PURITY OF F_4–TORSORS WITH TRIVIAL $\mathbf{g_3}$ INVARIANT

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Abstract. We prove that the functor $R \to \mathcal{P}f_n(R)$ of n-fold Pfister forms satisfies purity for regular local rings containing the field $\mathbf{Q}$ of rational numbers. As an application we show that purity holds for F_4-torsors with trivial g_3 invariant.

1. Main results

In the present note we address to the purity conjecture for F_4-torsors. The question on purity of torsors was raised in [CTS, Question 6.4, p. 124] by J.-L. Colliot-Thélène and J.-J. Sansuc. Until recently for exceptional groups the answer was known for type G_2 only (see [ChP]) and our aim is to consider the next open case of groups of type F_4 with trivial g_3 invariant.

Let us recall what the purity property for a functor is. Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of sets, and let R be a domain with field of fractions K. We say that an element $\xi \in \mathcal{F}(K)$ is *unramified* at a prime ideal $\mathfrak{P} \subset R$ of height 1 if

$$\xi \in \operatorname{Im}[\,\mathcal{F}(R_{\mathfrak{P}}) \longrightarrow \mathcal{F}(K)\,].$$

We say that ξ is *unramified* if it is unramified with respect to all prime ideals in R of height 1. It is clear that

$$\operatorname{Im}[\,\mathcal{F}(R) \to \mathcal{F}(K)\,] \subseteq \mathcal{F}(K)_{ur}$$

where $\mathcal{F}(K)_{ur}$ is the set of all unramified elements. We say that the functor $\mathcal{F}$ *satisfies purity for a domain* R if every $\xi \in \mathcal{F}(K)_{ur}$ is in the image of $\mathcal{F}(R)$, i.e. if

$$\bigcap_{\mathrm{ht}\,\mathfrak{P}=1} \operatorname{Im}[\mathcal{F}(R_{\mathfrak{P}}) \to \mathcal{F}(K)] = \operatorname{Im}[\mathcal{F}(R) \to \mathcal{F}(K)].$$

In what follows we assume that 2 is invertible in R. We say that a quadratic space over R is an n-fold Pfister space if the corresponding quadratic form is isomorphic to a form

$$\langle\langle\, a_1, \dots, , a_n \,\rangle\rangle = \langle\, 1, -a_1 \,\rangle \otimes \cdots \otimes \langle\, 1, -a_n \,\rangle$$

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where $a_1, \dots, a_n$ are units in R. We will consider the following two functors:

$$\mathcal{P}f_n(R) = \{\text{ isomorphism classes of } n\text{-fold Pfister spaces over } R\,\}.$$

and

$$\mathcal{T}(R) = H^1_{\text{ét}}(R, G_0) = \{\text{ isomorphism classes of } G_0\text{-torsors over } R\,\}$$

where G_0 is a split group of type F_4.

Recall that the set of isomorphism classes of group schemes of type F_4 over R (resp. over K) can be identified in a natural way with $H^1_{\text{ét}}(R, G_0)$ (resp. $H^1(K, G_0)$). So abusing notation we will identify a group G of type F_4 with the corresponding cocycle ξ and the isomorphism class $[G]$ with the equivalence class $[\xi]$. Given such a group G over a field K one can associate [S93], [GMS], [PetRac], [Ro] the cohomological invariants $f_3(G)$, $f_5(G)$ and $g_3(G)$ of G in $H^3(K, \mu_2)$, $H^5(K, \mu_2)$ and $H^3(K, \mathbb{Z}/3\mathbb{Z})$ respectively. The group G can be viewed as the automorphism group of a corresponding 27-dimensional Jordan algebra J. The invariant $g_3(G)$ vanishes if and only if J is reduced, i.e. J has zero divisors.

The main results of this paper are the following purity theorems:

1.1. **Theorem.** *Let R be a regular local ring containing the field $\mathbf{Q}$ of rational numbers and let K be its field of fractions. Let $[G] \in \mathcal{T}(K)_{ur}$ be such that $g_3(G) = 0$. Then $[G]$ is in the image of $\mathcal{T}(R) \to \mathcal{T}(K)$.*

1.2. **Theorem.** *The functor $R \mapsto \mathcal{P}f_n(R)$ satisfies purity for regular local rings containing $\mathbf{Q}$.*

1.3. **Theorem.** *Let φ_m (resp. φ_n) be an m-fold (resp. an n-fold) Pfister space over R. If $\varphi_{m,K}$ is a subform of the form $\varphi_{n,K}$, then there exists an $(n-m)$-fold Pfister space φ_{n-m} over R such that $\varphi_n \cong \varphi_m \otimes \varphi_{n-m}$ over R.*

1.4. **Remark.** For many groups of classical type the purity theorem is known; more precisely it is known for split groups of type A_n (unpublished); groups of the form $SL_{1,A}$, where A is a central simple algebra over a field [CTO]; split groups of type B_n [P]; split simply connected groups of type C_n (obvious); certain split groups of type D_n (like the special orthogonal group of a quadratic form) [P].

1.5. **Remark.** The characteristic restriction in the theorem is due to the fact that we use the main result in [P] on rationally isotropic quadratic spaces which was proven in characteristic zero only (the resolution of singularities is involved in that proof).

2. Rationally direct summands of a quadratic space

Throughout the paper R denotes a regular local ring containing the field $\mathbf{Q}$ of rational numbers. Note that R is a unique factorization domain ([Ma, Theorem 48, page 142]).

For the definition and basic properties of quadratic spaces over a commutative ring we refer to [K]. The aim of this section is to establish a criterion when one quadratic space over R can be realized as a direct summand of the second one. That criterion is stated in Proposition 2.2.

Let (V, f) be a quadratic space over R. We denote by $f(-,-)$ a symmetric bilinear form on V corresponding to f.

2.1. **Lemma.** *Let $v \in V$ be a vector such that $f(v)$ is a unit in R. Then we have $V \simeq \langle v \rangle \perp \langle v \rangle^{\perp}$.*

Proof. Obviously we have $\langle v \rangle \cap \langle v \rangle^{\perp} = 0$. So it suffices to show that every vector $w \in V$ can be written in the form $w = w_1 + w_2$ where $w_1 \in \langle v \rangle$ and $w_2 \in \langle v \rangle^{\perp}$. Let $w_1 = f(w, v) f(v)^{-1} v$. Then $w_2 = w - w_1$ is obviously orthogonal to v, so the result follows. □

2.2. **Proposition.** *Let (V, f) and (W, g) be two quadratic spaces over R. Then g is a direct summand of f if and only if g_K is a subform of f_K.*

Proof. If g is a direct summand of f then obviously g_K is a subform of f_K. Conversely, assume g_K is a subform of f_K. Let $m = \dim g$ and $n = \dim f$. We argue by induction on m. Consider first the case $m = 1$. Then $g = \langle a \rangle$ where $a \in R^{\times}$ and f_K represents a. By [P, Cor.2] there exists $v \in V$ such that $f(v) = a$. Clearly that v is unimodular. By Lemma 2.1 we have $V = \langle v \rangle \perp \langle v \rangle^{\perp}$ implying f can be decomposed as $f = \langle a \rangle \perp f'$.

Assume now that our statement is proven for all quadratic spaces of dimension $m = i$. Let (W, g) be a quadratic space of dimension $i + 1$. Take a decomposition $g = \langle a \rangle \perp g'$ where g' is a quadratic form of dimension $m - 1$ and a is a unit in R. By case $m = 1$, the quadratic form f can be decomposed as $f = \langle a \rangle \perp f'$. Since g_K is a subform of f_K, by Witt cancelation we get g'_K is a subform of f'_K. By induction g' is a direct summand of f', so the result follows. □

3. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let K be a quotient field of R and let f_K be an n-fold Pfister form over K unramified with respect to every prime ideal in R of height 1. If f_K is isotropic, then f_K is split over K and there is nothing to prove. So we may assume that f_K is anisotropic over K.

By [P, Cor.1] there exists a quadratic space f over R such that its fiber at the generic point of $\mathrm{Spec}\,(R)$ is isomorphic to f_K. Our aim is to show by induction that f can be decomposed as $f = g_i \perp h_i$ where $g_i = \langle\langle -a_1, -a_2, \ldots, -a_i \rangle\rangle$ is an i-fold Pfister form, h_i is a quadratic space over R and $i = 1, \ldots, n$. In view of dimension argument, taking $i = n$ we have $f = g_n$ and Theorem 1.2 follows.

Let first $i = 1$. Since f_K represents 1 over K, by Proposition 2.2 we may write $f = \langle 1 \rangle \perp f'$ for a suitable quadratic form $f' = \langle a_1, \ldots, a_{n-1} \rangle$ over R. Denote $g_1 = \langle 1, a_1 \rangle = \langle\langle -a_1 \rangle\rangle$. By our construction g_1 is a direct summand of f as required. Assume now that $f = g_i \perp h_i$ where $g_i = \langle\langle -a_1, \ldots, -a_i \rangle\rangle$ is an i-fold Pfister form and $i < n$. Consider a diagonalization $h_i = \langle b_{2^i+1}, \ldots, b_{2^n} \rangle$. Let $g_{i+1} = \langle\langle -a_1, \ldots, -a_i, -a_{i+1} \rangle\rangle$

where $a_{i+1} = b_{2^i+1}$. Clearly we have $f_K = g_{i,K} \otimes q$ for some $(n-i)$-fold Pfister form q over K so that $h_{i,K}$ is isomorphic over K to $g_{i,K} \otimes q'$ where q' is the pure subform of q. Since $h_{i,K}$ represents a_{i+1} so is $g_{i,K} \otimes q'$. Then by [KMRT, Theorem 1.10], $g_{i+1,K}$ is a subform of f_K. By Proposition 2.2 it follows g_{i+1} is a direct summand of f and we are done.

Proof of Theorem 1.3. For $m = n$ there is nothing to prove. Assume that $m < n$. Write φ_m as $\varphi_m = \langle\langle -a_1, \dots, -a_m \rangle\rangle$. Write φ_n as $\varphi_n = \varphi_m \perp h_m$. Consider a diagonalization $h_m = \langle b_{2^m+1}, \dots, b_{2^n} \rangle$. Arguing as in the proof of Theorem 1.2 we see that the space $\varphi_{m+1} := \langle\langle -a_1, \dots, -a_m, -a_{m+1} \rangle\rangle$ is subspace of the space φ_n, where $a_{m+1} = b_{2^m+1}$. Continuing this process we get $\varphi_n \cong \langle\langle -a_1, \dots, -a_m, \dots, -a_n \rangle\rangle$ for certain units $a_{m+1}, \dots, a_n$ in R as required.

4. Proof of Theorem 1.1

Let R be a regular local ring containing $\mathbf{Q}$ and let K be its field of fractions. Consider a group G of type F_4 over K unramified with respect to all prime ideals $\mathfrak{P} \subset R$ of height 1. It is a twisted form $G = {}^{\xi}G_{0,K}$ of a split group G_0 over R with some cocycle $\xi \in Z^1(K, G_{0,K})$.

If $T_0 \subset G_0$ is a maximal K-split torus we denote by $c \in \mathrm{Aut}(G_0) = G_0$ an element such that $c^2 = 1$ and $c(t) = t^{-1}$ for every $t \in T_0$ (it is known that such an automorphism exists, see e.g. [DG], Exp. XXIV, Prop. 3.16.2, p. 355). Let $\Sigma = \Sigma(G_0, T_0)$ be a root system of G_0 with respect to T_0. Fix its basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Denote by $\check{\alpha}_i : G_m \to T_0,\ i = 1, \dots, 4$, the cocharacters dual to $\alpha_1, \dots, \alpha_4$. Clearly, T_0 can be decomposed as a direct product $T_0 = \prod_i T_{\check{\alpha}_i}$ where $T_{\check{\alpha}_i}$ is the image of $\check{\alpha}_i$.

Let $u \in R$ be a unit which is not a square in K. Let $S = R(\sqrt{u})$ be the corresponding étale quadratic extension of R. Denote the nontrivial involution of S/R by τ.

4.1. **Lemma.** *Let $t = \prod_i \check{\alpha}_i(u_i) \in T_0(R)$ where u_1, u_2, u_3, u_4 are units in R and let $a_\tau = ct$. Then $\lambda = (a_\tau)$ is a cocycle in $Z^1(S/R, G_0(L))$.*

Proof. We need to check that $a_\tau \tau(a_\tau) = 1$. Indeed, we have

$$a_\tau \tau(a_\tau) = ct\,\tau(ct) = ctct = t^{-1}t = 1$$

as required. □

To complete the proof of Theorem 1.1 we are going to find the parameters u, u_1, u_2, u_3, u_4 such that the twisted group scheme $H = {}^{\lambda}G_0$ has generic fiber isomorphic to G. With this purpose we first remind a criterion when two groups of type F_4 over K are isomorphic. Note that both H and G have trivial g_3 invariant. Then by a result of Springer [Sp] (see also [Ch, Theorem 7.1]) we have G and H_K are K-isomorphic if and only if $f_3(G) = f_3(H_K)$ and $f_5(G) = f_5(H_K)$. This criterion suggests that the required parameters u, u_1, u_2, u_3, u_5 must be given in terms of $f_3(G)$ and $f_5(G)$.

4.2. **Proposition.** *Let G be as above. Then $f_3(G)$ and $f_5(G)$ are unramified with respect to all prime ideals $\mathfrak{P} \subset R$ of height* 1.

Proof. The invariants $f_3,\ f_5$ are symbols given in terms of the trace quadratic form of the Jordan algebra J corresponding to G and hence we may associate to them the 3-fold and 5-fold Pfister forms. Abusing notation we denote these Pfister forms by the same symbols $f_3(G)$ and $f_5(G)$.

Let $\mathfrak{P} \subset R$ be a prime ideal of height 1 and let $v = v_{\mathfrak{P}}$ be the corresponding discrete valuation on K. We need to show that $f_3(G),\ f_5(G)$ are in the images of $\mathcal{P}f_3\,(R_{\mathfrak{P}})) \to \mathcal{P}f_3\,(K)$ and $\mathcal{P}f_5\,(R_{\mathfrak{P}})) \to \mathcal{P}f_5\,(K)$ where $R_{\mathfrak{P}}$ is the localization of R at $\mathfrak{P}$. Equivalently, $f_3(G_{K_v}),\ f_5(G_{K_v})$ are in the images of $\mathcal{P}f_3\,(R_v) \to \mathcal{P}f_3\,(K_v)$ and $\mathcal{P}f_5\,(R_v) \to \mathcal{P}f_5\,(K_v)$ where R_v and K_v are completions of R and K with respect to v.

To see this, we consider a simple group scheme $\mathcal{H}_v$ of type F_4 over R_v such that its fiber at the generic point of $\mathrm{Spec}\,(R_v)$ is isomorphic to G_{K_v}. Note that such a scheme do exists because G is unramified at $\mathfrak{P}$ and hence G_{K_v} is also unramified. Since R_v is a regular local ring containing $\mathbf{Q}$ by [Ch, Theorems 6.1, 6.6, Remark 8.4] we have $f_3(G_{K_v})$ and $f_5(G_{K_v})$ are of the form $(v_1) \cup (v_2) \cup (v_3)$ and $(v_1) \cup (v_2) \cup (v_3) \cup (v_4) \cup (u_5)$ respectively where $v_1, \ldots, v_5$ are units in R_v. □

4.3. **Proposition.** *There exist units $v_1, \ldots, v_5 \in R^\times$ such that $f_3(G) = \langle\langle\, v_1, v_2, v_3\,\rangle\rangle$ and $f_5(G) = \langle\langle\, v_1, v_2, v_3, v_4, v_5\,\rangle\rangle$.*

Proof. By Theorem 1.2 and Proposition 4.2 there exist Pfister spaces $g = \langle\langle\, v_1, v_2, v_3\,\rangle\rangle$ and $h = \langle\langle\, w_1, w_2, w_3, w_4, w_5\,\rangle\rangle$ over R such that $f_3(G) = g_K$ and $f_5(G) = h_K$. Here $v_1, v_2, v_3, w_1, \ldots, w_5$ are units in R. Since $f_3(G)$ is a subform of $f_5(G)$ Theorem 1.3 shows that there are units v_3, v_4 in R such that $h = \langle\langle\, v_1, v_2, v_3, v_4, v_5\,\rangle\rangle$. We are done.

□

We are now in position to complete the proof of Theorem 1.1. Take the cocycle λ from Lemma 4.1 with the parameters $u = v_1,\ u_1 = v_2, \ldots, u_4 = v_5$ and the twisted group scheme $H = {}^{\lambda}G_0$ over R. Let τ be the nontrivial automorphism of $S = R(\sqrt{u})$ over R. Let

$$\{\, H_{\alpha_1}, \ldots, H_{\alpha_4}, X_\alpha,\ \alpha \in \Sigma\,\}$$

be a Chevalley basis of the Lie algebra of H over S with respect to T_0. The twisted action of τ on X_{α_i} is given by

$$\tau(X_{\alpha_i}) = ct(X_{\alpha_i})(ct)^{-1} = c\prod_j \check{\alpha}_j(u_j)(X_{\alpha_i})\prod_j \check{\alpha}_j(u_j^{-1})c^{-1} = u_i X_{-\alpha_i},$$

so that in the terminology of [Ch] the structure constants of H are $u_1, \ldots, u_4$. By [Ch, Theorem 6.1 and Theorem 6.6] the f_3 and f_5 invariants of H_K are $\langle\langle\, u, u_1, u_2\,\rangle\rangle$ and $\langle\langle\, u, u_1, u_2, u_3, u_4\,\rangle\rangle$. Thus H_K and G have the same $f_3,\ f_5$ invariants, hence they are isomorphic. Theorem 1.1 is proven.

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