VARIATIONS ON A THEME OF GROUPS SPLITTING BY A QUADRATIC EXTENSION AND GROTHENDIECK-SERRE CONJECTURE FOR GROUP SCHEMES F_4 WITH TRIVIAL g_3 INVARIANT

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Abstract. We study structure properties of reductive group schemes defined over a local ring and splitting over its étale quadratic extension. As an application we prove Serre–Grothendieck conjecture on rationally trivial torsors over a local regular ring containing a field of characteristic 0 for group schemes of type F_4 with trivial g_3 invariant.

Contents

1. Introduction

In the present paper we prove the Grothendieck-Serre conjecture on rationally trivial torsors for group schemes of type F_4 whose generic fiber has trivial g_3 invariant. The Grothendieck-Serre conjecture [Gr58], [Gr68], [S58] asserts that if R is a regular local ring and if G is a reductive group scheme defined over R then a G-torsor over R is trivial if and only if its fiber at the generic point of Spec (R) is trivial. In other words the kernel of a natural map $H^1_{ét}(R, G) \to H^1_{ét}(K, G)$ where K is a quotient field of R is trivial.

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Many people contributed to this conjecture by considering various particular cases. If R is a discrete valuation ring the conjecture was proven by Y. Nisnevich [N]. If R contains a field k and G is defined over k this is due to J.-L. Colliot-Thélène, M. Ojanguren [CTO] when k is infinite and it is due to M. S. Raghunathan [R94], [R95] when k is perfect. The case of tori was done by J.-L. Colliot-Thélène and J.-L. Sansuc [CTS]. For certain simple simply connected group of classical type the conjecture was proven by Ojanguren, Panin, Suslin and Zainoulline [PS], [OP], [Z], [OPZ]. For a recent progress on isotropic group schemes we refer to preprints [PSV], [P09], [PPS].

In the paper we deal with a still open case related to group schemes of type F_4. Remind that if G is a group of type F_4 defined over a field K of characteristic $\neq 2, 3$ one can associate (cf. [S93], [GMS03], [PetRac], [Ro]) cohomological invariants $f_3(G)$, $f_5(G)$ and $g_3(G)$ of G in $H^3(K,\mu_2)$, $H^5(K,\mu_2)$ and $H^3(K,\mathbb{Z}/3\mathbb{Z})$ respectively. The group G can be viewed as the automorphism group of a corresponding 27-dimensional Jordan algebra J. The invariant $g_3(G)$ vanishes if and only if J is reduced, i.e. it has zero divisors. The main result of the paper is the following.

1.1. **Theorem.** *Let R be a regular local ring containing a field of characteristic 0. Let G be a group scheme of type F_4 over R such that its fiber at the generic point of* $\mathrm{Spec}\,(R)$ *has trivial g_3 invariant. Then the canonical mapping $H^1_{\acute{e}t}(R,G) \to H^1_{\acute{e}t}(K,G)$ where K is a quotient field of R has trivial kernel.*

We remark that for a group scheme G of type F_4 we have $\mathrm{Aut}\,(G) \simeq G$, so that by the twisting argument the above theorem is equivalent to the following:

1.2. **Theorem.** *Let R be as above and let G_0 be a split group scheme of type F_4 over R. Let $H^1_{\acute{e}t}(R,G_0)_{\{g_3=0\}} \subset H^1_{\acute{e}t}(R,G_0)$ be the subset consisting of isomorphism classes $[\mathcal{T}]$ of G_0-torsors such that the corresponding twisted group $({}^{\mathcal{T}}G_0)_K$ has trivial g_3 invariant. Then a canonical mapping*

$$H^1_{\acute{e}t}(R,G_0)_{\{g_3=0\}} \to H^1_{\acute{e}t}(K,G_0)$$

is injective.

The characteristic restriction in the theorem is due to the fact that we use the main result in [P03] on rationally isotropic quadratic spaces which was proven in characteristic zero only (the resolution of singularities is involved in that proof). We remark that if the Panin's result is true in full generality (except probably characteristic 2 case) then our arguments can be easily modify in such way that the theorem holds for all regular local rings where 2 is invertible[1].

The proof of the theorem heavily depends on the fact that group schemes of type F_4 with trivial g_3 invariant are split by an étale quadratic extension

[1] I. Panin has informed the author recently that it suffices to require in his theorem that R contains an infinite perfect field.

of the ground ring R. This is why the main body of the paper consists of studying structure properties of simple group schemes of an arbitrary type over R (resp. K) splitting by an étale quadratic extension S/R (resp. L/K) which is of independent interest.

We show that the structure of such group schemes is completely determined by a finite family of units in R which we call structure constants of G. These constants depend on a chosen maximal torus $T \subset G$ defined over R and splitting over S. Such a torus is not unique in G. Giving two tori T and T' we find formulas which express structure constants of G related to T in terms of that of related to T' and this leads us quickly to the proof of the main theorem.

Of course we are using a group point view. It seems plausible that our proof can be carried over in terms of Jordan algebras and their trace quadratic forms, but we do not try to do it here.

The paper is divided into four parts. We begin by introducing notation, terminology that are used throughout the paper as well as by reminding properties of algebraic groups defined over a field and splitting by a quadratic field extension. This is followed by two sections on explicit formulas for cohomological invariants f_3 and f_5 in terms of structure constants for groups of type F_4 and their classification. In the third part of the paper we study structure properties of group schemes splitting by an étale quadratic extension of the ground ring. The proof of the main theorem is the content of the last section.

2. Notation and lemma on representability of units

Throughout the paper R denotes a ring where 2 is invertible. Also, all fields considered in the paper have characteristic $\neq 2$.

We let G_0 denote a split reductive group scheme over R and we let $T_0 \subset G_0$ denote a maximal split torus over R. We use standard terminology related to algebraic groups over rings. For the definition of reductive group schemes (and in particular split reductive group schemes), maximal tori, root systems of split group schemes and their properties we refer to [SGA3]. If G is a reductive algebraic group and $T \subset G$ is a maximal torus, we let $\Sigma(G, T)$ denote the root system of G with respect to T.

We number the simple roots of exceptional groups as in [Bourb68].

If G_0 is a K-split simple algebraic group and $T_0 \subset G_0$ is a maximal K-split torus we denote by $c \in \mathrm{Aut}(G_0)$ an element such that $c^2 = 1$ and $c(t) = t^{-1}$ for every $t \in T_0$ (it is known that such an automorphism exists, see e.g. [DG], Exp. XXIV, Prop. 3.16.2, p. 355). If G_0 has type D_4 or F_4 such an element can be chosen inside the normalizer $N_{G_0}(T_0)$ of T_0).

If R is a local ring with the maximal ideal M we let $k = \overline{R} = R/M$. Similarly, if V is a free module on rank n over R we let $\overline{V} = V \otimes_R \overline{R} = V \otimes_R k$ and for a vector $v \in V$ we set $\overline{v} = v \otimes 1$. If R is a regular local ring it is a unique factorization domain ([Ma, Theorem 48, page 142]). Throughout the paper a quotient field of R will be denoted by K.

Let $f = \sum_{i=1}^n a_i x_i^2$ be a quadratic form over R where $a_1, \dots, a_n \in R^\times$ given on a free module V. If $I \subset \{1, \dots, n\}$ is a subset we denote by $f_I = \sum_{i\in I} a_i x_i^2$ the corresponding subform of f. If $v = (v_1, \dots, v_n) \in V$ we set $f_I(v) = \sum_{i\in I} a_i v_i^2$. Finally, let $g = \prod_I f_i$ where the product is taken over all subsets of $\{1, \dots, n\}$. For a vector v we set $g(v) = \prod_I f_I(v)$.

2.1. **Lemma.** *Let f and g be as above. Assume k is infinite. Let $a \in R^\times$ be a unit such that $f(v) = a$ for some vector $v \in V$. Then there exists a vector $u \in V$ such that $f(u) = a$ and $g(u)$ is a unit.*

Proof. If $n = 1$, v has the required properties. Hence me may assume $n \geq 2$. If $w \in V$ is a vector whose length $f(w)$ with respect to f is a unit we denote by τ_w an orthogonal symmetry with respect to w given by

$$\tau_w(x) = x - 2f(x, w)f(w)^{-1}w$$

for all vectors x in V. Since orthogonal symmetries preserve length of vectors it suffices to find vectors $w_1, \dots, w_s \in V$ such that $g(\tau_{w_1} \cdots \tau_{w_s}(v))$ is a unit. For that, in turn, it suffices to find $\overline{w}_1, \dots, \overline{w}_s \in \overline{V}$ such that $\overline{g}(\tau_{\overline{w}_1} \cdots \tau_{\overline{w}_s}(\overline{v})) \neq 0$.

It follows that we can pass to a vector space $\overline{V}$ over k. Consider a quadric

$$Q_{\overline{a}} = \{\, x \in \overline{V} \mid \overline{f}(x) = \overline{a} \,\}$$

defined over k. We have $\overline{v} \in Q_{\overline{a}}(k)$, hence $Q_{\overline{a}}(k) \neq \emptyset$ implying $Q_{\overline{a}}$ is a rational variety over k.

Let $U \subset \overline{V}$ be an open subset given by $\overline{g}(x) \neq 0$. It is easy to see that $Q_{\overline{a}} \cap U \neq \emptyset$. Since k is infinite, k-points of $Q_{\overline{a}}$ are dense in $Q_{\overline{a}}$. Hence $Q_{\overline{a}}(k) \cap U$ is nonempty. Take a vector $\overline{w} \in Q_{\overline{a}}(k) \cap U$. Since the orthogonal group $\mathrm{O}(\overline{f})$ acts transitively on vectors of $Q_{\overline{a}}$ there exists $\overline{s} \in \mathrm{O}(\overline{f})$ such that $\overline{w} = \overline{s}(\overline{v})$. It remains to note that orthogonal symmetries generate $O(\overline{f})$. □

3. Algebraic groups splitting by quadratic field extensions

The aim of this section is to remind structure properties of a simple simply connected algebraic group G defined over a field K of characteristic $\neq 2$ and splitting over its quadratic extension L/K, say $L = K(\sqrt{d})$. There is nothing special in type F_4 and we will assume in this section that G is of an arbitrary type of rank n. The only technical restriction (which we need later on to simplify the exposition of the material) relates to the Weyl group W of G. Namely, we will assume that W contains -1, i.e. an element which takes an arbitrary root α into $-\alpha$. For the proofs of all results contained in this section without proofs we refer to [Ch].

Let τ be the nontrivial automorphism of L/K. If $B_L \subset G_L$ is a Borel subgroup over L in G_L in generic position then $B_L \cap \tau(B_L) = T$ is a maximal torus in G_L. Clearly, it is defined over K and splitting over L (because it is contained in B_L and all tori in B_L are L-split). In many cases (for instance, for a group of type F_4 which is the main target of this paper) the torus T is K-anisotropic. Indeed, if G_K is K-anisotropic so is T and there is nothing to prove. If G_K is K-isotropic then one needs to make the above mentioned additional assumption on the Weyl group of G which holds for type F_4.

3.1. **Lemma.** *Assume that $-1 \in W$. Then T is anisotropic over K.*

Proof. The Galois group of L/K acts in a natural way on characters of T and hence on the root system $\Sigma = \Sigma(G_K, T)$ of G_K with respect to T. Thus we have a natural embedding $\mathrm{Gal}\,(L/F) \hookrightarrow W$ which allows us to view τ as an element of W. Since the intersection of two Borel subgroups B_L and $\tau(B_L)$ is a maximal torus in G_L, one of them, say $\tau(B_L)$, is the opposite Borel subgroup to the second one B_L with respect to the ordering on Σ determined by the pair (T, B). One knows that W contains a unique element which takes B_L to $\tau(B_L) = B_L^-$. Since $-1 \in W$ such an element is necessary -1. Of course this implies $\tau = -1$, hence τ acts on characters of T as -1. In particular T is K-anisotropic. □

Our Borel subgroup B_L determines an ordering of the root system Σ of G, hence the system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Let Σ^+ (resp. Σ^-) be the set of positive (resp. negative) roots. Let us choose a Chevalley basis [St]

$$\{H_{\alpha_1}, \dots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\} \tag{3.2}$$

in the Lie algebra $\mathfrak{g}_L = \mathcal{L}(G_L)$ of G_L corresponding to the pair (T_L, B_L). This basis is unique up to signs and automorphisms of $\mathfrak{g}_L$ which preserve B_L and T_L (see [St], §1, Remark 1).

Since G_L is a Chevalley group over L, its L-structure as an abstract group, i.e. generators and relations, is well known. For more details and proofs of all standard facts about $G(L)$ used in this paper we refer to [St]. Recall that $G(L)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(u) \mid u \in L \rangle$, where $\alpha \in \Sigma$ and T is generated by the one-parameter subgroups

$$T_\alpha = T \cap G_\alpha = \langle\, h_\alpha(t) \mid t \in K^* \,\rangle.$$

Here G_α is the subgroup generated by $U_{\pm\alpha}$ and $h_\alpha : G_{m,L} \to T_L$ is the corresponding cocharacter (coroot) of T whose image is T_α. Furthermore, since G_L is a simply connected group, the following relations hold in G_L (cf. [St], Lemma 28 b), Lemma 20 c)):

(i) $T \simeq T_{\alpha_1} \times \cdots \times T_{\alpha_n}$;

(ii) for any two roots $\alpha, \beta \in \Sigma$ and $t, u \in L$ we have

$$h_\alpha(t)\, x_\beta(u)\, h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta, \alpha \rangle} u)$$

where $\langle \beta, \alpha \rangle = 2\,(\beta, \alpha)/(\alpha, \alpha)$ and

$$h_\alpha(t) X_\beta h_\alpha(t)^{-1} = t^{\langle \beta, \alpha \rangle} X_\beta \tag{3.3}$$

If $\Delta \subset \Sigma^+$ is a subset, we let G_Δ denote the subgroup generated by $U_{\pm\alpha}$, $\alpha \in \Delta$.

We shall now describe explicitly the K-structure of G, i.e. the action of τ on the generators $\{x_\alpha(u),\, \alpha \in \Sigma\}$ of G_L. As we already know $\tau(\alpha) = -\alpha$ for any $\alpha \in \Sigma$ and this implies $T_\alpha \simeq R^{(1)}_{L/K}(G_{m,L})$.

Let $\alpha \in \Sigma$. Since $\tau(\alpha) = -\alpha$ there exists a constant $c_\alpha \in L^\times$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$. It follows that the action of τ on $G(L)$ is determined completely by the family $\{c_\alpha,\, \alpha \in \Sigma\}$. We call these constants by *structure constants* of G with respect to T and Chevalley basis (3.2). Of course, they depend on the choice of T and a Chevalley basis. We summarize their properties in the following two lemmas (for their proofs we refer to [Ch]).

3.4. **Lemma.** *Let $\alpha \in \Sigma$. Then we have*

(i) $c_{-\alpha} = c_\alpha^{-1}$*;*

(ii) $c_\alpha \in K^\times$*;*

(iii) *if $\beta \in \Sigma$ is a root such that $\alpha + \beta \in \Sigma$, then $c_{\alpha+\beta} = -c_\alpha c_\beta$; in particular, the family $\{c_\alpha,\, \alpha \in \Sigma\}$ is determined completely by its subfamily $\{c_{\alpha_1}, \dots, c_{\alpha_n}\}$.*

3.5. **Lemma.** (i) $\tau[\,x_\alpha(u)\,] = x_{-\alpha}(c_\alpha \tau(u))$ *for any $u \in L$ and any $\alpha \in \Sigma$;*

(ii) *the subgroup G_α of G is isomorphic to* SL$\,(1, D)$ *where D is a quaternion algebra over K of the form $D = (d, c_\alpha)$.*

4. Moving tori

The family $\{c_\alpha,\, \alpha \in \Sigma\}$ determining the action of τ on $G(L)$ depends on a chosen Borel subgroup B and the corresponding Chevalley basis. Given another Borel subgroup and Chevalley basis we get another family of constants and we now are going to describe the relation between the old ones and the new ones.

Let $B' \subset G$ be a Borel subgroup over L such that the intersection $T' = B' \cap \tau(B')$ is a maximal K-anisotropic torus. Clearly both tori T and T' are isomorphic over K (because both of them are isomorphic to the direct product of n copies of $R^{(1)}_{L/K}(G_{m,L})$). Furthermore, there exists a K-isomorphism $\lambda : T \to T'$ preserving positive roots, i.e. which takes $(\Sigma')^+ = \Sigma(G, T')^+$ into $\Sigma^+ = \Sigma(G, T)^+$. Any such isomorphism can be extended to an inner automorphism

$$i_g : G \longrightarrow G, \quad x \to g\,x\,g^{-1}$$

for some $g \in G(K_s)$, where K_s is a separable closure of K, which takes B into B' (see [Hum], Theorem 32.1). Note that g is not unique since for any $t \in T(K_s)$ the inner conjugation by gt also extends λ and it takes B into B'.

4.1. **Lemma.** *The element g can be chosen in $G(L)$.*

Proof. Take an arbitrary g' with the above properties. Since the restriction $i_{g'}\,|_T$ is a K-defined isomorphism, we have

$$t_\sigma = (g')^{-1+\sigma} \in T(K_s)$$

for any $\sigma \in \mathrm{Gal}\,(K_s/F)$. The family $\{t_\sigma,\, \sigma \in \mathrm{Gal}\,(K_s/F)\}$ determines a cocycle $\xi = (t_\sigma) \in Z^1(K,T)$. Since T splits over L, $res_L(\xi)$ viewed as a cocycle in T is trivial, by Hilbert's Theorem 90. It follows there is $z \in T(K_s)$ such that $t_\sigma = z^{1-\sigma}$, $\sigma \in \mathrm{Gal}\,(K_s/L)$. Then $g = g'z$ is stable under $\mathrm{Gal}\,(K_s/L)$ implying $g \in G(L)$ and clearly $gBg^{-1} = B'$. □

Let g be an element from Lemma 4.1 and let $t = g^{-1+\tau}$. Since $t \in T(L)$, it can be written as a product $t = h_{\alpha_1}(t_1)\cdots h_{\alpha_n}(t_n)$, where $t_1,\dots,t_n \in L^\times$ are some parameters. Using the equality $t\,\tau(t) = 1$ and the fact that τ acts on characters of T as multiplication by -1 one can easily see that $t_1,\dots,t_n \in K^\times$.

The set

$$\{H'_{\alpha_1} = gH_{\alpha_1}g^{-1},\dots,H'_{\alpha_n} = gH_{\alpha_n}g^{-1},\ X'_\alpha = gX_\alpha g^{-1},\ \alpha \in \Sigma\} \tag{4.2}$$

is a Chevalley basis related to the pair (T',B'). Let $\{c'_\alpha,\ \alpha \in \Sigma\}$ be the corresponding structure constants of G with restect to T' and Chevalley basis (4.2).

4.3. **Lemma.** *For each $\alpha \in \Sigma$ one has $c'_\alpha = t_1^{-\langle\alpha,\alpha_1\rangle}\cdots t_n^{-\langle\alpha,\alpha_n\rangle}\cdot c_\alpha$.*

Proof. Apply τ to the equality $X'_\alpha = gX_\alpha g^{-1}$ and use relation (3.3). □

Our element g constructed in Lemma 4.1 has the property $g^{-1+\tau} \in T(L)$. It turns out that an arbitrary $g \in G(L)$ with this property gives rise to a new pair (B',T') and hence to new structure constants $\{c'_\alpha\}$.

4.4. **Lemma.** *Let $g \in G(L)$ be an element such that $t = g^{-1+\tau} \in T(L)$. Then $T' = gTg^{-1}$ is a K-defined maximal torus splitting over L and the restriction of the inner automorphism i_g to T is a K-defined isomorphism. The structure constants $\{c'_\alpha\}$ related to T' are given by the formulas in Lemma 4.3.*

Proof. This is clear. □

4.5. **Example.** Let G,T be as above and let $\Sigma = \Sigma(G,T)$. Take an element

$$g = x_\alpha\left(\frac{\tau(v)}{1-c_\alpha v\tau(v)}\right)x_{-\alpha}(c_\alpha v) \tag{4.6}$$

where $\alpha \in \Sigma$ is an arbitrary root and $v \in L^\times$ is such that $1 - c_\alpha v\tau(v) \neq 0$. One easily checks that

$$g^{1-\tau} = h_\alpha\left(\frac{1}{1-c_\alpha v\tau(v)}\right)$$

and hence g gives rise to a new torus $T' = gTg^{-1}$ and to a new structure constants.

In what follows, we say that we apply an *elementary transformation* of T with respect to a root α and the parameter $v \in L^\times$ when we move from T to $T' = g^{-1}Tg$ where g be given by (4.6).

4.7. **Remark.** The main property of an elementary transformation with respect to a root α is that the new structure constant c'_β with respect to T' doesn't change (up to squares) if β is orthogonal to α or $\langle \beta, \alpha \rangle = \pm 2$ and it is equal to $(1 - c_\alpha v\tau(v))c_\beta$ (up to squares) if $\langle \beta, \alpha \rangle = \pm 1$. Thus in the context of algebraic groups this an analogue of an elementary chain equivalence of quadratic forms.

4.8. **Remark.** An arbitrary reduced norm $x \in \operatorname{Nrd} D$ in the quaternion algebra $D = (d, c_\alpha)$ can be written as a product of two elements of the form $1 - c_\alpha v\tau(v)$, hence in the case $\langle \beta, \alpha \rangle = \pm 1$ we can change c_β by any reduced norm in D.

While considering cohomological invariants of G of type F_4 sometimes it is convenient to consider G as a twisting group. Let $\overline{G}$ be the corresponding adjoint group. Note that groups of type F_4 are simply connected and adjoint so that for them we have $G = \overline{G}$. Let $\overline{G}_0$ be a K-split adjoint group of the same type as $\overline{G}$ and let $\overline{T}_0 \subset \overline{G}_0$ be a maximal K-split torus. Remind that c denotes an automorphism of $\overline{G}_0$ of order 2 such that $ctc^{-1} = t^{-1}$ for every $t \in \overline{T}_0$. We assume additionally that $c \in N_{\overline{G}_0}(\overline{T}_0)$ (this is the case for types D_4 and F_4 considered below).

4.9. **Lemma.** *Let $t \in \overline{T}_0(K)$ and let $a_\tau = ct$. Then $\xi = (a_\tau)$ is a cocycle in $Z^1(L/K, \overline{G}_0(L))$.*

Proof. We need to check that $a_\tau \tau(a_\tau) = 1$. Indeed,

$$a_\tau \tau(a_\tau) = ct\, \tau(ct) = ctct = t^{-1}t = 1$$

as required. □

For further reference we note that every cocycle $\eta \in Z^1(K, \overline{G}_0)$ acts by inner conjugation on both G_0 and $\overline{G}_0$ and hence we can twist ${}^\eta G_0$, ${}^\eta \overline{G}_0$ both groups.

Since $\overline{G}_0$ is adjoint the character group of $\overline{T}_0$ is generated by simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of the root system $\Sigma = \Sigma(\overline{G}_0, \overline{T}_0)$ of $\overline{G}_0$ with respect to $\overline{T}_0$. Choose a decomposition $\overline{T}_0 = G_m \times \cdots \times G_m$ such that the canonical embeddings $\pi_i : G_m \to \overline{T}_0$ onto the ith factor, $i = 1, \ldots, n$, are the cocharacters dual to $\alpha_1, \ldots, \alpha_n$.

4.10. **Proposition.** *Let $\overline{G}$ be as above with structure constants $c_{\alpha_1}, \ldots, c_{\alpha_n}$. Let $\xi = (a_\tau)$ where $a_\tau = c \prod_i \pi_i(c_{\alpha_i})$. Then the twisted group ${}^\xi G_0$ is isomorphic to G over K.*

Proof. Since the cocharacters $\pi_1, \ldots, \pi_n$ are dual to the roots $\alpha_1, \ldots, \alpha_n$ it is easy to see that the twisted group ${}^\xi G_0$ has the same structure constants as G. It follows that the Lie algebras $\mathcal{L}(G)$ and $\mathcal{L}({}^\xi G_0)$ of G and ${}^\xi G_0$

have the same Galois descent data. This yields $\mathcal{L}(G) \simeq \mathcal{L}({}^{\xi}G_0)$ and as a consequence we obtain that their automorphism groups are isomorphic over K as well. □

4.11. **Remark.** Assume that R is a domain where 2 is invertible with a field of fractions K and G_0 is a split group scheme over R. Let $S = R(\sqrt{d})$ be an étale quadratic extension of R where d is a unit in R. Let τ be the generator of $\mathrm{Gal}\,(S/R)$. Assume that $c_{\alpha_1}, \dots, c_{\alpha_n} \in R^\times$. Then we may view $\xi = (a_\tau)$ where $a_\tau = c\prod_i \pi_i(c_{\alpha_i})$ as a cocycle in $Z^1(S/R, G_0(S))$ and hence the twisted group ${}^{\xi}G_0$ is a group scheme over R whose fiber at the generic point of $\mathrm{Spec}\,(R)$ is isomorphic to G_K.

As an application of the above proposition we get

4.12. **Lemma.** *Let G and G' be groups over K and splitting over L with structure constants $\{c_{\alpha_1}, \dots, c_{\alpha_n}\}$ and $\{c_{\alpha_1}u_1, \dots, c_{\alpha_n}u_n\}$ where $u_1, \dots, u_n$ are in $N_{L/K}(L^\times)$. Then G and G' are isomorphic over K.*

Proof. Let $u_i = N_{L/K}(v_i)$. By Proposition 4.10 it follows that G and G' are twisted forms of G_0 by means of cocycles $\xi = (a_\tau)$ and $\xi' = (a'_\tau)$ with coefficients in $\overline{G}_0$ where $a_\tau = c\prod_i \pi_i(c_{\alpha_i})$ and $a'_\tau = c\prod_i \pi_i(c_{\alpha_i}u_i)$. Furthermore, we have

$$a_\tau = \left(\prod_i \pi_i((v_i)^{-1})\right) a'_\tau \left(\prod_i \pi_i((v_i)^{-1})^{-\tau}\right)$$

implying ξ is equivalent to ξ'. □

The statement of the lemma can be equivalently reformulated as follows.

4.13. **Corollary.** *Let $T \subset G$ be a maximal torus with the structure constants $\{c_{\alpha_1}, \dots, c_{\alpha_n}\}$ and let $u_1, \dots, u_n \in N_{L/K}(L^\times)$. Then G contains a maximal torus T' whose structure constants are $\{c_{\alpha_1}u_1, \dots, c_{\alpha_n}u_n\}$.*

5. Strongly inner forms of type D_4

For later use we need some classification results on strongly inner forms of type ${}^1D_4$; in other words we need an explicit description of the set $H^1(K, G_0)$ where G_0 is a simple simply connected group over a field K of type D_4.

For an arbitrary cocycle $\xi' \in Z^1(K, G_0)$ the twisted group $G = {}^{\xi'}G_0$ is isomorphic to $\mathrm{Spin}(f)$ where f is in I^3. We may assume that f represents 1. Hence by dimension consideration f is a 3-fold Pfister form over K and as a consequence we obtain G is splitting over a quadratic extension L/K of K, say $L = K(\sqrt{d})$. By Proposition 4.10, the image of ξ' in $H^1(K, \overline{G}_0)$ up to equivalence equals to the image of $\xi = (a_\tau) \in Z^1(L/K, N_{G_0}(T_0)(L))$ where $a_\tau = ch_{\alpha_1}(u_1)\cdots h_{\alpha_4}(u_4)$ for some $u_1, \dots, u_4 \in K^\times$. Using an obvious twisting argument we find that the classes of ξ and ξ' are equal up to central cocycles.

The center Z of G_0 is isomorphic to $\mu_2 \times \mu_2$, hence it contains three elements of order 2. They give rise to three homomorphisms $\phi_i : G_0 \to$

$\mathrm{SO}(f_0)$ where $i = 1, 2, 3$ and f_0 is a split 8-dimensional quadratic form. The images $\phi_i(\xi),\ i = 1, 2, 3,$ of ξ in $Z^1(K, \mathrm{SO}(f_0))$ correspond to three quadratic form f_1, f_2, f_3 and we are going to give an explicit description of f_i in terms of the parameters u_1, u_2, u_3, u_4 and d.

One easily checks that the center of G_0 is generated by $h_{\alpha_1}(-1)h_{\alpha_3}(-1)$ and $h_{\alpha_1}(-1)h_{\alpha_4}(-1)$. Let us rewrite the cocycle $\xi = (a_\tau)$ in the form

$$a_\tau = c h_{\alpha_1}(v_1) h_{\alpha_2}(v_2) z_1 z_2$$

where $v_1 = u_1 u_3^{-1} u_4^{-1}$, $v_2 = u_2$ and

$$z_1 = h_{\alpha_1}(u_3)h_{\alpha_3}(u_3),\ \ z_2 = h_{\alpha_1}(u_4)h_{\alpha_4}(u_4).$$

Using relation (3.3) we find that the structure constants of G with respect to the twisted torus $T = {}^{\xi}T_0$ up to squares are $c_{\alpha_2} = v_1$ and $c_{\alpha_1} = c_{\alpha_3} = c_{\alpha_4} = v_2$. Also, following [ChS] we find that up to numbering we have $f_1 = u_3 f$, $f_2 = u_4 f$ and $f_3 = u_3 u_4 f$ where

$$f = \langle\langle\, d, v_1, v_2 \,\rangle\rangle = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle\,;$$

in particular G is split over a field extension E/K if and only if so is f_E.

We are now going to show that we don't change the equivalence class $[\xi]$ if we multiply the parameters u_3, u_4 in the expression for ξ by elements in $K^\times$ represented by f. Let V, V_1, V_2, V_3 be 8-dimensional vector space over K equipped with the quadratic forms f, f_1, f_2, f_3.

5.1. **Proposition.** *Let $w_1, w_2 \in V$ be two nonisotropic vectors and let $a = f(w_1)$, $b = f(w_2)$. Let $\xi' = (a'_\tau)$ where $a'_\tau = c h_{\alpha_1}(v_1) h_{\alpha_2}(v_2) z'_1 z'_2$ and*

$$z'_1 = h_{\alpha_1}(au_3)h_{\alpha_3}(au_3),\quad z'_2 = h_{\alpha_1}(bu_4)h_{\alpha_4}(bu_4).$$

Then ξ' is equivalent to ξ.

Proof. We have the canonical central embeddings $\psi_1 : \mu_2 \to G_0$ given by $-1 \to h_{\alpha_1}(-1)h_{\alpha_3}(-1)$ and $\psi_2 : \mu_2 \to G_0$ given by $-1 \to h_{\alpha_1}(-1)h_{\alpha_4}(-1)$. Up to numbering we may assume that

$${}^{\xi}G_0/\psi_1(\mu_2) \simeq \mathrm{SO}(f_1) \quad \text{and} \quad {}^{\xi}G_0/\psi_2(\mu_2) \simeq \mathrm{SO}(f_2).$$

We also have a canonical bijection $H^1(K, G_0) \to H^1(K, {}^{\xi}G_0)$ (translation by ξ) under which ξ' goes to $\eta = (h_{\alpha_1}(a)h_{\alpha_3}(a)h_{\alpha_1}(b)h_{\alpha_4}(b))$.

Note that η is the product of two cocycles $\eta_1 = (h_{\alpha_1}(a)h_{\alpha_3}(a))$ and $\eta_2 = (h_{\alpha_1}(b)h_{\alpha_4}(b))$ first of which being in the image of $\psi_1 : H^1(K, \mu_2) \to H^1(K, {}^{\xi}G_0)$ and the second one being to in the image of $\psi_2 : H^1(K, \mu_2) \to H^1(K, {}^{\xi}G_0)$. If we identify $H^1(K, \mu_2) = K^\times/(K^\times)^2$ then it is known that $\operatorname{Ker}\psi_1$ (resp. $\operatorname{Ker}\psi_2$) consists of spinor norms of f_1 (resp. f_2). Thus the statement of the Proposition is amount to saying that a, b are spinor norms for the twisted group $G = {}^{\xi}G_0$ with respect to the quadratic forms f_1 and f_2 respectively. It remains to remind that the subgroup in $K^\times$ consisting of spinor norms is generated by $f_i(s_1)f_i(s_2)$ where $s_1, s_2 \in V_i$ are nonisotropic vectors; so the result follows. □

5.2. **Remark.** Assume that R and S are as in Remark 4.11. Take a cocycle $\xi = (a_\tau)$ in $Z^1(S/R, G_0(S))$ given by $a_\tau = ch_{\alpha_1}(u_1)\cdots h_{\alpha_4}(u_4)$ where $u_1,\dots,u_4 \in R^\times$. Then arguing literally verbatim we find that the twisted group $G = {}^\xi G_0$ is isomorphic to $\mathrm{Spin}(f)$ where f is a 3-fold Pfister form given by $f = \langle\langle\, d, u_2, u_1u_3u_4\,\rangle\rangle$ and that for all units $a, b \in R^\times$ represented by f the cocycle ξ' from Proposition 5.1 is equivalent to ξ.

5.3. **Proposition.** *Let G be as above and let $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$ be the corresponding 3-fold Pfister form. Assume that f has another presentation $f = \langle\langle\, d, a, b\,\rangle\rangle$ over K. Then there exists a maximal torus $T' \subset G$ defined over K and splitting over L such that structure constants of G with respect to T' (up to squares) are $c'_{\alpha_1} = a$ and $c'_{\alpha_2} = b$.*

Proof. We construct a sequence of elementary transformations with respect to the roots α_1 and α_2 such that at the end we arrive to a torus with the required structure constants. Remind that applying elementary transformation with respect to α_1 (resp. α_2) we do not change c_{α_1} (resp. c_{α_2}) modulo squares and we multiply c_{α_2} (resp. c_{α_1}) by a reduced norm from the quaternion algebra (d, c_{α_1}) (resp. (d, c_{α_2})).

By Witt cancelation we may write a in the form $a = w_1c_{\alpha_1} + w_2c_{\alpha_2} - w_3c_{\alpha_1}c_{\alpha_2}$ where $w_1, w_2, w_3 \in N_{L/K}(L^\times)$. By Corollary 4.13, passing to another maximal torus and Chevalley basis (if necessary) we may assume without loss of generality that $w_1 = w_2 = 1$ and hence we may assume that a is of the form $a = c_{\alpha_1}(1 - w_3c_{\alpha_2}) + c_{\alpha_2}$ where w_3 is still in $N_{L/K}(L^\times)$.

If $1 - w_3c_{\alpha_2} = 0$ then $a = c_{\alpha_2}$ and we pass to the last paragraph of the proof. Otherwise applying a proper elementary transformation with respect to α_2 we pass to a new torus with structure constants $c'_{\alpha_1} = c_{\alpha_1}(1 - w_3c_{\alpha_2})$ and $c'_{\alpha_2} = c_{\alpha_2}$. Thus abusing notation without loss of generality we may assume

$$a = c_{\alpha_1} + c_{\alpha_2} = c_{\alpha_1}(1 - (-c_{\alpha_1})^{-1}c_{\alpha_2}).$$

Applying again a proper elementary transformation with respect to α_1 we can pass to a torus whose second structure constant is $(-c_{\alpha_1})^{-1}c_{\alpha_2}$, so that we may assume $a = c_{\alpha_1}(1 - c_{\alpha_2})$. Lastly, applying an elementary transformation with respect to α_2 we pass to a torus such that $a = c_{\alpha_1}$.

We finally observe that from

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle = \langle\langle\, d, a, b\,\rangle\rangle = \langle\langle\, d, c_{\alpha_1}, b\,\rangle\rangle$$

it follows that b is of the form $b = wc_{\alpha_2}$ where $w \in \mathrm{Nrd}\,(d, c_{\alpha_1})$. So a proper elementary transformation with respect to α_1 completes the proof. □

6. Alternative formulas for f_3 and f_5 invariants

We are going to apply the previous technique to produce explicit formulas for the f_3 and f_5 invariants of a group G of type F_4 over a field K of characteristic $\neq 2$ with trivial g_3 invariant. Recall (cf. [S93], [GMS03], [PetRac]) that given such G one can associate the cohomological invariants $f_3(G) \in$

$H^3(K,\mu_2)$ and $f_5(G) \in H^5(K,\mu_2)$ with the following properties (cf. [Sp], [Ra]):

(a) *The group G is split over a field extension E/K if and only if $f_3(G)$ is trivial over E;*

(b) *The group G is isotropic over a field extension E/K if and only if $f_5(G)$ is trivial over E.*

These two invariants f_3, f_5 are symbols given in terms of the trace quadratic form of the Jordan algebra J corresponding to G and hence we may associate to them the 3-fold and 5-fold Pfister forms. Abusing notation we denote these Pfister forms by the same symbols $f_3(G)$ and $f_5(G)$. It is well known that they are completely classify groups of type F_4 with trivial g_3 invariant (see [Sp], [S93]) and we would like to produce explicit formulas of $f_3(G)$ and $f_5(G)$ in group terms only in order to generalize them later on to the case of local rings.

It follows from (a) that our group G is splitting by a quadratic extension. Indeed, if $f_3(G) = (d) \cup (a) \cup (b)$ then passing to $L = K(\sqrt{d})$ we get G_L has trivial f_3 invariant and as a consequence G is L-split by property (a).

Let $T \subset G$ be a maximal torus defined over K and splitting over L. Fix a Chevalley basis of the Lie algebra $\mathfrak{g}$ of G with respect to T and let $c_{\alpha_1}, \dots, c_{\alpha_4}$ be the corresponding structure constants of G.

6.1. **Theorem.** *One has $f_3(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$.*

Proof. Let $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle$ and let E be the function field of f. Remind that two n-fold Pfister forms are isomorphic over a ground field if and only if one of them is hyperbolic over the function field of the second one. Applying (a) and taking into consideration that $f_3(G)$ is a 3-Pfister form we see that for the proof it suffices to show that G is split over E.

Let G_0 be a split group of type F_4 over E and let $T_0 \subset G_0$ be a maximal E-split torus. By Proposition 4.10 there exist $u_1, \dots, u_4 \in E^\times$ such that G_E is a twist of G_0 by a cocycle $\xi = (a_\tau)$ where $a_\tau = ct$ and $t = \prod h_{\alpha_i}(u_i)$. We now note that a subgroup H_0 in G_0 generated by the long roots of $\Sigma = \Sigma(G_0, T_0)$ is a simple simply connected group of type D_4 stable with respect to conjugation by a_τ, so that its twist H is a subgroup in G_E of type D_4. A basis of the root system of H is given in the proposition below. Looking at this basis and the corresponding structure constants of H we find with the use of results in Section 5 that $H \simeq \mathrm{Spin}(f_E)$. Hence H is split over E implying G_E is E-split as well. □

The following proposition shows that the structure constants c_{α_3} are c_{α_4} of G are well defined modulo values of $f = f_3(G) = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2} \,\rangle\rangle$.

6.2. **Proposition.** *Let $a, b \in K^\times$ be represented by f over K. Then there exists a maximal torus $T' \subset G$ defined over K and splitting over L such that modulo squares G has structure constants $c_{\alpha_1}, c_{\alpha_2}, ac_{\alpha_3}, bc_{\alpha_4}$ with respect to T'.*

Proof. We may view G as a twisted group ${}^{\xi}G_0$ where $\xi = (a_\tau)$, $a_\tau = c\prod_{i=1}^{4} h_{\alpha_i}(u_i)$ and $u_1,\dots,u_4 \in K^\times$. Looking at the tables in [Bourb68] we find that the subroot system in $\Sigma(G_0,T_0)$ generated by the long roots has type D_4 with a basis

$$\beta_1 = -\epsilon_1 - \epsilon_2, \quad \beta_2 = \alpha_1, \quad \beta_3 = \alpha_2, \quad \beta_4 = \epsilon_3 + \epsilon_4.$$

Since $\epsilon_3 + \epsilon_4 = \alpha_2 + 2\alpha_3$ and $\epsilon_1 + \epsilon_2 = 2\alpha_2 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, it follows from relations in Chevalley groups that

$$h_{\epsilon_3+\epsilon_4}(u) = h_{\alpha_2}(u)h_{\alpha_3}(u) \tag{6.3}$$

and

$$h_{\epsilon_1+\epsilon_2}(u) = h_{\alpha_1}(u^2)h_{\alpha_2}(u^3)h_{\alpha_3}(u^2)h_{\alpha_4}(u) \tag{6.4}$$

for all parameters $u \in L^\times$.

The relations (6.3) and (6.4) shows that a_τ can be rewritten in the form

$$a_\tau = ch_{\alpha_1}(v_1)h_{\alpha_2}(v_2)\,[h_{\epsilon_1+\epsilon_2}(v_3)h_{\alpha_2}(v_3)]\,[h_{\epsilon_3+\epsilon_4}(v_4)h_{\alpha_2}(v_4)] \tag{6.5}$$

where $v_1,v_2,v_3,v_4 \in K^\times$. Using (3.3) we easily find that modulo squares in $K^\times$ one has $c_{\alpha_3} = v_2v_3$ and $c_{\alpha_4} = v_4$ and c_{α_1}, c_{α_2} don't depend on v_3, v_4 modulo squares. According to Proposition 5.1 if multiply the parameters v_3, v_4 in the expression (6.5) by a, b respectively we obtain a cocycle equivalent to ξ, so the result follows. □

6.6. **Theorem.** *One has* $f_5(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2}) \cup (c_{\alpha_3}) \cup (c_{\alpha_4})$.

Proof. Arguing as in Theorem 6.1 and using (b) we may assume that the quadratic form $g = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4}\,\rangle\rangle$ is split and we have to prove that G is isotropic.

Since g is split we may write c_{α_4} in the form

$$c_{\alpha_4} = a^{-1}(1 - bc_{\alpha_3}) \tag{6.7}$$

where a, b are represented by $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$. Our aim is to pass to a new torus $T' \subset G$ defined over K and splitting over L such that a new structure constant c'_{α_4} related to T' is equal to 1 modulo squares. The last would imply that the corresponding subgroup G_{α_4} of G is isomorphic to SL_2 by Lemma 3.5 (ii) and this would show that G is isotropic as required.

By Proposition 6.2 there exists a maximal torus T' in G such that two last structure constants related to T' are $c'_{\alpha_3} = bc_{\alpha_3}$ and $c'_{\alpha_4} = ac_{\alpha_4}$. Then by (6.7) we have $c'_{\alpha_4} = 1 - c'_{\alpha_3}$. Applying a proper elementary transformation with respect to α_3 we pass to the third torus T'' for which $c''_{\alpha_4} = 1$ modulo squares and we are done. □

7. Classification of groups of type F_4 with trivial g_3 invariant

The theorem below is due to T. Springer [Sp]. In this section we produce an alternative proof which can be easily adjusted to the case of local rings.

7.1. **Theorem.** *Let G_0 be a split group of type F_4 over a field K. A mapping*

$$H^1_{ét}(K, G_0)_{\{g_3=0\}} \to H^3(K, \mu_2) \times H^5(K, \mu_2)$$

given by $G \to (f_3(G), f_5(G))$ is injective.

We need the following preliminary result.

7.2. **Proposition.** *Let G be a group of type F_4 defined over K and splitting over L with structure constants $c_{\alpha_1}, \dots, c_{\alpha_4}$ with respect to a torus T. Let $a \in K^\times$ be represented by $g = \langle\langle d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3} \rangle\rangle$ over K. Then there is a maximal torus $T' \subset G$ such that the corresponding structure constants are $c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, ac_{\alpha_4}$ modulo squares.*

Proof. Write a in the form $a = a_1(1 - a_2c_{\alpha_3})$ where a_1, a_2 are represented by $f = \langle\langle d, c_{\alpha_1}, c_{\alpha_2} \rangle\rangle$. By Proposition 6.2 the structure constants c_{α_3} and c_{α_4} are well defined modulo values of f. Hence we may pass to another maximal torus in G such that the first and the second structure constants are the same but the third structure constant is $a_2c_{\alpha_3}$ and the last one is $a_1c_{\alpha_4}$. Thus without loss of generality we may assume that $a_2 = 1$ and a is of the form $a = 1 - c_{\alpha_3}$. Since $1 - c_{\alpha_3}$ is a reduced norm in the quaternion algebra (d, c_{α_3}) a proper elementary transformation with respect to α_3 lead us to a torus whose first three structure constants are the same modulo squares and the last one is $(1 - c_{\alpha_3})c_{\alpha_4}$. □

Proof of Theorem 7.1. Let G, G' be two groups of type F_4 over K such that $f_3(G) = f_3(G')$ and $f_5(G) = f_5(G')$. Choose a quadratic extension L/K splitting $f_3(G)$. It splits both G and G'. Our strategy is to show that G, G' contain maximal tori defined over K and splitting over L with the same structure constants.

Choose arbitrary maximal tori $T \subset G$, $T' \subset G'$ defined over K and splitting over L. Let $c_{\alpha_1}, \dots, c_{\alpha_4}$ and $c'_{\alpha_1}, \dots, c'_{\alpha_4}$ be the corresponding structure constants. As we know, G, G' contain subgroups H, H' of type D_4 over K generated by the long roots. By Theorem 6.1 we have $f_3(G) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$ and $f_3(G') = (d) \cup (c'_{\alpha_1}) \cup (c'_{\alpha_2})$, hence

$$\langle\langle d, c_{\alpha_1}, c_{\alpha_2} \rangle\rangle = \langle\langle d, c'_{\alpha_1}, c'_{\alpha_2} \rangle\rangle.$$

Then according to Proposition 5.3 applied to H' and $f = \langle\langle d, c_{\alpha_1}, c_{\alpha_2} \rangle\rangle$ we may assume without loss of generality that $c_{\alpha_1} = c'_{\alpha_1}$ and $c_{\alpha_2} = c'_{\alpha_2}$.

We next show that up to choice of maximal tori in G and G' we also may assume that $c_{\alpha_3} = c'_{\alpha_3}$. Since $f_5(G) = f_5(G')$ we get

$$\langle\langle d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4} \rangle\rangle = \langle\langle d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4} \rangle\rangle. \tag{7.3}$$

By Witt cancelation we can write c'_{α_3} in the form $c'_{\alpha_3} = a_1c_{\alpha_3} + a_2c_{\alpha_4} - a_3c_{\alpha_3}c_{\alpha_4}$ where a_1, a_2, a_3 are values of f. By Proposition 6.2 we may assume without loss of generality that $a_1 = a_2 = 1$. Arguing as in Proposition 5.3

we may pass to another maximal torus in G' such that the corresponding structure constants are

$$c'_{\alpha_1} = c_{\alpha_1},\ c'_{\alpha_2} = c_{\alpha_2},\ c'_{\alpha_3} = c_{\alpha_3}.$$

Finally, from (7.3) it follows that $c'_{\alpha_4} = ac_{\alpha_4}$ for some $a \in K^\times$ represented by $g = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}\,\rangle\rangle$. Application of Proposition 7.2 completes the proof. □

8. Group schemes splitting by étale quadratic extensions

We now pass to a simple simply connected group scheme G of an arbitrary type of rank n defined over a local ring R where 2 is invertible and splitting by an étale quadratic extension $S = R(\sqrt{u}) \simeq R[t]/(t^2 - u)$ of R where $u \in R^\times$. We assume that R is a domain with a quotient field K and with a residue field k and we assume u is not square in $K^\times$. We also denote $L = S \otimes_R K$ and $l = S \otimes_R k$. Abusing notation we denote the nontrivial automorphisms of S/R, L/K and l/k by the same letter τ.

Let $\mathfrak{g}$ be the Lie algebra of G. As usual we set

$$\mathfrak{g}_S = \mathfrak{g} \otimes_R S,\quad \mathfrak{g}_K = \mathfrak{g} \otimes_R K,\quad \mathfrak{g}_L = \mathfrak{g} \otimes_R L$$

and

$$\overline{\mathfrak{g}} = \mathfrak{g}_k = \mathfrak{g} \otimes_R k,\quad \overline{\mathfrak{g}}_S = \mathfrak{g}_l = \mathfrak{g}_S \otimes_S l.$$

Let $\mathfrak{b}_S$ be a Borel subalgebra in $\mathfrak{g}_S$. We say that it is in *a generic position* if $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ is a Cartan subalgebra in $\overline{\mathfrak{g}}_l$. This amounts to saying that $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ has dimension n over l.

We will systematically use below the fact that in a split simple Lie algebra defined over a field the intersection of two Borel subalgebras contains a split Cartan subalgebra; in particular this intersection has dimension at least n.

8.1. **Lemma.** *The Lie algebra $\mathfrak{g}_S$ contains Borel subalgebras in generic position.*

Proof. Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be the varieties of Borel subalgebras in the split Lie algebras $\mathfrak{g}_S$ and $\mathfrak{g}_l$ respectively. Passing to residues we have a canonical mapping $\mathcal{B} \to \overline{\mathcal{B}}$ whose image is dense (because $\mathfrak{g}_S$ is split). Let $U \subset \overline{\mathcal{B}}$ be an open subset in Zariski topology consisting of Borel subalgebras $\mathfrak{b}_l$ such that $\mathfrak{b}_l \cap \tau(\mathfrak{b}_l)$ has dimension n. Since $\mathcal{B}(S)$ is dense in $\mathcal{B}$ there exists a Borel subalgebra $\mathfrak{b}_S$ in $\mathfrak{g}_S$ over S whose image in $\overline{\mathcal{B}}$ is contained in U. □

8.2. **Lemma.** *Let $\mathfrak{b}_S \subset \mathfrak{g}_S$ be a Borel subalgebra in generic position. Then a submodule $\mathfrak{t}_S = \mathfrak{b}_S \cap \tau(\mathfrak{b}_S)$ of $\mathfrak{b}_S$ has rank n.*

Proof. The subalgebra $\mathfrak{t}_S$ is given as an intersection of two free submodules in $\mathfrak{g}_S$ of codimensions m, where m is the number of positive roots in $\mathfrak{g}_S$, each of them being a direct summand in $\mathfrak{g}_S$. So $\mathfrak{t}_S$ consists of all solutions of a linear system of m equations in $m + n$ variables. The space of solutions of this system modulo M where $M \subset R$ is a maximal ideal coincides with the intersection $\overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$ and hence it has dimension n. This implies that the linear system has a minor of size $m \times m$ whose determinant is a unit in S and we are done. □

Our next aim is to show that the Galois descent data for the generic fiber G_K of G described in previous sections can be pushed down at the level of R. As usual we will assume that the Weyl group of G contains -1.

8.3. **Proposition.** *Let $\mathfrak{b}_S \subset \mathfrak{g}_S$ be a Borel subalgebra in generic position and let $\mathfrak{t}_S = \mathfrak{b}_S \cap \tau(\mathfrak{b}_S)$. Then $\mathfrak{t}_S$ is a split Cartan subalgebra of $\mathfrak{g}_S$ contained in $\mathfrak{b}_S$.*

Proof. Let $\mathfrak{u}_S$ be the ideal in $\mathfrak{b}_S$ consisting of nilpotent elements. It is complimented in $\mathfrak{b}_S$ by a split Cartan algebra and hence $\mathfrak{b}_S/\mathfrak{u}_S$ is isomorphic to a split Cartan subalgebra in $\mathfrak{b}_S$. We need to show that a canonical projection $p : \mathfrak{b}_S \to \mathfrak{b}_S/\mathfrak{u}_S$ restricted at $\mathfrak{t}_S$ is an isomorphism.

Let $\mathfrak{b}_L = \mathfrak{b}_S \otimes_S L$ be a generic fiber of $\mathfrak{b}_S$. We already know that $\mathfrak{t}_L = \mathfrak{b}_L \cap \tau(\mathfrak{b}_L)$ has dimension n over L, so it is a split Cartan algebra in $\mathfrak{g}_L$. Since $\mathfrak{t}_S$ imbeds into $\mathfrak{t}_L$, it is a commutative Lie subalgebra contained in $\mathfrak{b}_S$ and consisting of diagonalizable semisimple elements. So injectivity of p follows.

As for surjectivity, it suffices to prove it modulo maximal ideal $M \subset R$. In the course of proving of Lemma 8.2 we saw that $\mathfrak{t}_S$ is the space of solutions of the linear system of m equations in $m+n$ variables whose matrix modulo M has rank m. It follows $\mathfrak{t}_S$ modulo M has dimension n and we are done. □

Let now $\mathfrak{t}_S$ be as in Proposition 8.3 and let $\mathfrak{t} = \mathfrak{t}_S^{\langle \tau \rangle}$ be the invariant subspace. By descent we have $\mathfrak{t} \otimes_R S = \mathfrak{t}_S$, hence $\mathfrak{t}$ is an R-defined Cartan subalgebra splitting over S. Let B_S be a Borel subgroup in G_S corresponding to $\mathfrak{b}_S$. The connected component of the automorphism group of a pair $(\mathfrak{b}_S, \mathfrak{t}_S)$ gives rise to a maximal torus T_S in B_S. It is R-defined and S-split because so is $\mathfrak{t}$. Let us choose a Chevalley basis

$$\{H_{\alpha_1}, \dots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$$

in $\mathfrak{g}_S$ corresponding to (T_S, B_S). Since W contains -1, we know that τ acts on the root system $\Sigma = \Sigma(G_S, T_S)$ as -1. Now repeating verbatim the arguments in [Ch] we easily find that for every root $\alpha \in \Sigma$ there exists a constants $c_\alpha \in R$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$ and hence the action of τ on $G(S)$ is determined completely by the family $\{c_\alpha,\ \alpha \in \Sigma\}$. We call these constants by *structure constants* of G with respect to T.

As in [Ch] one checks that the structure constants satisfy the relations given in Lemmas 3.4, 3.5. Also, as in Example 4.5 we may obviously define the notion of an elementary transformation with respect to a root $\alpha \in \Sigma$ (because root subgroups U_α are defined over S).

8.4. **Remark.** We note that the structure constants $\{\, c_\alpha \mid \alpha \in \Sigma \,\}$ are units in R. Indeed, by our construction we have surjections $\mathfrak{b}_S \to \overline{\mathfrak{b}}_S$ and $\mathfrak{b}_S \cap \tau(\mathfrak{b}_S) \to \overline{\mathfrak{b}}_S \cap \tau(\overline{\mathfrak{b}}_S)$. Hence the residues of c_α are structure constants of $\overline{G} = G \otimes_R k$ in the corresponding basis.

9. Proof of Theorem 1.2

Let R be a ring satisfying all hypothesis in Theorem 1.2. As usual we denote its quotient field by K. Let G_0 be a split group of type F_4 over R and let $[\xi] \in H^1(R, G_0)_{\{g_3=0\}}$. We first claim that the twisted group $G = {}^{\xi}G_0$ is split by an étale quadratic extension of R. The proof is based on the following.

9.1. **Lemma.** *There exist $u, v, w \in R^\times$ such that $f_3(G_K) = (u) \cup (v) \cup (w)$.*

Proof. Let $f_3(G_K) = (a) \cup (b) \cup (c)$ where $a, b, c \in K^\times$. By [ChP] the functor of 3-fold Pfister forms satisfies purity, hence it suffices to show that $f_3(G)$ is unramified at prime ideals of R of height 1.

Let $\mathfrak{p} \subset R$ be a prime ideal of height 1 and let $v = v_{\mathfrak{p}}$ be the corresponding discrete valuation on K with the residue field $k(v) = R/\mathfrak{p}$. We need to show that the image of $f_3(G_K)$ under the boundary map $\partial_v : H^3(K, \mathbb{Z}/2) \to H^2(k(v), \mathbb{Z}/2)$ is trivial.

The image $\partial_v(f_3(G))$ coincides with that of under the composition

$$H^3(K, \mathbb{Z}/2) \longrightarrow H^3(K_v, \mathbb{Z}/2) \xrightarrow{\partial_v} H^2(k(v), \mathbb{Z}/2)$$

where by abusing notation the last boundary mapping is still denoted by ∂_v. Further, one knows that $f_3(G_{K_v}) = \mathcal{R}_{G_0}([\xi_{K_v}])$ where

$$\mathcal{R}_{G_0} : H^1(K_v, G_0) \to H^3(K_v, \mathbb{Z}/2)$$

is the 2-component of the Rost invariant for G_0. Let $\mathcal{O}_v$ be the ring of integers of K_v. The properties of the Rost invariant imply $\partial_v([\lambda]) = 0$ for every class $[\lambda] \in H^1(\mathcal{O}_v, G_0)$. Since the class of ξ_{K_v} is in the image of $H^1(R, G_0) \to H^1(\mathcal{O}_v, G_0) \to H^1(K_v, G_0)$ we are done. □

9.2. **Proposition.** *G is split by an étale quadratic extension of R.*

Proof. By Lemma 9.1 we have $f_3(G_K) = (u) \cup (v) \cup (w)$ where $u, v, w \in R^\times$. Take $S = R(\sqrt{u})$ and we claim G_S is split. One of the following two cases occurs.

If $u \in (K^\times)^2$ then we have $f_3(G_K) = 0$. It follows $\mathcal{R}_{G_0}([\xi_K]) = f_3(G_K) = 0$. Since the kernel of the Rost invariant for split groups of type F_4 defined over K is trivial by [Gar] (see also [Ch]), we have $[\xi_K] = 0$. Since by [CTO], [R94], [R95] Grothendieck–Serre conjecture holds for G_0 we conclude $\xi = 0$, i.e. G is already split over R.

Assume now that $u \notin (K^\times)^2$. Let L be a quotient field of S. Arguing along the same lines we first get $\mathcal{R}_{G_0}([\xi_L]) = 0$ and then G_S is split. □

The following lemma is an R-analogue of Corollary 4.13.

9.3. **Lemma.** *Let $T \subset G$ be a maximal torus with the structure constants $\{c_{\alpha_1}, \dots, c_{\alpha_4}\}$ and let $u_1, \dots, u_4 \in N_{S/R}(S^\times)$. Then G contains a maximal torus T' whose structure constants are $\{c_{\alpha_1}u_1, \dots, c_{\alpha_n}u_4\}$.*

Proof. Apply the same argument as in Lemma 4.12 with the use of Remark 4.11. □

Proof of Theorem 1.2. Let $[\xi], [\xi'] \in H^1(R, G_0)_{\{g_3=0\}}$ be two classes and let G, G' be the corresponding twisted group schemes over R. Assume that the generic fibers G_K, G'_K of G and G' are isomorphic over K. If G_K is K-split, there is nothing to prove, because Grothendieck-Serre conjecture is already proven for G_0, and so we may assume that G_K, G'_K are not split over K (and hence G, G' are not split over R) which amounts to saying that $f_3(G_K) \neq 0$ and $f_3(G'_K) \neq 0$.

By Proposition 9.2 there exists an étale quadratic extension $S = R(\sqrt{d})$, where $d \in R^\times$, splitting G. Of course, it is split G' as well. It now suffices to show that G, G' contain maximal tori T, T' defined over R and splitting over S and such that the corresponding structure constants for G_1 and G_2 are the same.

Let T, T' be arbitrary R-defined and S-splitting maximal tori in G, G'. Let $c_{\alpha_1}, \ldots, c_{\alpha_4}$ and $c'_{\alpha_1}, \ldots, c'_{\alpha_4}$ be structure constants of G, G' with respect to T and T'. By Theorem 6.1 we have $f_3(G_K) = (d) \cup (c_{\alpha_1}) \cup (c_{\alpha_2})$ and $f_3(G'_K) = (d) \cup (c'_{\alpha_1}) \cup (c'_{\alpha_2})$. Since $f_3(G_K) = f_3(G'_K)$ we get

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle_K \overset{K}{\simeq} \langle\langle\, d, c'_{\alpha_1}, c'_{\alpha_2}\,\rangle\rangle_K$$

and hence

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle \overset{R}{\simeq} \langle\langle\, d, c'_{\alpha_1}, c'_{\alpha_2}\,\rangle\rangle.$$

We first claim that up to choice of T and T' we may assume that $c_{\alpha_1} = c'_{\alpha_1}$ and $c_{\alpha_2} = c'_{\alpha_2}$. The proof of the claim is completely similar to that of Proposition 5.3. Namely, by Witt cancelation and by Lemma 2.1 we may write c'_{α_1} in the form $c'_{\alpha_1} = w_1 c_{\alpha_1} + w_2 c_{\alpha_2} - w_3 c_{\alpha_1} c_{\alpha_2}$ where $w_1, w_2, w_3 \in N_{S/R}(S^\times)$ and $w_1 c_{\alpha_1} - w_3 c_{\alpha_1} c_{\alpha_2}$ is a unit in R. By Lemma 9.3, passing to another maximal torus in G (if necessary) we may assume that $w_1 = w_2 = 1$ and then $c'_{\alpha_1} = c_{\alpha_1}(1 - w_3 c_{\alpha_2}) + c_{\alpha_2}$ where w_3 is still in $N_{S/R}(S^\times)$ and $1 - w_3 c_{\alpha_2}$ is a unit in R. The rest of the proof is the same as in Proposition 5.3.

We next claim that up to choice of T and T' we may additionally assume that $c_{\alpha_3} = c'_{\alpha_3}$. To prove it we are just copying the related part of the proof of Theorem 7.1. Arguing as in Proposition 4.10 we conclude that up to equivalence ξ and ξ' are of the form $\xi = (a_\tau)$ and $\xi' = (a'_\tau)$ where $a_\tau = c \prod_{i=1}^n h_{\alpha_i}(u_i)$ and $a'_\tau = c \prod_{i=1}^n h_{\alpha_i}(u'_i)$, so that, by Remark 5.2, G and G' contain simple simply connected subgroups H and H' generated by long roots such that $H \simeq H' \simeq \mathrm{Spin}\,(f)$ where $f = \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}\,\rangle\rangle$. Furthermore arguing as in Proposition 6.2 with the use of the second part of Remark 5.2 we see that the structure constants c_{α_3}, c_{α_4}, c'_{α_3}, c'_{α_4} are well defined modulo units in R represented by f.

Since $f_5(G_K) = f_5(G'_K)$ we get

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4}\,\rangle\rangle \overset{K}{\simeq} \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4}\,\rangle\rangle$$

and hence

$$\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}, c_{\alpha_4}\,\rangle\rangle \overset{R}{\simeq} \langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c'_{\alpha_3}, c'_{\alpha_4}\,\rangle\rangle. \tag{9.4}$$

By Witt cancelation we can write c'_{α_3} in the form $c'_{\alpha_3} = a_1c_{\alpha_3} + a_2c_{\alpha_4} - a_3c_{\alpha_3}c_{\alpha_4}$ where a_1, a_2, a_3 are units in R represented by f and $a_1c_{\alpha_3} - a_3c_{\alpha_3}c_{\alpha_4}$ is also a unit in R. Since c_{α_3}, c_{α_4} are defined modulo values of f passing to another maximal torus in G we may assume without loss of generality that $a_1 = a_2 = 1$. The rest of the proof is the same as in Proposition 5.3.

Finally we claim that we may assume that $c_{\alpha_4} = c'_{\alpha_4}$. Indeed, from (9.4) and Witt cancelation we conclude that c'_{α_4} is of the form $c'_{\alpha_4} = ac_{\alpha_4}$ where a is a unit in R represented by $\langle\langle\, d, c_{\alpha_1}, c_{\alpha_2}, c_{\alpha_3}\,\rangle\rangle$. Copying the proof of Proposition 7.2 we easily complete the proof of the claim. Thus Theorem 1.2 is proven. □

References

[Bourb68] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4,5 et 6, Hermann, Paris 1968.

[Ch] V. Chernousov, The kernel of the Rost invariant, Serre's Conjecture II and the Hasse principle for quasi-split groups ${}^{3,6}D_4$, E_6, E_7, Math. Annalen **326** (2003), 297– 330.

[ChS] V. Chernousov, J.-P. Serre, Lower bounds for essential dimensions via orthogonal representations, Journal of Algebra, **305** (2006), 1055 – 1070.

[ChP] V. Chernousov, I. Panin, Purity of G_2-torsors, C. R. Math. Acad. Sci. Paris **345** (2007), no. 6, 307–312.

[CTO] J.-L. Colliot-Thélène, M. Ojanguren, Espaces principaux homogènes localement triviaux, Publ. Math. IHES **75** (1992), no. 2, 97–122.

[CTS] J.-L. Colliot-Thélène, J.-J. Sansuc, Principal homogeneous spaces under flasque tori: Applications, Journal of Algebra **106** (1987), 148–205.

[DG] M. Demazure and A. Grothendieck, Structure des Schémas en Groupes Réductifs, SGA 3 III, LN 153, Springer-Verlag, 1970.

[Gar] R. S. Garibaldi, The Rost invariant has trivial kernel for quasi-split groups of low rank, To appear in Commentarii Mathematici Helvetici.

[GMS03] S. Garibaldi, A. Merkurjev, and J.-P. Serre, Cohomological invariants in Galois cohomology, University Lecture Series, vol. 28, Amer. Math. Soc., 2003.

[Gr58] A. Grothendieck, Torsion homologique et sections rationnelles, *Anneaux de Chow et applications*, Séminaire Chevalley, 2^e année, Secrétariat mathématique, Paris, 1958.

[Gr68] A. Grothendieck, Le groupe de Brauer II, in *Dix exposés sur la cohomologie des schémas*, Amsterdam, North-Holland, 1968.

[Hum] J. Humphreys, Linear Algebraic Groups, Springer-Verlag 1975.

[Ma] H. Matsumura, Commutative algebra, W.A. Benjamin Co., New York, 1970.

[N] Y. Nisnevich, Rationally trivial principal homogeneous spaces and arithmetic of reductive group schemes over Dedekind rings, C. R. Acad. Sci. Paris, Série I, **299** (1984), 5–8.

[OP] M. Ojangurem, I. Panin, Rationally trivial hermitian spaces are locally trivial, Math. Z. **237** (2001), 181–198.

[OPZ] M. Ojanguren, I. Panin, K. Zainoulline, On the norm principle for quadratic forms, J. Ramanujan Math. Soc. **19** (2004), 1–12.

[P03] I. Panin, Rationally isotropic quadratic spaces are locally isotropic, http://www.math.uiuc.edu/K-theory/0671/2003.

[P09] I. Panin, On Grothendieck-Serre conjecture concerning principal G-bundles over reductive group schemes: II, Preprint 2009.

[PSV] I. Panin, A. Stavrova, N. Vavilov, On Grothendieck-Serre conjecture concerning principal G-bundles over reductive group schemes: I, Preprint 2009.

[PPS] I. Panin, V. Petrov, A. Stavrova, On Grothendieck-Serre conjecture for simple adjoint group schemes of types E_6, E_7, Preprint 2009.

[PetRac] H. Petersson, M. Racine, On the invariants mod 2 of Albert Algebras, Journal of Algebra **174** (1995), 1049–1072.

[PS] I. Panin, A. Suslin, On a conjecture of Grothendieck concerning Azumaya algebras, St. Petersburg Math. J. **9** (1998), 851–858.

[Ra] M. Racine, A note on Quadratic Jordan Algebras of degree 3, Trans. Amer. Math. Soc. **164** (1972), 93–103.

[R94] M. S. Raghunathan, Principal bundles admitting section, Invent. Math. **116** (1994), 409– 423.

[R95] M. S. Raghunathan, Erratum: Principal bundles admitting section, Invent. Math. **121** (1995), 223.

[Ro] M. Rost, A (mod 3) invariant for exceptional Jordan algebras, C. R. Acad. Sci. Paris, Série I **313** (1991), 823–827.

[S58] J.-P. Serre, Espaces fibrés algébriques, in *Anneaux de Chow et applications*, Séminaire Chevalley, 2^e année, Secrétariat mathématique, Paris, 1958.

[S93] J.–P. Serre, Cohomologie galoisienne: progrès et problèmes, Sém. Bourbaki 1993–1994, exposé 783.

[Sp] T. A. Springer, The classification of reduced exceptional simple Jordan algebras, Indag. Math. **22** (1960), 414–422.

[SGA3] Séminaire de Géométrie algébrique de l'I.H.E.S., 1963–1964, Schémas en groupes, dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math. 151–153, Springer (1970).

[St] R. Steinberg, Lectures on Chevalley Groups, Yale University 1967.

[Z] , K. Zainoulline, On Grothendieck conjecture about principal homogeneous spaces for some classical algebraic groups, St. Petersburg Math. J. **12** (2001), 117–143.

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