Serre's conjecture II: a survey

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Abstract[1]**:** Our goal is to provide an up to date survey of Serre's conjecture II (1962) on the vanishing of Galois cohomology for simply connected semisimple groups defined over a field of cohomological dimension ≤ 2.

Keywords: Galois cohomology, linear algebraic groups.
MSC: 20G05

1 Introduction

Serre's original conjecture II (1962) states that the Galois cohomology set $H^1(k, G)$ vanishes for a semisimple simply connected algebraic group G defined over a perfect field k of cohomological dimension $\mathrm{cd}(k) \leq 2$ [62, §4.1] [63, II.3.1]. In other words, that all G-torsors (or principal homogeneous spaces) over $\mathrm{Spec}(k)$ are trivial.

For example, if A is a central simple algebra defined over a field k and $c \in k^\times$, the subvariety

$$X_c := \{\mathrm{nrd}(y) = c\} \subset \mathbf{GL}_1(A)$$

of elements of reduced norm c is a torsor under the special linear group $G = \mathbf{SL}_1(A)$ which is semisimple and simply connected. If $\mathrm{cd}(k) \leq 2$, we expect then that this G-torsor is trivial, i.e. $X_c(k) \neq \emptyset$. By considering all scalars c, we expect then that the reduced norm map $A^\times \to k^\times$ is surjective.

For imaginary number fields, the surjectivity of the reduced norm map goes back to Eichler in 1938 (see [48, §5.4]). For function fields of complex surfaces, this follows from the Tsen-Lang theorem given that the reduced norm is a homogeneous form of degree $\deg(A)$ in $\deg(A)^2$-indeterminates [63, II.4.5]. The general case of the surjectivity of reduced norm maps was established in 1981 by Merkurjev and Suslin

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[70, th. 24.8]. As we shall explain below, this fact essentially characterizes fields of cohomological dimension ≤ 2.

Throughout its history, the evidence for and progress towards establishing conjecture II has been gathered by either considering special classes of fields, or by looking at the implications that the conjecture would have on the classification of algebraic groups. We will explore both points of view in this survey.

¿From the groups point of view, the strongest evidence for the validity of the conjecture is given by the description of the classical groups established in 1995 by Bayer and Parimala [5]. ¿From the point of view of fields, we know that the conjecture holds in the case of imaginary number fields (Kneser [48], Harder [43], Chernousov [15], see [60, §6]), and more recently for function fields of complex surfaces. For exceptional groups with no factors of type E_8, the relevant reasonings and references are given in [21]. A general proof for all types using deformation methods was given in 2008 by He-de Jong-Starr [46]. This result has a clear geometric meaning: If $G/\mathbb{C}$ is a semisimple simply connected group and X a smooth complex surface, then a G-torsor over X (or a G-bundle) is locally trivial with respect to the Zariski topology (see §6.6).

There are previous surveys on Galois cohomology discussing Serre's conjecture II and related topics. Tits' lectures at Collège de France from 90-91 discuss the Hasse principle and group classification [74]. Serre's Bourbaki seminar deals among other things with progress and the status of conjecture II as of 1994. For classical groups, there is Bayer's survey [3]. For function fields of surfaces, see the surveys of Starr [68] and Lieblich [50].

For exceptional groups (trialitarian, type E_6, E_7 and E_8), the general conjecture is still open in spite of some considerable progress [17][21][23][35].

We finish the introduction by mentioning that Serre's conjecture II can be linked with analogous considerations in Topology within Morel-Voevodky's theory [58]. Indeed, if G is a semisimple simply connected complex group, we know that $\pi_1(G) = \pi_2(G) = 0$, hence G is 2–connected. Then for every CW-complex of dimension ≤ 2, the G-bundles over X are trivial (cf. [71, th. 11.34]).

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2 Fields of cohomological dimension ≤ 2

Let k be a field and l be a prime. Recall that k is of l–cohomological dimension at most d, written $\mathrm{cd}_l(k) \leq d$, if $H^i(k, A) = 0$ for every finite l–primary Galois module

A and for all $i \geq d+1$. We know that this assertion is equivalent to the vanishing of $H^{d+1}(L, \mathbb{Z}/l\mathbb{Z})$ for any finite separable extension L/k. Recall the following examples of fields of cohomological dimension 2.

Examples 2.1. (1) Imaginary number fields;

(2) Function fields of complex surfaces;

(3) Merkurjev's tower of fields F_∞, namely an extension of $\mathbb{C}(X_1, ..., X_{2n})$ such that the u-invariant is $u(F_\infty) = 2n$. This means that every $2n+1$-dimensional quadratic form over F_∞ is isotropic but the form $\langle X_1, X_2, \cdots X_{2n}\rangle$ remains anisotropic over F_∞. Furthermore the tensor product of the quaternion algebras (X_{2i-1}, X_{2i}) for $i = 1, ..., n$ is a division algebra over F_∞ [53][54, th. 3].

The third example shows that central simple algebras and quadratic forms are not in general low dimensional objects. We have already mentioned the following characterization which uses Merkurjev-Suslin's theorem [56].

Theorem 2.2. *[70, th. 24.8] Let l be an invertible prime in k. The following are equivalent:*

1. $\mathrm{cd}_l(k) \leq 2$.
2. *For any finite separable extension L/k and any l-primary central simple L–algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*
3. *For any finite extension L/k and any l-primary central simple L–algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*

We have added here the easy implication $2) \Longrightarrow 3)$ which follows of by a familiar transfer argument. We say that k is of cohomological dimension $\leq d$ if k is of l–cohomological dimension $\mathrm{cd}_l(k) \leq d$ for all primes l.

If k is of positive characteristic p, we always have $\mathrm{cd}_p(k) \leq 1$; this explains the necessary change in the following analogous statement.

Theorem 2.3. *[34, th. 7] Assume that* $\mathrm{char}(k) = p > 0$. *The following are equivalent:*

1. $H^3_p(L) = 0$ *for any finite separable extension L/k;*
2. *For any finite separable extension L/k and any l-primary central simple L–algebra A/L, the reduced norm* $\mathrm{nrd} : A^\times \to L^\times$ *is surjective.*

Here $H_p^3(k)$ is the cohomology group of Kato defined by means of logarithmic differential forms [47], see also [40, §9]. We shall say that k is of separable p–dimension $\leq d$ if $H_p^{d+1}(L) = 0$ for all finite separable extension L/k. This defines in the obvious way the separable dimension $\mathrm{sd}_p(k)$ of k. For $l \neq p$, we define $\mathrm{sd}_l(k) = \mathrm{cd}_l(k)$.[2] If k is perfect, then $H_p^i(L) = 0$ for every finite extension L/k and for every $i \geq 2$. Hence if k is perfect and of cohomological dimension ≤ 2, k is of separable dimension ≤ 2.

Examples 2.4. (1) The function field of a curve over a finite field is of separable dimension 2.
(2) The function field $k_0(S)$ of a surface over an algebraically closed field k_0 of characteristic $p \geq 0$ is of separable dimension 2.
(3) Given an arbitrary field F, Theorems 2.2 and 2.3 provide a way to construct a "generic" field extension E/F of separable dimension 2, see Ducros [25].

We can now state the strong form of Serre's conjecture II. For each simply connected group G, Serre defined the set $S(G)$ of primes in terms of the Cartan-Killing type of G, cf. [64, §2.2]. For absolutely simple almost groups, the sets $S(G)$ are as follows.

Table 1: $S(G)$ for absolutely almost simple groups

type	$S(G)$
A_n $(n \geq 1)$	2 and the prime divisors on $n+1$
B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n (non trialitarian for $n = 4$)	2
G_2	2
trialitarian D_4, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Conjecture 2.5. *Let G be a semisimple simply connected algebraic group. Assume that $\mathrm{sd}_l(k) \leq 2$ for every prime $l \in S(G)$, then $H^1(k, G) = 0$.*

In the original conjecture, k was assumed perfect and of cohomological dimension ≤ 2. In characteristic $p > 0$, Serre's strenghthened question assumed furthermore that $[k : k^p] \leq p^2$ if p belongs to $S(G)$ [64, §5.5]. In view of all known results, it would seem that there is no need for this assumption.

Conjecture 2.5 is indeed stronger than the original one. Theorems 2.2 and 2.3 show that this strong version of the conjecture holds for groups of inner type A, and that the hypothesis on k are sharp.

[2] Kato defined the p–dimension $\dim_p(k)$ as follows [47]. If $[k : k^p] = \infty$, define $\dim_p(k) = \infty$. If $[k : k^p] = p^r < \infty$, $\dim_p(k) = r$ if $H_p^{r+1}(L) = 0$ for any finite extension L/k, and $\dim_p(k) = r + 1$ otherwise.

3 Link between the conjecture and the classification of groups

The classification of semisimple groups reduces essentially to that of semisimple simply connected groups G which are absolutely almost simple [49, §31.5][72]. This means that $G \times_k k_s$ is isomorphic to $\mathbf{SL}_{n,k_s}$, $\mathbf{Spin}_{2n+1,k_s}$, $\mathbf{Sp}_{2n,k_s}$, $\mathbf{Spin}_{2n,k_s}$,...

Let G/k be such a k–group and let $G \to G_{ad}$ be the adjoint quotient of G. Denote by G^q its quasi-split form and by G^q_{ad} its adjoint quotient. Then G is an inner twist of G^q, i.e. there exists a cocycle $z \in Z^1(k_s/k, G^q_{ad}(k_s))$ such that $G \cong {}_zG^q$. We identify then G and ${}_zG^q$.

Converely, we know that there exists a unique class $\nu_G = [a] \in H^1(k, G_{ad})$ such that $G^q \cong {}_aG$ [49, 31.6]. We denote by $z^{op} \in Z^1(k, {}_zG^q_{ad})$ the opposite cocycle of z, it is defined by $\sigma \mapsto z_\sigma^{-1} \in {}_zG(k_s)$.

We have $G^q \cong {}_{z^{op}}\big({}_zG^q\big)$. Hence the image of ν_G under $H^1(k, G_{ad}) \xrightarrow{\sim} H^1(k, {}_zG^q{}_{ad})$ is nothing but $[z^{op}]$. We have an exact sequence

$$1 \to Z(G) \to G \to G_{ad} \to 1$$

of k–algebraic groups with respect to the $fppf$-topology (faithfully flat of finite presentation, see [24, III] or [66]). This gives rise to an exact sequence of pointed sets [6, app. B]

$$1 \to Z(G)(k) \to G(k) \to G_{ad}(k) \xrightarrow{\varphi_G}$$

$$H^1_{fppf}(k, Z(G)) \to H^1_{fppf}(k, G) \to H^1_{fppf}(k, G_{ad}) \xrightarrow{\delta_G} H^2_{fppf}(k, Z(G)).$$

The homomorphism φ_G is called the characteristic map and the mapping δ_G is the boundary. Since G (resp. G_{ad}) are smooth, the $fppf$-cohomology of G (resp. G_{ad}) coincide with Galois cohomology [65, XXIV.8], i.e. we have a bijection $H^1(k, G) \xrightarrow{\sim} H^1_{fppf}(k, G)$. Following [49, 31.6], one defines the Tits class T_G of G by the formula

$$t_G = -\delta_G\big(\nu_G\big) \in H^2_{fppf}(k, Z(G)).$$

By the compatibility property under the torsion bijection τ_z [41, IV.4.2][3]

$$\begin{array}{ccc} H^1(k, G_{ad}) & \xrightarrow{\delta_G} & H^2_{fppf}(k, Z(G)) \\ \tau_z \downarrow \wr & & ? + \delta_{G^q}([z]) \downarrow \wr \\ H^1(k, G^q_{ad}) & \xrightarrow{\delta_{G_{ad}}} & H^2_{fppf}(k, Z(G^q)), \end{array}$$

we see that $t_G = \delta_{G^q}([z])$ which is indeed Tits definition [74, §1].

[3]Note that $Z(G) = Z(G^q)$ since G^q_{ad} acts trivially on $Z(G^q)$.

Proposition 3.1. *Assume that $H^1(k, G) = 1$.*

1. *The boundary map $H^1(k, G_{ad}) \to H^2_{fppf}(k, Z(G))$ has trivial kernel.*
2. *Let G' be an inner k–form of G^q. Then G and G' are isomorphic if and only if $t_G = t_{G'}$.*

Proof. (1) This follows from the exact sequence above.
(2) Let $z' \in Z^1(k, G^q_{ad})$ be a cocycle such that $G' \cong {}_{z'}G$. We assume that $t_G = t_{G'}$. Hence $\delta_{G^q}([z]) = \delta_{G^q}([z']) \in H^2_{fppf}(k, Z(G^q))$. The compatibility above shows that

$$\tau_z^{-1}([z']) \in \ker\bigl(H^1(k, G_{ad})\bigr) \to H^2_{fppf}(k, Z(G))\bigr).$$

By 1), we have $\tau_z^{-1}([z']) = [1] \in H^1(k, G_{ad})$, hence $[z] = [z'] \in H^1(k, G_{ad})$. Thus G and G' are k–isomorphic. □

Summarizing then, the validity of Serre's conjecture II implies that semisimple k-groups are classified by their quasi-split forms and their Tits classes. For more precise results for classical groups, see Tignol-Lewis [51]. The classification is of special importance in view of the rationality question for groups (Chernousov-Platonov [19], see also Merkurjev [55]), and consequently also for the Kneser-Tits problem (Gille, [38]).

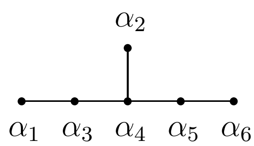
4 Approaches to the conjecture

We would like to describe some of the methods that have been used to attack the conjecture to date, and their limitations. We should point out that separating each of the methods and looking at them individually is a bit artificial. In practice, most work is carried by simultaneously combining the different methods.

4.1 Subgroup trick

We illustrate how this method works by using the following example due to Tits [75]. Let G/k be the split semisimple simply connected group of type

E_6



Assume here that k is infinite. Let $z \in Z^1(k_s/k, G)$, and consider the twisted group $G' = {}_zG$. Since $t_{G'} = 0$, the 27-dimensional standard representation of G of highest weight $\overline{\omega}_6$ descends to G' by [73]. We have then a representation $\rho' : G' \to \mathbf{GL}(V)$. The point is that G' has a dense orbit in the projective space $X = \mathbb{P}(V)$, so there exists a k-rational point $[x]$ in this orbit. The connected stabiliser $(G'_x)^0$ is then semisimple of type F_4 [28, 9.12]. Assuming that Conjecture 2.5 holds for groups of type F_4, it follows that $(G'_x)^0$ is split. Hence G' has relative rank ≥ 4 and a glance on Tits tables [72] tells us that G' is split. It is then easy to conclude that $[z] = 1 \in H^1(k, G)$.

The subgroup trick, and variations thereof, was fully investigated by Garibaldi in his Lens lectures [28]. The underlying topic is that of prehomogeneous spaces, namely projective G-varieties with a dense orbit.

Unfortunately, this trick works only in few cases. Tits has shown that the general form of type E_8 is “almost abelian” namely that it has no non trivial (connected) reductive subgroups other than maximal tori [75]. Garibaldi and the author have shown that the general trialitarian group is almost abelian [29].

4.2 Rost invariant

In this case, the idea is to derive Serre's conjecture II from a more general setting. The Rost invariant [31] generalizes the Arason invariant for 3-fold Pfister form which attaches (in characteristic $\neq 2$) to a Pfister form $\phi = \langle\langle a, b, c \rangle\rangle$ the cup-product $e_3(\phi) = (a) \cup (b) \cup (c) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$. We see it now as the cohomological invariant $H^1(k, \mathbf{Spin}_8) \to H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$. More generally, for G/k simply connected and absolutely almost simple, there is a cohomological invariant

$$r_k : H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

where the p-primary part has to be understood in Kato's setting [31]. If this invariant has trivial kernel, then $H^1(k, G) = 1$ for G/k satisfying the hypothesis of Conjecture 2.5. This is the case for $\mathbf{Spin}_8$ by Arason's theorem, namely the fact that the invariant $e_3(\phi)$ determines ϕ.

4.3 Serre's injectivity question

A special case of a question raised by Serre in 1962 ([62], see also [64, §2.4]) is the following.

Question 4.1. *Let G/k be a connected linear algebraic group. Let $(k_i)_{i=1,..,r}$ be a family of finite field extensions of k such that g.c.d.$([k_i : k]) = 1$. Is the kernel of the map*

$$H^1(k, G) \to \prod_{i=1,..,r} H^1(k_i, G)$$

trivial ?

Remarks 4.2. (1) The hypothesis of connectedness is necessary since there are counterexamples with finite constant groups [42][59]. Let us mention here another counterexample concocted by S. Garibaldi. Suppose that k supports central division algebras B_i of degree i for $i = 3, 5$. There are extensions k_i of k of degree i that split B_i. The groups $\mathbf{SL}_1(B_3 \otimes_k B_5)$ and $\mathbf{SL}_1(B_3^{op} \otimes_k B_5)$ are isomorphic over k_i for $i = 3, 5$ but they are not k-isomorphic. So Serre's injectivity fails for the group $\mathbf{Aut}\big(\mathbf{SL}(B_3 \otimes B_5)\big)$, which has just two connected components.

(2) Question 4.1 has been generalized by Totaro [76, question 0.2]. See also [30].

(3) If k is of positive characteristic p, there exists a complete DVR R with residue field k and an R–group scheme $\mathfrak{G}$ with special fiber G and such that the fraction field K of R is of characteristic zero. An answer for $\mathfrak{G}_K$ to Serre's question yields an answer for G. A fortiori, we can assume without lost of generality, that the extensions k_i/k are separable.

We shall rephrase question 4.1 in terms of special fields.

Definition 4.3. Let l be a prime. We say that a field k is l–special if every finite separable extension of k is of degree a power of l.

The subfield k_l of k consisting of elements fixed by a p-Sylow subgroup of $\mathrm{Gal}(k_s/k)$ is l-special. We call k_l a *co-l-closure* of k. If we restrict Serre's question for finite separable extensions k_i/k and consider all cases, it can be rephased by asking whether the map

$$H^1(k, G) \to \prod_l H^1(k_l, G)$$

has trivial kernel for l running over the primes. If the answer to this question is in the affirmative, then conjecture II becomes a question for l-special fields for primes l in $S(G)$.

Indeed there are very few cases for which the answer to Serre's question is known: unitary groups (Bayer-Lenstra [4]), groups of type G_2, quasi-split groups of type D_4, F_4, E_6, E_7 [33] [17] [27].

If we know that the Rost invariant has trivial kernel, then we easily deduce that the answer to Question 4.1 is yes. Thus we can answer Serre's question for groups of type G_2, and quasi-split semisimple simply connected groups of type D_4, F_4, E_6 and E_7.

5 Known cases in terms of groups

5.1 Classical groups

Recall that a semisimple simply connected group is classical if its factors are of type A, B, C or D, and there is no triality involved.

Theorem 5.1. *Let G be a semisimple simply connected classical group which is absolutely almost simple. Then $H^1(k, G) = 1$.*

If k is perfect or $\mathrm{char}(k) \neq 2$, this is the original Serre's conjecture II proven by Bayer-Parimala [5]. The general case is done in recent work by Berhuy-Frings-Tignol [6]. Their proof is based on Weil's presentation of classical group in terms of unitary groups of algebras with involutions [77], This proof is characteristic free. It provides quite a different approach to the conjecture than that present in Bayer and Parimala's work.

Possibly the most tricky case is that of outer groups of type A, namely unitary groups of central simple algebras equipped with an involution of the second kind. It is enough to think about the number field case using Landherr's theorem [48, §5.5] to see how complicated the case these outer groups is.

5.2 Quasi-split exceptional groups

For this type of groups, the best results to date have been obtained by investigating the Rost invariant.

Theorem 5.2. *Let G/k be a quasi-split semisimple simply connected group of Cartan-Killing type G_2, F_4, D_4, E_6 or E_7. Then the Rost invariant $H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ has trivial kernel.*

Note first that this kind of statement reduces to the characterictic zero case by a lifting argument [34]. For the cases G_2 and F_4, see [5] or [64]. As pointed out by Garibaldi, the D_4 case is done in the Book of Involutions but not stated in this form. We need to know that a trialitarian algebra whose underlying algebra is split arises as the endomorphism of a twisted composition [49, 44.16] and to use results on degree 3 invariants of twisted compositions (*ibid*, 40.16). For type E_6 and E_7, this is due independently to Chernousov [17] and Garibadi [27].

Thus Conjecture 2.5 holds for quasi-split groups of all types other than E_8. The author has given an independent proof based on Bruhat-Tits theory which is quite different in spirit from the one outlined above [35]. For the split group of type E_8, which will be denoted by E_8, the Rost invariant has in general a non trivial kernel (for the field of real numbers and also for suitable fields of cohomological dimension 4, see [37, appendix]). In characteristic 0, Semenov constructed recently a higher invariant

$$\ker\bigl[H^1(k, E_8) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))\bigr] \to H^5(k, \mathbb{Z}/2\mathbb{Z})$$

which is non trivial since it is does not vanish for the field of real numbers [67, §8]. Moreover, Semenov's invariant has trivial kernel for 2-special fields.

By means of norm group of varieties of Borel subgroups, the case of quasi-split groups is the key ingredient for proving the following.

Theorem 5.3. *[35, th. 6] Let G/k be a semisimple simply connected group which satisfies the hypothesis of Conjecture 2.5. Let $\mu \subset G$ be a finite central subgroup of G. Then the characteristic map*

$$(G/\mu)(k) \to H^1_{fppf}(k, \mu)$$

is surjective.

The flat cohomology (see [66], [6, app. B] or [41]) is the right set up where to phrase the problem if the order of μ is not invertible in k. If the order of μ i invertible in k then the flat and usual Galois cohomology coincide. By continuing the exact sequence of pointed sets

$$1 \to \mu(k) \to G(k) \to (G/\mu)(k) \to H^1_{fppf}(k, \mu) \to H^1_{fppf}(k, G),$$

we see that $H^1_{fppf}(k, \mu) \to H^1_{fppf}(k, G)$ is the trivial map. In other words, the center of G does not contribute to $H^1(k, G)$.[4]

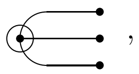
5.3 Other exceptional groups

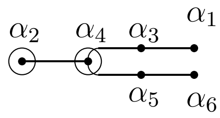
Theorem 5.4. *[17] [35] Let G/k be a semisimple group satisfying the hypothesis of Conjecture 2.5. Then $H^1(k, G) = 1$ in the following cases:*

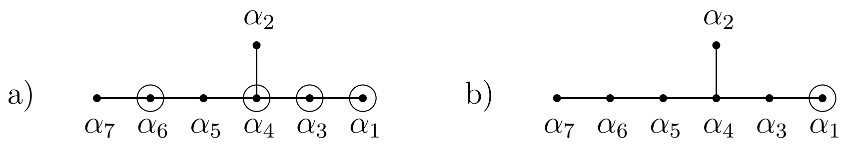
1. *G is trialitarian and its Allen algebra is of index ≤ 2.*
2. *G is of quasi-split type ${}^1E_6$ or ${}^2E_6$ and its Tits algebra is of index ≤ 3.*
3. *G is of type E_7 and its Tits algebra is of index ≤ 4.*

[4] The reason why we can avoid groups of type E_8 is because such groups have trivial center.

Furthermore the groups under consideration are either quasi-split or isotropic of Tits indices







where case a) (resp. b)) holds when the Tits algebra is of index 2 (resp. 4). One more reason why other exceptional groups are difficult to deal with is because they are anisotropic.

For function fields of surfaces, central simple algebras of period 2 (resp.3) are of index 2 (resp. 3) as pointed by Artin [1]. Thus Theorem 5.4 covers all groups of type ${}^{3,6}D_4$, ${}^{1,2}E_6$, and E_7 for these fields.

In joint work with with Colliot-Thélène and Parimala [21], we exploited Serre's conjecture II to study arithmetic properties of not necessarily simply connected groups. Our methods were inspired by Sansuc's paper [61] in the number field case. For more on this topic, see the paper by Borovoi and Kunyavskiĭ [11].

6 Known cases in terms of fields

6.1 l-special fields

(a) If $l = 2, 3, 5$ and k is an l-special field of separable dimension ≤ 2, Conjecture 2.5 holds for the split group of type E_8, see [16] for $l = 5$ and [35, §III.2].

(b) If $l = 3$ and k is an l-special field of characteristic $\neq 2$ and separable dimension ≤ 2, then Conjecture 2.5 holds for trialitarian groups. For $l = 3$, this follows of Theorem 5.2.

In both cases, a positive answer to Serre's injectivity question would provide Conjecture 2.5 for the groups under consideration.

6.2 Complete valued fields

Let K be a henselian valued field for a discrete valuation with perfect residue field κ. A consequence of the Bruhat-Tits decomposition for Galois cohomology over complete fields is the following.

Theorem 6.1. *(Bruhat-Tits [14, cor. 3.15]) Assume that κ is of cohomological dimension ≤ 1. Let G/K be a simply connected semisimple group. Then $H^1(K, G) = 1$.*

Note that the hypotheses imply that K is of separable dimension ≤ 2. Serre asked whether it can be generalized when assuming $[\kappa : \kappa^p] \leq p$ [64, 5.1]. The hypothesis $[\kappa : \kappa^p] \leq p$ alone is not enough here because $K = \mathbf{F}_p((x))((y))$ is of separable dimension 3 and is complete with residue field $\mathbf{F}_p((x))$.

But if κ is separably closed and $[\kappa : \kappa^p] \leq p$, then K is of separable dimension 1 and enough cases of the vanishing of $H^1(\kappa((x)), G)$ have been established in view of the proof of Tits conjectures on unipotent subgroups [36]. The general case, however, is still open.

Note also that the conjecture is proven for fraction fields of henselian two dimensional local rings with algebraically closed residue field of characteristic zero, e.g. $\mathbb{C}[[x, y]]$, as shown in [21]. For the E_8 case, a key point is that the derived group of the absolute Galois group is of cohomological dimension 1 [23, Th. 2.2].

6.3 Global fields

The number field case is due to Kneser for classical groups [48], Harder for exceptional groups of type other than E_8 [43, I, II], and Chernousov for type E_8 [15], see also [60]. The function field case due to Harder [43, III].

6.4 Function fields

He, de Jong and Starr have proven Conjecture 2.5 for split groups over function fields in a uniform way in arbitrary characteristic.

Theorem 6.2. *[46, cor. 1.5] Let k be an algebraically closed field and let K be the function field of a quasi-projective smooth surface S. Let G be a split semisimple simply connected group over k. Then $H^1(K, G) = 1$.*

Except E_8, all other cases of the conjecture were known by case by case considerations [21]. Hence Conjecture 2.5 is fully proven for function fields of surfaces. The proof of Theorem 6.2 is based on the existence of sections for fibrations in rationally simply connected varieties.[5]

Theorem 6.3. *[46, Th. 1.4] Let S/k as in Theorem 6.2. Let X/S be a projective morphism whose geometric generic fiber is a twisted flag variety. Assume that* $\mathrm{Pic}(X) \to \mathrm{Pic}(X \times_K \overline{K})$ *is surjective. Then $X \to S$ has a rational section.*

The assumption on the Picard group means that there is no "Brauer obstruction". When applied to higher Severi-Brauer schemes, this statement yields as a corollary de Jong's theorem "period=index" [45] for central simple algebras over such fields; see also [20]. This is the first classification-free work described in this survey.

6.5 Why Theorem 6.3 implies Theorem 6.2

We take this opportunity to reproduce how this argument goes.

Lemma 6.4. *Set G/F be a semisimple simply connected group over a field F. Let E/F be a G-torsor.*

1. $\mathrm{Pic}(E) = 0$ *and we have an exact sequence*

$$0 \to \mathrm{Br}(F) \to \mathrm{Br}(E) \to \mathrm{Br}(E \times_F F_s).$$

2. *Let P be a F-parabolic subgroup of G and let E/P the variety of parabolic subgroups of the twisted F-group $E(G)$ having the same type than P. Then* $\mathrm{Pic}(P) = 0$ *and we have an exact sequence*

$$0 \to \mathrm{Br}(F) \to \mathrm{Br}(E/P) \to \mathrm{Br}(E/P \times_F F_s)$$

and an isomorphism $\mathrm{Pic}(E/P) \xrightarrow{\sim} \mathrm{Pic}(E/P \times_F F_s)^{\mathrm{Gal}(F_s/F)}$.

Proof. (1) We have $H^1(F, (F_s)^\times) = 0$ and $\mathrm{Pic}(E \times_F F_s) \cong \mathrm{Pic}(G \times_F F_s) = 0$ since G is simply connected [26]. The first terms of the Hochschild-Serre spectral sequence $H^p(\mathrm{Gal}(F_s/F), H^q(E \times_F F_s, \mathbf{G}_m)) \Longrightarrow H^{p+q}(E, \mathbf{G}_m)$ show that $\mathrm{Pic}(E) = 0$ and that the sequence $0 \to \mathrm{Br}(F) \to \mathrm{Br}(E) \to \mathrm{Br}(E \times_F F_s)$ is exact.

[5]Integral versions of this result have recently been studied. The case when the function field $\mathbb{C}(t_1, t_2)$ is replaced by the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ appear in the study of infinite dimensional Lie algebras. In analogy with conjecture II it is natural to ask whether $H^1_{fppf}(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}], \mathfrak{G})$ vanishes for all simple simply connected group schemes $\mathfrak{G}$ over $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. An example of B. Margaux shows that the answer to this question is negative (see [39] §3.6.)

2) The fibration $G \to G/P$ is locally trivial for the Zariski topology (Borel-Tits). By a result of Sansuc applied to the fibration $G \to G/P$ [61, 6.10.2],[6] we have a surjective map $\mathrm{Pic}(G) \to \mathrm{Pic}(P)$, hence $\mathrm{Pic}(P) = 0$. By [61, 6.10.1] applied to the fibration $E \to E/P$, there is an exact sequence

$$0 = \mathrm{Pic}(P) \to \mathrm{Br}(E/P) \to \mathrm{Br}(E),$$

hence the map $\mathrm{Br}(E/P) \to \mathrm{Br}(E)$ is injective. We look at the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(E/P) & \longrightarrow & \mathrm{Br}(E/P \times_F F_s) \\
 & & || & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br}(E) & \longrightarrow & \mathrm{Br}(E \times_F F_s).
\end{array}$$

Since the bottom sequence is exact, we get by diagram chasing that the upper horizontal sequence is exact as well. The second isomorphism $\mathrm{Pic}(E/P) \xrightarrow{\sim} \mathrm{Pic}(E/P \times_F F_s)^{\mathrm{Gal}(F_s/F)}$ comes from the Hochschild-Serre spectral sequence. □

For complete results on Picard and Brauer groups of twisted flag varieties, see Merkurjev-Tignol [57, §2].

Proposition 6.5. *[46, Th. 1.4] Let S/k as in Theorem 6.2. Let G/K be a simple simply connected K-group which is an inner form and let P be a K-parabolic subgroup of G. Then the map $H^1(K, P) \to H^1(K, G)$ is bijective.*

Proposition 6.5 implies Theorem 6.2 by taking a Borel subgroup of G because $H^1(K, B) = 1$.

Proof. Injectivity is a general fact due to Borel-Tits ([9], théorème 4.13.a). Let E/K be a G-torsor of class $[E] \in H^1(K, G)$. Up to shrinking S, we can assume that G/K extends to a semisimple group scheme $\mathfrak{G}/S$, P/K extends to a S–parabolic subgroup scheme $\mathfrak{P}/S$, and that E/K extends to a $\mathfrak{G}$-torsor $\mathfrak{E}/S$ [52]. By étale descent, we can twist the S-group scheme $\mathfrak{G}/S$ by inner automorphisms, namely we can define the S-group scheme $\mathfrak{E}(\mathfrak{G})/S$. We define then $\mathfrak{V}/S := \mathfrak{E}/\mathfrak{P}$, i.e. the scheme of parabolic subgroup schemes of $\mathfrak{E}(\mathfrak{G})/S$ ([65], exp. XXVI) of the same type than P. The morphism $\pi : \mathfrak{V} \to X$ is projective, smooth and with geometrically integral fibers.

[6] Strictly speaking, Sansuc's result is proved under the assumption that k be perfect or that the group of the fibration be reductive. By dévissage, one can see that the result holds for fibrations under groups that are extensions of reductive k-group by split k–unipotent groups.

Set $V = \mathfrak{V} \times_S K$, this is a generalised twisted flag variety. Since G is assumed to be an inner form, $\mathrm{Pic}(V \times_K K_s)$ is a trivial $\mathrm{Gal}(K_s/K)$-module. By Lemma 6.4.2, the map

$$\mathrm{Pic}(V) \to \mathrm{Pic}(V \times_K K_s)$$

is onto. Thus the composite map $\mathrm{Pic}(\mathfrak{V}) \to \mathrm{Pic}(V) \to \mathrm{Pic}(V \times_K K_s)$ is onto. Theorem 6.2 applies and shows that $V(K) \neq \emptyset$. It means that the torsor E admits a reduction to P ([63], §I.5, proposition 37), that is $[E] \in \mathrm{Im}\big(H^1(K,P) \to H^1(K,G)\big)$. We conclude that the mapping $H^1(K,P) \to H^1(K,G)$ is surjective. □

The Grothendieck-Serre's conjecture on rationally trivial torsors has been proven by Colliot-Thélène and Ojanguren for torsors over a semisimple group defined over an algebraic closed field [22]. Thus He-de Jong-Starr's theorem has the following geometric application.

Corollary 6.6. *Let S/k be a smooth quasi-projective surface. Let G/k be a (split) semisimple simply connected group. Let E/S be a G-torsor. Then E is locally trivial for the Zariski topology.*

7 Remaining cases and open questions

• Provide a classification free proof of the case of totally imaginary number fields, at least in the quasi-split case.

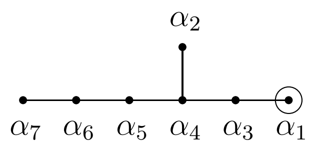
• The first remaining cases of Conjecture 2.5 are that of trialitarian groups, groups of type E_6 over a 3–special field, groups of type E_7 over a 2-special field and groups of type E_8.

• Let K be a function field of surface over an algebraically closed field. Are K−division algebras cyclic ? Is it true that $\mathrm{cd}(K_{ab}) = 1$ where K_{ab} stands for the abelian closure of K ?

In the global field case, class field theory answers both questions positively. The question on K_{ab} is due to Bogomolov and makes sense for arbitrary fields. As noticed by Chernousov, Reichstein and the author, a positive answer would provide a positive answer to Serre's conjecture II for groups of type E_8 [18].

• For the Kneser-Tits conjecture for perfect fields of cohomological dimension ≤ 2, there remains only the case of a group of Tits index [38, §8.2]

$E_{7,1}^{66}$



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