AN UPPER BOUND ON THE ESSENTIAL DIMENSION OF A CENTRAL SIMPLE ALGEBRA

AUREL MEYER[†] AND ZINOVY REICHSTEIN[††]

ABSTRACT. We prove a new upper bound on the essential p-dimension of the projective linear group PGL_n.

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1. INTRODUCTION

Let k be a base field; all other fields will be assumed to be extensions of k.

Given a central simple algebra A over a field K one can ask whether A can be written as $A = A_0 \otimes_{K_0} K$ where A_0 is a central simple algebra over some subfield K_0 of K. In that situation we say that A *descends* to K_0. The *essential dimension* of A, denoted $\mathrm{ed}(A)$, is the minimal transcendence degree over k of a field $K_0 \subset K$ such that A descends to K_0. It can be thought of as "the minimal number of independent parameters" required to define A.

For a prime number p, the related notion of essential dimension at p of an algebra A/K is defined as $\mathrm{ed}(A;p) = \min \mathrm{ed}(A_{K'})$, where K'/K runs over all finite field extensions of degree prime to p.

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We also define

$$\operatorname{ed}(\mathrm{PGL}_n) := \mathbf{max}\,\{\operatorname{ed}(A)\}\,,$$

and

$$\operatorname{ed}(\mathrm{PGL}_n; p) := \mathbf{max}\,\{\operatorname{ed}(A; p)\}\,,$$

where the maximum is taken over all fields K/k and over all central simple K-algebras A of degree n. The appearance of PGL_n in the symbols $\operatorname{ed}(\mathrm{PGL}_n)$ and $\operatorname{ed}(\mathrm{PGL}_n; p)$ has to do with the fact that central simple algebras of degree n are in a natural bijective correspondence with PGL_n-torsors. In fact, one can define $\operatorname{ed}(G)$ and $\operatorname{ed}(G; p)$ for every algebraic k-group G in a similar manner, using G-torsors instead of central simple algebras; see [Re$_2$], [RY] or [BF].

To the best of our knowledge, the problem of computing $\operatorname{ed}(\mathrm{PGL}_n)$ was first raised by C. Procesi in the 1960s. Procesi and S. Amitsur constructed so-called *universal division algebras* $\mathrm{UD}(n)$ and showed that $\mathrm{UD}(n)$ has various generic properties among central simple algebras of degree n. In particular, their arguments can be used to show that

$$\operatorname{ed}(\mathrm{UD}(n)) \geq \operatorname{ed}(A) \text{ and } \operatorname{ed}(\mathrm{UD}(n); p) \geq \operatorname{ed}(A; p)$$

for any prime integer p; cf. [LRRS, Remark 2.8]. Equivalently,

$$\operatorname{ed}(\mathrm{UD}(n)) = \operatorname{ed}(\mathrm{PGL}_n) \text{ and } \operatorname{ed}(\mathrm{UD}(n); p) = \operatorname{ed}(\mathrm{PGL}_n; p).$$

Since the center of $\mathrm{UD}(n)$ has transcendence degree $n^2 + 1$ over k, we conclude that $\operatorname{ed}(\mathrm{PGL}_n) \leq n^2 + 1$. Procesi showed (using different terminology) that in fact,

$$\operatorname{ed}(\mathrm{PGL}_n) \leq n^2\,;$$

see [Pr, Theorem 2.1].

The problem of computing $\operatorname{ed}(\mathrm{PGL}_n)$ was raised again by B. Kahn in the early 1990s. In particular, in 1992 Kahn asked the second author if $\operatorname{ed}(\mathrm{PGL}_n)$ grows sublinearly in n, i.e., whether

$$\operatorname{ed}(\mathrm{PGL}_n) \leq an + b$$

for some positive real numbers a and b. To the best of our knowledge, this question never appeared in print but it is implicit in [Ka, Section 2]. It remains open; the best known upper bound,

$$\operatorname{ed}(\mathrm{PGL}_n) \leq \begin{cases} \frac{(n-1)(n-2)}{2}, \text{ for every odd } n \geq 5 \text{ and} \\ n^2 - 3n + 1, \text{ for every } n \geq 4 \end{cases} \tag{1}$$

(see [LR], [LRRS, Theorem 1.1], [Le, Proposition 1.6] and [FF]), is quadratic in n and the best known lower bound,

$$\operatorname{ed}(\mathrm{PGL}_{p^r}) \geq \operatorname{ed}(\mathrm{PGL}_{p^r}; p) \geq 2r\,,$$

is logarithmic.

Note that if p^s is the largest power of p dividing n then one easily checks, using primary decomposition of central simple algebras, that $\operatorname{ed}(\mathrm{PGL}_n; p) =$

$\mathrm{ed}(\mathrm{PGL}_{p^s}; p)$. Thus for the purpose of computing $\mathrm{ed}(\mathrm{PGL}_n; p)$ it suffices to consider the case where $n = p^s$. In this case we have showed that

$$\mathrm{ed}(\mathrm{PGL}_{p^s}; p) \le p^{2s-1} - p^s + 1$$

for any $s \ge 2$; see [MR, Corollary 1.2]. The main result of this paper is the following stronger upper bound.

Theorem 1.1. *Let $n = p^s$ for some $s \ge 2$. Then*

$$\mathrm{ed}(\mathrm{PGL}_n; p) \le 2\frac{n^2}{p^2} - n + 1$$

A. S. Merkurjev [Me_2] recently showed that for $s = 2$ this bound is sharp, i.e., $\mathrm{ed}(\mathrm{PGL}_{p^2}; p) = p^2 + 1$. We conjecture that this bound is sharp for every $s \ge 2$; this would imply, in particular, that $\mathrm{ed}(\mathrm{PGL}_n)$ is not sublinear in n.

Our upper bound on $\mathrm{ed}(\mathrm{PGL}_n; p)$ is a consequence of the following result. Here n is not assumed to be a prime power.

Theorem 1.2. *Let A/K be a central simple algebra of degree n. Suppose A contains a field F, Galois over K and $\mathrm{Gal}(F/K)$ can be generated by $r \ge 1$ elements. If $[F : K] = n$ then we further assume that $r \ge 2$. Then*

$$\mathrm{ed}(A) \le r\frac{n^2}{[F : K]} - n + 1$$

Note that we always have $[F : K] \le n$. In the special case where equality holds, i.e., A is a crossed product in the usual sense, Theorem 1.2 reduces to [LRRS, Corollary 3.10(a)].

To deduce Theorem 1.1 from Theorem 1.2, let $n = p^s$ and $A = \mathrm{UD}(n)$. In [RS_1, 1.2], L. H. Rowen and D. J. Saltman showed that if $s \ge 2$ then there is a finite field extension K'/K of degree prime to p, such that $A' := A \otimes_K K'$ contains a field F, Galois over K' with $\mathrm{Gal}(F/K') \simeq \mathbb{Z}/p \times \mathbb{Z}/p$. Thus, if $s \ge 2$, Theorem 1.2 tells us that

$$\mathrm{ed}(\mathrm{PGL}_n; p) = \mathrm{ed}(A; p) \le \mathrm{ed}(A') \le 2\frac{n^2}{p^2} - n + 1\,.$$

This proves Theorem 1.1. □

The remainder of this paper will be devoted to proving Theorem 1.2. We reduce the problem to a question about G-lattices, using the same approach as in [LRRS, Sections 2–3], but our analysis is more delicate here, and the results (Theorems 1.2 and 4.1) are stronger.

2. G/H-crossed products

Lemma 2.1. *In the course of proving Theorem 1.2 we may assume without loss of generality that F is contained in a subfield L of A such that L/K is a separable extension of degree $n = \deg(A)$.*

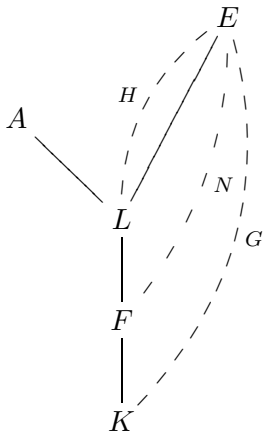
Proof. Note that we are free to replace K by $K(t)$, F by $F(t)$ and A by $A(t) = A \otimes_K K(t)$, where t is an independent variable. Indeed, $\mathrm{ed}_k\, A(t) = \mathrm{ed}_k(A)$; see, e.g., [LRRS, Lemma 2.7(a)]. Thus if the inequality of Theorem 1.2 is proved for $A(t)$, it will also hold for A.

The advantage of passing from A to $A(t)$ is that $K(t)$ if Hilbertian for any infinite field K; see, e.g., [FJ, Proposition 13.2.1]. Thus after adjoining two variables, t_1 and t_2 as above, we may assume without loss of generality that K is Hilbertian. (Note that a subfield $L \subset A$ of degree n over K may not exist without this assumption.)

Let $F \subset F'$ be maximal among separable field extension of F contained in A. We will look for L inside the centralizer $C_A(F')$. By the Double Centralizer Theorem, $C_A(F')$ is a central simple algebra with center F'. The maximality of F' tells us that $C_A(F')$ contains no non-trivial field extensions of F'. In particular, $C_A(F') = \mathrm{M}_r(F')$, where $r[F' : K] = n$.

On the other hand, since K is Hilbertian, so is its finite separable extension F'; cf. [FJ, 12.2.3]. Consequently, F' admits a finite separable extension L/F' of degree r. (To construct L/F', start with the field extension $L_r = F'(t_1, \dots, t_r)[x]/(f(x))$ of $F'(t_1, \dots, t_n)$ of degree r, where $f(x) = x^r + t_1 x^{r-1} + \dots + t_{n-1}x + t_n$ is the general polynomial of degree r. Then specialize $t_1, \dots, t_r$ in F', using the Hilbertian property, to obtain a field extension L/F' of degree r.) Any such L/F' can be embedded into $\mathrm{M}_r(F')$ via the regular representation of L on $L = (F')^r$; cf. [Pi, Lemma 13.1a]. By the maximality of F', we conclude that $L = F'$, i.e., $r = 1$ and $[L : K] = n$, as desired. □

Let us now assume that our central simple algebra A/K has a separable maximal subfield L/K, as in Lemma 2.1. We will denote the Galois closure of L over K by E and the associated Galois groups by $G = \mathrm{Gal}(E/K)$, $H = \mathrm{Gal}(E/L)$ and $N = \mathrm{Gal}(E/F)$, as in the diagram below.



In the terminology of [LRRS], A/K is a G/H-crossed product; cf. also [FSS, Appendix]. Note that since E/K is the smallest Galois extension containing

L/K, we have

$$\mathrm{Core}_G(H) = \bigcap_{g\in G} H^g = \{1\}\,. \tag{2}$$

where $H^g := gHg^{-1}$. We will assume that this condition is satisfied whenever we talk about G/H-crossed products.

Using the notation introduced above and remembering that $[G : H] = [L : K] = \deg(A) = n$, and $\dfrac{n}{[F : K]} = [L : F] = [N : H]$, we can restate Theorem 1.2 as follows.

Theorem 2.2. *Let A be a G/H-crossed product. Suppose H is contained in a normal subgroup N of G and G/N is generated by r elements. Furthermore, assume that either $H \neq \{1\}$ or $r \geq 2$. Then*

$$\mathrm{ed}(A) \leq r[G : H] \cdot [N : H] - [G : H] + 1\,.$$

3. G-lattices

In the sequel $H \leq G$ will be finite groups. Given $g \in G$ we will write $\overline{g}$ for the left coset gH of H. We will denote the identity element of G by 1.

Recall that a G-lattice M is a (left) $\mathbb{Z}[G]$-module, which is free of finite rank over $\mathbb{Z}$. In particular, any finite set X with a G-action gives rise to a G-lattice $\mathbb{Z}[X]$; G-lattices of this form are called *permutation*. For background material on G-lattices we refer the reader to [Lo].

Of particular interest to us will be the G-lattice $\omega(G/H)$, which is defined as the kernel of the natural augmentation map $\mathbb{Z}[G/H] \to \mathbb{Z}$, sending $n_1\overline{g_1} + \cdots + n_s\overline{g_s}$ to $n_1 + \cdots + n_s$.

The starting point for our proof of Theorem 2.2 (and hence, of Theorem 1.2) will be the following result from [LRRS].

Theorem 3.1. ([LRRS, Theorem 3.5]) *Let P be a permutation G-lattice and*

$$0 \to M \to P \to \omega(G/H) \to 0$$

be an exact sequence of G-lattices. If the G-action on M is faithful then

$$\mathrm{ed}(A) \leq \mathrm{rank}(M) - n + 1$$

for any G/H-crossed product A. □

The condition that G acts faithfully on M is not automatic. However, the following lemma shows that it is satisfied for many natural choices of P.

Lemma 3.2. *Let $G \neq \{1\}$ be a finite group $H \leq G$ be a subgroup of G, $H_1, \dots, H_r$ be subgroups of H and*

$$0 \to M \to \oplus_{i=1}^r \mathbb{Z}[G/H_i] \to \omega(G/H) \to 0 \tag{3}$$

be an exact sequence of G-lattices. Assume that H does not contain any nontrivial normal subgroup of G (i.e., H satisfies condition (2) above). Then the G-action on M fails to be faithful if and only if $s = 1$ and $H_1 = H$.

Here we are not specifying the map $\oplus_{i=1}^r \mathbb{Z}[G/H_i] \to \omega(G/H)$; the lemma holds for any exact sequence of the form (3). We also note that in the case where $H_1 = \cdots = H_r = \{1\}$, Lemma 3.2 reduces to [LRRS, Lemma 2.1].

Proof. To determine whether or not the G-action on M is faithful, we may replace M by $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$. After tensoring with $\mathbb{Q}$, the sequence (3) splits, and we have an isomorphism

$$\omega(G/H)_{\mathbb{Q}} \oplus M_{\mathbb{Q}} \simeq \oplus_{i=1}^r \mathbb{Q}[G/H_i]\,. \tag{4}$$

Case 1: $r \geq 2$. Then H_r is a subgroup of H, we have a natural surjective map $\mathbb{Q}[G/H_r] \to \mathbb{Q}[G/H]$. Using complete irreducibility over $\mathbb{Q}$ once again, we see that $\mathbb{Q}[G/H]$ (and hence $\omega(G/H)$) is a subrepresentation of $\mathbb{Q}[G/H_r]$. Thus (4) tells us that $\mathbb{Q}[G/H_{r-1}]$ is a subrepresentation of $M_{\mathbb{Q}}$. The kernel of the G-representation on $\mathbb{Q}[G/H_{r-1}]$ is a normal subgroup of G contained in H_{r-1} (and hence, in H); by our assumption on H, any such subgroup is trivial. This shows that G acts faithfully on $\mathbb{Q}[G/H_{r-1}]$ and hence, on M.

Case 2: Now assume $r = 1$. Our exact sequence now assumes the form

$$0 \to M_{\mathbb{Q}} \to \mathbb{Q}[G/H_1] \to \omega(G/H)_{\mathbb{Q}} \to 0\,.$$

If $H = H_1$ then $M \simeq \mathbb{Z}$, with trivial (and hence, non-faithful) G-action.

Our goal is thus to show that if $H_1 \subsetneq H$ then the G-action on $M_{\mathbb{Q}}$ is faithful. Denote by $\mathbb{Q}[1]$ the trivial representation (it will be clear from the context of which group). Observe that

$$\begin{aligned}
\mathbb{Q}[G/H_1] &\simeq \operatorname{Ind}_{H_1}^G \mathbb{Q}[1] \simeq \operatorname{Ind}_H^G \operatorname{Ind}_{H_1}^H \mathbb{Q}[1] \simeq \operatorname{Ind}_H^G \mathbb{Q}[H/H_1] \\
&\simeq \operatorname{Ind}_H^G(\omega(H/H_1)_{\mathbb{Q}} \oplus \mathbb{Q}[1]) \\
&\simeq \operatorname{Ind}_H^G \omega(H/H_1)_{\mathbb{Q}} \oplus \mathbb{Q}[G/H] \\
&\simeq \operatorname{Ind}_H^G \omega(H/H_1)_{\mathbb{Q}} \oplus \omega(G/H)_{\mathbb{Q}} \oplus \mathbb{Q}[1]
\end{aligned}$$

and we obtain

$$M_{\mathbb{Q}} \simeq \operatorname{Ind}_H^G \omega(H/H_1)_{\mathbb{Q}} \oplus \mathbb{Q}[1]\,.$$

If $H_1 \subsetneq H$ then the kernel of the G-representation $\operatorname{Ind}_H^G \omega(H/H_1)_{\mathbb{Q}}$ is a normal subgroup of G contained in H_1 (and hence, in H). By our assumption on H, this kernel is trivial. □

4. An upper bound

In this section we will prove the following upper bound on the essential dimension of a G/H-crossed product.

We will say that $g_1, \dots, g_s \in G$ generate G over H if $G = \langle g_1, \dots, g_s, H\rangle$.

Theorem 4.1. *Let A be a G/H-crossed product. Suppose that*

(i) $g_1, \dots, g_s \in G$ generate G over H, and

(ii) if G is cyclic then $H \neq \{1\}$.

Then $\operatorname{ed}(A) \leq \sum_{i=1}^s [G : (H \cap H^{g_i})] - [G : H] + 1$.

Remark 4.2. The index $[G : (H \cap H^{g_i})]$ appearing in the above formula can be rewritten as

$$[G : H] \cdot [H : (H \cap H^{g_i})] = [G : H] \cdot [(H \cdot H^{g_i}) : H]\,;$$

see, e.g., [Ro, 1.3.11(i)]. Note $H \cdot H^g := \{hh' \,|\, h \in H,\ h' \in H^g\}$ is a subset of G but may not be a subgroup, and $[(H \cdot H^g) : H]$ is defined as $\dfrac{|H \cdot H^g|}{|H|}$.

If H is contained in a normal subgroup N of G then clearly $H \cdot H^g$ lies in N, each $[H \cdot H^g : H] \leq [N : H]$ and thus Theorem 4.1 yields

$$\operatorname{ed}(A) \leq s[G : H] \cdot [N : H] - [G : H] + 1\,.$$

This is a bit weaker than the inequality of Theorem 2.2, even though the two look very similar. The difference is that we have replaced r in the inequality of Theorem 2.2 by s, where G is generated by s elements over H and by r elements over N. A priori r can be smaller than s. Nevertheless in the next section we will deduce Theorem 2.2 from Theorem 4.1 by a more delicate argument along these lines.

Our proof of Theorem 4.1 will rely on the following lemma.

Lemma 4.3. *Let V be a $\mathbb{Z}[G]$-submodule of $\omega(G/H)$. Then*

$$G_V := \{g \in G \,|\, \overline{g} - \overline{1} \in V\}$$

is a subgroup of G containing H.

Proof. The inclusion $H \subset G_V$ is obvious from the definition.

To see that G_V is closed under multiplication, suppose $g, g' \in G_V$. That is, both $\overline{g} - \overline{1}$ and $\overline{g'} - \overline{1}$ lie in V. Then

$$\overline{gg'} - \overline{1} = g \cdot (\overline{g'} - \overline{1}) + (\overline{g} - \overline{1})$$

also lies in V, i.e., $gg' \in G_V$, as desired. □

Proof of Theorem 4.1. We claim that the elements $\overline{g_1} - \overline{1}, \dots, \overline{g_s} - \overline{1}$ generate $\omega(G/H)$ as a $\mathbb{Z}[G]$-module.

Indeed, let V be the $\mathbb{Z}[G]$-submodule of $\omega(G/H)$ generated by these elements. Lemma 4.3 and condition (i) tell us that V contains $\overline{g} - \overline{1}$ for every $g \in G$. Translating these elements by G, we see that V contains $\overline{a} - \overline{b}$ for every $a, b \in G$. Hence, $V = \omega(G/H)$, as claimed.

For $i = 1, \dots, s$, let

$$S_i := \{g \in G \,|\, g \cdot (\overline{g_i} - \overline{1}) = \overline{g_i} - \overline{1}\}$$

be the stabilizer of $\overline{g_i} - \overline{1}$ in G. We may assume here that g_i is not in H, otherwise it could be removed since it is not needed to generate G over H. Then clearly $g \in S_i$ iff $\overline{gg_i} = \overline{g_i}$ and $\overline{g} = \overline{1}$. From this one easily sees that $S_i = H \cap H^{g_i}$. Thus we have an exact sequence

$$0 \to M \to \oplus_{i=1}^{s} \mathbb{Z}[G/S_i] \xrightarrow{\phi} \omega(G/H) \to 0$$

where ϕ sends a generator of $\mathbb{Z}[G/S_i]$ to $\overline{g_i} - \overline{1} \in \omega(G/H)$. By Theorem 3.1 it remains to show that G acts faithfully on M.

By Lemma 3.2 G fails to act faithfully on M if and only if $r = 1$ and $S_1 = H = H^{g_1}$. But this possibility is ruled out by (ii). Indeed, assume that $s = 1$ and $S_1 = H = H^{g_1}$. Then $G = \langle g_1, H\rangle$ and $H = H^{g_1}$. Hence, H is normal in G. Condition (2) then tells us that $H = \{1\}$. Moreover, in this case $G = \langle g_1, H\rangle = \langle g_1\rangle$ is cyclic, contradicting (ii). □

5. Proof Theorem 1.2

As we saw above, it suffices to prove Theorem 2.2.

Let $t_1, \dots, t_r \in G/N$ be a set of generators for G/N. Choose $g_1, \dots, g_r \in G$ representing $t_1, \dots, t_r$. and let $H' := \langle H, H^{g_1}, \dots, H^{g_r}\rangle$. Since $H \leq N$ and N is normal in G, $H' \leq N$. The group H' depends on the choice of $g_1, \dots, g_r \in G$, so that $g_i N = t_i$. Fix $t_1, \dots, t_r$ and choose $g_1, \dots, g_r \in G$ representing them, so that H' has the largest possible order or equivalently the smallest possible index in N. Denote this minimal possible value of $[N : H']$ by m. In particular

$$m = [N : H'] \leq [N : (H^{g_i g} \cdot H)] \tag{5}$$

for any $i = 1, \dots, r$ and any $g \in N$. Here $[N : (H^{g_i g} \cdot H)] = \dfrac{|N|}{|H^{g_i g} \cdot H|}$, as in Remark 4.2.

Choose a set of representatives $1 = n_1, n_2, \dots, n_m \in N$ for the distinct left cosets of H' in N. We claim that the elements

$$\{g_i n_j \,|\, i = 1, \dots, r;\ j = 1, \dots, m\}$$

generate G over H. Indeed, let G_0 be the subgroup of G generated by these elements and H. Since $n_1 = 1$, G_0 contains $g_1, \dots, g_r$. Hence, G_0 contains H'. Moreover, G_0 contains $n_j = g_1^{-1}(g_1 n_j)$ for every j; hence, G_0 contains all of N. Finally, since $t_1 = g_1 N, \dots, t_r = g_r N$ generate G/N, we conclude that G_0 contains all of G. This proves the claim.

We now apply Theorem 4.1 to the elements $\{g_i n_j\}$. Substituting

$$[G : H] \cdot [H : (H \cdot H^{g_i n_j})] \text{ for } [G : (H \cap H^{g_i n_j})],$$

as in Remark 4.2, we obtain

$$\begin{aligned}
\mathrm{ed}(A) \leq & \sum_{i=1}^{r}\sum_{j=1}^{m}[G:(H\cap H^{g_in_j})]-[G:H]+1\\
= & [G:H]\cdot\sum_{i=1}^{r}\sum_{j=1}^{m}[(H\cdot H^{g_in_j}):H]-[G:H]+1\\
= & [G:H]\cdot\sum_{i=1}^{r}\sum_{j=1}^{m}\frac{[N:H]}{[N:(H\cdot H^{g_in_j})]}-[G:H]+1\\
\leq & \quad\text{(by (5))}\quad [G:H]\cdot\sum_{i=1}^{r}\sum_{j=1}^{m}\frac{[N:H]}{m}-[G:H]+1\\
= & r[G:H]\cdot[N:H]-[G:H]+1
\end{aligned}$$

as desired. This completes the proof of Theorem 2.2 and thus of Theorem 1.2. □

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Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada