On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes:II

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Abstract

Let R be a regular semi-local ring containing an **infinite perfect subfield** and let K be its field of fractions. Let G be a reductive R-group scheme satifying a mild "isotropy condition". Then each principal G-bundle P which becomes trivial over K is trivial itself. If R is of geometric type, then it suffices to assume that R is of geometric type over an infinite field. Two main Theorems of Panin's, Stavrova's and Vavilov's [PSV] state the same results for semi-simple simply connected R-group schemes. Our proof is heavily based on those two Theorems of [PSV, Thm.1.1], on the main result of [C-T/S] and **on two purity Theorems** proven in the present preprint.

Those purity result look as follows. Given an R-torus C and **a smooth** R-group scheme morphism $\mu : G \to C$ one can form a functor from R-algebras to abelian groups $S \mapsto \mathcal{F}(S) := C(S)/\mu(G(S))$. We prove that this functor satisfies a purity theorem for R. If R is of geometric type, then it suffices to assume that R is of geometric type over an infinite field.

Examples to mentioned purity results are considered in the very end of the preprint.

1 Introduction

Recall that an R-group scheme G is called reductive (respectively, semi-simple; respectively, simple), if it is affine and smooth as an R-scheme and if, moreover, for each ring homomorphism $s : R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a reductive (respectively, semi-simple; respectively, simple) algebraic group over $\Omega(s)$. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple R-group scheme coincides with the notion of a simple semi-simple R-group scheme from Demazure—Grothendieck [D-G, Exp. XIX, Defn. 2.7 and Exp. XXIV, 5.3]. *Throughout the paper R denotes an integral domain and G denotes a semi-simple R-group scheme, unless explicitly stated otherwise.*

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A well-known conjecture due to J.-P. Serre and A. Grothendieck [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has trivial kernel.

Let R be a semi-local ring and G be a reductive group scheme over R. Recall an "isotropy condition " for G.

(I) *Each R-simple component of the derived group G_{der} of the group G is isotropic over R.*

The hypotheses **(I)** means more precicely the following : consider the derived group G_{der} of the group G and the simply-connected cover G^{sc}_{der} of G_{red}; that group G^{sc}_{der} is a product over R (in a unique way) of R-indecomposable groups, which are required to be isotropic.

Theorem 1.0.1. *Let R be regular semi-local domain containing an infinite perfect field and let K be its field of fractions. Let G be a reductive R-group scheme satisfying the condition (**I**). Then the map*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion R into K, has trivial kernel.

In other words, under the above assumptions on R and G each principal G-bundle P over R which has a K-rational point is itself trivial.

• *One can ask* whether two elements $\xi, \zeta \in H^1_{\text{ét}}(R, G)$ which are equal over K are equal over R. Theorem 1.0.1 *does not imply that in general.* However it *does imply that provided that* at least one of the two group schemes $G(\xi)$ or $G(\zeta)$ is isotropic.

• Clearly, this Theorem extends as the geometric case of the main result of J.-L. Colliot-Thélène and J.-J. Sansuc [C-T/S], so main results of I.Panin, A.Stavrova and N.Vavilov [PSV, Thm.1.1, Thm. 1.2]. However our proof of Theorem 1.0.1 **is based heavily on those results and on two purity results** (Theorems 1.0.3 and 1.0.4) proven in the present preprint.

• The case of arbitrary reductive group scheme over a discrete valuation ring is completely solved by Y.Nisnevich in [Ni].

• The case when G is an arbitrary tori over a regular local ring is done by J.-L. Colliot-Thélène and J.-J. Sansuc in [C-T/S].

• For simple group schemes of **classical** series this result follows from more general results established by the first author, A. Suslin, M. Ojanguren and K. Zainoulline [PS], [OP], [Z], [OPZ], [Pa]. In fact, unlike our Theorem 1.0.1, **no isotropy hypotheses** was imposed there. However our result **is new, say for simple adjoint groups of exceptional type** (and for many others).

• The case of **arbitrary** simple adjoint group schemes of type E_6 and E_7 is done by the first author, V.Petrov and A.Stavrova in [PaPS].

• There exists a folklore result, concerning type G_2. It gives affirmative answer in this case, also independent of isotropy hypotheses, see the paper by V. Chernousov and the first author [ChP].

• The case when the group scheme G comes from the ground field k is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, M. S. Raghunatan and O. Gabber in [C-T/O], when k is perfect, in [R1]; O. Gabber announced a proof for a general ground field k.

A geometric counterpart of Theorem 1.0.1 is the following result

Theorem 1.0.2. *Let k be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many points on a smooth irreducible k-variety X and let K be its field of fractions of $\mathcal{O}$. Let G be a reductive $\mathcal{O}$-group scheme satisfying the condition (**I**) above. Then the map*

$$H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion $\mathcal{O}$ into K, has trivial kernel.

In other words, under the above assumptions on R and G each principal G-bundle P over R which has a K-rational point is itself trivial.

To state our purity Theorems recall certain notions. Let $\mathcal{F}$ be a covariant functor from the category of commutative R-algebras to the category of abelian groups. For any domain R consider the sequence

$$\mathcal{F}(R) \to \mathcal{F}(K) \to \bigoplus_{\mathfrak{p}} \mathcal{F}(K)/\mathcal{F}(R_{\mathfrak{p}})$$

where p runs over the height 1 primes of R and K is the field of fractions of R. We say that $\mathcal{F}$ *satisfies purity for* R if this sequence — which is clearly a complex — is exact. The purity for R is equivalent to the following property

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} Im[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)] = Im[\mathcal{F}(R) \to \mathcal{F}(K)].$$

Theorem 1.0.3 (A)**.** *Let $\mathcal{O}$ be a semi-local ring of finitely many points on a k-smooth scheme X with an infinite field k. Let*

$$\mu : G \to C$$

be a smooth $\mathcal{O}$-morphism of reductive algebraic $\mathcal{O}$-group schemes, with a torus C. The functor

$$\mathcal{F} : S \mapsto C(S)/\mu(G(S))$$

defined on the category of $\mathcal{O}$-algebras satisfies purity for $\mathcal{O}$.

Theorem 1.0.4 (B). *Let R be a regular local ring containing an infinite perfect field k. Let*

$$\mu : G \to C$$

be a smooth R-group scheme morphism to a tori C. The functor

$$\mathcal{F} : S \mapsto C(S)/\mu(G(S))$$

defined on the category of R-algebras satisfies purity for R.

If K is the field of fractions of R, then this statement can be restated in an explicit way as follows: given an element $c \in C(K)$ suppose that for each height 1 prime ideal $\mathfrak{p}$ in A there exist $a_{\mathfrak{p}} \in C(R_{\mathfrak{p}})$, $g_{\mathfrak{p}} \in G(K)$ with $a = a_{\mathfrak{p}} \cdot \mu(g_{\mathfrak{p}}) \in C(K)$. Then there exist $g_{\mathfrak{m}} \in G(K)$, $a_{\mathfrak{m}} \in C(R)$ such that

$$a = a_{\mathfrak{m}} \cdot \mu(g_{\mathfrak{m}}) \in C(K).$$

After the pioneering articles [C-T/P/S] and [R] on purity theorems for algebraic groups, various versions of purity theorems were proved in [C-T/O], [PS], [Z], [Pa]. The most general result in the so called constant case was given in [Z, Exm.3.3]. This result follows now from our Theorem (A) by taking G to be a k-rational reductive group, $C = \mathbb{G}_{m,k}$ and $\mu : G \to \mathbb{G}_{m,k}$ a dominant k-group morphism. The papers [PS], [Z], [Pa] contain results for the nonconstant case. However they only consider specific examples of algebraic scheme morphisms $\mu : G \to C$.

It seems plausible to expect purity theorem in the following context. Let R be a regular local ring. Let $\mu : G \to T$ be a smooth of reductive R-group schemes with an R-torus T. Let $\mathcal{F}$ be the covariant functor from the category of commutative rings to the category of abelian groups given by $S \mapsto T(S)/\mu(G(S))$. Then $\mathcal{F}$ should satisfies purity for R.

Another functor for which one should expect purity is the following one. Let R be a regular local ring, G and G' reductive R-group schemes, $\pi : G \to G'$ a smooth central R-group scheme morphism. Assume $Z = ker(\pi)$ is finite étale of multiplicative type over R. The boundary operator $\delta_{\pi,R} : G'(R) \to H^1_{\text{ét}}(R, Z)$ makes sense and is a group homomorphism [Se, Ch.II, §5.6, Cor.2]. For an R-algebra S set

$$\mathcal{F}(S) = \mathrm{H}^1_{\text{ét}}(S, Z)/\mathrm{Im}(\delta_{\pi,S}). \tag{1}$$

It seems plausible that the functor $\mathcal{F}$ satisfies purity for R. **Theorem** 12.0.34 **states that this is the case if** R **contains an infinite perfect field** k.

Note that **we use transfers** for the functor $R \mapsto C(R)$, but **we do not use at all the norm principle** for the homomorphism $\mu : G \to C$.

The preprint is organized as follows. In Section 2 we construct norm maps following a method from [SuVo, Sect.6]. In Section 3 we prove the main geometric Lemma 3.0.8. In Section 4 we discuss unramified elements. A key point here is Lemma 4.0.13. In Section 5 we discuss specialization maps. A key point here is Corollary 5.0.20. In Section 7 certain Artin's results are generalized. In Section 8 a convenient technical tool is introduced. In

Section 10 Theorem (A) is proved. In Section 11 Theorem (B) is proved. In Section 12 we consider the functor (1) and prove a purity Theorem 12.0.34 for that functor. In Section 13 Theorems 1.0.1 and 1.0.2 are proved. Finally in Section 14 we collect several examples **illustrating our Purity Theorems**.

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2 Norms

Let $k \subset K \subset L$ be field extensions and assume that L is finite separable over K. Let K^{sep} be a separable closure of K and

$$\sigma_i : K \to K^{sep}, \;\; 1 \leq i \leq n$$

the different embeddings of K into L. As in §1, let C be a commutative algebraic group scheme defined over k. We can define a norm map

$$\mathcal{N}_{L/K} : C(L) \to C(K)$$

by

$$\mathcal{N}_{L/K}(\alpha) = \prod_i C(\sigma_i)(\alpha) \in C(K^{sep})^{\mathcal{G}(K)} = C(K)\;.$$

Following Suslin and Voevodsky [SuVo, Sect.6] we generalize this construction to finite flat ring extensions. Let $p : X \to Y$ be a finite flat morphism of affine schemes. Suppose that its rank is constant, equal to d. Denote by $S^d(X/Y)$ the d-th symmetric power of X over Y.

Lemma 2.0.5. *There is a canonical section*

$$\mathcal{N}_{X/Y} : Y \to S^d(X/Y)$$

which satisfies the following three properties:

(1) Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\mathcal{N}_{X'/Y'}} & S^d(X'/Y') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle S^d(Id_X \times f)} \\ Y & \xrightarrow{\mathcal{N}_{X/Y}} & S^d(X/Y) \end{array}$$

(2) *Additivity: If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are finite flat morphisms of degree d_1 and d_2 respectively, then, putting $X = X_1 \coprod X_2$, $f = f_1 \coprod f_2$ and $d = d_1 + d_2$, we have a commutative diagram*

$$\begin{array}{ccc} S^{d_1}(X_1/Y) \times S^{d_2}(X_2/Y) & \xrightarrow{\ \sigma\ } & S^d(X/Y) \\ & {\scriptstyle \mathcal{N}_{X_1/Y} \times \mathcal{N}_{X_2/Y}} \nwarrow \quad \nearrow {\scriptstyle \mathcal{N}_{X/Y}} & \\ & Y & \end{array}$$

where σ is the canonical imbedding.

(3) *Normalization: If $X = Y$ and p is the identity, then $\mathcal{N}_{X/Y}$ is the identity.*

Proof. We construct a map $\mathcal{N}_{X/Y}$ and check that it has the desired properties. Let $B = k[X]$ and $A = k[Y]$, so that B is a locally free A-module of finite rank d. Let $B^{\otimes d} = B \otimes_A B \otimes_A \cdots \otimes_A B$ be the d-fold tensor product of B over A. The permutation group $\mathfrak{S}_d$ acts on $B^{\otimes d}$ by permuting the factors. Let $S^d(B) \subseteq B^{\otimes d}$ be the A-algebra of all the $\mathfrak{S}_d$-invariant elements of $B^{\otimes d}$. We consider $B^{\otimes d}$ as an $S^d(B)$-module through the inclusion $S^d(B) \subseteq B^{\otimes d}$ of A-algebras. Let I be the kernel of the canonical homomorphism $B^{\otimes d} \to \bigwedge^d(B)$ mapping $b_1 \otimes \cdots \otimes b_d$ to $b_1 \wedge \cdots \wedge b_d$. It is well-known (and easily checked locally on A) that I is generated by all the elements $x \in B^{\otimes d}$ such that $\tau(x) = x$ for some transposition τ. If s is in $S^d(B)$, then $\tau(sx) = \tau(s)\tau(x) = sx$, hence sx is in $S^d(B)$ too. In other words, I is an $S^d(B)$-submodule of $B^{\otimes d}$. The induced $S^d(B)$-module structure on $\bigwedge^d(B)$ defines an A-algebra homomorphism

$$\varphi : S^d(B) \to End_A(\bigwedge^d(B)) .$$

Since B is locally free of rank d over A, $\bigwedge^d(B)$ is an invertible A-module and we can canonically identify $End_A(\bigwedge^d(B)$ with A. Thus we have a map

$$\varphi : S^d(B) \to A$$

and we define

$$Tr_{X/Y} : Y \to S^d(X)$$

as the morphism of Y-schemes induced by φ. The verification of properties (1), (2) and (3) is straightforward. □

Let now C be a commutative k-group scheme and $f : X \to C$ any morphism. We define the norm $N_{X/Y}(f)$ of f as the composite map

$$Y \xrightarrow{Tr_{X/Y}} S^d(X) \xrightarrow{S^d(f)} S^d(C) \xrightarrow{\times} C \qquad (2)$$

Here we write " $\times$ " for the group law on C. The norm maps $N_{X/Y}$ satisfy the following conditions

(1) Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X') \\ N_{X/Y}\downarrow & & \downarrow N_{X'/Y'} \\ C(Y) & \xrightarrow{f^*} & C(Y') \end{array}$$

(2) multiplicativity: if $X = X_1 \amalg X_2$ then the diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X_1) \times C(X_2) \\ N_{X/Y}\downarrow & & \downarrow N_{X_1/Y} N_{X_2/Y} \\ C(Y) & \xrightarrow{id} & C(Y) \end{array}$$

(3) normalization: if $X = Y$ and the map $X \to Y$ is the identity then $N_{X/Y} = id_{C(X)}$.

3 Geometric lemmas

The aim of this Section is to prove an analog of three Lemmas from [**?**] making them characteristic free. Lemma 3.0.8 is a refinement of [OP, Lemma 2].

Lemma 3.0.6. *Let k be an infinite field and let S be an k-smooth equi-dimensional k-algebra of dimension 1. Let $f \in S$ be a non-zero divisor*

Let $\mathfrak{m}_0$ be a maximal ideal with $S/\mathfrak{m}_0 = k$. Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ be different maximal ideals of S (it might be that $\mathfrak{m}_0 = \mathfrak{m}_i$ for an index i). Then there exists a non-zero divisor $\bar{s} \in S$ such that S is finite over $k[\bar{s}]$ and

(1) the ideals $\mathfrak{n}_i := \mathfrak{m}_i \cap k[\bar{s}]$ $(i = 1, 2, \dots, n)$ are all different from each other and different from $\mathfrak{n}_0 := \mathfrak{m}_0 \cap k[\bar{s}]$ provided that $\mathfrak{m}_0$ is different from $\mathfrak{m}_i$'s;

(2) the extension $S/k[\bar{s}]$ is étale at each $\mathfrak{m}_i$'s and at $\mathfrak{m}_0$;

(3) $k[\bar{s}]/\mathfrak{n}_i = S/\mathfrak{m}_i$ for each $i = 1, 2, \dots, n$;

(4) $\mathfrak{n}_0 = \bar{s}k[\bar{s}]$.

Proof. For each index i let x_i be the point on $\mathrm{Spec}(S)$ corresponding to the ideal $\mathfrak{m}_i$. Consider a closed imbedding $\mathrm{Spec}(S) \hookrightarrow \mathbf{A}^n_k$ and find a generic k-defined linear projection $p : \mathbf{A}^n_k \to \mathbf{A}^1_k$ to such that for each index i one has

(1) for each $i \geq 0, j \geq 0$ one has $p(x_i) \neq p(x_j)$ provided that $x_i \neq x_j$;

(2) for each index $i \geq 0$ the map $p|_{\mathrm{Spec}(S)} : \mathrm{Spec}(S) \to \mathbf{A}^1$ is étale at the point x_i;

(3) the separable degree extension $k(x_i)/k(p(x_i))$ is one.

With this in hand one gets for each the equality $k(p(x_i)) = k(x_i)$. In fact the extension $k(x_i)/k(p(x_i))$ is separable by the item (2). By the item (3) we conclude that $k(p(x_i)) = k(x_i)$. The Lemma follows.

□

Lemma 3.0.7. *Under the hypotheses of Lemma 3.0.6 let $f \in S$ be a non-zero divisor which does not belong to a maximal ideal different of $\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_n$. Let $N(f) = N_{S/k[s]}(f)$ be the norm of f. Then for an element $\bar{s} \in S$ satisfying* (1) *to* (4) *of Lemma 3.0.6 one has*

(a) $N(f) = fg$ for an element $g \in S$;

(b) $fS + gS = S$;

(c) the map $k[\bar{s}]/(N(f)) \to S/(f)$ is an isomorphism.

Proof. It is straightforward.

□

Lemma 3.0.8. *Let R be a semi-local essentially smooth k-algebra with maximal ideals $\mathfrak{p}_i$'s, where i runs from 1 to r. Let $R[t] \subset A$ be an $R[t]$-algebra which is smooth as an R-algebra and is finite over the polynomial algebra $R[t]$. Assume that for each index i the $R/\mathfrak{p}_i$-algebra $A_i = A/\mathfrak{p}_i A$ is equi-dimensional of dimension one. Let $\epsilon : A \to R$ be an R-augmentation and $I = ker(\epsilon)$. Given an $f \in A$ with $0 \neq \epsilon(f) \in A$ and such that the R-module A/fA is finite, one can find an $u \in A$ satisfying the following conditions*

(1) A is finite projective as an $R[u]$-module;

(2) $A/uA = A/I \times A/J$ for certain ideal J;

(3) $J + fA = A$;

(4) $(u-1)A + fA = A$;

(5) if $N(f) = N_{A/R[u]}(f)$, then $N(f) = fg \in A$ for certain $g \in A$

(6) $fA + gA = A$;

(7) the composition map $\varphi : R[u]/(N(f)) \to A/(N(f)) \to A/(f)$ is an isomorphism.

Proof. Replacing t by $t - \epsilon(t)$ we may assume that $\epsilon(t) = 0$. Since A is finite over $R[t]$ we conclude that it is a finite projective $R[t]$-module by a theorem of Grothendieck [E, Cor.17.18]. Since A is finite over $R[t]$ and A/fA is finite over R we conclude that $R[t]/(N(f))$ is finite over R, hence $R/(tN(f))$ is finite over R too. So setting $v = tN(f)$ we get an integral extension A over $R[v]$ and one has equalities

$$v = tN_{A/R[t]}(f) = (ft)g = tfg.$$

We claim that $A/R[v]$ is integral, $\epsilon(v) = 0$ and $u \in fA$. In fact, $u = tN_{A/R[t]}(f) = t(fg)$ and whence $\epsilon(v) = \epsilon(t)\epsilon(fg) = 0$. Finally, and $v \in fA$.

We use below "bar" and lower script i to denote the reduction modulo the ideal $\mathfrak{p}_i A$. Let $l_i = \bar{R}_i = R/\mathfrak{p}_i$. By the assumption of the lemma the l_i-algebra $\bar{A}_i$ is an l_i-smooth equi-dimensional of dimension 1. Let $\mathfrak{m}_1^{(i)}, \mathfrak{m}_2^{(i)}, \dots, \mathfrak{m}_n^{(i)}$ be different maximal ideals of $\bar{A}_i$ dividing $\bar{f}_i$ and let $\mathfrak{m}_0^{(i)} = ker(\bar{\epsilon}_i)$. Let $\bar{s}_i \in \bar{A}_i$ be such that the extension $\bar{A}_i/l_i[\bar{s}_i]$ satisfies the conditions (1) to (4) of Lemma 3.0.6.

Let $s \in A$ be a common lift of $\bar{s}_i$'s , that is for each index i one has $\overline{s}_i = \bar{s}_i$ in $\bar{A}_i$. Replacing s by $s - \epsilon(s)$ we may assume that $\epsilon(s) = 0$ and still for each index i one has $\overline{s}_i = \bar{s}_i$.

Let $s^n + p_1(v)s^{n-1} + \dots + p_n(v) = 0$ be an integral dependence equation for s. Let N be an integer large than the $max\{2, deg(p_i(v))\}$, where $i = 1, 2, \dots, n$. Then for any $r \in k^\times$ the element $u = s - rv^N$ is such that v is integral over $R[u]$. Thus for any $r \in k^\times$ the ring R is integral over $A[u]$.

On the other hand for each index i the element $\bar{u}_i = \bar{s}_i - r\bar{v}_i{}^N$ still satisfies the conditions (1) to (4) of Lemma 3.0.6 because for each index $j = 0, 1, 2, \dots, n$ one has $\bar{v}_i \in \mathfrak{m}_j^{(i)}$.

We claim that the element $u \in R$ is the required element (for almost all $r \in k^\times$).

In fact, for almost all $r \in k^\times$ the element u satisfies the conditions (1) to (4) of Lemma [OP, Lemma 5.2]. It remains to show that the conditions (5) to (7) are satisfied for all $r \in k^\times$.

Since A is finite over $R[u]$ we conclude that it is a finite projective $R[u]$-module by a theorem of Grothendieck [E, Cor.18.17] To prove (5) consider the characteristic polynomial of the operator $A \xrightarrow{f} A$ as an $R[u]$-module operator. This polynomial vanishes on f and its free coefficient is $\pm N(f)$ (the norm of f). Thus $f^n - a_1 f^{n-1} + \dots \pm N(f) = 0$ and $N(f) = fg$ for an element $g \in R$.

To prove (6) one has to check that this g is a unit modulo the ideal fA. It suffices to check that for each index i the element $\bar{g}_i \in \bar{A}_i$ is a unit modulo the ideal $\bar{f}_i\bar{A}_i$. The field $l_i = R/\mathfrak{p}_i$, the l_i-algebra $S_i = \bar{A}_i$, its maximal ideals $\mathfrak{m}_0^{(i)}, \mathfrak{m}_1^{(i)}, \dots, \mathfrak{m}_n^{(i)}$ and the element $\bar{u}_i$ satisfy the hypotheses of Lemma 3.0.7 with u replaced by $\bar{u}_i$. Now $\bar{g}_i$ is a unit modulo the ideal $\bar{f}_i\bar{R}_j$ by the item (b) of Lemma 3.0.7.

To prove (7) note that $R[u]/(N_{A/k[X]}(f))$ are A/fA finite A-modules. So it remains to check that the map $\varphi : R[u]/(N_{A/k[X]}(f)) \to A/fA$ is an isomorphism modulo each maximal ideal $\mathfrak{m}_i^{(j)}$. For that it suffices to verify that the map $\bar{\varphi}_i : l_i[\bar{u}_i]/(N(\bar{f}_i)) \to \bar{A}_i/\bar{f}_i\bar{A}_i$ is an isomorphism for each index i, where $N(\bar{f}_i) := N_{\bar{A}_i/l_i[\bar{u}]}(\bar{f}_i)$. Now $\bar{\varphi}_i$ is an isomorphism by the item (c) of Lemma 3.0.7. The lemma follows.

□

Corollary 3.0.9. *Under the hypotheses of Lemma 3.0.8 let K be the field of fractions of R, $A_K = K \otimes_R A$ and $\epsilon_K = id_K \otimes \epsilon : A_K \to K$. Consider the inclusion $K[u] \subset A_K$. Then the norm $N(f) = N_{A_K/K[u]}(f) \in K[u]$ does not vanish at the points 1 and 0 of the affine line $\mathbf{A}^1_K$.*

Proof. The condition (4) of 3.0.8 implies that $N(f)$ does not vanish at the point 1. Since $\epsilon_K(f) \neq 0 \in K$ the conditions (2) and (3) imply that $N(f)$ does not vanish at 0 either. ∎

4 Unramified elements

We work in this section with *the category of commutative R-algebras*. Let $\mathcal{F}$ be a covariant functor from the category of commutative R-algebras to the category of abelian groups. Let K be a field containing k and $S \subset K$ be a k-subalgebra whose field of fractions is K. We define the *subgroup of S-unramified elements of K* as

$$\mathcal{F}_{nr,S}(K) = \bigcap_{\mathfrak{p} \in Spec(S)^{(1)}} Im[\mathcal{F}(S_{\mathfrak{p}}) \to \mathcal{F}(K)], \tag{3}$$

where $Spec(S)^{(1)}$ is the set of hight 1 prime ideals in S. Obviously the image of $\mathcal{F}(R)$ in $\mathcal{F}(K)$ is contained in $\mathcal{F}_{nr,S}(K)$. In most cases $\mathcal{F}(S_{\mathfrak{p}})$ injects into $\mathcal{F}(K)$ and $\mathcal{F}_{nr,S}(K)$ is simply the intersection of all $\mathcal{F}(S_p)$.

A basic functor we are interested in is the following one. Let $\mu : G \to C$ be the morphism of reductive R-group schemes from Theorem 1.0.3. For a commutative R-algebra S set

$$\mathcal{F}(S) = C(S)/\mu(G(S)). \tag{4}$$

For an element $\alpha \in C(S)$ we will write $\bar{\alpha}$ for its image in $\mathcal{F}(S)$. In this section we will write $\mathcal{F}$ for the functor (7), the only exception being Lemma 4.0.14. We will repeatedly use the following result

Theorem 4.0.10 ([Ni]). *Let R be a discrete valuation ring with fraction field K. The map $\mathcal{F}(R) \to \mathcal{F}(K)$ is injective.*

Proof. Let H be the kernel of μ. Then the boundary map $\partial : C(R) \to \mathrm{H}^1_{\text{ét}}(R, H)$ fits in a commutative diagram

$$\begin{array}{ccc} C(R)/\mu(G(R)) & \longrightarrow & C(K)/\mu(G(K)) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_{\text{ét}}(R,H) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,H). \end{array}$$

in which the vertical arrows have trivial kernels. The bottom arrow has trivial kernel by a Theorem from [Ni]. Thus the top arrow has trivial kernel too. ∎

Lemma 4.0.11. *Let $\mu : G \to C$ be the morphism of our reductive group schemes. Let $H = \ker(\mu)$. Then for an R-algebra K, where K is a field, the boundary map $\partial : C(K)/\mu(G(K)) \to H^1_{\acute{e}t}(K, H)$ is injective.*

Proof. For a K-rational point $t \in C$ set $H_t = \mu^{-1}(t)$. The action by left multiplication of H on H_t makes H_t into a left principal homogeneous H-space and moreover $\partial(t) \in \mathrm{H}^1_{\text{ét}}(K, H)$ coincides with the isomorphism class of H_t. Now suppose that $s, t \in C(K)$ are such that $\partial(s) = \partial(t)$. This means that H_t and H_s are isomorphic as principal homogeneous H-spaces. We must check that for certain $g \in G(K)$ one has $t = sg$.

Let K^{sep} be a separable closure of K. Let $\psi : H_s \to H_t$ be an isomorphism of left H-spaces. For any $r \in H_s(K^{sep})$ and $h \in H_s(K^{sep})$ one has

$$(hr)^{-1}\psi(hr) = r^{-1}h^{-1}h\psi(r) = r^{-1}\psi(r).$$

Thus for any $\sigma \in Gal(K^{sep}/K)$ and any $r \in H_s(K^{sep})$ one has

$$r^{-1}\psi(r) = (r^{\sigma})^{-1}\psi(r^{\sigma}) = (r^{-1}\psi(r))^{\sigma}$$

which means that the point $u = r^{-1}\psi(r)$ is a $Gal(K^{sep}/K)$-invariant point of $G(K^{sep})$. So $u \in G(K)$. The following relation shows that the ψ coincides with the right multiplication by u. In fact, for any $r \in H_s(K^{sep})$ one has $\psi(r) = rr^{-1}\psi(r) = ru$. Since ψ is the right multiplication by u one has $t = s\mu(u)$, which proves the lemma.

□

Let K be the field of fractions of R and $\mathcal{K}$ be a field containing K and $x : \mathcal{K}^* \to \mathbb{Z}$ be a discrete valuation vanishing on K. Let A_x be the valuation ring of x. Let $\hat{A}_x$ and $\hat{\mathcal{K}}_x$ be the completions of A_x and $\mathcal{K}$ with respect to x. Let $i : K \hookrightarrow \hat{K}_x$ be the inclusion. By Lemma 4.0.10 the map $\mathcal{F}(\hat{A}_x) \to \mathcal{F}(\hat{K}_x)$ is injective. We will identify $\mathcal{F}(\hat{A}_x)$ with its image under this map. Set

$$\mathcal{F}_x(\mathcal{K}) = i_*^{-1}(\mathcal{F}(\hat{A}_x)).$$

The inclusion $A_x \hookrightarrow \mathcal{K}$ induces a map $\mathcal{F}(A_x) \to \mathcal{F}(\mathcal{K})$ which is injective by Lemma 4.0.10. So both groups $\mathcal{F}(A)$ and $\mathcal{F}_x(\mathcal{K})$ are subgroups of $\mathcal{F}(\mathcal{K})$. The following lemma shows that $\mathcal{F}_x(\mathcal{K})$ coincides with the subgroup of $\mathcal{F}(\mathcal{K})$ consisting of all elements *unramified* at x.

Lemma 4.0.12. $\mathcal{F}(A_x) = \mathcal{F}_x(\mathcal{K})$.

Proof. We only have to check the inclusion $\mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x)$. Let $a_x \in \mathcal{F}_x(\mathcal{K})$ be an element. It determines the elements $a \in \mathcal{F}(\mathcal{K})$ and $\hat{a} \in \mathcal{F}(\hat{A}_x)$ which coincide when regarded as elements of $\mathcal{F}(\hat{\mathcal{K}}_x)$. We denote this common element in $\mathcal{F}(\hat{\mathcal{K}}_x)$ by $\hat{a}_x$. Let $H = \ker(\mu)$ and let $\partial : C(-) \to \mathrm{H}^1_{\text{ét}}(-, H)$ be the boundary map.

Let $\xi = \partial(a) \in \mathrm{H}^1_{\text{ét}}(\mathcal{K}, H)$, $\hat{\xi} = \partial(\hat{a}) \in \mathrm{H}^1_{\text{ét}}(\hat{A}_x, H)$ and $\hat{\xi}_x = \partial(\hat{a}_x) \in \mathrm{H}^1_{\text{ét}}(\hat{\mathcal{K}}_x, H)$ Clearly, $\hat{\xi}$ and ξ both coincide with $\hat{\xi}_x$ when regarded as elements of $\mathrm{H}^1_{\text{ét}}(\hat{\mathcal{K}}_x, H)$. Thus one can glue ξ and $\hat{\xi}$ to get a $\xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$ which maps to ξ under the map induced by the inclusion $A_x \hookrightarrow \mathcal{K}$ and maps to $\hat{\xi}$ under the map induced by the inclusion $A_x \hookrightarrow \hat{A}_x$.

We now show that ξ_x has the form $\partial(a'_x)$ for a certain $a'_x \in \mathcal{F}(A_x)$. In fact, observe that the image ζ of ξ in $\mathrm{H}^1_{\text{ét}}(\mathcal{K}, G)$ is trivial. By Theorem [Ni] the map

$$\mathrm{H}^1_{\text{ét}}(A_x, G) \to \mathrm{H}^1_{\text{ét}}(\mathcal{K}, G)$$

has trivial kernel. Therefore the image ζ_x of ξ_x in $\mathrm{H}^1_{\text{ét}}(A_x, G)$ is trivial. Thus there exists an element $a'_x \in \mathcal{F}(A_x)$ with $\partial(a'_x) = \xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$.

We now prove that a'_x coincides with a_x in $\mathcal{F}_x(\mathcal{K})$. Since $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{K})$ are both subgroups of $\mathcal{F}(\mathcal{K})$, it suffices to show that a'_x coincides with the element a in $\mathcal{F}(\mathcal{K})$. By Lemma 4.0.11 the map

$$\mathcal{F}(\mathcal{K}) \xrightarrow{\partial} \mathrm{H}^1_{\text{ét}}(\mathcal{K}, H) \tag{5}$$

is injective. Thus it suffices to check that $\partial(a'_x) = \partial(a)$ in $\mathrm{H}^1_{\text{ét}}(\mathcal{K}, H)$. This is indeed the case because $\partial(a'_x) = \xi_x$ and $\partial(a) = \xi$, and ξ_x coincides with ξ when regarded over $\mathcal{K}$. We have proved that $a'_x \in \mathcal{F}(A_x)$ coincides with a_x in $\mathcal{F}_x(\mathcal{K})$. Thus the inclusion $\mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x)$ is proved, whence the lemma.

□

Lemma 4.0.13. *Let $A \subset B$ be a finite extension of Dedekind K-algebras. Let $0 \neq f \in B$ be such that B/fB is finite over K and reduced.*

Suppose $N_{B/A}(f) = fg \in B$ for a certain $g \in B$ coprime with f. Suppose the composite map $A/N(f)A \to B/N(f)B \to B/fB$ is an isomorphism. Let F and E be the field of fractions of A and B respectively. Let $\beta \in C(B_f)$ be such that $\bar{\beta} \in \mathcal{F}(E)$ is B-unramified. Then, for $\alpha = N_{E/F}(\beta)$, the class $\bar{\alpha} \in \mathcal{F}(F)$ is A-unramified.

Proof. The only primes at which $\bar{\alpha}$ could be ramified are those which divide $N(f)$. Let $\mathfrak{p}$ be one of them. Check that $\bar{\alpha}$ is unramified at $\mathfrak{p}$.

To do this we consider all primes $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ in B lying over $\mathfrak{p}$. Let $\mathfrak{q}_1$ be the unique prime dividing f and lying over $\mathfrak{p}$. Then

$$\hat{B}_{\mathfrak{p}} = \hat{B}_{\mathfrak{q}_1} \times \prod_{i \neq 1} \hat{B}_{\mathfrak{q}_i}$$

with $\hat{B}_{\mathfrak{q}_1} = \hat{A}_{\mathfrak{p}}$. If F, E are the fields of fractions of A and B then

$$E \otimes \hat{F}_{\mathfrak{p}} = \hat{E}_{\mathfrak{q}_1} \times \cdots \times \hat{E}_{\mathfrak{q}_n}$$

and $\hat{E}_{\mathfrak{q}_1} = \hat{F}_{\mathfrak{p}}$. We will write $\hat{E}_i$ for $\hat{E}_{\mathfrak{q}_i}$ and $\hat{B}_i$ for $\hat{B}_{\mathfrak{q}_i}$. Let $\beta \otimes 1 = (\beta_1, \ldots, \beta_n) \in C(\hat{E}_1) \times \cdots \times C(\hat{E}_n)$. Clearly for $i \geq 2$ $\beta_i \in C(\hat{B}_i)$ and $\beta_1 = \mu(\gamma_1)\beta'_1$ with $\beta'_1 \in C(\hat{B}_1) = C(\hat{A}_{\mathfrak{p}})$ and $\gamma_1 \in G(\hat{E}_1) = G(\hat{F}_{\mathfrak{p}})$. Now $\alpha \otimes 1 \in C(\hat{F}_{\mathfrak{p}})$ coincides with the product

$$\beta_1 N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2) \cdots N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n) = \mu(\gamma_1)[\beta'_1 N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2) \cdots N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n)].$$

Thus $\overline{\alpha \otimes 1} = \bar{\beta}'_1 \overline{N_{\hat{L}_2/\hat{K}_{\mathfrak{p}}}(\beta_2)} \cdots \overline{N_{\hat{L}_n/\hat{K}_{\mathfrak{p}}}(\beta_n)} \in \mathcal{F}(\hat{A}_{\mathfrak{p}})$. Let $i : F \hookrightarrow \hat{F}_{\mathfrak{p}}$ be the inclusion and $i_* : \mathcal{F}(F) \to \mathcal{F}(\hat{F}_{\mathfrak{p}})$ be the induced map. Clearly $i_*(\bar{\alpha}) = \overline{\alpha \otimes 1}$ in $\mathcal{F}(\hat{F}_{\mathfrak{p}})$. Now Lemma 4.0.12 shows that the element $\bar{\alpha} \in \mathcal{F}(F)$ belongs to $\mathcal{F}(A_{\mathfrak{p}})$. Hence $\bar{\alpha}$ is A-unramified.

□

Lemma 4.0.14 (Unramifiedness Lemma)**.** *Let $\mathcal{F}$ be a covariant functor from the category of commutative R-algebras to the category of abelian groups. Let S' and R' be two R-algebras which are noetherian domains with fields of fractions K' and L' respectively. Let*

$S' \xrightarrow{i} R'$ *be an injective flat R-algebra homomorphism of finite type and let $j : K' \to L'$ be the induced inclusion of the field of fractions. Then for each localization $R'' \supset R'$ of R' the map*

$$j_* : \mathcal{F}(K') \to \mathcal{F}(L')$$

takes S'-unramified elements to R''-unramified elements.

Proof. Let $v \in \mathcal{F}(K')$ and let $\mathfrak{r}$ be height 1 primes of R''. Then $\mathfrak{q} = R' \cap \mathfrak{r}$ is a height 1 prime of R'. Let $\mathfrak{p} = S' \cap \mathfrak{q}$. Since the S'-algebra R' is flat of finite type one has $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p})$. Thus $\mathrm{ht}(\mathfrak{p})$ is 1 or 0. The commutative diagram

$$\begin{array}{ccc} \mathcal{F}(K') & \longrightarrow & \mathcal{F}(L') \\ \uparrow & & \uparrow \\ \mathcal{F}(S'_{\mathfrak{p}}) & \longrightarrow & \mathcal{F}(R''_{\mathfrak{r}}) \end{array}$$

shows that the class $j_*(v)$ is in the image of $\mathcal{F}(R'_{\mathfrak{r}})$ and hence the class $j_*(v) \in \mathcal{F}(L)$ is R''-unramified.

□

5 Specialization maps

Let K be an infinite field. Let

$$\mu : G \to C \tag{6}$$

be a smooth K-morphism of reductive algebraic K-group schemes, with a torus C. We work in this section with *the category of commutative K-algebras* and with the functor

$$\mathcal{F} : S \mapsto C(S)/\mu(G(S)) \tag{7}$$

defined on the category of A-algebras. So, we assume in this Section that each ring from this Section is equipped with a distinguished K-algebra structure and each ring homomorphism from this Section respects that structures.

For a regular ring R with the fraction field $\mathcal{K}$ and each height one prime $\mathfrak{p}$ in R we construct **specialization maps** $s_{\mathfrak{p}} : \mathcal{F}_{nr,R}(\mathcal{K}) \to \mathcal{F}((\mathfrak{p}))$, where $\mathcal{K}$ is the field of fractions of R and $k(\mathfrak{p})$ is the residue field of R at the prime $\mathfrak{p}$.

Definition 5.0.15. *Let $Ev_{\mathfrak{p}} : C(R_{\mathfrak{p}}) \to C(k(\mathfrak{p}))$ and $ev_{\mathfrak{p}} : \mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(k(\mathfrak{p}))$ be the maps induced by the canonical ring homomorphism $R_{\mathfrak{p}} \to k(\mathfrak{p})$. Define a homomorphism $s_{\mathfrak{p}} : \mathcal{F}_{nr,R}(\mathcal{K}) \to \mathcal{F}((\mathfrak{p}))$ by $s_{\mathfrak{p}}(\alpha) = ev_{\mathfrak{p}}(\tilde{\alpha})$, where $\tilde{\alpha}$ is a lift of α in $\mathcal{F}(R_{\mathfrak{p}})$. Theorem 4.0.10 shows that the map $s_{\mathfrak{p}}$ is well-defined. It is called the specialization map. The map $ev_{\mathfrak{p}}$ is called the evaluation map at the prime $\mathfrak{p}$.*

Obviously for $\alpha \in C(R_{\mathfrak{p}})$ one has $s_{\mathfrak{p}}(\bar{\alpha}) = \overline{Ev_{\mathfrak{p}}(\alpha)} \in \mathcal{F}(k(\mathfrak{p}))$.

Lemma 5.0.16 ([H]). *Let H be a smooth linear algebraic group over the field K. Let R be a K-algebra which is a Dedekind domain with field of fractions $\mathcal{K}$. If $\xi \in H^1_{ét}(\mathcal{K}, H)$ is an R-unramified element for the functor $H^1_{ét}(-, H)$ (see (3) for the Definition), then ξ can be lifted to an element of $H^1_{ét}(R, H)$.*

Proof. Patching.

□

Theorem 5.0.17. *The canonical map $H^1_{ét}(K, H) \to H^1_{ét}(K[t], H)$ is bijective.*

Proof. Let $i : \{0\} \hookrightarrow \mathbf{A}^1_K$ be the origin of the affine line. Let $\mathrm{H}^1_{\text{ét}}(K[t], H) \xrightarrow{i^*} \mathrm{H}^1_{\text{ét}}(K, H)$ be the pull-back map. To prove the theorem we need the following

Claim 5.0.18. *The following two statements are equivalent.*

(1) For any linear algebraic group H over K the map i^* is injective.

(2) For any linear algebraic group H over K the map i^* has the trivial kernel.

Clearly the first statement implies the second one. Now suppose the second statement holds and prove the first one. For that consider $\xi, \xi' \in \mathrm{H}^1_{\text{ét}}(K[t], H)$ and set $\xi_0 = i^*(\xi), \xi'_0 = i^*(\xi')$. Let H_0 be the inner form of the group H with respect to the H-torsor $\mathcal{H}_0$. For any K-scheme S there is a well-known bijection $\phi_S : \mathrm{H}^1_{\text{ét}}(S, H) \to \mathrm{H}^1_{\text{ét}}(S, H_0)$ between non-pointed sets. It takes the H-torsor $\mathcal{H}_0 \times_K S$ to the trivial H_0-torsor $H_0 \times_K S$, where $\mathcal{H}_0$ is an H-torsor over K representing the class ξ_0. These bijections respect K-scheme morphisms. Suppose $\xi_0 = \xi'_0$, then

$$i^*(\phi_{\mathbf{A1}}(\xi)) = \phi_K(\xi_0) = * = \phi_K(\xi'_0) = i^*(\phi_{\mathbf{A1}}(\xi')) \in \mathrm{H}^1_{\text{ét}}(K, H_0).$$

So $\phi_{\mathbf{A1}}(\xi) = * = \phi_{\mathbf{A1}}(\xi') \in \mathrm{H}^1_{\text{ét}}(K[t], H_0)$. Whence $\xi = \xi'$ and the injectivity follows. With this Claim in hand prove the theorem as follows.

The surjectivity of the map i^* is obvious. To prove its injectivity just use the Claim and mimic the proof of [C-T/O, Thm.2.1, Property P2]. Let $p : \mathbf{A1}_K \to \mathrm{Spec}(K)$ be the structural morphism. Since $p \circ i = id$ and i^* is bijective p^* is bijective.

□

We need the following theorem.

Theorem 5.0.19 (Homotopy invariance). *Let K be the field and $\mu : G \to C$ be the algebraic group morphism (6). Let $K(t)$ be the rational function field in one variable. Let $R \mapsto \mathcal{F}(R)$ be the functor defined by the formulae (7). Then one has*

$$F(K) = F_{nr,K[t]}(K(t)).$$

Proof. The injectivity is clear, since the composition

$$\mathcal{F}(K) \to \mathcal{F}_{nr,K[t]}(K(t)) \xrightarrow{s_0} \mathcal{F}(K)$$

coincides with the identity (here s_0 is the specialization map at the point zero defined in 4.6).

It remains to check the surjectivity. Let $a \in \mathcal{F}_{nr}(K(t))$ and let $H = ker(\mu)$. Then by Lemma 4.0.10 the element $\partial(a) \in H^1_{et}(K(t), H)$ is a class which for every $x \in \mathbf{A}^1_K$ belongs to the image of $H^1_{et}(\mathcal{O}_x, H)$. Thus by Lemma 5.0.16, $\partial(a)$ can be represented by an element $\xi \in H^1_{et}(K[t], H)$, where $K[t]$ is the polynomial ring. By Theorem 5.0.17, the map

$$H^1_{et}(K, H) \to H^1_{et}(K[t], H)$$

is an isomorphism. Then $\xi = \rho(\xi_0)$ for an element $\xi_0 \in H^1_{et}(K, H)$. Consider the diagram

$$\begin{array}{ccccccc}
 & & a & \longmapsto & \xi & \longmapsto & * \\
 & & & & & & \\
1 & \longrightarrow & \mathcal{F}(K(t)) & \xrightarrow{\partial} & H^1_{et}(K(t), H) & \longrightarrow & H^1_{et}(K(t), G) \\
 & & \uparrow{\scriptstyle \epsilon} & & \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \eta} \\
1 & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\partial} & H^1_{et}(K, H) & \longrightarrow & H^1_{et}(K, G) \\
 & & & & & & \\
 & & a_0 & \longmapsto & \xi_0 \ , & &
\end{array}$$

in which all the maps are canonical and all the vertical arrows have trivial kernels. Since ξ goes to the trivial element in $H^1_{et}(K(t), G)$, one concludes that ξ_0 goes to the trivial element in $H^1_{et}(K, G)$. Thus there exists an element $a_0 \in \mathcal{F}(K)$ such that $\partial(a_0) = \xi_0$. The map $\mathcal{F}(K(t)) \to H^1_{et}(K(t), H)$ is injective by Lemma 4.0.11. Thus $\epsilon(a_0) = a$.

□

Corollary 5.0.20. *Let $S \mapsto \mathcal{F}(S)$ be the functor defined in (7). Let*

$$s_0, s_1 : \mathcal{F}_{nr,K[t]}(K(t)) \to \mathcal{F}(K)$$

be the specialization maps at zero and at one (at the primes (t) and (t-1)). Then $s_0 = s_1$.

Proof. It is an obvious consequence of Theorem 5.0.19. □

6 Equating Groups

The aim of this Section is to sketch a proof of Theorem **??**. The following Proposition is a straightforward analog of [OP, Prop.7.1]

Proposition 6.0.21. *Let S be a regular semi-local irreducible scheme. Let $\mu_1 : G_1 \to C_1$ and $\mu_2 : G_2 \to C_2$ be two smooth S-group scheme morphisms with tori C_1 and C_2. Let $T \subset S$ be a non-empty closed sub-scheme of S, and $\varphi : G_1|_T \to G_2|_T$, $\psi : C_1|_T \to C_2|_T$ be S-group scheme isomorphisms such that $(\mu_2|_T) \circ \varphi = \psi \circ (\mu_1|_T)$. Then there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \tilde{S}$ over T and $\tilde{S}$-group scheme isomorphisms $\Phi : \pi^* G_1 \to \pi^* G_2$ and $\Psi : \pi^* C_1 \to \pi^* C_2$ such that* *(i)* $\delta^*(\Phi) = \varphi$,

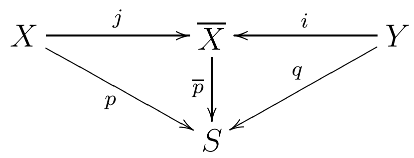
(ii) $\delta^*(\Psi) = \psi$,

(iii) $\pi^*(\mu_2) \circ \Phi = \Psi \circ \pi^*(\mu_1) : \pi^*(G_1) \to \pi^*(C_2)$.

7 Elementary fibrations

In this Section we extend a result of Artin from [A] concerning existence of nice neighborhoods. The following notion is due to Artin [A, Exp.XI,Defn.3.1]

Definition 7.0.22. *An elementary fibration is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram*



(8)

satisfying the following conditions

(i) *j is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$;*

(ii) *$\overline{p}$ is a smooth projective such that all fibres are geometrically irreducible of dimension one;*

(iii) *q is a finite étale morphism such that each fibre is non-empty.*

The following result is an extension of Artin's result [A, Exp.XI,Thm.2.1]

Theorem 7.0.23. *Let k be an infinite field, $V \subset \mathbf{P}^n_k$ a locally closed subscheme of pure dimension r. Let $V' \subset V$ be an open subscheme consisting of all points $x \in V$ such that V is k-smooth at x. Let $p_1, p_2, \ldots, p_m \in \mathbf{P}^n_k$ be a family of closed points with $p_i \neq p_j$ for $i \neq j$. For a family $H_1(d), H_2(d), \ldots, H_s(d)$ $(s \leq r)$ of hyperplanes of degree d containing all points p_i $(i = 1, 2, \ldots, m)$ set $Y = H_1(d) \cap H_2(d) \cap \cdots \cap H_s(d)$.*

There exists an integer d depending of the family $p_1, p_2, \ldots, p_m$ such that if the family $H_1(d), H_2(d), \ldots, H_s(d)$ with $s \leq r$ is general enough, then Y crosses V transversally at each point of $Y \cap V'$.

If V is irreducible (geometrically irreducible) and $s < r$ then for the same integer d and for a general enough family $H_1(d), H_2(d), \ldots, H_s(d)$ the intersection $Y \cap V$ is additionally irreducible (geometrically irreducible).

Using this Bertini type Theorem one can prove the following result, which is a slight extension of Artin's result [A, Exp.XI,Prop.3.3]

Proposition 7.0.24. *Let k be an infinite field, X/k be a smooth geometrically irreducible variety, $x_1, x_2, \ldots, x_n \in X$ be closed points. Then there exists a Zarisky open neighborhood X^0 of the family $\{x_1, x_2, \ldots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where S is an open subscheme of the projective space $\mathbf{P}^{dimX-1}$. Additionally, if Z is a closed codimension one subvariety in X, then one can choose X^0 and p such that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.*

Proposition 7.0.25. *Let $p : X \to S$ be an elementary fibration. If S is a regular semi-local scheme, then there exists a commutative diagram of S-schemes such that the left hand side square is Cartesian.*

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \overline{X} & \xleftarrow{\ i\ } & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow \\ \mathbf{A}^1 \times S & \xrightarrow{\ in\ } & \mathbf{P}^1 \times S & \xleftarrow{\ i\ } & \{\infty\} \times S \end{array} \qquad (9)$$

where j and i are from Definition 7.0.22 and $pr_S \circ \pi = p$, where pr_S is the projection $\mathbf{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbf{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X (resp. $\mathbf{A}^1 \times S$) is regarded as an S-scheme via the morphism p (resp. via the projection pr_S).

We omit for now the proof of this proposition.

8 Nice triples

We study in the present Section certain packages of geometric data and morphisms of that packages. The concept of "nice triples" is very closed to the one of "standard triples " [Vo, Defn.4.1] and is inspired by the latter one. Let k be an infinite field, X/k be a smooth geometrically irreducible variety, $x_1, x_2, \ldots, x_n \in X$ be closed points. Let $\mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$ be the respecting semi-local ring.

Definition 8.0.26. *Let $U := Spec(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$. A nice triple over U consists of the following family of data:*

(i) *a smooth morphism $q_U : \mathcal{X} \to U$,*

(iii) *an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$*

(ii) *a section Δ of the morphism q_U.*

These data must satisfy the following conditions:

(a) *each component of each fibre of the morphism q_U has dimension one,*

(b) *the $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a finite $\Gamma(U, \mathcal{O}_U) = O_{X,\{x_1,x_2,\ldots,x_n\}}$-module,*

(c) there exists a finite surjective U-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$,

(d) $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.

A morphism of two nice triples $(\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ is an étale morphism of U-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that

(1) $q'_U = q_U \circ \theta$,

(2) $f' = \theta^*(f) \cdot g'$ for an element $g' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$
(in particular, $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $O_{X,\{x_1,x_2,\dots,x_n\}}$-module),

(3) $\Delta = \theta \circ \Delta'$.

(Stress that there are no conditions concerning an interaction of Π' and Π).

Theorem 8.0.27. *Let U be as in Definition 8.0.26. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Let $G_{\mathcal{X}}$ be a reductive $\mathcal{X}$-group scheme and $G_U := \Delta^*(G_{\mathcal{X}})$ and G_{const} be the pull-back of G_U to $\mathcal{X}$. Let $C_{\mathcal{X}}$ be an $\mathcal{X}$-tori and $C_U := \Delta^*(C_{\mathcal{X}})$ and C_{const} be the pull-back of C_U to $\mathcal{X}$. Then there exist a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples and isomorphisms*

$$\Phi : \theta^*(G_{const}) \to \theta^*(G_{\mathcal{X}}),\ \ \Psi : \theta^*(C_{const}) \to \theta^*(C_{\mathcal{X}})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = id_{G_U}$, $(\Delta')^*(\Phi) = id_{G_U}$ and

$$\theta^*(\mu_2) \circ \Phi = \Psi \circ \theta^*(\mu_1) \tag{10}$$

This Theorem is a consequence of Proposition 6.0.21. The proof of this theorem is omitted for now.

9 A basic nice triple

With Propositions 7.0.24 and 7.0.25 at our disposal we may form *a basic nice triple*, namely the triple (14). This is the main aim of the present Section. Namely, fix a smooth irreducible affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and a non-zero function $\mathrm{f} \in k[X]$. We *always assume* that the set $\{x_1, x_2, \dots, x_n\}$ is contained in the vanishing locus of the function f.

Replacing k with its algebraic closure in $k[X]$, we may assume that X is a geometrically irreducible k-variety. By Proposition 7.0.24 there exist a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \dots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{dim X-1}$ such that

$$p|_{\{\mathrm{f}=0\} \cap X^0} : \{\mathrm{f} = 0\} \cap X^0 \to S$$

is finite surjective. Let $s_i = p(x_i) \in S$, for each $1 \leq i \leq n$. Shrinking S, we may assume that S is *affine* and still contains the family $\{s_1, s_2, \dots, s_n\}$. Clearly, in this case

$p^{-1}(S) \subseteq X^0$ contains the family $\{x_1, x_2, \dots, x_n\}$. Now we replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an elementary fibration $p : X \to S$ such that

$$\{x_1, \dots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{dimX-1}$, and the restriction of $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ to the vanishing locus of f is a finite surjective morphism. In other words, $k[X]/(\mathrm{f})$ is finite as a $k[S]$-module.

As an open affine sub-scheme of the projective space $\mathbf{P}^{dimX-1}$ the scheme S is regular. By Proposition 7.0.25 one can shrink S such that S is still affine, contains the family $\{s_1, s_2, \dots, s_n\}$ and there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = pr_S \circ \pi$. Clearly, in this case $p^{-1}(S) \subseteq X$ contains the family $\{x_1, x_2, \dots, x_n\}$. Now we replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an elementary fibration $p : X \to S$ such that

$$\{x_1, \dots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{dimX-1}$, and the restriction of $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ to the vanishing locus of f is a finite surjective morphism. And eventually there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = pr_S \circ \pi$.

Now set $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$, and denote by $can : U \hookrightarrow X$ the canonical inclusion of schemes, and $p_U = p \circ can : U \to S$. Further, we consider the fibre product $\mathcal{X} := U \times_S X$. Then the canonical projections $q_U : \mathcal{X} \to U$ and $q_X : \mathcal{X} \to X$ and the diagonal morphism $\Delta : U \to \mathcal{X}$ can be included in the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q_X} & X \\ {\scriptstyle q_U}\downarrow\uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle can} & \\ U & & \end{array} \tag{11}$$

where

$$q_X \circ \Delta = can \tag{12}$$

and

$$q_U \circ \Delta = id_U. \tag{13}$$

Note that q_U is *a smooth morphism with geometrically irreducible fibres of dimension one*. Indeed, observe that q_U is a base change via p_U of the morphism p which has the desired properties. Taking the base change via p_U of the finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$ we get *a finite surjective morphism*

$$\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$$

such that $q_U = pr_U \circ \Pi$. Set $f := q_X^*(\mathrm{f})$. The $\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$-module $\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is finite, since the $k[S]$-module $k[X]/\mathrm{f} \cdot k[X]$ is finite.

Note that the data

$$(q_U : \mathcal{X} \to U, f, \Delta) \tag{14}$$

form an example of a *nice triple*, as defined in Definition 8.0.26.

Claim 9.0.28. *Note that the schemes $\Delta(U)$ and $\{f = 0\}$ are both semi-local and that the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.*

This holds since the set $\{x_1, x_2, \dots, x_n\}$ is contained in the vanishing locus of the function f.

10 Proof of Theorem (A)

Proof. We begin with the following data. Fix a smooth irreducible affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_r$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\dots,x_r\}}$ and $U := \mathrm{Spec}(\mathcal{O})$. Replacing k with its algebraic closure in $\Gamma(X, \mathcal{O}_X)$ we may and will assume that X is k-smooth and geometrically irreducible. Further, consider the reductive U-group scheme G, the U-tori C and the smooth U-group scheme morphism

$$\mu : G \to C.$$

Let K be the fraction field of $\mathcal{O}$. Let $\xi \in C(K)$ be such that the element $\bar{\xi} \in \mathcal{F}(K)$ is $\mathcal{O}$-unramified (see (3)).

Shrinking X if necessary, we may secure the following properties:

(i) The points $x_1, x_2, \dots, x_r$ are still in X;

(ii) The group schemes G, C and the group morphism μ are defined over X, G is reductive over X, C is a tori over X and μ is a smooth X-group morphism. We will often denote this X-group scheme G by G_X and write G_U for the original U-group scheme G; We will often denote this X-group scheme C by C_X and write C_U for the original U-group scheme C;

(iii) The element ξ is defined over X_{f}, that is $\xi \in C(k[X]_{\mathrm{f}})$ for a non-zero function $\mathrm{f} \in k[X]$ such that f vanishes at each x_i's and the k-algebra $k[X]/(\mathrm{f})$ is reduced;

(iv) we may assume further that $\bar{\xi} \in \mathcal{F}(k[X]_{\mathrm{f}})$ is $k[X]$-unramified.

So, we are given now with a geometically irreducible k-smooth scheme X, a finite family points $x_1, x_2, \dots, x_r$ on X and a non-zero function $\mathrm{f} \in k[X]$ vanishing at each x_i's. Beginning with these data the nice triple (14)

$$(q_U : \mathcal{X} \to U, f, \Delta)$$

was constructed in Section 9.

Set $G_{\mathcal{X}} = q_X^*(G_X)$ and $G_U = \Delta^*(G_{\mathcal{X}}) = can^*(G_X)$. Let G_{const} be the pull-back of G_U to $\mathcal{X}$.

Set $C_{\mathcal{X}} = q_X^*(C_X)$ and $C_U := \Delta^*(C_{\mathcal{X}}) = can^*(C_X)$. Let C_{const} be the pull-back of C_U to $\mathcal{X}$.

By Theorem 8.0.27 there exist a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples and isomorphisms

$$\Phi : \theta^*(G_{\text{const}}) \to \theta^*(G_{\mathcal{X}}),\ \Psi : \theta^*(C_{\text{const}}) \to \theta^*(C_{\mathcal{X}})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = id_{G_U}$, $(\Delta')^*(\Phi) = id_{G_U}$ and

$$\theta^*(\mu_2) \circ \Phi = \Psi \circ \theta^*(\mu_1) : \theta^*(G_{const}) \to \theta^*(C_{\mathcal{X}}) \qquad (15)$$

The equality (15) implies that for each U-scheme $\mathcal{Y}'$ and each U-scheme morphism $t : \mathcal{Y}' \to \mathcal{X}'$ **the group isomorphism** $\Psi(\mathcal{Y}') : \theta^*(C_{const})(\mathcal{Y}') \to \theta^*(C_{\mathcal{X}})(\mathcal{Y}')$ induces a group isomorphism

$$C_U(\mathcal{Y}')/\mu(G_U(\mathcal{Y}')) = \mathcal{F}_{const}(\mathcal{Y}') \cong \mathcal{F}_{\mathcal{X}}(\mathcal{Y}') = C(\mathcal{Y}')/(\mu(G(\mathcal{Y}'))) \qquad (16)$$

Group isomorphisms $\Psi(\mathcal{Y}')$ form a natural functor transformation of functors defined on the category of U-schemes and U-scheme morphisms. The same holds concerning isomorphisms (16).

Taking the open inclusion $\mathcal{X}'_{f'} \hookrightarrow \mathcal{X}'$ and the element $\zeta := (q'_X)^*(\xi) \in \theta^*(C_{\mathcal{X}})(\mathcal{X}'_{f'})$ and applying the equality (15) we get an element $\zeta_U \in \theta^*(C_{const})(\mathcal{X}'_{f'})$.

Let $U = \text{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_r\}})$ be as in Definition 8.0.26. Write R for $\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$. It is a local essentially smooth k-algebra with maximal ideals $\mathfrak{m}_i$'s where i runs from 1 to r. Let $(\mathcal{X}', f', \Delta')$ be the above mentioned nice triple over U. Show that it gives rise to certain data subjecting the hypotheses of Lemma 3.0.8.

Let $A = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. It is an R-algebra via the ring homomorphism $(q'_U)^* : R \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Moreover this R-algebra is smooth. The triple $(\mathcal{X}', f', \Delta')$ is a nice triple. Thus there exists a finite surjective U-morphism $\Pi : \mathcal{X}' \to \mathbf{A}^1_U$. It induces the respecting R-algebras inclusion $R[t] \hookrightarrow \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = A$ such that the A is finitely generated as an $R[t]$-module. For each index i the $R/\mathfrak{m}_i$-algebra $A/\mathfrak{m}_i A$ is equi-dimensional of dimension one since $(\mathcal{X}', f', \Delta')$ is a nice triple. Let $\epsilon = (\Delta')^* : A \to R$ be an R-algebra homomorphism induced by the section Δ' of the morphism q'_U. Clearly, that ϵ is an augmentation. Further, $\epsilon(f') \neq 0 \in R$ since $(\mathcal{X}', f', \Delta')$ is a nice triple. Set $I = ker(\epsilon)$. The R-module $A/f'A$ is finite since $(\mathcal{X}', f', \Delta')$ is a nice triple. So, we are under the hypotheses of Lemma 3.0.8. Thus we may use the conclusion of that Lemma.

So, there exists an element $u \in A$ subjecting (1) to (7) of Lemma 3.0.8. The function u defines a finite morphism $\pi : \mathcal{X}' \to U \times \mathbf{A}^1$. Since $\mathcal{X}'$ and $\mathcal{Y}' := U \times \mathbf{A}^1$ are regular schemes and π is finite surjective it is finite and flat by a theorem of Grothendieck [E, Cor.18.17]. So, for any map $t : \mathcal{Y}'' \to \mathcal{Y}'$ of affine schemes, putting $\mathcal{X}'' = \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Y}''$ we have the norm map given by (2)

$$C(\mathcal{X}'') \to C(\mathcal{Y}'').$$

Set $\mathcal{D} = \mathrm{Spec}(R/J)$ and $\mathcal{D}_1 = \mathrm{Spec}(R/(s-1)R)$. Clearly $\mathcal{D}_1$ is the scheme theoretic pre-image of $U \times \{1\}$ and the disjoint union $\mathcal{D}_0 := \mathcal{D} \coprod \delta(U)$ is the scheme theoretic pre-image of $U \times \{0\}$ under the morphism π. In particular $\mathcal{D}$ is finite flat over U. We will write Δ' for $\Delta'(U)$. Recall that $\xi \in C(k[X]_{\mathrm{f}})$ and set $\zeta = (q'_X)^*(\xi) \in C(A_{f'})$. Since the function $f' \in A$ is co-prime to the ideals J and $(s-1)R$ one can form the following element

$$\xi' = N_{\mathcal{D}_1/U}(\zeta|_{\mathcal{D}_1}) N_{\mathcal{D}/U}(\zeta|_{\mathcal{D}})^{-1} \in C(U) := C(R). \tag{17}$$

Claim 10.0.29 (Main)**.** *One has $\bar{\xi}'_K = \bar{\xi}_K$ in $\mathcal{F}(K)$, where K is the fraction field of both $k[X]$ and $k[U] = A$.*

To complete the proof, it remains to prove the claim. To do this it is convenient to fix some notations. For an A-module M set $M_K = K \otimes_R M$, for an U-scheme $\mathcal{Z}$ set $\mathcal{Z}_K = \mathrm{Spec}(K) \times_U \mathcal{Z}$, for a U-morphism $\varphi : \mathcal{Z} \to \mathcal{W}$ set $\varphi_K = \mathrm{Spec}(K) \times_U \varphi$. Clearly one has $R_K = K$, $U_K = \mathrm{Spec}(K)$, $\mathcal{X}'_K = \mathrm{Spec}(A_K)$ and $(U \times \mathbf{A}^1)_K$ is just the affine line $\mathbf{A}^1_K$. Its closed subschemes $U_K \times \{1\}$ and $U_K \times \{0\}$ coincide with the points 1 and 0 of $\mathbf{A}^1_K$. The morphism $(q'_U)_K : \mathcal{X}'_K \to U_K = \mathrm{Spec}(K)$ is smooth. In fact, the morphism θ is étale and $q_U : \mathcal{X} \to U$ is smooth. The morphism π_K is finite flat and fits in the commutative triangle

$$\begin{array}{ccc} \mathcal{X}'_K & \xrightarrow{\pi_K} & \mathbf{A}^1_K \\ & {\scriptstyle (q'_U)_K}\searrow \quad \swarrow{\scriptstyle pr_K} & \\ & \mathrm{Spec}(K) & \end{array} \tag{18}$$

One has $\mathcal{D}_K = \mathrm{Spec}(R_K/J_K)$, $\mathcal{D}_{1,K} = \mathrm{Spec}(R_K/(s-1)R_K)$. Clearly $\mathcal{D}_{1,K}$ is the scheme theoretic pre-image of the point $\{1\} \in \mathbf{A}^1_K$ and the disjoint union $\mathcal{D}_{0,K} := \mathcal{D}_K \coprod \Delta_K$ is the scheme theoretic pre-image of the point $\{0\} \in \mathbf{A}^1_K$ under the morphism π_K. Let

$$\zeta_u := N_{K(\mathcal{X}'_K)/K(\mathbf{A}^1)}(\zeta) \in C(K(\mathbf{A}^1)) = C(K(u)). \tag{19}$$

To prove Claim 10.0.29 we need the following one:

Claim 10.0.30. *The class $\bar{\zeta}_u \in \mathcal{F}(K(s))$ is $K[s]$-unramified.*

Assuming this claim we complete the proof of Claim 10.0.29. By Claim 10.0.30 we can apply the specialization maps to $\bar{\zeta}_u$. By Corollary 5.0.20 the specializations at 0 and 1 of the element $\bar{\zeta}_u$ coincide, that is

$$s_1(\bar{\zeta}_u) = s_0(\bar{\zeta}_u) \in \mathcal{F}(K). \tag{20}$$

By Corollary 3.0.9 the function $N_{A_K/K[u]}(f) \in K[s]$ does not vanishes as at 1, so at 0 and by the condition (5) of Lemma 3.0.8 one has $\zeta_u \in C(K[u]_{N(f)})$. Using the relation between specialization and evaluation maps described in Definition 5.0.15 one has a chain of relations

$$\overline{Ev_1(\zeta_u)} = s_1(\bar{\zeta}_u) = s_0(\bar{\zeta}_u) = \overline{Ev_0(\zeta_u)} \in \mathcal{F}(K). \tag{21}$$

Base change and the multiplicativity properties of the norm map (2) imply relations

$$Ev_1(\zeta_u) = N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}}) \tag{22}$$

and

$$Ev_0(\zeta_u) = N_{\mathcal{D}_K \coprod \Delta_K / U_K}(\zeta|_{\mathcal{D}_K \coprod \Delta_K}) = N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K}) N_{\Delta_K/U_K}(\zeta|_{\Delta_K}) \qquad (23)$$

So, we have a chain of relations in $\mathcal{F}(K)$

$$\overline{N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}})} = \overline{Ev_1(\zeta_u)} = \overline{Ev_0(\zeta_u)} = \overline{N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})} \cdot \overline{N_{\Delta_K/U_K}(\zeta|_{\Delta_K})}$$

By the normalization property of the norm map (2) one has $N_{\Delta_K/U_K}(\zeta|_{\Delta_K}) = \delta^*_K(\zeta)$. Since $\delta^*_K(\zeta) = \xi_K \in C(K)$ we have a chain of relations in $\mathcal{F}(K)$

$$\bar{\xi}_K = \overline{\delta^*_K(\zeta)} = \overline{N_{\Delta_K/U_K}(\zeta|_{\Delta_K})} = \overline{N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}})} \cdot (\overline{N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})})^{-1}$$

By the base change property of the norm map (2) one has

$$\xi'_K = N_{\mathcal{D}_{1,K}/U_K}(\zeta|_{\mathcal{D}_{1,K}}) N_{\mathcal{D}_K/U_K}(\zeta|_{\mathcal{D}_K})^{-1} \in C(U_K) := C(K). \qquad (24)$$

So $\bar{\xi}'_K = \bar{\xi}_K$. Claim 10.0.29 follows.

It remains to prove Claim 10.0.30. Recall that the class $\bar{\xi} \in \mathcal{F}(k[X]_{\mathrm{f}})$ is $k[X]$-unramified. Since the morphism $p : X \to S$ is smooth so is the morphism $r : U \hookrightarrow X \to S$. Since the $p_X : \mathcal{X} \to X$ is the base change of r it coincides with a composition map $\mathcal{X} \hookrightarrow X \times_S X \xrightarrow{id \times pr_X} X$ where $id \times pr_X$ is a smooth morphism. Whence $id \times pr_X$ is flat and of finite type. By Lemma 4.0.14 the class $\bar{\zeta} \in \mathcal{F}(A_f)$ is R-unramified. Thus the same class $\bar{\zeta}$ when regarded in $\mathcal{F}(K[\mathcal{X}'_K]_f)$ is $K[\mathcal{X}'_K]$-unramified.

Now check that the inclusion of K-algebras $K[u] \subset A_K = K[\mathcal{X}'_K]$ and the function $f' \in A$ satisfy the hypotheses of Lemma 4.0.13. We first check that the K-algebra $A_K/f'A_K$ is reduced. Recall that the ring $k[X]/(\mathrm{f})$ is reduced. As we mentioned above, the morphism $q'_{\mathcal{X}'} : \mathcal{X}' \to X$ is essentially smooth, hence the ring $A/(f')$ is reduced too and thus its localization $A_K/f'A_K$ is reduced. Since the extension $K[u] \subset A_K = K[\mathcal{X}'_K]$ and the functions $f' \in A$ satisfy the conditions (5) to (7) of Lemma 3.0.8 they also satisfy the hypotheses of Lemma 4.0.13 for the functor $\mathcal{F}_{const}$ from equality (16). Thus by Lemma 4.0.13 the class $\bar{\zeta}_u$ is $K[u]$-unramified. This implies Claim 10.0.30. The proof of **the Theorem A** is completed.

□

11 Proof of Theorem (B)

Proof. To prove Theorem (B) we now recall a celebrated result of Dorin Popescu (see [P] or, for a self-contained proof, [Sw]).

Let k be a field and R a local k-algebra. We say that R is *geometrically regular* if $k' \otimes_k R$ is regular for any finite extension k' of k. A ring homomorphism $A \to R$ is called *geometrically regular* if it is flat and for each prime ideal $\mathfrak{q}$ of R lying over $\mathfrak{p}$, $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = k(\mathfrak{p}) \otimes_A R_{\mathfrak{q}}$ is geometrically regular over $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Observe that any regular semi- local ring containing a **perfect** field k is geometrically regular over k.

Theorem 11.0.31 (Popescu's theorem)**.** *A homomorphism $A \to R$ of noetherian rings is geometrically regular if and only if R is a filtered direct limit of smooth A-algebras.*

Proof of Theorem B.

Let R be a regular semi-local ring containing an infinite **perfect** field k. Since k is perfect one can apply Popescu's theorem. So, R can be presented as a filtered direct limit of smooth k-algebras A_α over the infinite field k. We first observe that we may replace the direct system of the A_α's by a system of essentially smooth semi-local k-algebras. In fact, if $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$ are all maximal ideal of R and $S = R - (\cup\, \mathfrak{m}_i)$ we can replace each A_α by $(A_\alpha)_{S_\alpha}$, where $S_\alpha = S \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \to R$ are local (sends a maximal ideal to a maximal one) and every A_α is a regular semi-local ring, in particular a factorial ring.

Let now L be the field of fractions of R and, for each α, let K_α be the field of fractions of A_α. For each index α let $\mathfrak{a}_\alpha$ be the kernel of the map $\varphi_\alpha : A_\alpha \to R$ and $B_\alpha = (A_\alpha)_{\mathfrak{a}_\alpha}$. Clearly, for each α, K_α is the field of fractions of B_α. The composition map $A_\alpha \to R \to L$ factors through B_α and hence it also factors through the residue field k_α of B_α. Since R is a filtering direct limit of the A_α's we see that L is a filtering direct limit of the B_α's.

Let $\xi \in C(L)$ be such that the class $\bar{\xi} \in \mathcal{F}(L)$ is R-unramified. We need the following two lemmas.

Lemma 11.0.32. *Let B be a local ring which is a domain and let K be its field of fractions. Let $\mathfrak{m}$ be a maximal ideal of B and $\bar{B} = B/\mathfrak{m}$. For an element $\theta \in \mathcal{F}(B)$ write $\bar{\theta}$ for its image in $\mathcal{F}(\bar{B})$ and θ_K for its image in $\mathcal{F}(K)$. Let $\eta, \rho \in \mathcal{F}(B)$ be such that $\eta_K = \rho_K \in \mathcal{F}(K)$. Then $\bar{\eta} = \bar{\rho} \in \mathcal{F}(\bar{B})$.*

Lemma 11.0.33. *There exists an index α and an element $\xi_\alpha \in C(B_\alpha)$ such that $\varphi(\xi_\alpha) = \xi$ and the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified.*

Assuming these two claims we complete the proof as follows. Consider a commutative diagram

$$\begin{array}{ccccc} A_\alpha & & \xrightarrow{\varphi_\alpha} & & R \\ \downarrow & & & & \downarrow \\ B_\alpha & \longrightarrow & k_\alpha & \longrightarrow & L \\ \downarrow & & & & \\ K_\alpha. & & & & \end{array}$$

By Lemma 11.0.33 the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified. Hence by Theorem A there exists an element $\eta \in C(A_\alpha)$ such that $\bar{\xi}_\alpha = \bar{\eta} \in \mathcal{F}(K_\alpha)$. By Lemma 11.0.32 the elements $\bar{\xi}_\alpha$ and $\bar{\eta}$ have the same image in $\mathcal{F}(k_\alpha)$. Hence $\bar{\xi} \in \mathcal{F}(L)$ coincides with the image of the element $\varphi_\alpha(\bar{\eta})$ in $\mathcal{F}(L)$. It remains to prove the two Lemmas.

Proof of Lemma 11.0.32. Induction on $dim(B)$. The case of dimension 1 follows from Theorem 4.0.10 applied to the local ring B. To prove the general case choose an $f \in B$ such that $\eta = \rho \in \mathcal{F}(B_f)$. Let $\pi \in B$ be such that π is regular parameter in B, having no common factors with f. Let $\overline{B} = B/\pi B$. Then for the image $\overline{\eta - \rho}$ of $\eta - \rho$ in

$\mathcal{F}(\overline{B})$ we have $(\overline{\eta-\rho})_f = \overline{(\eta-\rho)_f} = 0 \in \mathcal{F}(\overline{B}_{\overline{f}})$. By the inductive hypotheses one has $\overline{\eta-\rho} = 0 \in \mathcal{F}(\bar{B})$. Thus $\bar{\eta} = \bar{\rho} \in \mathcal{F}(\bar{B})$.

Proof of Lemma 11.0.33.

Choose an $f \in R$ such that ξ is defined over R_f. Then ξ is ramified at most at those hight one primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ which contains f. Since the class $\bar{\xi} \in \mathcal{F}(L)$ is R-unramified there exists, for any $\mathfrak{p}_i$, an element $\sigma_i \in G(L)$ and an element $\xi_i \in C(R_{\mathfrak{p}_i})$ such that $\xi = \mu(\sigma_i)\xi_i \in C(L)$. We may assume that ξ_i is defined over R_{h_i} for some $h_i \in R - \mathfrak{p}_i$ and that σ_i is defined over R_{g_i} for some $g_i \in R$.

We can find an index α such that A_α contains lifts $f_\alpha, h_{1,\alpha}, \dots, h_{r,\alpha}, g_{1\alpha}, \dots, g_{r,\alpha}$ and moreover

(1) $C(A_{\alpha,f_\alpha})$ contains a lift ξ_α of ξ,

(2) $C(A_{\alpha,h_{i,\alpha}})$ contains a lift of $\xi_{i,\alpha}$ of ξ_i,

(3) $G(A_{\alpha,g_{i,\alpha}})$ contains a lift of $\sigma_{i,\alpha}$ of σ_i.

Since none of the $f_\alpha, h_{1,\alpha}, \dots, h_{r,\alpha}, g_{1\alpha}, \dots, g_{r,\alpha}$ vanishes in R, the elements $\xi_\alpha, \xi_{1,\alpha}, \dots, \xi_{ir,\alpha}$ and $\sigma_{1,\alpha}, \dots, \sigma_{r,\alpha}$ may be regarded as elements of $C(B_\alpha)$ and $G(B_\alpha)$ respectively.

We know that $\xi_{i,\alpha}\mu(\sigma_{i,\alpha})$ and ξ_α map to the same element in $C(L)$. Hence replacing α by a larger index, we may assume that $\xi_\alpha = \xi_{i,\alpha}\mu(\sigma_{i,\alpha}) \in C(B_\alpha)$. We claim that the class $\bar{\xi}_\alpha \in \mathcal{F}(K_\alpha)$ is A_α-unramified. To prove this note that the only primes at which $\bar{\xi}_\alpha$ could be ramified are those which divide f_α. Let $\mathfrak{q}_\alpha$ be one of them. Check that $\bar{\xi}_\alpha$ is unramified at $\mathfrak{q}_\alpha$. Let $q_\alpha \in A_\alpha$ be a prime element such that $q_\alpha A_\alpha = \mathfrak{q}_\alpha$. Then $q_\alpha r_\alpha = f_\alpha$ for an element r_α. Thus $qr = f \in R$ for the images of q_α and r_α in R. Since the homomorphism $\varphi_\alpha : A_\alpha \to R$ is local, $q \in \mathfrak{m}_R$. The relation $qr = f$ shows that $q \in \mathfrak{p}_i$ for some index i. Thus $q_\alpha \in \varphi_\alpha^{-1}(\mathfrak{p}_i)$ and $\mathfrak{q}_\alpha \subset \varphi_\alpha^{-1}(\mathfrak{p}_i)$. On the other hand $h_{i,\alpha} \in A_\alpha - \varphi_\alpha^{-1}(\mathfrak{p}_i)$, because $h_i \in R - \mathfrak{p}_i$. Thus $h_{i,\alpha} \in A_\alpha - \mathfrak{q}_\alpha$. Now the relation $\xi_\alpha = \xi_{i,\alpha}\mu(\sigma_{i,\alpha}) \in C(B_\alpha)$ with $\xi_{i,\alpha} \in C(A_{\alpha,h_{i,\alpha}})$ shows that $\bar{\xi}_\alpha$ is unramified at $\mathfrak{q}_\alpha$. Thus $\bar{\xi}_\alpha$ is unramified at each hight one prime in A_α containing f_α. Since $\xi_\alpha \in C(A_{\alpha,f_\alpha})$ we conclude that $\bar{\xi}_\alpha$ is A_α-unramified. The lemma follows. The theorem is proved.

□

12 Applications

In this Section we prove a purity theorem for reductive groups. Let G be a reductive group over the characteristic zero field k and $Z \xrightarrow{i} G$ a closed central subgroup of G. Let $G' = G/Z$ be the factor group, $\pi : G \to G'$ be the projection. For any k-algebra R consider the boundary operator $\delta_{\pi,R} : G'(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$. It is a group homomorphism [Se, Ch.II, §5.6, Cor.2]. Set

$$\mathcal{F}(R) = \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_{\pi,R}).$$

Theorem 12.0.34. *Let R be a regular local ring containing the field k. The functor $\mathcal{F}$ satisfies purity for R. If K is the fraction field of R this statement can be restated in an explicit way as follows:*

given an element $\xi \in H^1_{ét}(K, Z)$ suppose that for each height 1 prime ideal $\mathfrak{p}$ in R there exist $\xi_{\mathfrak{p}} \in H^1_{ét}(R_{\mathfrak{p}}, Z)$, $g_{\mathfrak{p}} \in G'(K)$ with $\xi = \xi_{\mathfrak{p}} + \delta_\pi(g_{\mathfrak{p}}) \in H^1_{ét}(K, Z)$. Then there exist $\xi_{\mathfrak{m}} \in H^1_{ét}(R, Z)$, $g_{\mathfrak{m}} \in G'(K)$, such that

$$\xi = \xi_{\mathfrak{m}} + \delta_\pi(g_{\mathfrak{m}}) \in H^1_{ét}(K, Z).$$

Proof. Since G is reductive the group Z is of multiplicative type. So we can find a commutative separable k algebra l and a closed embedding $Z \hookrightarrow R_{l/k}(\mathbb{G}_{m,\ l})$ into the permutation torus $T^+ = R_{l/k}(\mathbb{G}_{m,\ l})$. Let $G^+ = (G \times T^+)/Z$ and $T = T^+/Z$, where Z is embedded in $G \times T^+$ diagonally. Clearly $G^+/G = T$. Consider a commutative diagram

$$\begin{array}{ccccccccc}
 & & \{1\} & & \{1\} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & G' & \xrightarrow{id} & G' & & & & \\
 & & \pi\uparrow & & \pi^+\uparrow & & & & \\
\{1\} & \longrightarrow & G & \xrightarrow{j^+} & G^+ & \xrightarrow{\mu^+} & T & \longrightarrow & \{1\} \\
 & & i\uparrow & & i^+\uparrow & & \uparrow id & & \\
\{1\} & \longrightarrow & Z & \xrightarrow{j} & T^+ & \xrightarrow{\mu} & T & \longrightarrow & \{1\} \\
 & & \uparrow & & \uparrow & & & & \\
 & & \{1\} & & \{1\} & & & &
\end{array}$$

with exact rows and columns. For a local k-algebra A one has $\mathrm{H}^1_{\text{ét}}(A, T^+) = \{*\}$ by Hilbert 90 and this diagram gives rise to a commutative diagram of pointed sets

$$\begin{array}{ccccccc}
 & & & & \mathrm{H}^1_{\text{ét}}(A, G') & \xrightarrow{id} & \mathrm{H}^1_{\text{ét}}(A, G') \\
 & & & & \pi_*\uparrow & & \pi_*^+\uparrow \\
G^+(A) & \xrightarrow{\mu_A^+} & T(A) & \xrightarrow{\delta_A^+} & \mathrm{H}^1_{\text{ét}}(A, G) & \xrightarrow{j_*^+} & \mathrm{H}^1_{\text{ét}}(A, G^+) \\
i_*^+\uparrow & & id\uparrow & & i_*\uparrow & & i_*^+\uparrow \\
T^+(A) & \xrightarrow{\mu_A} & T(A) & \xrightarrow{\delta_A} & \mathrm{H}^1_{\text{ét}}(A, Z) & \xrightarrow{\mu} & \{*\} \\
 & & & & \delta_\pi\uparrow & & \\
 & & & & G'(A) & &
\end{array}$$

with exact rows and columns. It follows that π_*^+ has trivial kernel and one has a chain of group isomorphisms

$$\mathrm{H}^1_{\text{ét}}(A, Z)/Im(\delta_{\pi,A}) = ker(\pi_*) = ker(j_*^+) = T(A)/\mu^+(G^+(A)).$$

Clearly these isomorphisms respect k-homomorphisms of local k-algebras. The functor $A \mapsto T(A)/\mu^+(G^+(A))$ satisfies purity for the regular local k-algebra R by Theorem (B). Hence the functor $A \mapsto \mathrm{H}^1_{\text{ét}}(A, Z)/Im(\delta_{\pi,A})$ satisfies purity for R. □

13 Proof of Theorems 1.0.1 and 1.0.2

Proof of Theorem 1.0.1. Consider a short exact sequence of reductive R-group schemes

$$\{1\} \to G_{der} \xrightarrow{i} G \xrightarrow{\mu} C \to \{1\}, \tag{25}$$

where G_{der} is the derived R-group scheme of G and C be a tori over R. Prove that

$$ker[H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)] = * \tag{26}$$

provided that one has

$$ker[H^1_{\text{ét}}(R, G_{der}) \to H^1_{\text{ét}}(K, G_{der})] = * \tag{27}$$

This sequence of R-group schemes gives a short exact sequence of the corresponding sheaves in the étale topology on the big étale site. That sequence of sheaves gives rise to a commutative diagram with exact arrows of pointed sets

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & C(R)/\mu(G(R)) & \xrightarrow{\partial} & H^1_{\text{ét}}(R, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(R, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(R, C) \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
\{1\} & \longrightarrow & C(R_{\mathfrak{p}})/\mu(G(R_{\mathfrak{p}})) & \xrightarrow{\partial} & H^1_{\text{ét}}(R_{\mathfrak{p}}, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(R_{\mathfrak{p}}, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(R_{\mathfrak{p}}, C) \\
 & & \downarrow{\scriptstyle \alpha_{\mathfrak{p}}} & & \downarrow{\scriptstyle \beta_{\mathfrak{p}}} & & \downarrow{\scriptstyle \gamma_{\mathfrak{p}}} & & \downarrow{\scriptstyle \delta_{\mathfrak{p}}} \\
\{1\} & \longrightarrow & C(K)/\mu(G(K)) & \xrightarrow{\partial} & H^1_{\text{ét}}(K, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(K, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(K, C)
\end{array} \tag{28}$$

Set $\alpha_K = \alpha_{\mathfrak{p}} \circ \alpha$, $\beta_K = \beta_{\mathfrak{p}} \circ \beta$, $\gamma_K = \gamma_{\mathfrak{p}} \circ \gamma$, $\delta_K = \delta_{\mathfrak{p}} \circ \delta$. By a theorem of Nisnevich [Ni] one has

$$ker(\alpha_{\mathfrak{p}}) = ker(\beta_{\mathfrak{p}}) = ker(\gamma_{\mathfrak{p}}) = ker(\delta_{\mathfrak{p}}) = * \tag{29}$$

Let $\xi \in ker(\gamma_K)$, then $\mu(\xi) \in ker(\delta_K)$. By [C-T/S] $ker(\delta_K) = *$, whence $\mu(\xi) = *$ and $\xi = i_*(\zeta)$ for an $\zeta \in H^1_{\text{ét}}(R, G_{der})$. By a Theorem of Nisnevich [Ni,] $ker(\gamma_{\mathfrak{p}}) = *$. Since $\gamma_K(\xi) = *$ we see that $\gamma(\xi) = *$. Whence $i_*(\beta(\zeta)) = *$ and $\beta(\zeta) = \partial(\epsilon_{\mathfrak{p}})$ for an $\epsilon_{\mathfrak{p}} \in C(R_{\mathfrak{p}})/\mu(G(R_{\mathfrak{p}}))$. A diagram chaise shows that there exists a unique element $\epsilon_K \in C(K)/\mu(G(K))$ for each hight one prime ideal $\mathfrak{p}$ of R one has $\alpha_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}) = \epsilon_K \in C(K)/\mu(G(K))$. By Purity Theorem 1.0.4 there exists an element $\epsilon \in C(R)/\mu(G(R))$ such that $\alpha_K(\epsilon) = \epsilon_K$. The element ϵ_K has the property that $\partial(\epsilon_K) = \beta_K(\zeta)$. Whence $\partial(\epsilon) = \zeta$ and $\xi = i(\partial(\epsilon)) = *$. The proof equality (26) is reduced to the proof of the equality (31).

To prove (31) recall that G_{der} is a semi-simple R-group scheme. Let G^{sc}_{red} be the corresponding simply-connected semi-simple R-group scheme and let $\pi : G^{sc}_{red} \to G$ be the corresponding R-group scheme morphism. Let $Z' = ker(\pi)$. It is known that Z' is contained in the center of G^{sc}_{red} and Z' is a finite group scheme of multiplicative type. Consider a short exact sequence of R-group schemes

$$\{1\} \to Z' \to G^{cs}_{der} \xrightarrow{\pi} G_{der} \to \{1\}, \tag{30}$$

By the main result of [PSV, Thm.1.1] one has

$$ker[H^1_{\text{ét}}(R, G^{sc}_{der}) \to H^1_{\text{ét}}(K, G^{sc}_{der})] = * \tag{31}$$

The short exact sequence (30) give rises to an exact sequence of pointed sets

$$H^1_{\text{ét}}(R, Z')/\partial(G_{red}) \to H^1_{\text{ét}}(R, G^{sc}_{der}) \to H^1_{\text{ét}}(R, G_{der}) \to H^2_{\text{ét}}(R, Z') \tag{32}$$

and furthermore to a commutative diagram like (28) with exact arrows. Now a diagram chaise similar to the one above in this proof completes the proof of Theorem 1.0.4.

□

Proof of Theorem 1.0.2. Repeat literally the proof of Theorem 1.0.1.

□

14 Examples

We follow here the notation of The Book of Involutions [KMRT]. The field k is a characteristic zero field. The functors (33) to (47) satisfy purity for regular local rings containing k as follows either from Theorem 12.0.34 or from Theorem (B).

(1) Let G be a simple algebraic group over the field k, Z a central subgroup, $G' = G/Z$, $\pi : G \to G'$ the canonical morphism. For any k-algebra A let $\delta_{\pi,R} : G'(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/\mathrm{Im}(\delta_{\pi,R}). \tag{33}$$

(2) Let (A, σ) be a finite separable k-algebra with an orthogonal involution. Let $\pi : \mathrm{Spin}(A, \sigma) \to \mathrm{PGO}^+(A, \sigma)$ be the canonical morphism of the spinor k-group scheme to the projective orthogonal k-group scheme. Let $Z = ker(\pi)$. For a k-algebra R let $\delta_R : \mathrm{PGO}^+(A, \sigma)(R) \to \mathrm{H}^1_{\text{ét}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto \mathrm{H}^1_{\text{ét}}(R, Z)/Im(\delta_R). \tag{34}$$

In $(2a)$ and $(2b)$ below we describe this functor somewhat more explicitly following [KMRT].

(2a) Let $C(A,\sigma)$ be the Clifford algebra. Its center l is an étale quadratic k-algebra. Assume that $deg(A)$ is divisible by 4. Let $\Omega(A,\sigma)$ be the extended Clifford group [KMRT, Definition given just below (13.19)]. Let $\underline{\sigma}$ be the canonical involution of $C(A,\sigma)$ as it is described in [KMRT, just above (8.11)]. Then $\underline{\sigma}$ is either orthogonal or symplectic by [KMRT, Prop.8.12]. Let $\underline{\mu}: \Omega(A,\sigma) \to R_{l/k}(\mathbb{G}_{m,l})$ be the multiplier map defined in [KMRT, just above (13.25)] by $\underline{\mu}(\omega) = \underline{\sigma}(\omega)\cdot\omega$. Set $R_l = R\otimes_k l$. For a field or a local ring R one has $\mathrm{H}^1_{\text{ét}}(R,Z)/Im(\delta_R) = R_l^{\times}/\underline{\mu}(\Omega(A,\sigma)(R))$ by [KMRT, the diagram in (13.32)]. Consider the functor

$$R \mapsto R_l^{\times}/\underline{\mu}(\Omega(A,\sigma)(R)). \tag{35}$$

It coincides with the functor $R \mapsto \mathrm{H}^1_{\text{ét}}(R,Z)/Im(\delta_R)$ on local rings containing k.

(2b) Now let $deg(A) = 2m$ with odd m. Let $\tau : l \to l$ be the involution of l/k. The kernel of the morphism $R_{l/k}(\mathbb{G}_{m,l}) \xrightarrow{id-\tau} R_{l/k}(\mathbb{G}_{m,l})$ coincides with $\mathbb{G}_{m,k}$. Thus $id-\tau$ induces a k-group scheme morphism which we denote $\overline{id-\tau} : R_{l/k}(\mathbb{G}_{m,l})/\mathbb{G}_{m,k} \hookrightarrow R_{l/k}(\mathbb{G}_{m,l})$. Let $\underline{\mu}: \Omega(A,\sigma) \to R_{l/k}(\mathbb{G}_{m,l})$ be the multiplier map defined in [KMRT, just above (13.25)] by $\underline{\mu}(\omega) = \underline{\sigma}(\omega)\cdot\omega$. Let $\kappa : \Omega(A,\sigma) \to R_{l/k}(\mathbb{G}_{m,l})/\mathbb{G}_{m,k}$ be the k-group scheme morphism described in [KMRT, Prop.13.21]. The composition $\overline{(id-\tau)}\circ\kappa$ lands in $R_{l/k}(\mathbb{G}_{m,l})$. Let $U \subset \mathbb{G}_{m,k}\times R_{l/k}(\mathbb{G}_{m,l})$ be a closed k-subgroup consisting of all (α,z) such that $\alpha^4 = N_{l/k}(z)$.

Set $\mu_* = (\underline{\mu}, [\overline{(id-\tau)}\circ\kappa]\cdot\underline{\mu}2) : \Omega(A,\sigma) \to \mathbb{G}_{m,k}\times R_{l/k}(\mathbb{G}_{m,l})$. This k-group scheme morphism lands in U. So we get a k-group scheme morphism $\mu_* : \Omega(A,\sigma) \to \mathrm{U}$. On the level of k-rational points it coincides with the one described in [KMRT, just above (13.35)]. For a field or a local ring one has

$$\mathrm{H}^1_{\text{ét}}(R,Z)/Im(\delta_R) = \mathrm{U}(R)/[\{(N_{l/k}(\alpha),\alpha^4)|\alpha\in R_l^{\times}\}\cdot\mu_*(\Omega(A,\sigma)(R))].$$

Consider the the functor

$$R \mapsto \mathrm{U}(R)/[\{(N_{l/k}(\alpha),\alpha^4)|\alpha\in R_l^{\times}\}\cdot\mu_*(\Omega(A,\sigma)(R))]. \tag{36}$$

It coincides with the functor $R \mapsto \mathrm{H}^1_{\text{ét}}(R,Z)/Im(\delta_R)$ on local rings containing k.

(3) Let $\Gamma(A,\sigma)$ be the Clifford group k-scheme of (A,σ). Let $\mathrm{Sn} : \Gamma(A,\sigma) \to \mathbb{G}_{m,k}$ be the spinor norm map. It is dominant. Consider the functor

$$R \mapsto R^{\times}/\mathrm{Sn}(\Gamma(A,\sigma)(R)). \tag{37}$$

Purity for this functor was originally proved in [Z, Thm.3.1]. In fact, $\Gamma(A,\sigma)$ is k-rational.

(4) We follow here the Book of Involutions [KMRT, §23]. Let A be a separable finite dimensional k-algebra with center l and k-involution σ such that k coincides with all

σ-invariant elements of l, that is $k = l^\sigma$. Consider the k-group schemes of similitudes of (A, σ):

$$\mathrm{Sim}(A,\sigma)(R) = \{a \in A_R^\times \ |a \cdot \sigma_R(a) \in l_K^\times\}.$$

We have a k-group scheme morphism $\mu : \mathrm{Sim}(A,\sigma) \to \mathbb{G}_{m,k},\ a \mapsto a \cdot \sigma(a)$. It gives an exact sequence of algebraic k-groups

$$\{1\} \to \mathrm{Iso}(A,\sigma) \to \mathrm{Sim}(A,\sigma) \to \mathbb{G}_{m,k} \to \{1\}.$$

One has a the functor

$$R \mapsto R^\times/\mu(\mathrm{Sim}(A,\sigma)(R)). \tag{38}$$

Purity for this functor was originally proved in [Pa, Thm.1.2]. Various particular cases are obtained considering unitary, symplectic and orthogonal involutions.

(4a) In the case of an orthogonal involution σ the connected component $\mathrm{GO}^+(A,\sigma)$ [KMRT, (12.24)] of the similitude k-group scheme $\mathrm{GO}(A,\sigma) := \mathrm{Sim}(A,\sigma)$ has the index two in $\mathrm{GO}(A,\sigma)$. The restriction of μ to $\mathrm{GO}^+(A,\sigma)$ is still a dominant morphism to $\mathbb{G}_{m,k}$. One has a functor

$$R \mapsto R^\times/\mu(\mathrm{GO}^+(A,\sigma)(R)) \tag{39}$$

It seems that its purity does not follow from [Pa, Thm.1.2]. In fact we do not know whether the norm principle holds for $\mu : \mathrm{GO}^+(A,\sigma) \to \mathbb{G}_{m,k}$ or not.

(5) Let A be a central simple algebra (csa) over k and $\mathrm{Nrd} : GL_{1,A} \to \mathbb{G}_{m,k}$ the reduced norm morphism. One has a functor

$$R \mapsto R^\times/\mathrm{Nrd}(\mathrm{GL}_{1,A}(R)). \tag{40}$$

Purity for this functor was originally proved in [C-T/O, Thm.5.2].

(6) Let (A,σ) be a finite separable k-algebra with a unitary involution such that its center l is a quadratic extension of k. Let $U(A,\sigma)$ be the unitary k-group scheme. Let $U_l(1)$ be an algebraic tori given by $N_{l/k} = 1$. One has a functor

$$R \mapsto \mathrm{U}_l(1)(R)/\mathrm{Nrd}(\mathrm{U}_{A,\sigma}(R)) = \{\alpha \in R_l^\times | N_{l/k}(\alpha) = 1\}/\mathrm{Nrd}(\mathrm{U}_{A,\sigma}(R)) \tag{41}$$

where Nrd is the reduced norm map. Purity for this functor was originally proved in [Z, Thm.3.3].

(7) With the notation of example (5) choose an integer d and consider the k-group scheme morphism $\mu : \mathrm{GL}_{1,A} \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ given by $(\alpha, a) \mapsto \mathrm{Nrd}(\alpha) \cdot a^d$. One has a functor

$$R \mapsto R^\times/\mu[\mathrm{GL}_{1,A} \times \mathbb{G}_{m,k})(R)] = R^\times/\mathrm{Nrd}(A_R^\times) \cdot R^{\times d} \tag{42}$$

Purity for this functor was originally proved in [Z, Thm.3.2].

(8) With the notation of example (5) choose an integer d and consider the functor

$$R \mapsto U_l(1)(R)/[\mathrm{Nrd}(U_{A,\sigma}(R)) \cdot \{\alpha \in R_l^\times | N_{l/k}(\alpha) = 1\}^d] \tag{43}$$

Purity for this functor was originally proved in [Z, Thm.3.2].

(9) Let G_1,G_2,C be affine k-group schemes. Assume that C is commutative and let $\mu_1 : G_1 \to C$, $\mu_2 : G_2 \to C$ be k-group scheme morphisms and $\mu : G_1 \times G_2 \to C$ be given by $\mu(g_1, g_2) = \mu_1(g_1) \cdot \mu_2(g_2)$. One has a functor

$$R \mapsto C(R)/\mu[(G_1 \times G_2)(R)] = C(R)/[\mu_1(G_1(R)) \cdot \mu_2(G_2(R))] \tag{44}$$

In this style one could get a lot of curious examples of functors, one of which is given here.

(10) Let (A_1, σ_1) be a finite separable k-algebra with an orthogonal involution. Let A_2 be a csa over k. Let μ_1 be the multiplier map for (A_1, σ_1) and Nrd_2 be the reduced norm for A_2. One has a functor

$$R \mapsto R^\times/[\mu_1(\mathrm{GO}^+(A_1, \sigma_1)(R)) \cdot \mathrm{Nrd}_2(A_{2,R}^\times) \cdot {R^\times}^d]. \tag{45}$$

(10) Let A be a csa of degree 3 over k, Nrd the reduced norm and Trd be the reduced trace. Consider the cubic form on the 27-dimensional k-vector space $A \times A \times A$ given by $N := \mathrm{Nrd}(x) + \mathrm{Nrd}(y) + \mathrm{Nrd}(z) - \mathrm{Trd}(xyz)$. Let $\mathrm{Iso}(A, N)$ be the k-group scheme of isometries of N and $\mathrm{Sim}(N)$ be the k-group scheme of similitudes of N. It is known that $\mathrm{Iso}(N)$ is a normal algebraic subgroup in $\mathrm{Sim}(A, N)$ and the factor group coincides with $\mathbb{G}_{m,k}$. So we have a canonical k-group morphism (the multiplier) $\mu : \mathrm{Sim}(N) \to \mathbb{G}_{m,k}$. Now one has a functor

$$R \mapsto R^\times/\mu(\mathrm{Sim}(N)(R)). \tag{46}$$

Note that the connected component of $\mathrm{Iso}(N)$ is a simply connected algebraic k-group of the type E_6.

(11) Let (A, σ) be a csa of degree 8 over k with a symplectic involution. Let $V \subset A$ be the subspace of all skew-symmetric elements. It is of dimension 28. Let Pfd be the reduced Pfaffian on V and Trd be the reduced trace on A. Consider the degree 4 form on the space $V \times V$ given by $F := \mathrm{Pfr}(x) + \mathrm{Pfr}(y) - 1/4\mathrm{Trd}((xy)2) - 1/16\mathrm{Trd}(xy)^2$. Consider the symplectic form on $V \times V$ given by $\phi((x_1, y_1), (x_2, y_2)) = \mathrm{Trd}(x_1y_2 - x_2y_1)$. Let $\mathrm{Iso}(F)$ (resp. $\mathrm{Iso}(\phi)$) be the k-group scheme of isometries of the pair F (resp. of ϕ)). Let $\mathrm{Sim}(F)$ (resp. $\mathrm{Sim}(\phi)$) be the k-group scheme of similitudes of F (resp. of ϕ). Set $G = \mathrm{Iso}(F) \cap \mathrm{Iso}(\phi)$ and $G^+ = \mathrm{Sim}(F) \cap \mathrm{Sim}(\phi)$. It is known that G is a normal algebraic subgroup in G^+ and the factor group coincides with $\mathbb{G}_{m,k}$. So we have a canonical k-group morphism $\mu : G^+ \to \mathbb{G}_{m,k}$. Now one has a functor

$$R \mapsto R^\times/\mu(G^+(R)). \tag{47}$$

Note that G is a simply-connected group of the type E_7.

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