ISOTROPY OF ORTHOGONAL INVOLUTIONS

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Abstract. An orthogonal involution on a central simple algebra becoming isotropic over any splitting field of the algebra, becomes isotropic over a finite odd degree extension of the base field (provided that the characteristic of the base field is not 2).

Our aim is a proof of the following result, generalizing the hyperbolicity statement of [5]:

Theorem 1. *Let F be a field of characteristic not 2, A a central simple F-algebra, σ an orthogonal involution on A. The following two conditions are equivalent:*

1. *σ becomes isotropic over any splitting field of A;*
2. *σ becomes isotropic over some finite odd degree extension of the base field.*

The proof of Theorem 1 is given in the very end of the paper; a sketch of the proof is given shortly below.

For F with no finite field extensions of odd degree, Theorem 1 proves [8, Conjecture 5.2].

The general reference on central simple algebras and involutions is [11].

The implication (2) $\Rightarrow$ (1) is a consequence of the Springer theorem on quadratic forms. We only prove the implication (1) $\Rightarrow$ (2). Note that condition (2) is equivalent to the condition that σ becomes isotropic over some generic splitting field of the algebra, such as the function field of the Severi-Brauer variety of any central simple algebra Brauer-equivalent to A.

We prove this theorem over all fields simultaneously using an induction on the index $\operatorname{ind} A$ of A. The case of $\operatorname{ind} A = 1$ is trivial. The case of $\operatorname{ind} A = 2$ is done in [15] (with "σ is isotropic (over F)" in place of condition (2)). From now on we are assuming that $\operatorname{ind} A > 2$. Therefore $\operatorname{ind} A = 2^r$ for some integer $r \geq 2$.

Let us list our basic notation: F is a field of characteristic different from 2; r is an integer ≥ 2; A is a central simple F-algebra of the index 2^r; σ is an orthogonal involution on A; D is a central division F-algebra (of degree 2^r) Brauer-equivalent to A; V is a right D-module with an isomorphism $\operatorname{End}_D(V) \simeq A$; v is the D-dimension of V (therefore $\operatorname{rdim} V = \deg A = 2^r \cdot v$, where $\operatorname{rdim} V := \dim_F V/\deg D$ is the reduced dimension of V); we fix an orthogonal involution τ on D; h is a hermitian (with respect to τ) form on V such that the involution σ is adjoint to h; $\mathfrak{X} = X(2^r; (V,h))$ is the variety of totally

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isotropic submodules in V of the reduced dimension 2^r which is isomorphic (via Morita equivalence) to the variety $X(2^r;(A,\sigma))$ of right totally isotropic ideals in A of the same reduced dimension; $\mathcal{Y} = X(2^{r-1}; D)$ is the variety of right ideals in D of reduced dimension 2^{r-1}.

We assume that the hermitian form h (and therefore, the involution σ) becomes isotropic over the function field of the Severi-Brauer variety $X(1;D)$ of D, and we want to show that h (and σ) becomes isotropic over a finite odd degree extension of F. By [9], the Witt index of h (which coincides with the Witt index of σ) over this function field is at least $2^r = \operatorname{ind} A$. In particular, $v \geq 2$. If the Witt index is bigger than 2^r, we replace V by a submodule in V of D-codimension 1 (that is, of the reduced dimension $2^r(v-1)$) and we replace h by its restriction on this new V. The Witt index of $h_{F(X(1;D))}$ drops by at most 2^r or stays unchanged. We repeat the procedure until the Witt index becomes equal to 2^r (we come down eventually to the Witt index 2^r because the Witt index is at most 2^r for V with $\dim_D V = 2$).

If $\dim_D V = 2$, then h becomes hyperbolic over $F\big(X(1;D)\big)$. Therefore, by the main result of [5], h is hyperbolic over F and we are done. By this reason, we assume that $\dim_D V \geq 3$, that is, $v \geq 3$. In particular, the variety $\mathfrak{X}$ is projective *homogeneous* (in the case of $v = 2$, the variety $\mathfrak{X}$ has two connected components each of which is homogeneous).

The variety $\mathfrak{X}$ has an $F\big(X(1;D)\big)$-point and $\operatorname{ind} D_{F(\mathcal{Y})} = 2^{r-1}$. Consequently, by the induction hypothesis, the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has an odd degree closed point. We prove Theorem 1 by showing that the variety $\mathfrak{X}$ has an odd degree closed point. Here is a sketch of the proof:

Sketch of Proof of Theorem 1. First we show that the Chow motive with coefficients in $\mathbb{F}_2$ of the variety $\mathfrak{X}$ contains a summand isomorphic to a shift of the *upper* indecomposable summand $M_{\mathcal{Y}}$ of the motive of $\mathcal{Y}$ (Corollary 7). (Here we use the 2-incompressibility of $\mathcal{Y}$ which is due to [12].) Moreover, the corresponding projector on $\mathfrak{X}$ can be *symmetrized* (Proposition 8). This makes it possible to compute the degree of the 0-cycle class on $\mathfrak{X}\times\mathfrak{X}$, given by the value of a Steenrod operation on this projector. Namely (see Corollary 9), this degree is identified with the *rank* of $M_{\mathcal{Y}}$ and therefore is 2 modulo 4 by a result of [6]. Finally, a computation of Steenrod operations on split orthogonal grassmannians (Proposition 10) allows to show that the above 0-cycle class is divisible by 2. This shows that the variety $\mathfrak{X} \times \mathfrak{X}$ (and therefore also $\mathfrak{X}$ itself) has a 0-cycle of odd degree (and therefore, of degree 1). $\square$

We need an enhanced version of [5, Proposition 4.6]. This is a statement about the *Grothendieck Chow motives* (see [4, Chapter XII]) with coefficients in a prime field $\mathbb{F}_p$ (which we shall apply to $p = 2$). We write Ch for Chow groups with coefficients in $\mathbb{F}_p$ and we write $M(X)$ for the motive of a complete smooth F-variety X.. The base field F may have arbitrary characteristic in this statement:

Proposition 2. *Let Y be a geometrically split, geometrically irreducible F-variety satisfying the nilpotence principle and X be a smooth complete F-variety. Assume that there exists a field extension E/F such that*

(1) *for some field extension* $\overline{E(Y)}/E(Y)$*, the image of the change of field homomorphism* $\mathrm{Ch}(X_{E(Y)}) \to \mathrm{Ch}(X_{\overline{E(Y)}})$ *coincides with the image of the change of field homomorphism* $\mathrm{Ch}(X_{F(Y)}) \to \mathrm{Ch}(X_{\overline{E(Y)}})$*;*

(2) *the* E*-variety* Y_E *is* p*-incompressible;*

(3) *a shift of the upper indecomposable summand of* $M(Y)_E$ *is a summand of* $M(X)_E$*.*

Then the same shift of the upper indecomposable summand of $M(Y)$ *is a summand of* $M(X)$*.*

Proof. The only difference with the original version is in the condition (1): the field extension $E(X)/F(X)$ is assumed to be purely transcendental in the original version. However, only the new condition (1), a consequence of the pure transcendentality, is used in the original proof. □

Everywhere below, the prime p is 2. We are going to apply Proposition 2 (with $p = 2$) to $Y = \mathcal{Y}$, $X = \mathfrak{X}$, and $E = F(\mathfrak{X})$. We do not know if the field extension $E(\mathcal{Y})/F(\mathcal{Y})$ is purely transcendental because we do not know whether the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has a rational point (we only know that this variety has an odd degree closed point).

Next we are going to check that conditions (1)–(3) are satisfied for these Y, X, E. First of all, we need a motivic decomposition of $\mathfrak{X}_{F(\mathfrak{X})}$. This is the decomposition of [2] arising from the fact that $\mathfrak{X}(F(\mathfrak{X})) \neq \emptyset$. More generally, the "same" decomposition holds for $F(\mathfrak{X})$ replaced by any field K/F with $\mathfrak{X}(K) \neq \emptyset$. Over such K, the hermitian form h decomposes in the orthogonal sum of the hyperbolic D_K-plane and a hermitian form h' on a right D_K-module V' with $\mathrm{rdim}\, V' = 2^r(v-2)$.

It requires some work to derive the decomposition from the general theorem of [2]. We use a ready answer from [7], where the projective homogeneous varieties under the *classical* semisimple affine algebraic groups has been treated:

Lemma 3 ([7, Corollary 15.4])**.** $M(\mathfrak{X}_K) =$

$$\bigoplus_{i,j} M\big(X(i, i+j; D_K) \times X(j; (V', h'))\big)\big(i(i-1)/2 + j(i+j) + i(\mathrm{rdim}\, V' - j)\big),$$

where $X(i, i+j; D_K)$ *is the variety of flags given by a right ideal in the* K*-algebra* D_K *of the reduced dimension* i *contained in a right ideal of the reduced dimension* $i+j$ *(this is a non-empty variety if and only if* $0 \le i \le i+j \le \deg D$*).*

In particular, a shift of the motive of the variety $\mathcal{Y}_{F(\mathfrak{X})}$ is a motivic summand of $\mathfrak{X}_{F(\mathfrak{X})}$: namely, the summand of Lemma 3 given by $i = 2^{r-1}$ and $j = 0$ (with $K = F(\mathfrak{X})$). This summand has as the shifting number the integer

$$n := 2^{r-2}(2^{r-1} - 1) + 2^{2r-1}(v-2). \tag{4}$$

We note that $\dim \mathfrak{X} = 2^{r-1}(2^r - 1) + 2^{2r}(v-2)$, $\dim \mathcal{Y} = 2^{2r-2}$, and therefore

$$n = (\dim \mathfrak{X} - \dim \mathcal{Y})/2.$$

By [12] (see [6] for a generalization), the variety $\mathcal{Y}_{F(\mathfrak{X})}$ is 2-incompressible if (and only if) the division algebra D remains division over the field $F(\mathfrak{X})$. This is indeed the case:

Lemma 5. $\mathrm{ind}\, D_{F(\mathfrak{X})} = \mathrm{ind}\, D$*.*

Proof. Of course, the statement can be checked using the index reduction formulas of [13] (in the inner case, that is, in the case when the discriminant of h is trivial) and of [14] (in the outer case). However, we prefer to do it in a different way which is more internal with respect to the methods of this paper.

Assume that $\operatorname{ind} D_{F(\mathfrak{X})} < \operatorname{ind} D$. Then $\mathcal{Y}(F(\mathfrak{X})) \neq \emptyset$. Since in the same time the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has an odd degree closed point, it follows (by the main property of the upper motives established in [6, Corollary 2.15]) that the upper indecomposable motivic summand of $\mathcal{Y}$ is a motivic summand of $\mathfrak{X}$. This implies (because the variety $\mathcal{Y}$ is 2-incompressible) that the complete motivic decomposition of the variety $\mathfrak{X}_{F(\mathcal{Y})}$ contains the Tate summand $\mathbb{F}_2(\dim \mathcal{Y}) = \mathbb{F}_2(2^{2r-2})$. On the other hand, all the summands of the motivic decomposition of Lemma 3 (applied to the field $K = F(\mathcal{Y})$) are shifts of the motives of anisotropic varieties besides the following three: $\mathbb{F}_2$ (given by $i = j = 0$), $\mathbb{F}_2(\dim \mathfrak{X}) = \mathbb{F}_2\big(2^{r-1}(2^r-1) + 2^{2r}(v-2)\big)$ (given by $i = 2^r$ and $j = 0$), and $M(\mathcal{Y}_{F(\mathcal{Y})})(n)$ (given by $i = 2^{r-1}$ and $j = 0$) with n defined in (4). Here a variety is called anisotropic, if all its closed points are of even degree. The motive of an anisotropic variety does not contain Tate summands by [6, Lemma 2.21]. Taking into account the Krull-Schmidt principle of [3] (see also [10, §2]), we get a contradiction because $0 < 2^{2r-2} < n$ (the assumption $v \geq 3$ is used here). □

Lemma 6. *Let L/K be a finite odd degree field extension of a field K containing F. Let $\bar{L}$ be an algebraically closed field containing L. Then*

$$\operatorname{Im}\big(\operatorname{res}_{\bar{L}/L} : \operatorname{CH}(\mathfrak{X}_L) \to \operatorname{CH}(\mathfrak{X}_{\bar{L}})\big) = \operatorname{Im}\big(\operatorname{res}_{\bar{L}/K} : \operatorname{CH}(\mathfrak{X}_K) \to \operatorname{CH}(\mathfrak{X}_{\bar{L}})\big).$$

Proof. We write I_L and I_K for these images and we evidently have $I_K \subset I_L$.

Inside of $\bar{L}$, the variety $\mathfrak{X}_K$ has a finite 2-primary splitting field K'/K.

If $\operatorname{disc} h_K = 1$, then $[L : K] \cdot I_L \subset I_K$. Since moreover $[K' : K] \cdot \operatorname{CH}(\mathfrak{X}_{\bar{L}}) \subset I_K$ and $[K' : K]$ is coprime with $[L : K]$, it follows that $I_L \subset I_K$.

If $\operatorname{disc} h_K \neq 1$ then also $\operatorname{disc} h_L \neq 1$ and the group $G := \operatorname{Aut}(\bar{L}/K)$, acting on $\operatorname{CH}(\mathfrak{X}_{\bar{L}})$, acts trivially on I_L. Therefore we still have $[L : K]\cdot I_L \subset I_K$. Besides, $[K' : K]\cdot\operatorname{CH}(\mathfrak{X}_{\bar{L}})^G \subset I_K$, and it follows that $I_L \subset I_K$. □

We write $M_{\mathcal{Y}}$ for the upper indecomposable motivic summand of $\mathcal{Y}$.

Corollary 7. *$M_{\mathcal{Y}}(n)$ is a motivic summand of $\mathfrak{X}$.*

Proof. As planned, we apply Proposition 2 to $p = 2$, $Y = \mathcal{Y}$, $X = \mathfrak{X}$, and $E = F(\mathfrak{X})$. There exists a finite odd degree extension $L/F(Y)$ such that $X(L) \neq \emptyset$. The field extension $L(X)/L$ is purely transcendental. Since $E(Y) \subset L(X)$, condition (1) is satisfied by Lemma 6.

Condition (2) is satisfied by Lemma 5. Finally, condition (3) is satisfied by Lemma 3. □

We need the following enhancement of Corollary 7:

Proposition 8. *There exists a* symmetric *projector $\pi_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the motive $(\mathfrak{X}, \pi_{\mathfrak{X}})$ is isomorphic to $M_{\mathcal{Y}}(n)$.*

Proof. Let us start by checking that the motive $M_{\mathcal{Y}}$ can be given by a symmetric projector $\pi_{\mathcal{Y}}$ on $\mathcal{Y}$. The proof we give is valid for any projective homogeneous 2-incompressible variety in place of the variety $\mathcal{Y}$. Let π be a projector on $\mathcal{Y}$ such that $(\mathcal{Y},\pi) \simeq M_{\mathcal{Y}}$. Since our Chow groups are with finite coefficients, there exists an integer $l \geq 1$ such that $\pi_{\mathcal{Y}} := (\pi^t \circ \pi)^{\circ l}$ is a (symmetric) projector, where π^t is the transposition of π. Since the variety $\mathcal{Y}$ is 2-incompressible, $\operatorname{mult}\pi^t = 1$. It follows that $\operatorname{mult}\pi_{\mathcal{Y}} = 1$ and therefore the motive $(\mathcal{Y},\pi_{\mathcal{Y}})$ is non-zero. In the same time, it is a direct summand of the indecomposable motive $(\mathcal{Y},\pi)$ (the morphisms to and from $(\mathcal{Y},\pi)$ having the identical composition are given, for instance, by $\pi\circ\pi_{\mathcal{Y}}$ and simply $\pi_{\mathcal{Y}}$). Therefore $M_{\mathcal{Y}} \simeq (\mathcal{Y},\pi_{\mathcal{Y}})$ for the symmetric projector $\pi_{\mathcal{Y}}$.

Now let $\alpha : (\mathcal{Y},\pi_{\mathcal{Y}})(n) \to M(\mathfrak{X})$ and $\beta : M(\mathfrak{X}) \to (\mathcal{Y},\pi_{\mathcal{Y}})(n)$ be morphisms with $\beta\circ\alpha = \pi_{\mathcal{Y}} = \mathrm{id}_{(\mathcal{Y},\pi_{\mathcal{Y}})}$ (existing because $(\mathcal{Y},\pi_{\mathcal{Y}})(n)$ is a motivic summand of $\mathfrak{X}$). Note that α^t is a morphism

$$M(\mathfrak{X}) \to (\mathcal{Y},\pi^t_{\mathcal{Y}})(\dim\mathfrak{X} - \dim\mathcal{Y} - n) = (\mathcal{Y},\pi_{\mathcal{Y}})(n)$$

because $\pi^t_{\mathcal{Y}} = \pi_{\mathcal{Y}}$ and $2n = \dim\mathfrak{X} - \dim\mathcal{Y}$. There exists an integer $l \geq 1$ such that $(\alpha^t\circ\alpha)^{\circ l}$ is a projector. If $\operatorname{mult}(\alpha^t\circ\alpha) \neq 0$, then $(\alpha^t\circ\alpha)^{\circ l} = \pi_{\mathcal{Y}}$. Therefore $(\alpha\circ\alpha^t)^{\circ l}$ is a (symmetric) projector on $\mathfrak{X}$ and $\alpha : (\mathcal{Y},\pi_{\mathcal{Y}})(n) \to \big(\mathfrak{X},(\alpha\circ\alpha^t)^{\circ l}\big)$ is an isomorphism of motives, so that we are done in this case.

Similarly, if $\operatorname{mult}(\beta\circ\beta^t) \neq 0$, then $\beta^t : (\mathcal{Y},\pi_{\mathcal{Y}})(n) \to \big(\mathfrak{X},(\beta^t\circ\beta)^{\circ l}\big)$ for some (other) l is an isomorphism, and we are done in this case also.

In the remaining case we have $\operatorname{mult}(\alpha^t\circ\alpha) = 0 = \operatorname{mult}(\beta\circ\beta^t)$. Let $\mathbf{pt} \in \mathrm{Ch}_0(\mathcal{Y}_{F(\mathcal{Y})})$ be the class of a rational point. The compositions $\alpha\circ([\mathcal{Y}_{F(\mathcal{Y})}]\times\mathbf{pt})\circ\beta$ and $\beta^t\circ([\mathcal{Y}_{F(\mathcal{Y})}]\times\mathbf{pt})\circ\alpha^t$ are orthogonal projectors on $\mathfrak{X}_{F(\mathcal{Y})}$, and each of two corresponding motives is isomorphic to $\mathbb{F}_2(n)$. It follows that the complete motivic decomposition of $\mathfrak{X}_{F(\mathcal{Y})}$ contains two exemplars of $\mathbb{F}_2(n)$. However, as shown in the end of the proof of Lemma 5, the complete motivic decomposition of $\mathfrak{X}_{F(\mathcal{Y})}$ contains only one exemplar of $\mathbb{F}_2(n)$ (because the motive of $\mathcal{Y}_{F(\mathcal{Y})}$ contains only one exemplar of $\mathbb{F}_2$). □

From now on we are assuming that all closed points on the variety $\mathfrak{X}$ have even degrees. Then all closed points on the product $\mathfrak{X}\times\mathfrak{X}$ also have even degrees. Therefore the homomorphism $\deg/2 : \mathrm{Ch}_0(\mathfrak{X}\times\mathfrak{X}) \to \mathbb{F}_2$ is defined (as in [5, §5]).

Corollary 9. *Let $\pi_{\mathfrak{X}}$ be as in Proposition 8. Then $\pi^2_{\mathfrak{X}}$ is a 0-cycle class on $\mathfrak{X}\times\mathfrak{X}$ for which we have $(\deg/2)(\pi^2_{\mathfrak{X}}) = 1 \in \mathbb{F}_2$.*

Proof. For any symmetric projector π on $\mathfrak{X}$, we have $(\deg/2)(\pi^2) = \mathrm{rk}(\mathfrak{X},\pi)/2 \pmod 2$, where rk is the rank of the motive (the number of the Tate summands in the complete decomposition over a splitting field). Indeed, taking a complete motivic decomposition of $\bar{\mathfrak{X}}$ (here and below $\bar{\mathfrak{X}}$ is $\mathfrak{X}$ over a splitting field of $\mathfrak{X}$) which is a refinement of the decomposition $M(\mathfrak{X}) \simeq (\mathfrak{X},\pi)\oplus(\mathfrak{X},\Delta_{\mathfrak{X}}-\pi)$, we get a homogeneous basis B of $\mathrm{Ch}(\bar{\mathfrak{X}})$ such that $\bar{\pi} = \sum_{b\in B_\pi} b\times b^*$, where B_π is a subset of B and $\{b^*\}_{b\in B}$ is the dual basis. Note that $\mathrm{rk}(\mathfrak{X},\pi) = \#B_\pi$. For every $b\in B$, let us fix an integral representative $\mathfrak{b} \in \mathrm{CH}(\bar{\mathfrak{X}})$ of b and an integral representative $\mathfrak{b}^* \in \mathrm{CH}(\bar{\mathfrak{X}})$ of b^*. Then the sum $\sum_{b\in B_\pi}\mathfrak{b}\times\mathfrak{b}^*$, as well as the sum $\sum_{b\in B_\pi}\mathfrak{b}^*\times\mathfrak{b}$, is an integral representative of $\bar{\pi}$, and for the integral degree homomorphism $\deg : \mathrm{CH}_0(\bar{\mathfrak{X}}) \to \mathbb{Z}$ we have:

$$\deg\left(\Big(\sum_{b\in B_\pi} \mathfrak{b}\times\mathfrak{b}^*\Big)\Big(\sum_{b\in B_\pi} \mathfrak{b}^*\times\mathfrak{b}\Big)\right) \equiv \#B_\pi \pmod 4.$$

Now the rank of the motive $(\mathfrak{X}, \pi_{\mathfrak{X}})$ coincides with the rank of the motive $(\mathcal{Y}, \pi_{\mathcal{Y}}) \simeq M_{\mathcal{Y}}$ which is shown to be 2 modulo 4 in [6, Theorem 4.1]. □

The following Proposition is a general statement on the action of the cohomological Steenrod operation $\mathrm{Sq}^\bullet$ (see [4, Chapter XI]) on the Chow groups modulo 2 of a split orthogonal grassmannian G (which we shall apply to $G = \bar{\mathfrak{X}}$):

Proposition 10. *Let d be an integer ≥ 1, m an integer satisfying $0 \leq m \leq d-1$, G the variety of the totally isotropic $(m+1)$-dimensional subspaces of a hyperbolic $(2d+2)$-dimensional quadratic form q (over a field of characteristic $\neq 2$). Then for any integer $i > (d-m)(m+1)$ we have $\mathrm{Sq}^i\,\mathrm{Ch}_i(G) = 0$.*

Proof. Let Q be the projective quadric of q, Φ the variety of flags consisting of a line contained in a totally isotropic m-dimensional subspace of q, and $pr_G : \Phi \to G$, $pr_Q : \Phi \to Q$ the projections. We write $h \in \mathrm{CH}^1(Q)$ for the (integral) hyperplane section class and we write $l_i \in \mathrm{CH}_i(Q)$, where $i = 0, \dots, d$, for the (integral) class of an i-dimensional linear subspace in Q (for $i = d$ we *choose* one of the two classes, call it l_d, and write l'_d for the other). As in [16, §2], we define the integral classes

$$W_i \in \mathrm{CH}^i(G) \quad \text{for} \quad i = 1, \dots, d-m \quad \text{by} \quad W_i := (pr_G)_*\, pr_Q^*(h^{m+i})$$

and we define the integral classes

$$Z_i \in \mathrm{CH}^i(G) \quad \text{for} \quad i = d-m, \dots, 2d-m \quad \text{by} \quad Z_i = (pr_G)_*\, pr_Q^*(l_{2d-m-i}).$$

The elements $W_1, \dots, W_{d-m}, Z_{d-m}, \dots, Z_{2d-m}$ generate the ring $\mathrm{CH}(G)$ by [16, Proposition 2.9]. We call them *the generators of* $\mathrm{CH}(G)$. We refer to $W_1, \dots, W_{d-m}$ as W-generators, and we refer to $Z_{d-m}, \dots, Z_{2d-m}$ as Z-generators.

Note that $Z_{d-m} = (pr_G)_*\, pr_Q^*(l_d)$. We also set $Z'_{d-m} = (pr_G)_*\, pr_Q^*(l'_d)$. Since $l_d + l'_d = h^d$, we have $Z_{d-m} + Z'_{d-m} = W_{d-m}$.

Note that any element of $\mathrm{O}_{2d+2}(F) \setminus \mathrm{SO}_{2d+2}(F)$ gives an automorphism of G such that the corresponding automorphism of the ring $\mathrm{CH}(G)$ acts trivially on all the generators but Z_{d-m} which is interchanged with Z'_{d-m}.

For any $i \geq 0$, let $c_i \in \mathrm{CH}^i(G)$ be the ith Chern class of the quotient bundle on G. According to [16, Proposition 2.1], $c_i = W_i$ for any i for which W_i is defined, and $c_i = 2Z_i$ for $i \neq d-m$.

A computation similar to [4, (86.15)] (see also [1, (44) and (45) in Theorem 3.2]) shows that for any $i = d-m, \dots, 2d-m$, the generators of $\mathrm{CH}(G)$ satisfy the following relation

$$Z_i^2 - Z_i c_i + Z_{i+1} c_{i-1} - Z_{i+2} c_{i-2} + \dots .$$

(This is not and we do not need a complete list of relations.)

We denote the images of the generators of $\mathrm{CH}(G)$ under the epimorphism $\mathrm{CH}(G) \to \mathrm{Ch}(G)$ to the modulo 2 Chow group using the small letters w and z (with the same indices), and call them *the generators of* $\mathrm{Ch}(G)$. We say that an element of $\mathrm{Ch}(G)$ is of level l, if it can be written as a sum of products of generators such that the number of the z-factors in each product is at most l (so, any level l element is also of level $l+1$). A z-generator in a power k is counted k times here, that is, we are looking at the total degree assigning to each z-generator the weight 1 (and to each w-generator the weight 0). For instance, the monomial z_d^2 is of level 2 (but because of the relation $z_d^2 = z_dc_d - z_{d+1}c_{d-1} + \dots$, the element z_d^2 is also of level 1).

By [16, Proposition 2.8], the value of the total cohomological Steenrod operation $\mathrm{Sq}^\bullet : \mathrm{Ch}(G) \to \mathrm{Ch}(G)$ on any single z-generator is of level 1. Similar computation shows that the value of $\mathrm{Sq}^\bullet$ on any w-generator is of level 0. Since $\mathrm{Sq}^\bullet$ is a ring homomorphism, it follows that for any $l \geq 0$, the image under $\mathrm{Sq}^\bullet$ of a level l element is also of level l.

The above relations on the generators show that any element of $\mathrm{Ch}(G)$ is a polynomial of the generators such that the exponent of any z-generator in any monomial of the polynomial is at most 1. Since the dimension of such (biggest-dimensional level $m+1$) monomial $z_{d-m}\dots z_d$ is equal to

$$\dim G - \Big((d-m) + \dots + d\Big) = \dim G - \Big((d-m)(m+1) + m(m+1)/2\Big) = (d-m)(m+1),$$

any homogeneous element $\alpha \in \mathrm{Ch}(G)$ of dimension $i > (d-m)(m+1)$ is of level m. Therefore $\mathrm{Sq}^i(\alpha) \in \mathrm{Ch}_0(G)$ if also of level m.

We finish by showing that any level m element in $\mathrm{Ch}_0(G)$ is 0. For this we turn back to the integral Chow group $\mathrm{CH}(G)$ and show that any odd degree element $\beta \in \mathrm{CH}_0(G)$ is not of level m. The integral version of the notion of level used here is defined in the same way as the above modulo 2 version (using the generators of $\mathrm{CH}(G)$ instead of the generators of $\mathrm{Ch}(G)$).

Since the description of the ring $\mathrm{CH}(G)$ does not depend on the base field F, we may assume that $G = G'_F$, where G' is the grassmannian of a generic quadratic form defined over a subfield $F' \subset F$. We say that an element of $\mathrm{CH}(G)$ is *rational*, if it is in the image of the change of field homomorphism $\mathrm{res}_{F/F'} : \mathrm{CH}(G') \to \mathrm{CH}(G)$.

For any $i \geq 0$, the element c_i is rational. Therefore, for any $l \geq 0$, the 2^l-multiple of any level l element in $\mathrm{CH}_0(G)$ is rational. Indeed, this statement is a consequence of the formulas $W_i = c_i$ for any i such that W_i is defined, and the formulas $Z_i + \sigma Z_i = c_i$ for any i such that Z_i is defined, where σ is the ring automorphism of $\mathrm{CH}(G)$ given by an element of $\mathrm{O}_{2d+2}(F) \setminus \mathrm{SO}_{2d+2}(F)$ (note that σ is the identity on $\mathrm{CH}_0(G)$). The degree of any closed point on G' is divisible by 2^{m+1}. Therefore the element $2^m\beta$ is not rational, and it follows that β is not of level m. □

Remark 11. The statement of Proposition 10 also holds in the case of $m = d$, that is, in the case of a split *maximal* orthogonal grassmannian. The proof is even simpler and also the given proof of Proposition 10 can be easily modified to cover this case. Using this, one can cover the case of $v = 2$, excluded in the very beginning, and obtain this way a new proof for the hyperbolicity result of [5].

Corollary 12. *For any integer $i \geq n$ (where n is as in (4)) we have* $\mathrm{Sq}^i \mathrm{Ch}_i(\bar{\mathfrak{X}}) = 0$.

Proof. We apply Proposition 10 to $G = \bar{\mathfrak{X}}$. We have $d = 2^{r-1}v - 1$ and $m = 2^r - 1 \leq d - 1$ (because $v \geq 3$). Therefore $(d-m)(m+1) = 2^{2r-1}(v-2)$ and

$$n := 2^{r-2}(2^{r-1} - 1) + 2^{2r-1}(v-2) > (d-m)(m+1)$$

(because $r \geq 2$). $\square$

Example 13. Corollary 12 fails for $r = 1$. For instance, if $v = 6$ (and therefore $d = 5$), we have: $n = 8$, $z_4 z_5 \in \mathrm{Ch}^9(\bar{\mathfrak{X}}) = \mathrm{Ch}_8(\bar{\mathfrak{X}})$, and $\mathrm{Sq}^8(z_4 z_5) \neq 0$. Therefore, an additional argument is needed to prove the quaternion case by the method of this paper.

Proof of Theorem 1. We are going to show that $(\deg/2)(\pi_{\mathfrak{X}}^2) = 0$. This will contradict to Corollary 9 thus proving Theorem 1.

Since $\pi_{\mathfrak{X}}^2 = \mathrm{Sq}^{\dim \mathfrak{X}} \pi_{\mathfrak{X}}$, we have $(\deg/2)(\pi_{\mathfrak{X}}^2) = (\deg/2)(\mathrm{Sq}^\bullet \pi_{\mathfrak{X}})$. Let $\alpha : M(\mathcal{Y})(n) \to M(\mathfrak{X})$ and $\beta : M(\mathfrak{X}) \to M(\mathcal{Y})(n)$ be morphisms with $\alpha \circ \beta = \pi_{\mathfrak{X}}$ and let

$$pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}} : \mathfrak{X} \times \mathcal{Y} \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$$

be the projection. Since $\alpha \circ \beta = (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\big(([\mathfrak{X}] \times \alpha) \cdot (\beta \times [\mathfrak{X}])\big)$, we have

$$\mathrm{Sq}^\bullet \pi_{\mathfrak{X}} = (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\Big(\big([\mathfrak{X}] \times \mathrm{Sq}^\bullet(\alpha)\big) \cdot \big(\mathrm{Sq}^\bullet(\beta) \times [\mathfrak{X}]\big) \cdot \big([\mathfrak{X}] \times c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big),$$

where $T_{\mathcal{Y}}$ is the tangent bundle of $\mathcal{Y}$ and $c_\bullet$ is the total Chern class modulo 2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral representatives of $\mathrm{Sq}^\bullet(\alpha)$ and $\mathrm{Sq}^\bullet(\beta)$. It suffices to show that the degree of the integral cycle class

$$\mathfrak{c} := (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\Big(\big([\mathfrak{X}] \times \mathfrak{a}\big) \cdot \big(\mathfrak{b} \times [\mathfrak{X}]\big) \cdot \big([\mathfrak{X}] \times c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big)$$

is divisible by 4 (now $c_\bullet$ stands for the *integral* total Chern class).

We have

$$(pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}\mathfrak{X}})_*\Big(\big([\mathfrak{X}] \times \mathfrak{a}\big) \cdot \big(\mathfrak{b} \times [\mathfrak{X}]\big) \cdot \big([\mathfrak{X}] \times c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big) = \mathfrak{a} \cdot \big((pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b}) \times [\mathfrak{X}]\big) \cdot \big(c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)$$

and

$$(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\Big(\mathfrak{a} \cdot \big((pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b}) \times [\mathfrak{X}]\big) \cdot \big(c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big) = (pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\mathfrak{a}) \cdot (pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b}) \cdot c_\bullet(-T_{\mathcal{Y}}).$$

Therefore

$$\deg(\mathfrak{c}) = \deg\Big((pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\mathfrak{a}) \cdot (pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b}) \cdot c_\bullet(-T_{\mathcal{Y}})\Big)$$

and it suffices to show that the cycle classes $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\bar{\mathfrak{a}})$ and $(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\bar{\mathfrak{b}})$ are divisible by 2.

The (modulo 2) cycle class $\bar{\alpha}$ is a sum of $a' \times a$ with some $a' \in \mathrm{Ch}(\bar{\mathcal{Y}})$ and some homogeneous $a \in \mathrm{Ch}(\bar{\mathfrak{X}})$ of dimension $\geq n$. By Corollary 12, $\deg \mathrm{Sq}^\bullet(a) = 0 \in \mathbb{F}_2$ for such a. Therefore $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\big(\mathrm{Sq}^\bullet(\bar{\alpha})\big) = 0$ and the integral cycle class $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\bar{\mathfrak{a}})$, which represents the modulo 2 cycle class $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\big(\mathrm{Sq}^\bullet(\bar{\alpha})\big)$, is divisible by 2. Similarly, the cycle class $\bar{\beta}$ is a sum of $b \times b'$ with some $b' \in \mathrm{Ch}(\bar{\mathcal{Y}})$ and some homogeneous $b \in \mathrm{Ch}(\bar{\mathfrak{X}})$ of dimension $\geq n$, and it follows that the cycle class $(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\bar{\mathfrak{b}})$ is also divisible by 2. $\square$

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