GROTHENDIECK—SERRE CONJECTURE FOR ADJOINT GROUPS OF TYPES E_6 AND E_7

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Abstract. Assume that R is a semi-local regular ring containing an infinite perfect field, or that R is a semi-local ring of several points on a smooth scheme over an infinite field. Let K be the field of fractions of R. Let H be a strongly inner adjoint simple algebraic group of type E_6 or E_7 over R. We prove that the kernel of the map

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K is trivial. This continues the recent series of papers [PaSV], [Pa] on the Grothendieck—Serre conjecture [Gr, Rem. 1.11].

In what follows we use the notation and terminology of [PS]. Our numbering of vertices of Dynkin diagrams follows [B].

Lemma 1. *Let R be a regular semi-local domain and let K be the the field of fractions of R. Let H be a simple group scheme of inner type over R such that $H \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,K$ is of strongly inner type. Then H is of strongly inner type.*

Proof. Let A be a Tits algebra of H. By the condition, the class $[A \otimes_R K]$ is trivial in $\mathrm{Br}(K)$. By [Gr, Corollaire 1.10] $[A]$ is trivial in $\mathrm{Br}(R)$. So all Tits algebras of H are trivial, and therefore H is strongly inner. □

Lemma 2. *Let R be a semi-local domain, and let H be a strongly inner simply connected simple group scheme of type E_6 (resp., E_7) over R. There exists an inner simply connected simple group scheme G of type E_7 (resp., E_8) over R, together with a maximal parabolic subgroup P of type $\{1,2,3,4,5,6\}$ (resp., $\{1,2,3,4,5,6,7\}$), such that H is isomorphic to the derived subgroup of a Levi subgroup of P. Such a group scheme G is unique up to an isomorphism.*

Proof. Let H_0 be a split simply-connected algebraic group over R of the same type as H, and let G_0 be a split simply-connected algebraic group over R of type E_7 (respectively, E_8) if H is of type E_6 (respectively, E_7). Let P_0 be a standard maximal parabolic subgroup of G_0 corresponding to the 7th (respectively, the 8th) vertex of the Dynkin diagram of G_0. Then H_0 is isomorphic to the derived subgroup of a standard Levi subgroup L_0 of P_0. By [PS, Th. 2 (2)] for any strongly inner form H of H_0 there exist an inner form G of G_0, a parabolic subgroup P of G of the same type as P_0 in G_0, and a Levi subgroup L of P, such that H is isomorphic to the derived subgroup of L. By [PS, Th. 2 (3)] if $\mathrm{Pic}(R) = 0$, which is our case, such a G is unique up to an isomorphism. □

Theorem 1. *Let R be a semi-local domain. Assume moreover that R is regular and contains a infinite perfect field k, or that R is a semi-local ring of several points on a k-smooth scheme over an infinite field k. Let K be the field of fractions of R. Let H be an adjoint strongly inner simple group scheme of type E_6 or E_7 over R. Then the map*

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K has trivial kernel.

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Proof. Let H_0 be a split simply-connected algebraic group over R of the same type as H. The elements of $\mathrm{H}^1_{ét}(R,H)$ parametrize isomorphism classes of inner forms of H_0. Let H_1 and H_2 be two inner forms of H_0, and assume that

$$H_1 \times_{\operatorname{Spec} R} \operatorname{Spec} K \cong H_2 \times_{\operatorname{Spec} R} \operatorname{Spec} K.$$

Assume that H_1 is of strongly inner type. Then H_2 is also of strongly inner type by Lemma 1. Now by Lemma 2 there exist two simply connected R-group schemes G_i, $i=1,2$, of inner type E_7 (resp., E_8), if H is of type E_6 (resp., E_7), together with parabolic subgroups P_i, $i=1,2$, of type $\{1,2,3,4,5,6\}$ (resp., $\{1,2,3,4,5,6,7\}$), such that H_i, $i=1,2$, is isomorphic to the derived subgroup of a Levi subgroup L_i of P_i. Since $H_1 \times_{\operatorname{Spec} R} \operatorname{Spec} K \cong H_2 \times_{\operatorname{Spec} R} \operatorname{Spec} K$, the uniqueness part of Lemma 2 applied to $R=K$ implies that

$$G_1 \times_{\operatorname{Spec} R} \operatorname{Spec} K \cong G_2 \times_{\operatorname{Spec} R} \operatorname{Spec} K.$$

Since the groups G_i, $i=1,2$, are isotropic, and are inner forms of each other, by [Pa, Th. 1.0.1, Th. 1.0.2] they are isomorphic over R. We can assume $G_1=G_2$. Since R is semi-local, by [SGA, Exp. XXVI Cor. 5.5 (iv)] there exists $g \in G_1(R)$ such that $gP_1g^{-1}=P_2$ and $gL_1g^{-1}=L_2$. Then the derived subgroups of L_1 and L_2 are isomorphic, that is, $H_1 \cong H_2$ over R. □

Remark. Actually, the proof is also valid for adjoint (semi)simple algebraic groups of classical type.

References

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