INCOMPRESSIBILITY OF ORTHOGONAL GRASSMANNIANS OF RANK 2

BRYANT G. MATHEWS

Abstract. For a nondegenerate quadratic form φ on a vector space V of dimension $2n+1$, let X_d be the variety of d-dimensional totally isotropic subspaces of V. We give a sufficient condition for X_2 to be 2-incompressible, generalizing in a natural way the known sufficient conditions for X_1 and X_n. Key ingredients in the proof include the Chernousov-Merkurjev method of motivic decomposition as well as Pragacz and Ratajski's characterization of the Chow ring of $(X_2)_E$, where E is a field extension splitting φ.

1. Preliminaries

Before stating our results in the next section, we begin by recalling the notions of canonical p-dimension, p-incompressibility, and higher Witt index.

Let X be a scheme over a field F, and let p be a prime or zero. A field extension K of F is called a *splitting field of* X (or is said to *split* X) if $X(K) \neq \emptyset$. A splitting field K is called *p-generic* if, for any splitting field L of X, there is an F-place $K \rightharpoonup L'$ for some finite extension L'/L of degree prime to p. In particular, K is 0-generic if for any splitting field L there is an F-place $K \rightharpoonup L$.

The canonical p-dimension of a scheme X over F was originally defined [1, 7] as the minimal transcendence degree of a p-generic splitting field K of X. When X is a smooth complete variety, the original algebraic definition is equivalent to the following geometric one [7, 10].

Definition 1.1. Let X be a smooth complete variety over F. The *canonical p-dimension* $\operatorname{cdim}_p(X)$ *of* X is the minimal dimension of the image of a morphism $X' \to X$, where X' is a variety over F admitting a dominant morphism $X' \to X$ with $F(X')/F(X)$ finite of degree prime to p. The canonical 0-dimension of X is thus the minimal dimension of the image of a rational morphism $X \dashrightarrow X$.

In the case $p = 0$, we will drop the p and speak simply of *generic* splitting fields and canonical *dimension* $\operatorname{cdim}(X)$.

For a third definition of canonical p-dimension as the essential p-dimension of the detection functor of a scheme X, we refer the reader to Merkurjev's comprehensive exposition [10] of essential dimension.

For a smooth complete variety X, the inequalities

$$\operatorname{cdim}_p(X) \leq \operatorname{cdim}(X) \leq \dim(X)$$

are clear from Definition 1.1. Note also that if X has a rational point, then $\operatorname{cdim}(X) = 0$ (though the converse is not true).

Definition 1.2. When a smooth complete variety X has canonical p-dimension as large as possible, namely $\operatorname{cdim}_p(X) = \dim(X)$, we say that X is *p-incompressible.*

It follows immediately that if X is p-incompressible, it is also *incompressible* (i.e. 0-incompressible).

We next recall the definitions of absolute and relative higher Witt indices, introduced by Knebusch in [8]. Our discussion follows [5, §90]. The *Witt index* $\mathfrak{i}_0(\varphi)$ of a quadratic form φ is the number of copies of the hyperbolic plane $\mathbb{H}$ which appear in the Witt decomposition of φ. Now let φ be a nondegenerate quadratic form over a field F and set $F_0 := F$ and $\varphi_0 := \varphi_{an}$, the anisotropic part of φ. We proceed to recursively define $F_k := F_{k-1}(\varphi_{k-1})$, $\varphi_k := (\varphi_{F_k})_{an}$ for $k = 1, 2, \ldots$, stopping at F_h such that $\dim \varphi_h \leq 1$.

Definition 1.3. For $k \in \{0, 1, \ldots, h\}$, the *$k$-th absolute higher Witt index* $\mathfrak{j}_k(\varphi)$ *of* φ is defined to be $\mathfrak{i}_0(\varphi_{F_k})$. For $k \in \{1, 2, \ldots, h\}$, the *$k$-th relative higher Witt index* $\mathfrak{i}_k(\varphi)$ *of* φ is defined to be the difference

$$\mathfrak{i}_k(\varphi) := \mathfrak{j}_k(\varphi) - \mathfrak{j}_{k-1}(\varphi).$$

The *0-th relative higher Witt index of* φ is the usual Witt index $\mathfrak{i}_0(\varphi)$.

It follows from the definition that

$$0 \leq \mathfrak{j}_0(\varphi) < \mathfrak{j}_1(\varphi) < \cdots < \mathfrak{j}_h(\varphi) = [(\dim \varphi)/2]\,.$$

Moreover, it can be shown that the set $\{\mathfrak{j}_0(\varphi), \ldots, \mathfrak{j}_h(\varphi)\}$ of absolute higher Witt indices of φ is equal to the set of all Witt indices $\mathfrak{i}_0(\varphi_K)$ for K an extension field of F.

2. Introduction

Let φ be a nondegenerate quadratic form on a vector space V of dimension $2n+1$ over a field F. Associated to φ there are smooth projective varieties $X_1, X_2, \ldots,$ X_n, where X_d is the variety of d-dimensional totally isotropic subspaces of V. The variety X_1 is simply the projective quadric hypersurface associated to the quadratic form φ.

We recall the following result proved in [6] and also in [5, Ch. XIV and §90].

Theorem 2.1 (Karpenko, Merkurjev)**.** *If the quadric X_1 is anisotropic, then*

$$\operatorname{cdim}_2(X_1) = \operatorname{cdim}(X_1) = \dim(X_1) - \mathfrak{i}_1(\varphi) + 1.$$

In particular, X_1 is 2-incompressible if and only if $\mathfrak{i}_1(\varphi) = 1$.

At the other extreme is the variety X_n of maximal totally isotropic subspaces of V. In [5, Ch. XVI], building on a result of Vishik from [12], the canonical 2-dimension of X_n is computed in terms of the J-invariant of φ. The following result is a corollary.

Theorem 2.2 (Karpenko, Merkurjev)**.** *If* $\deg \mathrm{CH}(X_n) = 2^n\mathbb{Z}$*, then X_n is 2-incompressible.*

To compute the canonical 2-dimension of a general X_d appears to be difficult because of the complexity of the Chow ring when $d \notin \{1, n\}$. In this paper, we complete a small piece of this general program by determining a sufficient condition for the variety X_2 to be 2-incompressible. We assume everywhere that $n \geq 3$, the $n = 2$ case having already been dealt with.

Theorem 2.3. *If* $\deg \mathrm{CH}(X_2) = 4\mathbb{Z}$ *and* $\mathfrak{i}_2(\varphi) = 1$*, then* X_2 *is* 2*-incompressible. In particular,*

$$\mathrm{cdim}_2(X_2) = \mathrm{cdim}(X_2) = \dim(X_2) = 4n - 5.$$

This result concerning X_2 is a natural generalization of what is already known about X_1 and X_n. To see this, note that X_1 being anisotropic means that $\deg \mathrm{CH}(X_1) = 2\mathbb{Z}$. Furthermore, $\deg \mathrm{CH}(X_n) = 2^n\mathbb{Z}$ implies that $\mathfrak{j}_{n-1}(\varphi) = n-1$, from which it immediately follows that $\mathfrak{i}_n(\varphi) = 1$. One might then conjecture, for general d, that

$$\deg \mathrm{CH}(X_d) = 2^d\mathbb{Z}, \quad \mathfrak{i}_d(\varphi) = 1$$

are sufficient conditions for X_d to be 2-incompressible.

3. Higher Witt indices

In this section we collect two results concerning higher Witt indices which will be needed later.

Proposition 3.1. *If* $\deg \mathrm{CH}(X_2) = 4\mathbb{Z}$*, then* $\mathfrak{j}_1(\varphi) = 1$.

Proof. Let K be a field of degree 2 over F such that the anisotropic part of φ has a rational point over K. Then $\mathfrak{i}_0(\varphi_K) > \mathfrak{i}_0(\varphi)$. By [5, Prop. 25.1], it follows that $\mathfrak{i}_0(\varphi_K) \geq \mathfrak{j}_1(\varphi)$. If $\mathfrak{j}_1(\varphi) \geq 2$ then so is $\mathfrak{i}_0(\varphi_K)$, which implies that the variety X_2 has a rational point over K. Since K has degree 2 over F, this contradicts the assumption. Thus $\mathfrak{j}_1(\varphi) = 1$ and $\mathfrak{j}_0(\varphi) = 0$. □

From this proposition, we see that the hypothesis of our Theorem 2.3 implies

$$\mathfrak{j}_2(\varphi) = \mathfrak{j}_1(\varphi) + \mathfrak{i}_2(\varphi) = 1 + 1 = 2.$$

Proposition 3.2. *If* $\mathfrak{j}_2(\varphi) = 2$*, then* $\mathfrak{i}_0(\varphi_{F(X_2)}) = 2$.

In fact, one need only assume that *some* absolute higher Witt index of φ is equal to 2 (possibly $\mathfrak{j}_0(\varphi)$ or $\mathfrak{j}_1(\varphi)$), but we don't need this generality for our purposes.

Proof. The variety X_2 has a rational point over $F(X_2)$, so $\mathfrak{i}_0(\varphi_{F(X_2)}) \geq 2$.

We prove that $\mathfrak{i}_0(\varphi_{F(X_2)}) \leq 2$ by contradiction. If $\tilde{\varphi}$ is the anisotropic part of $\varphi_{F(\varphi)}$, then $\varphi_{F(\varphi)} \simeq \mathbb{H} \perp \tilde{\varphi}$, since our assumption $\mathfrak{j}_2(\varphi) = 2$ implies that $\mathfrak{j}_1(\varphi) = 1$. We define two varieties over $F' := F(\varphi)$. Let Y_1 be the projective quadric corresponding to $\tilde{\varphi}$, and let Y_2 be the variety of totally isotropic subspaces of dimension 2 with respect to $\tilde{\varphi}$. Since

$$\mathfrak{i}_0(\varphi_{F'(Y_1)}) = \mathfrak{i}_0(\varphi_{F(\varphi)(\tilde{\varphi})}) = \mathfrak{j}_2(\varphi) = 2,$$

the variety X_2 has a rational point over $F'(Y_1)$. If $\mathfrak{i}_0(\varphi_{F(X_2)}) \geq 3$, then $\mathfrak{i}_0\left(\tilde{\varphi}_{F'((X_2)_{F'})}\right) \geq 2$, so Y_2 has a rational point over $F'\left((X_2)_{F'}\right)$. We thus have two rational morphisms between varieties over F':

$$Y_1 \dashrightarrow (X_2)_{F'} \dashrightarrow Y_2.$$

By [2, Lem. 6.1], since X_2 is smooth and Y_2 is complete, there exists a rational morphism from Y_1 to Y_2, i.e. $(Y_2)_{F'(Y_1)}$ has a rational point. But this is a contradiction, since $\mathfrak{j}_1(\tilde{\varphi}) = \mathfrak{j}_2(\varphi) - 1 = 1$. □

4. Shapes and multiplicities

A quadratic form φ is *split* if $\mathfrak{i}_0(\varphi) = [(\dim \varphi)/2]$, the greatest possible value. Given a quadratic form φ over F, we fix an extension field E/F such that φ_E is split and define $\bar{\varphi} := \varphi_E$ and $\bar{X}_d := (X_d)_E$. Let $\overline{\mathrm{CH}}(X_d)$, called the *reduced Chow group*, denote the image of the change of field homomorphism $\mathrm{CH}(X_d) \to \mathrm{CH}(\bar{X}_d)$. Elements of $\overline{\mathrm{CH}}(X_d)$ will be called *rational* cycles.

In this section we prove a technical lemma, based on a characterization given by Pragacz and Ratajski in [11] of the Chow ring of the variety $\bar{X}_2$.

Lemma 4.1. *If $\gamma \in \mathrm{CH}^r(\bar{X}_2)$ with $r \in \{2n-3, 2n-2\}$, then 2γ is a rational cycle.*

We begin by fixing notation and recalling some definitions. Let $X_B := \bar{X}_2$, the variety of 2-dimensional isotropic subspaces of V with respect to the *split* nondegenerate quadratic form $\bar{\varphi}$. Recall that V has dimension $2n+1$. Let X_C denote the variety of 2-dimensional isotropic subspaces of a vector space W of dimension $2n$ with respect to a nondegenerate alternating form ψ on W.

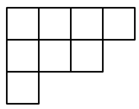
The next group of definitions are adapted for our purposes from Macdonald's [9].

Definition 4.2. A *partition* is a finite, strictly decreasing sequence of positive integers

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_s^*).$$

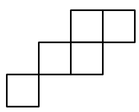
The $*$ in our notation will take on the value t for "top" or b for "bottom," depending on the role the partition plays in a shape, defined below. The *length* $l(\lambda^*)$ of the partition above is just s, while the *weight* of the partition is defined to be $|\lambda^*| := \sum_{k=1}^{s} \lambda_k^*$. The empty partition, denoted $\emptyset$, is the sequence with no terms.

Partitions are visualized as diagrams of boxes. The *diagram* D_λ^* of a partition λ^* has λ_k^* boxes in the kth row, beginning with the top row. For example, the partition $(4,3,1)$ has diagram:



Note that the length of a partition is just the number of rows in its diagram, while the weight is just the number of boxes.

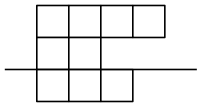
A *skew diagram* $D_\mu^* \backslash D_\lambda^*$ is obtained by removing the boxes in the intersection of D_μ^* and D_λ^* from D_μ^*. For example, the skew diagram $(4,3,1)\backslash(2,1)$ is as follows:



The remaining definitions are adapted from [11].

Definition 4.3. With n fixed as above, a pair $\lambda = (\lambda^t // \lambda^b)$ of partitions λ^t and λ^b is called a *shape* if $\lambda_1^t, \lambda_1^b \le n$, $l(\lambda^t) \le n-2$, $l(\lambda^b) \le 2$, and $\lambda_{n-2}^t > l(\lambda^b)$.

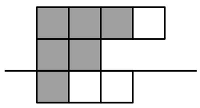
The *diagram* D_λ of a shape is drawn by stacking the diagram D^t_λ of the first partition on top of the diagram D^b_λ of the second, with a horizontal line in between. For example, the shape $((4,2)//(3))$ has diagram:



The inequalities in the definition of a shape λ amount to imposing three requirements on its diagram D_λ:

(1) D^t_λ must fit into a rectangle with height $n-2$ and width n
(2) D^b_λ must fit into a rectangle with height 2 and width n
(3) D_λ must contain the "triangle" of boxes lying on and above the diagonal beginning at the box in the diagram's lower-left corner and proceeding to the "northeast".

We display once again the diagram of the shape $((4,2)//(3))$, this time shading in the triangle of boxes required by the third condition above:



The *weight* of a shape $\lambda = (\lambda^t//\lambda^b)$ is defined in terms of the weights of the partitions λ^t, λ^b as

$$|\lambda| := |\lambda^t| + |\lambda^b| - \binom{n-1}{2}.$$

This definition is chosen so that the shape $\pi_0 := ((n-2, n-3, \ldots, 1)//\emptyset)$ of minimal weight will have weight 0.

We refer to [11] for the definitions of extremal and related components, $(\mu - \lambda)$-boxes, and compatible shapes.

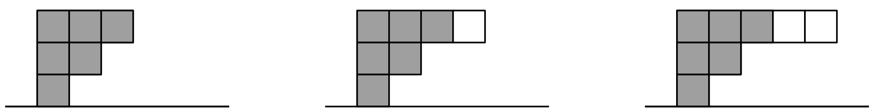
The set of all shapes, denoted $\mathcal{P}_2$, can be mapped bijectively onto bases for each of $\mathrm{CH}(X_B)$ and $\mathrm{CH}(X_C)$. We name the maps

$$\sigma : \mathcal{P}_2 \longrightarrow \mathrm{CH}(X_B)$$

$$\tau : \mathcal{P}_2 \longrightarrow \mathrm{CH}(X_C)$$

and we call cycles in the images of the maps *basic* cycles.

For $i = 0, 1, 2$, consider the shapes $\pi_i := ((n-2+i, n-3, n-4, \ldots, 1)//\emptyset)$. When $n = 5$, their diagrams are as follows:



π_0 π_1 π_2

If we set $\sigma_i := \sigma(\pi_i)$, $\tau_i := \tau(\pi_i)$, then σ_0, τ_0 are the respective multiplicative identities of the Chow rings, while σ_1, σ_2 (resp. τ_1, τ_2), called *special* cycles, are the nontrivial Chern classes of the tautological bundle over X_B (resp. X_C). Pragacz and Ratajski prove that τ_1, τ_2 algebraically generate $\mathrm{CH}(X_C)$ (Cor. 1.8 and Lem. 3.2), while σ_1, σ_2 only generate $\mathrm{CH}(X_B)$ after tensoring with $\mathbb{Z}[1/2]$ (Thm. 10.1).

With the weight $|\lambda|$ of a shape defined as above, we have $\sigma(\lambda) \in \mathrm{CH}^{|\lambda|}(X_B)$ and $\tau(\lambda) \in \mathrm{CH}^{|\lambda|}(X_C)$. In particular, $\mathrm{codim}(\sigma_i) = |\pi_i| = i$ and $\mathrm{codim}(\tau_i) = |\pi_i| = i$.

The multiplication rules in $\mathrm{CH}(X_B)$ and $\mathrm{CH}(X_C)$ are very similar, differing only by some factors of 2 in certain multiplicities. Indeed, for any shape $\lambda \in \mathcal{P}_2$, $i = 1, 2$, we have the Pieri-type formulas

$$\sigma(\lambda) \cdot \sigma_i = \sum 2^{e_B(\lambda,\mu)} \sigma(\mu)$$

$$\tau(\lambda) \cdot \tau_i = \sum 2^{e_C(\lambda,\mu)} \tau(\mu)$$

for multiplying a basic cycle by a special cycle [11, Thms. 2.2 and 10.1]. Here, the sums are over all μ compatible with λ satisfying $|\mu| = |\lambda| + i$, and $e_B(\lambda, \mu), e_C(\lambda, \mu)$ are the cardinalities of certain sets of components of the skew diagram $D^b_\mu \backslash D^b_\lambda$. For our purposes, what we need is that for compatible λ, μ with $|\mu| = |\lambda| + i$, the difference $e_B(\lambda, \mu) - e_C(\lambda, \mu)$ equals the number of *extremal* components of $D^b_\mu \backslash D^b_\lambda$. (This follows from the fact that an extremal component is not related and has no $(\mu - \lambda)$-boxes lying over it, by parts 2 and 4 of the definition [11, Def. 2.1] of compatible shapes.) The skew diagram $D^b_\mu \backslash D^b_\lambda$ clearly has at most one extremal component. There is exactly one extremal component if and only if $l(\mu^b) > l(\lambda^b)$, which by part 5 of the definition of compatible shapes is equivalent to $l(\mu^b) = l(\lambda^b) + 1$. There are no extremal components if and only if $l(\mu^b) = l(\lambda^b)$. Putting all of this together, we conclude that

$$e_B(\lambda, \mu) - e_C(\lambda, \mu) = l(\mu^b) - l(\lambda^b) \in \{0, 1\}. \tag{1}$$

To describe the product of several special cycles, we need to extend the notion of compatibility to sequences of shapes. For nonnegative integers a_1, a_2, define a *compatible* (a_1, a_2)*-chain* to be a sequence of shapes

$$\Lambda = (\pi_0 = \lambda_0, \lambda_1, \ldots, \lambda_{a_1+a_2})$$

such that for $i = 1, 2, \ldots, (a_1+a_2)$, the shapes λ_i and λ_{i-1} are compatible, $|\lambda_i| - |\lambda_{i-1}| \in \{1, 2\}$, and $|\lambda_{a_1+a_2}| = a_1 + 2a_2$.

We now can write down formulas for an arbitrary product of special cycles in $\mathrm{CH}(X_B)$ or $\mathrm{CH}(X_C)$. In $\mathrm{CH}(X_B)$, the formula is

$$\sigma_1^{a_1} \cdot \sigma_2^{a_2} = \sum_{\substack{\text{compatible } (a_1,a_2)\text{-chains} \\ \Lambda=(\lambda_0,\lambda_1,\ldots,\lambda_{a_1+a_2})}} 2^{b_\Lambda} \sigma(\lambda_{a_1+a_2}),$$

where

$$b_\Lambda = e_B(\lambda_0, \lambda_1) + e_B(\lambda_1, \lambda_2) + \cdots + e_B(\lambda_{a_1+a_2-1}, \lambda_{a_1+a_2}),$$

and in $\mathrm{CH}(X_C)$, the formula is

$$\tau_1^{a_1} \cdot \tau_2^{a_2} = \sum_{\substack{\text{compatible } (a_1,a_2)\text{-chains} \\ \Lambda=(\lambda_0,\lambda_1,\ldots,\lambda_{a_1+a_2})}} 2^{c_\Lambda} \tau(\lambda_{a_1+a_2}),$$

where

$$c_\Lambda = e_C(\lambda_0, \lambda_1) + e_C(\lambda_1, \lambda_2) + \cdots + e_C(\lambda_{a_1+a_2-1}, \lambda_{a_1+a_2}).$$

It follows from equation (1) that

$$\begin{aligned} b_\Lambda - c_\Lambda &= [l(\lambda_1^b) - l(\lambda_0^b)] + [l(\lambda_2^b) - l(\lambda_1^b)] + \cdots + [l(\lambda_{a_1+a_2}^b) - l(\lambda_{a_1+a_2-1}^b)] \\ &= l(\lambda_{a_1+a_2}^b) - l(\emptyset) \\ &= l(\lambda_{a_1+a_2}^b). \end{aligned} \tag{2}$$

We now are ready to prove the lemma.

Proof. It is enough to take γ to be a basic cycle, say $\gamma = \sigma(\lambda)$ for some shape λ of weight r. Since τ_1 and τ_2 generate the ring $\mathrm{CH}(X_C)$, there exist integers u_j such that

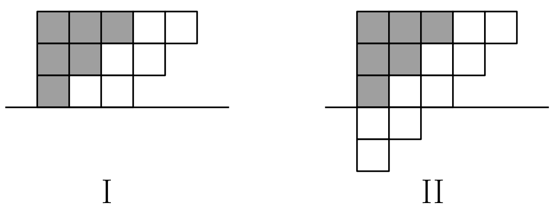
$$\tau(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \left(\tau_1^{r-2j} \cdot \tau_2^j\right) \in \mathrm{CH}^r(X_C).$$

Define

$$\gamma' := \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \left(\sigma_1^{r-2j} \cdot \sigma_2^j\right) \in \mathrm{CH}^r(X_B).$$

This is a rational cycle, since σ_1, σ_2 are Chern classes of the tautological bundle over $X_B = \bar{X}_2$ and are therefore defined over the base field F.

It remains to show that $\gamma' = 2\gamma$. The key is that for any shape $\lambda = (\lambda^t // \lambda^b)$ of weight $2n-3$ or $2n-2$, $l(\lambda^b) = 1$. Indeed, it follows easily from the conditions imposed in the definition of a shape λ that $l(\lambda^b) = 0$ implies $|\lambda| \leq 2n-4$, and $l(\lambda^b) = 2$ (the greatest value possible) implies $|\lambda| \geq 2n-1$. The case $n = 5$ is illustrated below, where we shade the "π_0 boxes" which don't contribute to the weight $|\lambda|$. Diagram I corresponds to the shape of maximal weight $2n-4 = 6$ among shapes λ with $l(\lambda^b) = 0$. Diagram II corresponds to the shape of minimal weight $2n-1 = 9$ among shapes λ with $l(\lambda^b) = 2$.



Expanding the products in the expression for $\tau(\lambda)$, we get

$$\tau(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\substack{\text{compatible } (r-2j,\, j)\text{-chains} \\ \Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-j})}} 2^{c_\Lambda} \tau(\lambda_{r-j}).$$

The maps σ and τ induce a group isomorphism "$\sigma \circ \tau^{-1}$" : $\mathrm{CH}(X_C) \to \mathrm{CH}(X_B)$ which when applied to the equation above yields

$$\sigma(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\substack{\text{compatible } (r-2j,j)\text{-chains} \\ \Lambda=(\lambda_0,\lambda_1,\ldots,\lambda_{r-j})}} 2^{c_\Lambda} \sigma(\lambda_{r-j}). \tag{3}$$

On the other hand, expanding the expression for γ' yields

$$\gamma' = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\substack{\text{compatible } (r-2j,j)\text{-chains} \\ \Lambda=(\lambda_0,\lambda_1,\ldots,\lambda_{r-j})}} 2^{b_\Lambda} \sigma(\lambda_{r-j}), \tag{4}$$

which differs from the expression for $\sigma(\lambda)$ only in that the exponent c_Λ has been changed to b_Λ. By equation (2) and our length computation above, $b_\Lambda - c_\Lambda = l(\lambda^b_{r-j}) = 1$ for any compatible $(r-2j, j)$-chain Λ, since

$$|\lambda_{r-j}| = (r-2j) + 2j = r \in \{2n-3, 2n-2\}.$$

Thus each term on the right-hand side of (4) is twice the corresponding term on the right-hand side of (3) and $\gamma' = 2\sigma(\lambda) = 2\gamma$. □

5. Proof of 2-incompressibility

We now have the ingredients necessary for a proof of the main theorem, whose statement we repeat below.

Theorem 5.1. *If* $\deg \mathrm{CH}(X_2) = 4\mathbb{Z}$ *and* $\mathrm{i}_2(\varphi) = 1$*, then* X_2 *is* 2*-incompressible. In particular,*

$$\mathrm{cdim}_2(X_2) = \mathrm{cdim}(X_2) = \dim(X_2) = 4n - 5.$$

We briefly recall some terminology from [5, §62 and §75]. Let X and Y be schemes with $\dim X = e$. A *correspondence of degree zero* $\delta : X \rightsquigarrow Y$ *from* X *to* Y is just a cycle $\delta \in \mathrm{CH}_e(X \times Y)$. The *multiplicity* $\mathrm{mult}(\delta)$ of such a δ is the integer satisfying $\mathrm{mult}(\delta) \cdot [X] = p_*(\delta)$, where p_* is the push-forward homomorphism

$$p_* : \mathrm{CH}_e(X \times Y) \to \mathrm{CH}_e(X) = \mathbb{Z} \cdot [X].$$

The exchange isomorphism $X \times Y \to Y \times X$ induces an isomorphism

$$\mathrm{CH}_e(X \times Y) \to \mathrm{CH}_e(Y \times X)$$

sending a cycle δ to its *transpose* δ^t.

Proof. To prove that a variety X is 2-incompressible, it suffices to show that for any correspondence $\delta : X \rightsquigarrow X$ of degree zero,

$$\mathrm{mult}(\delta) \equiv \mathrm{mult}(\delta^t) \pmod 2. \tag{5}$$

Indeed, suppose we have $f : X' \to X$ and a dominant $g : X' \to X$ with $F(X')/F(X)$ finite of odd degree. Let $\delta \in \mathrm{CH}(X \times X)$ be the pushforward of the class $[X']$ along the induced morphism $(g, f) : X' \to X \times X$. By assumption, $\mathrm{mult}(\delta)$ is odd, so by (5) we have that $\mathrm{mult}(\delta^t)$ is odd. It follows that $f_*([X'])$ is an odd multiple of $[X]$ and in particular is nonzero, so f is dominant.

We will check that the condition (5) holds for the variety X_2. A correspondence of degree zero $\delta : X_2 \rightsquigarrow X_2$ is just an element of $\mathrm{CH}_{4n-5}(X_2 \times X_2)$. Using the method of Chernousov and Merkurjev described in [3], we can decompose the motive of $X_2 \times X_2$ as follows. See also [4] for examples of similar computations.

We first realize X_2 as a projective homogeneous variety. Let G denote the special orthogonal group corresponding to the quadratic form φ on V. Let Π be a set of simple roots for the root system Σ of G. If $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$, we may take

$$\Pi = \{\alpha_1 := e_1 - e_2, \ldots, \alpha_{n-1} := e_{n-1} - e_n, \alpha_n := e_n\}.$$

Then X_2 is a projective G-homogeneous variety, namely the variety of all parabolic subgroups of G of type S, for the subset $S = \Pi \backslash \{\alpha_2\}$ of the set of simple roots.

Let W denote the Weyl group of the root system Σ. When $n \geq 4$, there are six double cosets $D \in W_P \backslash W / W_P$ with representatives w listed in the first column below (where we write α_k when we mean the reflection w_{α_k}). The second column lists the effect of w^{-1} on the first four e_i (the rest not being affected). The third column gives the subset of Π associated to w. When $n = 3$, there are only five double cosets. In this case, the table may be amended by deleting the final row and removing all mention of e_4 from the remaining rows.

$\boldsymbol{w}$ (α_k here means w_{α_k})	$\boldsymbol{w^{-1}(e_1, e_2, e_3, e_4)}$	$\boldsymbol{R_D}$
1	(e_1, e_2, e_3, e_4)	$\Pi \backslash \{\alpha_2\}$
$\alpha_2 \cdots \alpha_n \cdots \alpha_2$	$(e_1, -e_2, e_3, e_4)$	$\Pi \backslash \{\alpha_1, \alpha_2\}$
$(\alpha_2 \cdots \alpha_n \cdots \alpha_2)\alpha_1(\alpha_2 \cdots \alpha_n \cdots \alpha_2)$	$(-e_2, -e_1, e_3, e_4)$	$\Pi \backslash \{\alpha_2\}$
α_1	(e_1, e_3, e_2, e_4)	$\Pi \backslash \{\alpha_1, \alpha_2, \alpha_3\}$
$\alpha_2\alpha_1(\alpha_3 \cdots \alpha_n \cdots \alpha_2)$	$(e_3, -e_2, e_1, e_4)$	$\Pi \backslash \{\alpha_1, \alpha_2, \alpha_3\}$
$(\alpha_2\alpha_1)(\alpha_3\alpha_2)$	(e_3, e_4, e_1, e_2)	$\Pi \backslash \{\alpha_2, \alpha_4\}$

From [3, Thm. 6.3], we deduce the following decomposition of the motive of $X_2 \times X_2$, where the last summand is removed for the case $n = 3$. We denote by $X_{d_1, d_2, \ldots, d_s}$ the variety of flags of totally isotropic subspaces of V of dimensions $d_1, d_2, \ldots, d_s$.

$$\begin{aligned} M(X_2 \times X_2) \simeq M(X_2) &\oplus M(X_{1,2})(2n-3) \oplus M(X_2)(4n-5) \\ &\oplus M(X_{1,2,3})(1) \oplus M(X_{1,2,3})(2n-2) \oplus \Big[M(X_{2,4})(4)\Big] \end{aligned}$$

This in turn yields a decomposition of the middle-dimensional component of the Chow group of $X_2 \times X_2$.

$$\begin{aligned} \mathrm{CH}_{4n-5}(X_2 \times X_2) \simeq \mathrm{CH}_{4n-5}(X_2) &\oplus \mathrm{CH}_{2n-2}(X_{1,2}) \oplus \mathrm{CH}_0(X_2) \\ &\oplus \mathrm{CH}_{4n-6}(X_{1,2,3}) \oplus \mathrm{CH}_{2n-3}(X_{1,2,3}) \oplus \Big[\mathrm{CH}_{4n-9}(X_{2,4})\Big] \end{aligned}$$

It now suffices to check the congruence $\mathrm{mult}(\delta) \equiv \mathrm{mult}(\delta^t) \pmod 2$ for δ in the image of any of these summands. The embedding of the first summand $\mathrm{CH}_{4n-5}(X_2)$ is induced by the diagonal morphism $X_2 \to X_2 \times X_2$, so the multiplicities are equal by symmetry.

Any element δ of the third summand $\mathrm{CH}_0(X_2)$ has degree divisible by 4 by assumption, hence its image in the Chow group $\mathrm{CH}_0(\bar{X}_2)$ is divisible by 4. (Here we use that $\mathrm{CH}_0(\bar{X}_2)$ is generated by a single element of degree 1.) The image of δ in

$\mathrm{CH}_{4n-5}(\bar{X}_2 \times \bar{X}_2)$ is then also divisible by 4, and since multiplicity does not change under field extension, $\mathrm{mult}(\delta) \equiv 0 \equiv \mathrm{mult}(\delta^t) \pmod 4$.

The second summand requires our work from the previous section. Since $X_{1,2}$ is a projective bundle over X_2, there is a motivic decomposition $M(X_{1,2}) \simeq M(X_2) \oplus M(X_2)(1)$, so that

$$\mathrm{CH}_{2n-2}(X_{1,2}) \simeq \mathrm{CH}_{2n-2}(X_2) \oplus \mathrm{CH}_{2n-3}(X_2).$$

It is enough to consider δ equal to the image of some $\beta \in \mathrm{CH}_r(X_2)$, where $r \in \{2n-3, 2n-2\}$. By the same reasoning as in the previous paragraph, it suffices to show that the image of β in $\overline{\mathrm{CH}}_r(X_2) \subset \mathrm{CH}_r(\bar{X}_2)$ is divisible by 2 in $\mathrm{CH}_r(\bar{X}_2)$. Suppose it is not. Then the image $\hat{\beta}$ of β in the modulo-2 Chow group

$$\mathrm{Ch}_r(\bar{X}_2) := \mathrm{CH}_r(\bar{X}_2)/2\,\mathrm{CH}_r(\bar{X}_2)$$

is nonzero. By [7, Rem. 5.6], the "cellular" variety $\bar{X}_2$ is "2-balanced," i.e. the bilinear form $(\hat{\beta}, \hat{\gamma}) \mapsto \deg(\hat{\beta} \cdot \hat{\gamma})$ on $\mathrm{Ch}(\bar{X}_2)$ is nondegenerate. Hence there exists $\gamma \in \mathrm{CH}^r(\bar{X}_2)$ such that

$$\deg(\beta \cdot \gamma) \equiv 1 \pmod 2.$$

Since 2γ is rational by our Lemma 4.1, we have

$$\deg \overline{\mathrm{CH}}_0(X_2) \ni \deg(\beta \cdot 2\gamma) \equiv 2 \pmod 4.$$

Degree does not change under field extension, so this contradicts our assumption that $\deg \mathrm{CH}(X_2) = 4\mathbb{Z}$.

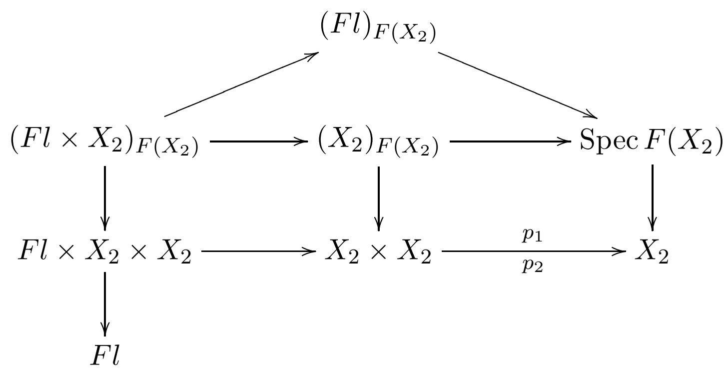
The last three summands of the decomposition are dealt with by the following proposition, whose proof uses our results on higher Witt indices. This will complete the proof of the theorem. □

Proposition 5.2. *Let $Fl := X_{d_1,d_2,\dots,d_s}$ be a variety of totally isotropic flags with $d_s > 2$ and let the correspondence $\alpha : Fl \rightsquigarrow X_2 \times X_2$ induce an embedding*

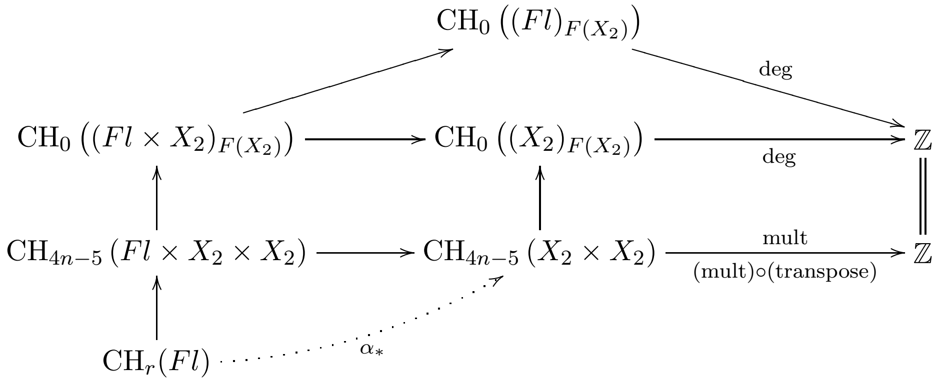
$$\alpha_* : \mathrm{CH}_r(Fl) \hookrightarrow \mathrm{CH}_{4n-5}(X_2 \times X_2).$$

Then for any δ in the image of α_*, $\mathrm{mult}(\delta) \equiv 0 \equiv \mathrm{mult}(\delta^t) \pmod 2$.

Proof. Consider the diagram below of fiber products, where we select either of the projections p_i and choose the other morphisms accordingly.



Taking push-forwards and pull-backs, we get the following diagram which commutes except for the triangle at the bottom. The push-forward by p_i takes a cycle $\delta \in \mathrm{CH}_{4n-5}(X_2 \times X_2)$ to $\mathrm{mult}(\delta)$ if we chose the first projection p_1 or to $\mathrm{mult}(\delta^t)$ if we chose the second projection p_2.



Any $\delta \in \mathrm{im}(\alpha_*)$ also lies in the image of $\mathrm{CH}_{4n-5}\left(Fl \times X_2 \times X_2\right)$, by the definition of the push-forward. Chasing through the diagram, one sees that $\mathrm{mult}(\delta)$ (and similarly $\mathrm{mult}(\delta^t)$) must lie in $\deg \mathrm{CH}_0\left((Fl)_{F(X_2)}\right)$. By Proposition 3.1, we know that $\mathfrak{j}_2(\varphi) = 2$. Hence by Proposition 3.2, $\varphi_{F(X_2)} \simeq 2\mathbb{H} \perp \psi$ for some anisotropic quadratic form ψ over $F(X_2)$. In order for the variety $Fl_{F(X_2)}$ to have a rational point over a field extension K of $F(X_2)$, ψ must be isotropic over K, due to the assumption $d_s > 2$. By Springer's Theorem, if the degree of the extension K is finite then it must be divisible by 2, so

$$\deg \mathrm{CH}_0\left((Fl)_{F(X_2)}\right) \subset 2\mathbb{Z}.$$

□

This completes the proof of the theorem.

References

[1] G. Berhuy and Z. Reichstein, *On the notion of canonical dimension for algebraic groups*, Adv. in Math. 198 (2005), no. 1, 128–171.
[2] P. Brosnan, Z. Reichstein, and A. Vistoli, *Essential dimension and algebraic stacks.*
[3] V. Chernousov and A. Merkurjev, *Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem*, Transformation Groups, 11 (2006), no. 3, 371–386.
[4] V. Chernousov, S. Gille, and A. Merkurjev, *Motivic decomposition of isotropic projective homogeneous varieties*, Duke Math. J., 126 (2005), 137–159.
[5] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, 56. American Mathematical Society, Providence, RI, 2008.
[6] N. Karpenko and A. Merkurjev, *Essential dimension of quadrics*, Invent. Math., 153 (2003), no. 2, 361–372.
[7] N. Karpenko and A. Merkurjev, *Canonical p-dimension of algebraic groups*, Adv. Math. 205 (2006), no. 2, 410–433.
[8] M. Knebusch, *Generic splitting of quadratic forms, I*, Proc. London Math. Soc. 33 (1976), no. 1, 65–93.
[9] I. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press Inc., New York, 1979.
[10] A. Merkurjev, *Essential dimension*, Contemporary Mathematics, to appear.
[11] P. Pragacz and J. Ratajski, *A Pieri-type theorem for Lagrangian and odd Orthogonal Grassmannians*, J. Reine Angew. Math., 476 (1996), 143–189.
[12] A. Vishik, *On the Chow groups of quadratic Grassmannians*, Documenta Math., 10 (2005), 111–130.

Department of Mathematics, University of California, Los Angeles, CA 90095-1555
E-mail address: bmathews@math.ucla.edu