MAXIMAL REPRESENTATION DIMENSION FOR GROUPS OF ORDER p^n

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Abstract. The representation dimension $\mathrm{rdim}(G)$ of a finite group G is the smallest positive integer m for which there exists an embedding of G in $\mathrm{GL}_m(\mathbb{C})$. In this paper we find the largest value of $\mathrm{rdim}(G)$, as G ranges over all groups of order p^n, for a fixed prime p and a fixed exponent $n \geq 1$.

1. Introduction

The representation dimension of a finite group G, denoted by $\mathrm{rdim}(G)$, is the minimal dimension of a faithful complex linear representation of G. In this paper we determine the maximal representation dimension of a group of order p^n. We are motivated by a recent result of N. Karpenko and A. Merkurjev [KM07, Theorem 4.1], which states that if G is a finite p-group then the essential dimension of G is equal to $\mathrm{rdim}(G)$. For a detailed discussion of the notion of essential dimension for finite groups (which will not be used in this paper), see [BR97] or [JLY02, §8]. We also note that a related invariant, the minimal dimension of a faithful complex *projective* representation of G, has been extensively studied for finite simple groups G; for an overview, see [TZ00, §3].

Let G be a p-group of order p^n and r be the rank of the centre $C(G)$. A representation of G is faithful if and only if its restriction to $C(G)$ is faithful. Using this fact it is easy to see that a faithful representation ρ of G of minimal dimension decomposes as a direct sum

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r \tag{1}$$

of exactly r irreducibles; cf. [MR09, Theorem 1.2]. Since the dimension of any irreducible representation of G is $\leq \sqrt{[G : C(G)]}$ (see, e.g., [W03, Corollary 3.11]) and $|C(G)| \geq p^r$, we conclude that

$$\mathrm{rdim}(G) \leq rp^{\lfloor (n-r)/2 \rfloor}. \tag{2}$$

2000 *Mathematics Subject Classification.* 20C15, 20D15.

Key words and phrases. p-group, representation dimension, symplectic subspace, generalized Heisenberg group.

The authors gratefully acknowledge financial support by the Natural Sciences and Engineering Research Council of Canada.

Let

$$f_p(n) := \max_{r\in\mathbb{N}}(rp^{\lfloor (n-r)/2\rfloor}).$$

It is easy to check that $f_p(n)$ is given by the following table:

n	p	$f_p(n)$
even	arbitrary	$2p^{(n-2)/2}$
odd	odd	$p^{(n-1)/2}$
odd, ≥ 3	2	$3p^{(n-3)/2}$
1	2	1

We are now ready to state the main result of this paper.

Theorem 1. *Let p be a prime and n be a positive integer. For almost all pairs (p, n), the maximal value of* $\mathrm{rdim}(G)$*, as G ranges over all groups of order p^n, equals $f_p(n)$. The exceptional cases are*

$$(p, n) = (2, 5),\ (2, 7) \text{ and } (p, 4), \text{ where } p \text{ is odd.}$$

In these cases the maximal representation dimension is 5, 10, *and* $p + 1$, *respectively.*

The proof will show that the maximal value of $\mathrm{rdim}(G)$, as G ranges over all groups of order p^n, is always attained for a group G of nilpotency class ≤ 2. Moreover, if (p, n) is non-exceptional, $n \geq 3$ and $(p, n) \neq (2, 3), (2, 4)$, the maximum is attained on a special class of p-groups of nilpotency class 2, which we call *generalized Heisenberg groups.*

The rest of this paper is structured as follows. In §2 we introduce generalized Heisenberg groups and study their irreducible representations. Theorem 1 is proved in §3.

Acknowledgement. We would like to thank Hannah Cairns, Robert Guralnick, Chris Parker, Burt Totaro, and Robert Wilson for helpful discussions.

2. Generalized Heisenberg groups

2.1. **Spaces of alternating forms.** Let V be a finite dimensional vector space over an arbitrary field F. Let $K \subset \Lambda^2(V)^*$ be a subspace. We will say that K is *symplectic* if every nonzero element of K is a non-degenerate alternating map. Clearly nontrivial symplectic subspaces of $\Lambda^2(V)^*$ can exist only if $\dim(V)$ is even.

Lemma 2. *Suppose V is an F-vector space of dimension $2m$. If F admits a field extension of degree m then there exists an m-dimensional symplectic subspace $K \subset \Lambda^2(V)^*$.*

Proof. Choosing a basis of V, we can identify $\Lambda^2(V)^*$ with the space of skew-symmetric $2m \times 2m$-matrices. Let $f\colon \mathrm{M}_m(F) \to \Lambda^2(V)^*$ be the linear map

$$A \mapsto \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}.$$

If W is a linear subspace of $\mathrm{M}_m(F) = \mathrm{End}_F(F^n)$ such that $W\backslash\{0\} \subset \mathrm{GL}_m(F)$ then $K = f(W)$ is a symplectic subspace.

It thus remains to construct an m-dimensional linear subspace W of $\mathrm{M}_m(F)$ such that $W\backslash\{0\} \subset \mathrm{GL}_m(F)$. Let E be a degree m field extension of F. Then E acts on itself by left multiplication. This gives an F-vector space embedding of $\Psi\colon E \hookrightarrow \mathrm{End}_F(E)$ such that $\Psi(e)$ is invertible for all $e \neq 0$. □

2.2. **Groups associated to spaces of alternating forms.** Let $\omega_K : \Lambda^2(V) \to K^*$ denote the dual of the natural injection $K \hookrightarrow \Lambda^2(V)^*$. Note that there exists a bilinear map $\beta : V \otimes V \to K^*$ such that

$$\omega_K(v,w) = \beta(v,w) - \beta(w,v) \qquad \forall\, v,w \in V. \tag{3}$$

Indeed, choose a basis $\{e_1,\dots,e_n\}$ for V, and define β by

$$\beta(e_i,e_j) = \begin{cases} \omega_K(e_i,e_j) & \text{if } i > j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3. To the data (V,K,β) as above, we attach a group $H = H(V,K,\beta)$. As a set $H = V \times K^*$; the group operation is given by

$$(v,t)\cdot(v',t') = (v+v', t+t'+\beta(v,v')). \tag{4}$$

If K is a symplectic subspace, we will refer to H as a *generalized Heisenberg group.*

It is easy to see that (4) is indeed a group law with the inverse given by $(v,t)^{-1} = (-v, -t+\beta(v,v))$ and the commutator given by

$$[(v_1,t_1),(v_2,t_2)] = (0, \omega_K(v_1,v_2)). \tag{5}$$

Remark 4. Let $H = H(V,K,\beta)$, as in Definition 3. Since the inclusion $K \hookrightarrow \Lambda^2(V)^*$ is, by definition, injective, its dual $\omega_K\colon \Lambda^2(V) \to K^*$ is surjective. Formula (5) now tells us that $[H,H] = K^*$.

Moreover, (5) also shows that $K^* \subset C(H)$, and that equality holds unless the intersection $\cap_{k\in K} \ker(k)$ is nontrivial. In particular, $C(H) = K^*$ if K contains a symplectic form.

Remark 5. The reason for the term *generalized Heisenberg group* is that in the special case, where $F = \mathbb{F}_p$, p is an odd prime, K is a one-dimensional symplectic subspace and $\beta = \frac{1}{2}\omega_K$, the group $H(V,K,\beta)$ is often called *the Heisenberg group.*

Note that β is not uniquely determined by K; it is only unique up to adding a symmetric bilinear form $V \times V \to K^*$. If β and β' both satisfy (3) then $H(V,K,\beta)$ may not be isomorphic to $H(V,K,\beta')$. For example, let V be a 2-dimensional vector space over $F = \mathbb{F}_2$, K be the one-dimensional (symplectic) subspace generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and β, β' be bilinear forms on V defined by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

respectively. Then β and β' both satisfy (3), but $H(V, K, \beta)$ is isomorphic to the quaternion group while $H(V, K, \beta')$ is isomorphic to the dihedral group of order 8.

Remark 6. Two groups S and T are isoclinic if there are isomorphisms $f : S/C(S) \to T/C(T)$ and $g : [S, S] \to [T, T]$ such that if $a, b \in S$ and $a', b' \in T$ with $f(aC(S)) = a'C(T)$ and $f(bC(S)) = b'C(T)$, then we have $g([a, b]) = [a', b']$; see [PH40]. Let K be a subspace of $\Lambda^2(V)^*$, and suppose β and β' are bilinear forms on V satisfying (3). Then $H = H(V, K, \beta)$ and $H' = H(V, K, \beta')$ are isoclinic (in this case f and g are identity maps).

2.3. **Representations.** Let p be an arbitrary prime, $F = \mathbb{F}_p$ be the finite field of p elements, V be a vector space over F and K be a subspace of $\Lambda^2(V)^*$. Let $\omega = \omega_K : \Lambda^2(V) \to K^*$ denote the dual of $K \hookrightarrow \Lambda^2(V)^*$.

Let ψ be a morphism $K^* \to \mathbb{C}^\times$. Identify (non-canonically) $\mathbb{F}_p$ with the group of p^{th} roots of unity in $\mathbb{C}^\times$, so that we can view ψ as being in $(K^*)^*$. Using the canonical isomorphism between $(K^*)^*$ and K we associate to ψ an element $k \in K$ such that $k = \psi \circ \omega$. In particular, k is non-degenerate if and only if $\psi \circ \omega$ is non-degenerate; this condition does not depend on the way we identify $\mathbb{F}_p$ with the group of p^{th} roots of unity in $\mathbb{C}^\times$. Conversely, to each $k \in K$ we can associate a character ψ of K^* such that if we view $\psi \in (K^*)^*$, we have $k = \psi \circ \omega$.

Lemma 7. *Let $G = V(V, K, \beta) = V \times K^*$ be as in Definition 3. Let ρ be an irreducible representation of G such that K^* acts by ψ. Assume $\psi \circ \omega : V \otimes V \to \mathbb{C}^\times$ is non-degenerate.*

(a) If $g \in G$, $g \notin K^*$, then $\mathrm{Tr}(\rho(g)) = 0$.
(b) $\dim(\rho) = \sqrt{|V|}$.
(c) ρ is uniquely determined (up to isomorphism) by ψ.

Proof. (a) Let $g \in G \backslash K^*$. Since $\psi \circ \omega$ is non-degenerate there exists $h \in G$ such that $\psi \circ \omega(gK^*, hK^*) \neq 1$. Observe that $\rho([g, h]) = \psi([g, h]) \,\mathrm{Id}$, and that $\rho(h^{-1}gh) = \rho(g)\rho([g, h])$. Taking the trace of both sides, we have $\mathrm{Tr}(\rho(g)) = \psi([g, h]) \,\mathrm{Tr}(\rho(g))$. Since $\psi([g, h]) \neq 1$ we must have $\mathrm{Tr}(\rho(g)) = 0$.

(b) Since ρ is irreducible, and the trace of ρ vanishes outside of K^*, we have:

$$\begin{aligned} 1 &= \frac{1}{|G|}\sum_{g\in G} \mathrm{Tr}(\rho(g))\overline{\mathrm{Tr}(\rho(g))} \\ &= \frac{1}{|G|}\sum_{g\in K^*} \mathrm{Tr}(\rho(g))\overline{\mathrm{Tr}(\rho(g))} \\ &= \frac{1}{|G|}\dim(\rho)^2 \sum_{g\in K^*} \mathrm{Tr}(\psi(g))\overline{\mathrm{Tr}(\psi(g))} \\ &= \dim(\rho)^2\frac{|K^*|}{|G|} \end{aligned}$$

Thus $\dim\rho = \sqrt{|G|/|K^*|} = \sqrt{|V|}$.

(c) We have completely described the character of ρ, and it follows that ρ is uniquely determined by ψ. Indeed,

$$\mathrm{Tr}(\rho(g)) = \begin{cases} \sqrt{|V|}\cdot\psi(g), & \text{if } g\in K^* \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

□

Henceforth, let K be a symplectic subspace of $\Lambda^2(V)^*$, $H = H(V, K, \beta) = V \times K^*$ be a generalized Heisenberg group, for some β as in (3). The proposition below is a direct consequence of Lemma 7.

Proposition 8. *The irreducible representations of H are exhausted by the following list:*

(i) $|V|$ one-dimensional representations, one for every character of V.
(ii) $|K|-1$ representations of dimension $\sqrt{|V|}$, one for every nontrivial character $\psi : K^* \to \mathbb{C}^\times$.

The next corollary is also immediate upon observing that $C(H) = K^*$; see Remark 4.

Corollary 9. *The representation dimension of H equals* $\dim(K)\sqrt{|V|}$.

If G is a finite Heisenberg group in the usual sense (as in Remark 5) then for each nontrivial character χ of $C(G)$ there is a unique irreducible representation ψ of G whose central character is χ; cf. [GH07, §1.1]. This is a finite group variant of the celebrated Stone-von Neumann Theorem. For a detailed discussion of the history and the various forms of the Stone-von Neumann theorem we refer the reader to [R04]. Proposition 8 tells us that, in fact, every generalized Heisenberg group over $\mathbb{F}_p$ has the Stone-von Neumann property. This observation, stated as Corollary 10 below, will not be used in the sequel.

Corollary 10. *Two irreducible representations of H with the same nontrivial central character are isomorphic.*

3. Proof of Theorem 1

The case where $n \leq 2$ is trivial; clearly $\mathrm{rdim}(G) = \mathrm{rank}\,(G)$ if G is abelian. We will thus assume that $n \geq 3$.

In the non-exceptional cases of the theorem, in view of the inequality (2), it suffices to construct a group G of order p^n with $\mathrm{rdim}(G) = f_p(n)$. Here $f_p(n)$ is the function defined just before the statement of Theorem 1.

If $(p, n) = (2, 3)$ or $(2, 4)$, we take G to be the elementary abelian group $\mathbb{F}_2^3$ and $\mathbb{F}_2^4$, yielding the desired representation dimension of 3 and 4, respectively. For all other non-exceptional pairs (p, n), we take G to be a generalized Heisenberg group as described in the table below. Here $H(V, K)$ stands for $H(V, K, \beta)$, for some β as in (3). In each instance, the existence of a symplectic subspace K of suitable dimension is guaranteed by Lemma 2 and the value of $\mathrm{rdim}(H(V, K))$ is given by Corollary 9.

n	p	$\dim(V)$	$\dim(K)$	$\mathrm{rdim}(H(V,K))$
even, ≥ 6	arbitrary	$n-2$	2	$2p^{(n-2)/2}$
odd, ≥ 3	odd	$n-1$	1	$p^{(n-1)/2}$
odd, ≥ 9	2	$n-3$	3	$3p^{(n-3)/2}$

This settles the generic case of Theorem 1. We now turn our attention to the exceptional cases. We will need the following upper bound on $\mathrm{rdim}(G)$, strengthening (2).

Let $C(G)_p$ be the subgroup of central elements $g \in C(G)$ such that $g^p = 1$. If $\rho\colon G \to \mathrm{GL}(V)$ is an irreducible representation then $C(G)$ (and hence, its subgroup $C(G)_p$) acts on V by scalar multiplication, $g \cdot v \mapsto \chi(g)v$, where χ is a multiplicative character of $C(G)$. Following [MR09, Lemma 2.2], we will call $\chi\colon C(G)_p \to \mathbb{C}^\times$ the *associated character* (to ρ).

Lemma 11. *Let G be a p-group and $r = \mathrm{rank}\,(C(G)) = \mathrm{rank}\,(C(G)_p)$.*

(a) Suppose there exists an irreducible representation ρ_1 such that $\mathrm{Ker}(\rho_1)$ *does not contain $C(G)_p$. Then there are irreducible representations $\rho_2, \dots, \rho_r$ of G such that $\rho_1 \oplus \cdots \oplus \rho_r$ is faithful. In particular,*

$$\mathrm{rdim}(G) \leq \dim(\rho_1) + (r-1)\sqrt{[G : C(G)]}\,.$$

(b) If $C(G)_p$ is not contained in $[G, G]$, then

$$\mathrm{rdim}(G) \leq 1 + (r-1)\sqrt{[G : C(G)]}\,.$$

The lemma can be deduced from [KM07, Remark 4.7] or [MR09, Theorem 1.2]; for the sake of completeness we give a self-contained proof.

Proof. (a) Let χ_1 be the character of $C(G)_p$ associated to ρ_1. By our assumption χ_1 is nontrivial. Complete χ_1 to a basis $\chi_1, \chi_2, \dots, \chi_r$ of the r-dimensional $\mathbb{F}_p$-vector space $C(G)_p^*$ and choose an irreducible representation ρ_i with associated character χ_i. (The representation ρ_i can be taken to be any irreducible component of the induced representation $\mathrm{Ind}_{C(G)_p}^G(\chi)$.) The restriction of $\rho\colon = \rho_1 \oplus \cdots \oplus \rho_r$ to $C(G)_p$ is faithful. Hence, ρ

is a faithful representation of G. As we mentioned in the introduction $\dim(\rho_i) \le \sqrt{[G : C(G)]}$ for every $i \ge 2$, and part (a) follows.

(b) By our assumption there exists one-dimensional representation ρ_1 of G whose restriction to $C(G)_p$ is nontrivial. Now apply part (a). □

We are now ready to prove Theorem 1 in the three exceptional cases.

3.1. **Exceptional case 1: p is odd and $n = 4$.**

Lemma 12. *Let p be an odd prime and G be a group of order p^4.*

(a) Then $\operatorname{rdim}(G) \le p + 1$.

(b) Suppose $C(G) \simeq \mathbb{F}_p^2$ and $G/C(G) \simeq \mathbb{F}_p^2$. Then $\operatorname{rdim}(G) = p + 1$.

Proof. (a) We argue by contradiction. Assume there exists a group of order p^4 such that $\operatorname{rdim}(G) \ge p + 2$. If $|C(G)| \ge p^3$ or $G/C(G)$ is cyclic then G is abelian and $\operatorname{rdim}(G) = \operatorname{rank}(G) \le 4 \le p + 1$, a contradiction. If $C(G)$ is cyclic then $\operatorname{rdim}(G) \le p$ by (2), again a contradiction.

Thus $C(G) \simeq G/C(G) \simeq \mathbb{F}_p^2$. This reduces part (a) to part (b).

(b) Here $C(G)_p = C(G)$ has rank 2. Hence, a faithful representation ρ of G of minimal dimension is the sum of two irreducibles $\rho_1 \oplus \rho_2$, as in (1), each of dimension 1 or p.

Clearly $\dim(\rho_1) = \dim(\rho_2) = 1$ is not possible, since in this case G would be abelian, contradicting $[G : C(G)] = p^2$. It thus remains to show that $\operatorname{rdim}(G) \le p + 1$. Since $G/C(G)$ is abelian, $[G, G] \subset C(G)$. Hence, by Lemma 11(b) we only need to establish that $[G, G] \subsetneq C(G)$.

To show that $[G, G] \subsetneq C(G)$, note that the commutator map

$$\begin{array}{rcl} \Psi : G/C(G) \times G/C(G) & \to & [G, G] \\ (gC(G), g'C(G)) & \mapsto & [g, g'] \end{array}$$

can be thought of as an alternating bilinear map from $\mathbb{F}_p^2$ to itself. Viewed in this way, Ψ can be written as $\Psi(v, v') = (w_1(v, v'), w_2(v, v'))$ for alternating maps w_1 and w_2 from $\mathbb{F}_p^2$ to $\mathbb{F}_p$. Since $\Lambda^2(\mathbb{F}_p^2)^*$ is a one-dimensional vector space over $\mathbb{F}_p$, w_1 and w_2 are scalar multiples of each other. Hence, the image of Ψ is a cyclic group of order p, and $[G, G] \subsetneq C(G)$, as claimed. □

To finish the proof in this case, note that $G = \mathbb{F}_p \times G_0$, where G_0 is a non-abelian group of order p^3, satisfies the conditions of Lemma 12(b). Thus the maximal representation dimension of a group of order p^4 is $p + 1$, for any odd prime p.

3.2. **Exceptional case 2: $p = 2$ and $n = 5$.**

Lemma 13. *Let G be a group of order* 32*. Then* $\operatorname{rdim}(G) \le 5$.

Proof. We argue by contradiction. Assume there exists a group of order 32 and representation dimension ≥ 6. Let $r = \operatorname{rank}(C(G))$. Then $1 \le r \le 5$ and (2) shows that $\operatorname{rdim}(G) \le 5$ for every $r \ne 3$.

Thus we may assume $r = 3$. If $|C(G)| \geq 16$ or $G/C(G)$ is cyclic then G is abelian, and $\mathrm{rdim}(G) = \mathrm{rank}\,(G) \leq 5$. We conclude that $C(G) \simeq \mathbb{F}_2^3$ and $G/C(G) \simeq \mathbb{F}_2^2$. Applying the same argument as in the proof of Lemma 12(b), we see that $[G, G] \subsetneq C(G)$, and hence $\mathrm{rdim}(G) \leq 5$ by Lemma 11(b), a contradiction. □

To finish the proof in this case, note that the elementary abelian group of order 2^5 has representation dimension 5. Thus the maximal representation dimension of a group of order 2^5 is 5.

3.3. **Exceptional case 3:** $p = 2$ **and** $n = 7$.

Lemma 14. *If* $|G| = 128$ *then* $\mathrm{rdim}(G) \leq 10$.

Proof. Again, we argue by contradiction. Assume there exists a group G of order 128 and representation dimension ≥ 11. Let r be the rank of $C(G)$. By (2), $r = 3$; otherwise we would have $\mathrm{rdim}(G) \leq 10$.

As we explained in the introduction, this implies that a faithful representation ρ of G of minimal dimension is the direct sum of three irreducibles ρ_1, ρ_2 and ρ_3, each of dimension $\leq \sqrt{2^7/|C(G)|}$. If $|C(G)| > 8$, then $\dim(\rho_i) \leq 2$ and $\mathrm{rdim}(G) = \dim(\rho_1) + \dim(\rho_2) + \dim(\rho_3) \leq 6$, a contradiction.

Therefore, $C(G) \cong (\mathbb{F}_2)^3$ and $\dim(\rho_1) = \dim(\rho_2) = \dim(\rho_3) = 4$. By Lemma 11(a) this implies that the kernel of every irreducible representation of G of dimension 1 or 2 must contain $C(G)$. In other words, any such representation factors through the group $G/C(G)$ of order 16. Consequently, if m_i is the number of irreducible representations of G of dimension i then $m_1 + 4m_2 = 16$. We can now appeal to [JNO90, Tables I and II], to show that no group of order 2^7 has these properties. From Table I we can determine which groups G (up to isoclinism, cf. Remark 6) have $|C(G)| = 8$ and using Table II we can determine m_1 and m_2 for these groups. There is no group G with $|C(G)| = 8$ and $m_1 + 4m_2 = 16$. □

We will now construct an example of a group G of order 2^7 with $\mathrm{rdim}(G) = 10$. Let $V = \mathbb{F}_2^4$ and let K be the 3-dimensional subspace of $\Lambda^2(V)^*$ generated by the following three elements:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let $G := H(V, K, \beta) = V \times K^*$ for some β as in (3). Note that K contains only one non-zero degenerate element (the sum of the three generators). In other words, there is only one character χ of K^* such that $\chi \circ \omega : V \times V \to \mathbb{C}^\times$ is degenerate. By Remark 4

$$[G, G] = C(G) = K^* .$$

Let ρ be a faithful representation of G of minimal dimension. As we explained in the Introduction, ρ is the sum of $\mathrm{rank}\,(C(G)) = 3$ irreducibles.

Denote them by ρ_1, ρ_2, and ρ_3, and their associated characters by χ_1, χ_2 and χ_3, respectively. Since ρ is faithful, χ_1, χ_2 and χ_3 form an $\mathbb{F}_2$-basis of $C(G)_p^* \simeq \mathbb{F}_2^3$. By Lemma 7, for each nontrivial character χ of K^* except one, there is a unique irreducible representation ψ of G such that χ is the associated character to ψ, and $\dim\psi = 4$. Thus at least 2 of the irreducible components of ρ, say, ρ_1 and ρ_2 must have dimension 4. By Lemma 14, $\dim(\rho) \leq 10$, i.e., $\dim(\rho_3) \leq 2$. But every one-dimensional representation of G has trivial associated character. We conclude that $\dim(\rho_3) = 2$ and consequently $\mathrm{rdim}(G) = \dim(\rho) = 4 + 4 + 2 = 10$.

Thus the maximal representation dimension of a group of order 2^7 is 10.

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