathcal{O}} M$ over K is isotropic, and M is hyperbolic iff $K \otimes_{\mathcal{O}} M$ is hyperbolic. \square*

Fact 4.9. *(Witt decomposition). If M is nondegenerate, then*

$$M \cong M_0 \perp r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with M_0 anisotropic (and nondegenerate) and $r \in \mathbb{N}_0$. \square

It follows from 4.7 that the number r and the isometry class of M_0 are uniquely determined by M . We call r the (Witt-) *index* of M and M_0 the *kernel module*, or the *anisotropic part* of M , and we write $r = \text{ind}(M), M_0 = \ker(M)$. We often denote the hyperbolic plane $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by H (regardless which ring \mathcal{O} is under consideration).

Notice also that, by 4.8, we have $\text{ind}(K \otimes_{\mathcal{O}} M) = \text{ind}(M)$ and $\ker(K \otimes_{\mathcal{O}} M) = K \otimes_{\mathcal{O}} \ker(M)$.

Fact 4.10. *If $M = (M, q)$ is strictly regular then*

$$(M, q) \perp (M, -q) \cong (\dim M) \times H.$$

\square

If we write M for (M, q) we usually write $-M$ for $(M, -q)$, following the same practice as for bilinear modules.

Definition 4.11. We call two nondegenerate quadratic modules M and N over \mathcal{O} *Witt-equivalent*, and write $M \sim N$ if there exist natural numbers s, t such that

$$M \perp s \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cong N \sim t \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Due to 4.9 this happens iff $\ker(M) \cong \ker(N)$. The *Witt class* of M , i.e., the equivalence of M under \sim , will be denoted by $\{M\}$. \square

It is now easy to verify:

Fact 4.12. *If M, M' are strictly regular and N, N' are non degenerate quadratic modules over \mathcal{O} with $M \sim M'$ and $N \sim N'$, then $M \perp N$ and $M' \perp N'$ are non degenerate and $M \perp N \sim M' \perp N'$.* \square

Definition 4.13. We denote the set of Witt classes of nondegenerate quadratic \mathcal{O} -modules by $\widetilde{\text{Wq}}(\mathcal{O})$. We denote the subsets of Witt classes of regular, resp. strictly regular quadratic \mathcal{O} -modules by $\text{Wqr}(\mathcal{O})$, resp. $\text{Wq}(\mathcal{O})$. \square

Due to 4.12 we have a well-defined “addition” of classes $\{M\} \in \text{Wq}(\mathcal{O})$ with classes $\{N\} \in \widetilde{\text{Wq}}(\mathcal{O})$,

$$\{M\} + \{N\} := \{M \perp N\}.$$

Restricting also $\{N\}$ to $\text{Wq}(\mathcal{O})$ we obtain on $\text{Wq}(\mathcal{O})$ the structure of an abelian group. This groups $\text{Wq}(\mathcal{O})$ operates by addition on the set $\widetilde{\text{Wq}}(\mathcal{O})$. The subset $\text{Wqr}(\mathcal{O})$ is a union of orbits.

Definition 4.14. We call $\text{Wq}(\mathcal{O})$ the *quadratic Witt group* of \mathcal{O} , $\widetilde{\text{Wq}}(\mathcal{O})$ the *quadratic Witt set* of \mathcal{O} , and $\text{Wqr}(\mathcal{O})$ the *regular quadratic Witt set* of \mathcal{O} . \square

In the case $\mathcal{O} = K$ we go further and define Witt classes of arbitrary (finite dimensional) quadratic modules over K as follows.

Starting with such a module $M = (M, q)$ we define the “defect” of M by

$$\delta(M) = \{x \in M^\perp : q(x) = 0\}.$$

The form q gives us a quadratic form \bar{q} on $M/\delta(M)$ in the obvious way, hence a quadratic space $(M/\delta(M), \bar{q})$, which we call the *quadratic space associated to M* and denote by \widehat{M} . Clearly

$$M \cong \widehat{M} \perp \delta(M) \cong \widehat{M} \perp s \times [0]$$

with $s := \dim \delta(M)$. We have

$$M \cong M_0 \perp r \times H$$

with \widehat{M}_0 anisotropic and $r \in \mathbb{N}_0$.

Again M_0 and r are uniquely determined by M , and again we call M_0 the *kernel module* $\ker(M)$ of M and r the *index* $\text{ind}(M)$ of M . Notice that now M_0 can be isotropic. Notice that in contrast to strictly regular quadratic spaces we do not have cancellation in general. For example $[a] \perp [a] \cong [0] \perp [a]$ for any $a \in K$.

We call two quadratic modules M, N over K *Witt equivalent*, and write $M \sim N$, if $M \perp s \times H \cong N \perp t \times H$ for some number s, t . This means the same as $\ker(M) \cong \ker(N)$. We denote the set of Witt equivalence classes $\{M\}$ of quadratic K -modules M by $\widehat{Wq}(K)$, and we call $\widehat{Wq}(K)$ the *defective quadratic Witt set* of K . The value $\{M\} + \{N\} = \{M \perp N\}$ makes $\widehat{Wq}(K)$ an abelian semigroup with neutral element $\{0\} = \{H\}$. It contains $\widetilde{Wq}(K)$ as a subset and $Wq(K)$ as a subgroup.

The reason why we need $\widehat{Wq}(K)$ instead of just $\widetilde{Wq}(K)$ is lack of functoriality of the latter set. If $K' \supset K$ is a field extension, we have a well defined semigroup homomorphism $\widehat{Wq}(K) \rightarrow \widehat{Wq}(K')$ mapping a class $\{M\}$ to $\{K' \otimes_K M\}$. This homomorphism does not map $\widetilde{Wq}(K)$ to $\widetilde{Wq}(K')$ in general (if $\text{char } K = 2$), since for a space M over K the quasilinear part of $K' \otimes_K M$ may be isotropic.

We return to an arbitrary valuation ring \mathcal{O} . If $M_1 = (M_1, B_1)$ is a bilinear \mathcal{O} -module and $M_2 = (M_2, q_2)$ is a quadratic \mathcal{O} -module, we can install on the tensor product $M_1 \otimes_{\mathcal{O}} M_2$ a quadratic form $q := B_1 \otimes q_2$ by choosing a (non symmetric) bilinear form β_2 with $\beta_2(z, z) = q_2(z)$ for all $z \in M_2$, taking the tensor product $\beta := B_1 \otimes \beta_2$ on $M_1 \otimes M_2$, and putting $q(x) := \beta(x, x)$. The quadratic form q is independent of the choice of β , and can be characterized by the rules

$$B_q = B_1 \otimes B_{q_2}, \quad q(x_1 \otimes x_2) = B_1(x_1, x_1)q_2(x_2)$$

$$(x_1 \in M_1, x_2 \in M_2).$$

We denote the quadratic module $(M_1 \otimes_{\mathcal{O}} M_2, q)$ by $M_1 \otimes M_2$ for short. If $M_1 = \langle a_1, \dots, a_n \rangle$, then

$$M_1 \otimes M_2 \cong (M_2, a_1 q_2) \perp \dots \perp (M_2, a_n q_2).$$

In particular, for any $a \in K$,

$$\langle a \rangle \otimes M_2 \cong (M_2, a q_2).$$

If M_1 is non degenerate and M_2 is strictly regular then $M_1 \otimes M_2$ is strictly regular. It is now straightforward to verify that we have a well defined product of Witt classes

$$\{M_1\} \cdot \{M_2\} = \{M_1 \otimes M_2\},$$

which turns $Wq(\mathcal{O})$ into a module over the ring $W(\mathcal{O})$. { Notice in particular that, if $M_1 \approx M'_1$ then $M_1 \otimes M_2 \cong M'_1 \otimes M_2$. } Unfortunately there seems to be no good way to let $W(\mathcal{O})$ operate on $\widehat{Wq}(\mathcal{O})$.

5. WEAK SPECIALIZATION AND GOOD REDUCTION

As in previous sections $\lambda : K \rightarrow L \cup \infty$ is a place, $\mathcal{O} = \mathcal{O}_\lambda$ is the valuation domain of λ , \mathfrak{m} its maximal ideal and $k = \mathcal{O}/\mathfrak{m}$ its residue class field. Let $E = (E, q)$ be a quadratic space over K .

Definition 5.1. We say that E has *good reduction* (abbreviated: GR) with respect to λ if $E \cong K \otimes_{\mathcal{O}} M$ with M a *non degenerate* quadratic \mathcal{O} -module. \square

In this situation we obtain from E a quadratic L -module

$$\lambda_*(E) := L \otimes_\lambda M = L \otimes_{\overline{\lambda}} M/\mathfrak{m}M \quad (14)$$

(Notations analogous to those in §3). Notice that the “reduced” quadratic module $M/\mathfrak{m}M$ over k is non degenerate, but $L \otimes_{\overline{\lambda}} M/\mathfrak{m}M$ may be degenerate, since the quasilinear part of $M/\mathfrak{m}M$ may become isotropic over L .

We would like to prove that the quadratic module $\lambda_*(E)$ is independent of the choice of M .

Only then the notation $\lambda_*(E)$ will be justified. If E is strictly regular this can be done by the same sort of geometric argument as used in §3 in the bilinear case. To prove it in the general case we would like to use an additive map $\lambda_W : \widehat{Wq}(K) \rightarrow \widehat{Wq}(L)$, similar to the map $\lambda_W : W(K) \rightarrow W(L)$ from §2, and then to proceed in a similar way as in §2 and §3 for bilinear spaces. But now a new path has to be taken, since we do not have a presentation of $\widehat{Wq}(K)$ by generators and relations which fits well with the place λ .

Let \mathcal{O}^h denote the henselization of \mathcal{O} , K^h its field of quotient (= the henselization of K with respect to \mathcal{O}). λ extends to a place $\lambda^h : K^h \rightarrow L \cup \infty$ with valuation ring \mathcal{O}^h , since \mathcal{O}^h has the same residue class field $\mathcal{O}^h/\mathfrak{m}^h = \mathcal{O}/\mathfrak{m}$ as \mathcal{O} . If M is a non degenerate quadratic \mathcal{O} -module, $M^h := \mathcal{O}^h \otimes_{\mathcal{O}} M$ is again non degenerate and

$$L \otimes_\lambda M = L \otimes_{\lambda^h} M^h.$$

Thus we can retreat to the case that \mathcal{O} is henselian.

Here the following lemma offers help.

Lemma 5.2. *Assume that \mathcal{O} is henselian. Let $E = (E, q)$ be an anisotropic quadratic space over K .*

(a) *The sets*

$$\mu(E) := \{x \in E \mid q(x) \in \mathcal{O}\} \text{ and } \mu_+(E) := \{x \in E \mid q(x) \in \mathfrak{m}\}$$

are \mathcal{O} -submodules of E .

(b) *For any $x \in \mu(E)$ and $y \in \mu_+(E)$ we have $q(x + y) - q(x) \in \mathfrak{m}$ and $B_q(x, y) \in \mathfrak{m}$.*

□

By this lemma

$$\rho(E) := \mu(E)/\mu_+(E)$$

is a k -vector space in a natural sense ($k = \mathcal{O}/\mathfrak{m}$). We define a function $\overline{q} : \rho(E) \rightarrow k$ as follows:

$$\overline{q}(\overline{x}) := \overline{q(x)} \quad (x \in \mu(E)),$$

where \overline{x} denotes the image of $x \in \mu(E)$ in $\rho(E)$ and \overline{a} denotes the image of $a \in \mathcal{O}$ in k . Lemma 4.1 tells us that the map \overline{q} is well defined, and, using the lemma further, one proves easily that \overline{q} is a quadratic form on the k -vector space $\rho(E)$ with associated bilinear form $\overline{B} = B_{\overline{q}}$ given by

$$\overline{B}(\overline{x}, \overline{y}) = \overline{B(x, y)}.$$

The quadratic k -module $(\rho(E), \bar{q})$ is clearly anisotropic.

If \mathcal{O} is not necessarily henselian we are motivated by this lemma to make the following Ansatz in order to associate to a space E over K a Witt class $\lambda_W\{E\}$ over L :

$$\lambda_W\{E\} := \{L \otimes_{\bar{\lambda}} \rho(\ker(K^h \otimes E))\}, \quad (15)$$

where, as before, $\bar{\lambda}: k \hookrightarrow L$ is the field embedding determined by λ .

All good and well, if only we know whether the vector space

$$\rho(\ker(K^h \otimes E))$$

has finite dimension! To guarantee this we have to confine the class of allowed quadratic modules E .

As explicated in §3 we choose a system S of representatives of $\Gamma/2\Gamma$ in K (with $1 \in S$), where $\Gamma = K^*/\mathcal{O}^*$ is the value group of the natural valuation associated to \mathcal{O} .

Definition 5.3. A quadratic space E over K is *obedient* with respect to λ if E has an orthogonal decomposition

$$E = \perp_{s \in S} E_s, \quad (16)$$

such that each space $(E_s, s \cdot (q|_{E_s}))$ has GR under λ , hence

$$E_s = KM_s \cong \langle s \rangle \otimes (K \otimes_{\mathcal{O}} M_s) \quad (17)$$

with M_s a non degenerate quadratic \mathcal{O} -submodule of E_s .³ Then (16) is called a λ -*modular decomposition* of E , and (16), (17) is called a λ -*modular representation* of E . Instead of “ λ -modular” we also use the term “ \mathcal{O} -modular”. \square

It is no big deal to verify the following two lemmas.

Lemma 5.4. *Assume that \mathcal{O} is henselian and E is an obedient anisotropic quadratic space over K , with \mathcal{O} -modular representation (16), (17). Then $(M_1, q|_{M_1})$ is the only non degenerate quadratic \mathcal{O} -submodule of E_1 , and*

$$\begin{aligned} \mu(E) &= M_1 \perp \perp_{s \neq 1} \mu_+(E_s), \\ \mu_+(E) &= \mathfrak{m}M_1 \perp \perp_{s \neq 1} \mu_+(E_s). \end{aligned}$$

Thus

$$(\rho(E), \bar{q}) \cong (M_1/\mathfrak{m}M_1, \bar{q}_1)$$

with $q_1 := q|_{M_1}$. \square

³Of course, $E_s \neq 0$ only for finitely many $s \in S$.

Lemma 5.5. *Let \mathcal{O} be henselian. Let s_1, \dots, s_r be different elements of S and M_1, \dots, M_r anisotropic nondegenerate quadratic \mathcal{O} -modules. Then*

$$E := \perp_{i=1}^n \langle s_i \rangle \otimes (K \otimes_{\mathcal{O}} M_i)$$

is an anisotropic quadratic space over K . □

We arrive at the main theorem of this section.

Theorem 5.6. *Let E be a quadratic space over K , obedient with respect to \mathcal{O} . Let*

$$E = \perp_{s \in S} E_s = \perp_{s \in S} F_s$$

be two \mathcal{O} -modular decompositions of E , and also let M_1, N_1 be nondegenerate quadratic \mathcal{O} -modules with $E_1 \cong K \otimes_{\mathcal{O}} M_1, F_1 \cong K \otimes_{\mathcal{O}} N_1$. Then the quadratic spaces $M_1/\mathfrak{m}M_1$ and $N_1/\mathfrak{m}N_1$ over $k = \mathcal{O}/\mathfrak{m}$ are Witt equivalent.

For the proof one passes from K to K^h , chooses Witt decompositions of the quadratic \mathcal{O}^h -modules M_i^h, N_j^h , and then computes the kernel space of E^h in two different ways applying Lemma 5. Then Lemma 4 gives the result.

Definition 5.7. Let E be a quadratic space over K , obedient with respect to λ . If $E = \perp_{s \in S} E_s$ is a λ -modular decomposition of E , and M_1 is a non degenerate quadratic \mathcal{O} -module with $E_1 \cong K \otimes_{\mathcal{O}} M_1$, then we call the quadratic space

$$L \otimes_{\lambda} M_1 = L \otimes_{\overline{\lambda}} M_1/\mathfrak{m}M_1$$

a *weak specialization* of E with respect to λ . (As before, \otimes_{λ} denotes a base extension with respect to the homomorphism $\lambda|_{\mathcal{O}} : \mathcal{O} \rightarrow L$.)

By Theorem 5.6, the space $L \otimes_{\lambda} M_1$ is uniquely determined by E and λ , up to Witt equivalence. We denote its Witt class by $\lambda_W(E)$, i.e.,

$$\lambda_W(E) := \{L \otimes_{\lambda} M_1\} \in \widehat{\text{Wq}}(L).$$

(“W” as in “Witt” or “weak”.) □

Remark. If E is strictly regular, then E_1 is strictly regular, hence M_1 is strictly regular, and we conclude that $\lambda_W(E) \in \text{Wq}(L)$. In particular this happens if $\text{char}K \neq 2$. If E is only regular then M_1 is still regular, hence $\lambda_W(E) \in \text{Wqr}(L)$, since now the quasilinear part of $L \otimes_{\lambda} M_1$ has at most dimension 1, hence is anisotropic. If $\text{char}L \neq 2$ then quadratic spaces over K resp. L can be identified with bilinear spaces over K resp. L , and the present weak specialization coincides with the weak specialization of §2 and §3. □

Corollary 5.8. *If E and E' are quadratic spaces over K , both obedient with respect to λ , and if $E \sim E'$, then $\lambda_W(E) = \lambda_W(E')$.*

Proof. This can be quickly deduced from Theorem 5.6. Suppose without loss of generality that $\dim E \leq \dim E'$. Then $E' \cong E \perp r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for a certain $r \in \mathbb{N}_0$. If we choose a non degenerate \mathcal{O} -module M_1 for E_1 , as in Definition 5.7, the

$M'_1 := M_1 \perp r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a possible choice for E' . Therefore, $L \otimes_\lambda M'_1 \sim L \otimes_\lambda M_1$. \square

Remark 5.9. *Let E and F be quadratic spaces over K , obedient with respect to λ , and suppose that E is strictly regular.*

Obviously we then have

$$\lambda_W(E \perp F) = \lambda_W(E) + \lambda_W(F).$$

\square

{The addition of an element of $\text{Wq}(K)$ and an element of $\widetilde{\text{Wq}}(K)$ has been explained in §4.}

We do not exploit here the full power of weak specializations but use them only to justify the Ansatz (15) from the beginning of the section for specializing a space with good reduction.

Scholium 5.10. *Assume that E has GR under λ , $E \cong K \otimes_{\mathcal{O}} M$ with M a non-degenerate quadratic \mathcal{O} -module. Then $L \otimes_\lambda M$ is uniquely determined by E and λ up to isometry. Indeed, the Witt class of $L \otimes_\lambda M$ does not depend on the choice of M by Theorem 5.6, and $\dim L \otimes_\lambda M = \dim E$.*

Definition 5.11. If E has GR under λ we define

$$\lambda_*(E) := L \otimes_\lambda M,$$

and we call $\lambda_*(E)$ the *specialization* of E under λ .

If E and F are quadratic K -spaces with GR under λ and E is strictly regular, then $E \perp F$ has GR under λ and clearly

$$\lambda_*(E \perp F) = \lambda_*(E) \perp \lambda_*(F).$$

By arguments analogous to the proof of Theorem 3.6 one now obtain the following important fact.

Theorem 5.12. *Let F and G be quadratic spaces over K . Suppose that F is strictly regular. If F and $F \perp G$ have GR with respect to λ , then G also has GR with respect to λ .*

We mention that – under some precaution – weak specialization is compatible with the tensor product of a bilinear and quadratic space. For example the following holds.

Remark 5.13. *Let F be a bilinear space and G a strictly regular quadratic space over K . Suppose that G has GR under λ . Then $F \otimes G$ is obedient with respect to λ , and*

$$\lambda_W(F \otimes G) = \lambda_W(F)\lambda_W(G) = \lambda_W(F)\{\lambda_*(G)\}.$$

\square

6. GENERIC SPLITTING OF QUADRATIC FORMS

In the following quadratic \mathcal{O} -modules leave the stage and will act only from the background. Openly we only deal with quadratic spaces over fields. Thus we switch to the language of quadratic forms (= homogeneous polynomials of degree 2) over fields, freely using the terminology of §4 and §5 for forms instead of spaces.

If $\varphi = \varphi(x_1, \dots, x_n)$ is a form⁴ over a field k and $k \subset K$ is a field extension then $\varphi \otimes K$ denotes the polynomial φ as an element of $K[x_1, \dots, x_n]$ instead of $k[x_1, \dots, x_n]$.

It turns out that we can extend the well known generic splitting theory of forms over fields of characteristic $\neq 2$ to arbitrary fields, as long as we can guarantee that under the relevant places $\lambda : K \rightarrow L \cup \infty$ a given form φ over K with GR with respect to λ^5 has a specialization $\lambda_*(\varphi)$ which is again nondegenerate, i.e., the specialization $\lambda_*(QL(\varphi))$ of the quasilinear part $QL(\varphi)$ remains anisotropic { Slogan: “Do not destroy the quasilinear part!” }

Definition 6.1. Let φ be a nondegenerate form over a field k . We call a field extensions $k \subset K$ φ -conservative if $K \otimes \varphi$ is again nondegenerate, i.e., $K \otimes QL(\varphi)$ is anisotropic. \square

Notice that a separable field extension $k \subset K$ is φ -conservative for every φ , since an anisotropic quasilinear form over k remains anisotropic over K .

Notice also that, if φ is regular then every field extension $k \subset K$ is φ -conservative, since forms of dimension ≤ 1 cannot become isotropic.

The generic splitting theory of a non degenerate form φ over k will deal with the Witt decomposition of $K \otimes \varphi$ for K varying in the class of all φ -conservative field extension of k .

The following observation is crucial here.

Theorem 6.2. Let $\lambda : K \rightarrow L \cup \infty$ be a place and φ a form over K which has GR with respect to λ . Suppose that also $\lambda_*(\varphi)$ is nondegenerate. Suppose further that $K' \supset K$ is a field extension and that $\mu : K' \rightarrow L \cup \infty$ is a place extending λ . Then the form $\varphi \otimes K'$ has GR with respect to μ and

$$\mu_*(\varphi \otimes K') = \lambda_*(\varphi).$$

Proof. Let $\mathcal{O} := \mathcal{O}_\lambda$, $\mathcal{O}' := \mathcal{O}_\mu$, and let k and k' denote the residue class fields of \mathcal{O} and \mathcal{O}' respectively. The field extension $\bar{\lambda} : k \hookrightarrow L$ is a composition of the extensions $k \hookrightarrow k'$ and $\bar{\mu} : k' \hookrightarrow L$, where the first extension is induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}'$.

Let E be a quadratic space for φ and M a nondegenerate quadratic \mathcal{O} -module with $E \cong K \otimes_{\mathcal{O}} M$. Then $K' \otimes E = K' \otimes_{\mathcal{O}'} M'$ with $M' := \mathcal{O}' \otimes_{\mathcal{O}} M$. The

⁴ “Form” will always mean “quadratic form”.

⁵ This assumption presupposes that φ is nondegenerate (cf. §5, Def. 1).

quasilinear quadratic k -module $G := k \otimes_{\mathcal{O}} QL(M)$ is anisotropic. By assumption, $L \otimes_{\overline{\lambda}} G = QL(L \otimes_{\lambda} M)$ is also anisotropic. Therefore

$$k' \otimes_k G = k' \otimes_{\mathcal{O}'} QL(M')$$

is anisotropic. This proves that M' is a nondegenerate quadratic \mathcal{O}' -module. Hence $\varphi \otimes K'$ is nondegenerate and has GR with respect to μ . Furthermore $\mu_*(\varphi \otimes K')$ corresponds to the quadratic space

$$L \otimes_{\mu} M' = L \otimes_{\mu} (\mathcal{O}' \otimes_{\mathcal{O}} M) = L \otimes_{\lambda} M.$$

Hence $\mu_*(\varphi \otimes K') = \lambda_*(\varphi)$. \square

In the following φ is a nondegenerate form over a field k .

Scholium 6.3. *Let $K \supset k, L \supset k$ be field extensions of k , and let $\lambda : K \rightarrow L \cup \infty$ be a place over k , i.e., a place extending the trivial place $k \hookrightarrow L$. Assume that L is φ -conservative. Then Theorem 6.2 tells us that K is φ -conservative, $\varphi \otimes K$ has GR with respect to λ , and $\lambda_*(\varphi \otimes K) = \varphi \otimes L$.*

Let $\varphi \otimes K \cong \varphi_1 \perp r_1 \times H$ be the Witt decomposition of φ . By Theorem 5.12 it follows that φ_1 has GR with respect to λ , and hence

$$\varphi \otimes L = \lambda_*(\varphi \otimes K) = \lambda_*(\varphi_1) \perp r \times H.$$

{We denote the hyperbolic plane $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over any field (or ring) by H .}

Thus $\text{ind}(\varphi \otimes L) \geq \text{ind}(\varphi \otimes K)$, and, in case of equality,

$$\ker(\varphi \otimes L) = \lambda_*(\ker(\varphi \otimes K)).$$

It now follows that, if K and L are specialization equivalent over k , i.e., there exists also a place over k from L to K , then

$$\text{ind}(\varphi \otimes L) = \text{ind}(\varphi \otimes K),$$

and

$$\ker(\varphi \otimes L) = \lambda_*(\ker \varphi \otimes K)$$

for any place λ from K to L over k . \square

Definition 6.4. We call a field extension $K \supset k$ a *generic zero field* of φ , if K is φ -conservative, and there exists a place from K to L over φ for any φ -conservative field extension $L \supset k$ such that $\varphi \otimes L$ is isotropic. \square

Notice that $\varphi \otimes L$ is isotropic iff $\text{ind}(\varphi \otimes L) \geq 1$, since $\varphi \otimes L$ is nondegenerate.

Any two generic zero fields of φ are specialization equivalent over k .

Fortunately generic zero fields of φ exist whenever this makes sense.

Theorem 6.5. *Assume that φ is anisotropic and not quasilinear, $n := \dim \varphi \geq 2$.*

- a) *The function field $k(\varphi)$ of the affine quadric $\varphi(x_1, \dots, x_n) = 0$, i.e., the quotient field of the integral domain*

$$k[x_1, \dots, x_n]/(\varphi(x_1, \dots, x_n))$$

is a generic zero field of φ . {N.B. $k(\varphi)$ is separable over k .}

- b) *More generally the following holds. If $\gamma : k \rightarrow L \cup \infty$ is a place such that φ has GR under γ , and if $\gamma_*(\varphi)$ is nondegenerate and isotropic, there exists a place $\lambda : k(\varphi) \rightarrow L \cup \infty$ extending γ .*

□

We now can build a *generic splitting tower* $(K_r \mid 0 \leq r \leq h)$ of φ in the way well known from the case that $\text{char } K \neq 2$ (cf. [K], [KS], [S], ...) and from the case that $\text{char } k = 2$, but φ regular (cf. [KR]).

Take $K_0 = k$, or more generally, let K_0 be field extension of k such that there exists a place from K_0 to k over k , with corresponding Witt decomposition

$$\varphi \otimes K_0 \cong \varphi_0 \perp i_0 \times H$$

(N.B.: $i_0 = \text{ind}(\varphi)$). If φ_0 is quasilinear, we stop.

Otherwise we choose a generic zero field $K_1 \supset K_0$ of φ_0 , and then have a Witt decomposition

$$\varphi_0 \otimes K_1 \cong \varphi_1 \perp i_1 \times H$$

etc. We could take $K_0 = k, K_1 = k(\varphi_0)$, etc. But for various problems it is useful to allow other generic splitting towers.

We retain the terminology from the generic splitting theory in characteristic $\neq 2$. In particular we call i_r the *r-th higher index* of φ and h the *height* of φ . The form φ_h is quasilinear.

Precisely as in the characteristic $\neq 2$ case we obtain from the above theorems immediately:

Theorem 6.6. *Let φ be a non degenerate form over k . Let $(K_r \mid 0 \leq r \leq h)$ be a generic splitting tower of φ with associated higher indices i_r and higher kernel forms φ_r . Let $\gamma : k \rightarrow L \cup \infty$ be a place such that φ has GR with respect to γ and $\gamma_*(\varphi)$ is non degenerate. We choose a place $\lambda : K_m \rightarrow L \cup \infty$ extending γ such that either $m = r$ as $m < r$, but λ cannot be extended to a place from K_{m+1} to L . Then φ_m has GR with respect to λ . The form $\lambda_*(\varphi)$ has the kernel form $\lambda_*(\varphi_m)$ and the Witt index $i_0 + \dots + i_m$.* □

If $L \supset k$ is a φ -conservative field extension we may apply the theorem to the trivial place $\gamma : k \hookrightarrow L$ and obtain precise information about the Witt decomposition of $\varphi \otimes L$.

7. EPILOGUE

- A) Perhaps the most urgent open problem in generic splitting theory is to determine all forms of height 1. Assume that φ is anisotropic and $h(\varphi) = 1$. If φ is strictly regular, then it turns out that φ is, up to a scalar factor, a quadratic Pfister form. (cf. [Spez, §20]). If $QL(\varphi)$ has dimension 1 then φ is, up to scalar factor, a certain “close neighbor” of a quadratic Pfister form (cf. [Spez, §22]), analogous to the pure part of a Pfister form in the case of characteristic $\neq 2$. But, if $\dim QL(\varphi) \geq 2$ there exist more forms of height 1 than those which are Witt equivalent to close Pfister neighbors.

- B) Let K and L be fields with $\text{char } K = 0$, $\text{char } L = 2$. Given a place $\lambda : K \rightarrow L \cup \infty$, it deserves interest to “lift” a nondegenerate quadratic form ψ over L to a form φ over K , i.e., to exhibit a quadratic form φ over K with GR with respect to λ and $\lambda_*(\varphi) \cong \psi$. Then one can hope to deduce properties of ψ from properties of φ .

In the specialization theory outlined above such a lifting is only possible if ψ is strictly regular. Indeed, since a nondegenerate form φ over K is automatically strictly regular, also $\lambda_*(\varphi)$ has to be strictly regular.

Fortunately there exists a more general specialization theory than the one explicated in §5.

Given a place $\lambda : K \rightarrow L \cup \infty$ with valuation ring \mathcal{O} , we say that a quadratic space $E = (E, q)$ over K has *fair reduction* with respect to λ , if E contains a free \mathcal{O} -submodule M with $E = KM$ and $q(M) \subset \mathcal{O}$, such that $(M/\mathfrak{m}M, \overline{q|_M})$ is a quadratic space over \mathcal{O}/\mathfrak{m} , while $(M, q|_M)$ may be degenerate. One can prove that then

$$\lambda_*(E) := L \otimes_{\overline{\lambda}} (M/\mathfrak{m}M) = L \otimes_{\lambda} M$$

is still well defined up to isometry by E and λ . This is the basis of a “fair specialization theory” which parallels our theory in §5, cf. [Spez, II, §11].

It is now well possible to find for a non degenerate form ψ over L a form φ over K with fair reduction with respect to λ and $\lambda_*(\varphi) \cong \psi$. For fair specializations there also exists a theorem completely analogous to the generic splitting theorem 6.6 above, cf. [Spez, II, §12].

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