WITT GROUPS OF HERMITIAN FORMS OVER A BRAUER SEVERI VARIETY

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Abstract. Let X be a Brauer Severi variety over a field k of characteristic not 2 and let D be a division algebra over k with a k-linear involution. We investigate Witt groups of certain hermitian forms over $D \otimes_k \mathcal{O}_X$.

Introduction

Let k be a field. For any Brauer Severi variety over k with structure morphism $\tau : X \to \operatorname{Spec}(k)$, the base change morphism $\tau^* : W(k) \to W(X)$ between the Witt rings of k and of X was shown to be surjective in [Pu 2, 3], provided that $\operatorname{char} k \neq 2$. The Witt groups of symmetric bilinear forms over X with values in a line bundle which generates $\operatorname{Pic} X$ were calculated in [Pu4, 5]. In the present paper, we see that the method involved in both proofs, i.e. the killing of certain cohomology groups, carries over to the setting of hermitian forms over finite separable field extensions of k with a k-linear involution. Moreover, the method employed in [Pu1] to prove that $\tau^* : W(k) \to W(X)$ is an isomorphism if X is the Brauer Severi variety associated to a central simple algebra of odd index, generalizes to Witt groups of ϵ-hermitian forms.

The content of the paper is as follows. Let A be an algebra over k together with a k-linear involution σ. After the preliminaries in Section 1, Section 2 deals with the injectivity and surjectivity of the group homomorphism $U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathcal{O}_X)$ in certain special cases. The Extension Theorem in Section 3 generalizes [A, Erster Schritt] to hermitian spaces and Theorem 8 generalizes Horrock's Theorem [B-H], proved in Section 4. Together with the results on extension groups in Section 5 the extension theorem is used to prove that for a separable field extension l/k with a k-linear involution σ, $\operatorname{char} k \neq 2$,

$$U_\tau : W^1(l) \to W^1(l \otimes_k \mathcal{O}_X)$$

is surjective. This result can be found in Section 6. We finish with a brief look at the case that $X = \mathbb{P}^1_k$, k a field of characteristic not 2 and D a division algebra over k with a k-linear involution σ in Section 7. Then $U_\tau : W^\epsilon(D) \to W^\epsilon(D \otimes \mathcal{O}_X)$ is bijective for $\epsilon = \pm 1$. A strategy for a possible proof of the same result for $X = \mathbb{P}^n_k$ is discussed in 7.2.

For the basic terminology and results on extension groups, the reader is referred to [H] and [Hi-S].

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1. Basic terminology

1.1. Let X be a scheme. By an "$\mathcal{O}_X$-algebra" we will always mean an associative $\mathcal{O}_X$-algebra which is unital and locally free of finite constant rank as $\mathcal{O}_X$-module. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra with an $\mathcal{O}_X$-linear involution σ. Let $\varepsilon \in H^0(X, \mathcal{A})$ be an element of the center of $\mathcal{A}$ such that $\varepsilon\sigma(\varepsilon) = 1$. Let $\mathcal{M}$ be a vector bundle over X which is locally free of finite rank as a right $\mathcal{A}$-module. Put $\mathcal{M}^* = \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ for the dual sheaf considered as a right $\mathcal{A}$-module $ma = \sigma(a)m$ through the involution σ for all a in $\mathcal{A}$, m in $\mathcal{M}$. Then $*$ is an exact contravariant duality functor [K, p. 75]. We canonically identify $\mathcal{M}$ and $\mathcal{M}^{**}$.

A isomorphism $h : \mathcal{M} \to \mathcal{M}^*$ is called a (nondegenerate) *ε-hermitian form* if $h = \varepsilon h^*$ and $(\mathcal{M}, h)$ is called an *ε-hermitian space over $\mathcal{A}$*. Two ε-hermitian spaces $(\mathcal{M}, h)$ and $(\mathcal{M}', h')$ over X are *isometric*, written as $(\mathcal{M}, h) \cong (\mathcal{M}', h')$ if there is an $\mathcal{O}_X$-linear isomorphism $f : \mathcal{M} \to \mathcal{M}'$ such that $f^*hf = h'$. For two ε-hermitian spaces $(\mathcal{M}_i, h_i)$, $i = 1, 2$, the *orthogonal sum* $(\mathcal{M}_1, h_1) \perp (\mathcal{M}_2, h_2)$ of $(\mathcal{M}_1, h_1)$ and $(\mathcal{M}_2, h_2)$ is defined as the ε-hermitian space

$$(\mathcal{M}_1 \oplus \mathcal{M}_2, \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}),$$

with the element

$$\begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \in \mathrm{Hom}(M_1 \oplus M_2, M_1^* \oplus M_2^*)$$

denoted by $h_1 \perp h_2$. Given an ε-hermitian space $(\mathcal{M}, h)$ and a right $\mathcal{A}$-submodule $\mathcal{N} \subset \mathcal{M}$, always assumed to be locally a direct summand of $\mathcal{M}$ which is locally free of finite rank as a right $\mathcal{A}$-module, with inclusion $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$,

$$\mathcal{A}^\perp = \ker(\mathcal{M} \xrightarrow{h} \mathcal{M}^* \xrightarrow{\iota^*} \mathcal{N}^*)$$

is a right $\mathcal{A}$-submodule of $\mathcal{M}$, the *orthogonal complement* of $\mathcal{N}$ in $(\mathcal{M}, h)$. A ε-hermitian space $(\mathcal{M}, h)$ is called *metabolic* if $\mathcal{M}$ contains a subbundle $\mathcal{N}$ which is locally free of finite rank as a right $\mathcal{A}$-module such that $\mathcal{N} = \mathcal{N}^\perp$, making the short exact sequence

$$0 \longrightarrow \mathcal{N} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\iota^* h} \mathcal{N}^* \longrightarrow 0$$

exact. Given a locally free right $\mathcal{A}$-module of finite rank $\mathcal{P}$,

$$H^\epsilon(\mathcal{P}) = (\mathcal{P} \oplus \mathcal{P}^*, \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix})$$

is a metabolic space, the *hyperbolic space* of $\mathcal{P}$. An ε-hermitian space $(\mathcal{M}, h)$ is *hyperbolic* if $(\mathcal{M}, h) \cong H^\epsilon(\mathcal{P})$ for a suitable $\mathcal{P}$. Two ε-hermitian spaces $(\mathcal{M}, h)$ and $(\mathcal{M}', h')$ over X are *Witt equivalent*, written as $(\mathcal{M}, h) \sim (\mathcal{M}', h')$ if there exist metabolic ε-hermitian spaces $(\mathcal{M}_1, h_1)$ and $(\mathcal{M}_2, h_2)$ such that

$$(\mathcal{M}, h) \perp (\mathcal{M}_1, h_1) \cong (\mathcal{M}', h') \perp (\mathcal{M}_2, h_2).$$

Witt-equivalence is an equivalence relation and the set of equivalence classes

$$W^\varepsilon(\mathcal{A}) = \{\ [(\mathcal{M}, h)] \,|\, (\mathcal{M}, h) \text{ an } \varepsilon\text{-hermitian space } \}$$

together with the addition canonically induced by the orthogonal sum is a group, the *Witt group of ε-hermitian spaces.*

1.2. Let Y be a scheme and $\tau : Y \to X$ a morphism of schemes. For a vector bundle $\mathcal{F}$ over X, $\tau^*\mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a vector bundle over Y, $\tau^*\mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is an algebra over Y with involution $\sigma \otimes 1$, and for every locally free right $\mathcal{A}$-module $\mathcal{M}$ of finite rank, $\tau^*\mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a locally free right $\tau^*\mathcal{A}$-module of finite rank. Given an ε-hermitian space $(\mathcal{M}, h)$ over $\mathcal{A}$, $\tau^*(\mathcal{M}, h) \cong (\tau^*\mathcal{M}, \tau^*h)$ is an $\tau^*\varepsilon$-hermitian space over $\tau^*\mathcal{A}$. τ induces a group homomorphism

$$U_\tau : W^\varepsilon(\mathcal{A}) \longrightarrow W^{\varepsilon'}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y), \quad (\mathcal{M}, h) \to (\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

where $\varepsilon' = \tau^*\varepsilon$. If $\pi : Z \to Y$ is another morphism of schemes then $U_{\tau\circ\pi} = U_\pi \circ U_\tau$.

1.3. Affine schemes. Let $X = \operatorname{Spec} R$ be an affine scheme. Under the usual categorical equivalence, vector bundles over X can be identified with finitely generated projective R-modules. For an algebra A over R with an R-linear involution σ, A always assumed to be finitely generated projective of constant rank as an R-module, $W^\varepsilon(A)$ canonically identifies with $W^\varepsilon(\widetilde{A})$, the Witt group of ε-hermitian forms over the $\mathcal{O}_X$-algebra $\widetilde{A}$, the sheaf of $\mathcal{O}_X$-algebras associated to A. Under this identification, the base change homomorphism

$$W^\varepsilon(\widetilde{A}) \longrightarrow W^\varepsilon(\widetilde{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

for a morphism $Y = \operatorname{Spec} R' \longrightarrow X = \operatorname{Spec} R$, corresponds to the base change

$$W^\varepsilon(A) \longrightarrow W^\varepsilon(A \otimes_R R')$$

from R to the R-algebra R'.

1.4. Brauer Severy varieties. Let k be a field. If B is a central simple algebra over k of $\dim_k B = n^2$, then $B \cong \operatorname{Mat}_s(D)$ for a central division algebra D over k. Let $r = \exp A$ be the order of B in the Brauer group $\operatorname{Br} k$. Let k'/k be a finite separable field extension which is a maximal subfield of D, so $[k' : k] = d$. Let X be the Brauer Severi variety associated with B and $X' = X \times_k k'$. Then $X' \cong \mathbb{P}^{n-1}_{k'}$. We know that $\operatorname{Pic} X \cong \mathbb{Z}$ and that there is an element $\mathcal{L}$ generating $\operatorname{Pic} X$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{O}_{X'}(r)$. $X \cong \mathbb{P}^{n-1}_k$ if and only if $r = 1$, if and only if X has a rational point [A-V]. In that case $\mathcal{L} = \mathcal{O}_X(1)$. We define $\mathcal{L}(0) = \mathcal{O}_X$, $\mathcal{L}(m) = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ (m-times) for $m > 0$ and $\mathcal{L}(m) = \mathcal{L}^\vee \otimes \cdots \otimes \mathcal{L}^\vee$ ($(-m)$-times) for $m < 0$, $m \in \mathbb{Z}$.

1.5. Some facts on vector bundles over proper schemes. Let X be a proper scheme over k and l/k an algebraic field extension. The Theorem of Krull-Schmidt holds for vector bundles over X, i.e., every vector bundle on X can be decomposed as a direct sum of indecomposable vector bundles, unique up to isomorphism and order of sumands [AEJ, p. 1324]. Moreover, non-isomorphic vector bundles on X extend to non-isomorphic vector bundles on $X_l = X \times_k l$, for every separable algebraic field extension l/k [AEJ, p. 1325].

Let l/k be a separable finite field extension of degree $s = [l : k]$. For a vector bundle $\mathcal{N}$ on X_l, the direct image $\pi_*\mathcal{N}$ of $\mathcal{N}$ under the projection morphism $\pi\colon X_l \to X$ is a vector bundle on X denoted by $tr_{l/k}(\mathcal{N})$ [AEJ, p. 1362 and p. 1329].

The canonical projection $\pi : X_l \to X$ is an affine flat morphism [AEJ, p. 1329] and the direct image $\mathcal{B} = \pi_* \mathcal{O}_{X_l}$ is an $\mathcal{O}_X$-algebra which is locally free of rank s as an $\mathcal{O}_X$-module, i.e.

$$tr_{l/k}(\mathcal{O}_{X_l}) = \pi_* \mathcal{O}_{X_l} \cong \mathcal{O}_X^s.$$

The assignment $\mathcal{F} \to \pi_* \mathcal{F}$ gives an equivalence of categories from quasi-coherent $\mathcal{O}_{X_l}$-modules to quasi-coherent $\mathcal{O}_X$-modules that are $\mathcal{B}$-modules at the same time [H, p. 145, Ex. 5.17]. This equivalence matches locally free $\mathcal{O}_{X_l}$-modules of finite rank with locally free $\mathcal{B}$-modules of finite rank and, in particular, $\mathrm{Pic}(X_l)$ with $\mathrm{Pic}(\mathcal{B})$.

2. Certain special cases

2.1. On the injectivity of U_τ. Let A be an algebra over k together with a k-linear involution σ. Let $n \geq 2$.

Theorem 1. *Let X be a k-scheme with a rational point. Then*

$$U_\tau : W^\varepsilon(A) \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_X)$$

is injective.

Proof. Pick a k-rational point in X, i.e. a k-morphism $\delta : \mathrm{Spec}\, k \to X$. Then $\tau\delta = \mathrm{id}$ on $\mathrm{Spec}\, k$, hence $U_\delta U_\tau = \mathrm{id}$ on $W^\varepsilon(A \otimes_k \mathcal{O}_X)$, implying that U_τ is injective. □

A similar trick as used in [Pu1] gives us the next result:

Theorem 2. *Let X be a Brauer Severi variety associated to a central simple algebra of odd index. Then*

$$U_\tau : W^\varepsilon(A) \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_X)$$

is injective.

Proof. Let $B \cong \mathrm{Mat}_s(D)$ be the central simple algebra associated to X and k'/k be a finite separable field extension which is a maximal subfield of the division algebra D, hence of odd degree. Define $X' = X \times_k k'$. Let (M_1, h_1) and (M_2, h_2) be two ϵ-hermitian spaces over A such that

$$(M_1, h_1) \otimes_A (A \otimes_k \mathcal{O}_X) \sim (M_2, h_2) \otimes_A (A \otimes_k \mathcal{O}_X).$$

Then

$$(M_1, h_1) \otimes_A (A \otimes_k \mathcal{O}_{X'}) \sim (M_2, h_2) \otimes_A (A \otimes_k \mathcal{O}_{X'})$$

which implies

$$(M_1, h_1) \otimes_A (A \otimes_k k') \sim (M_2, h_2) \otimes_A (A \otimes_k k')$$

by Theorem 1. The assertion now follows from [K, (10.3.1), p. 62]. □

Theorem 3. *Let X be a Brauer Severi variety associated to a central simple algebra of odd index. Let A be a division algebra over k and suppose* $\mathrm{char}\, k \neq 2$. *Let (M_1, h_1) and (M_2, h_2) be two ϵ-hermitian spaces over A which become isometric over $A \otimes_k \mathcal{O}_X$. Then*

$$(M_1, h_1) \cong (M_2, h_2).$$

Proof. Since

$$(M_1, h_1) \otimes_A (A \otimes_k \mathcal{O}_X) \cong (M_2, h_2) \otimes_A (A \otimes_k \mathcal{O}_X)$$

we have $(M_1, h_1) \sim (M_2, h_2)$ by Theorem 1. By [K, (10.3.3), p. 63], this yields $(M_1, h_1) \cong (M_2, h_2)$. □

2.2. On the surjectivity of U_τ. Let A be an algebra over k (e.g. quadratic étale or central simple) together with a k-linear involution σ. Let X be a scheme over k and let k'/k be a separable odd degree field extension. Let $X' = X \times_k k'$ and $A' = A \otimes_k k'$. Observe that $A' \otimes_{k'} \mathcal{O}_{X'} \cong A \otimes_k \mathcal{O}_{X'}$.

Theorem 4. *If*

$$U_\tau : W^\varepsilon(A') \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_{X'})$$

is surjective, then

$$U_\tau : W^\varepsilon(A) \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_X)$$

is surjective.

Proof. Let $tr_{k'/k} : k' \to k$ be the trace of the extension k'/k. Its A-linear extension $\mathrm{id} \otimes tr_{k'/k} : A \otimes_k k' \to A$ is an involution trace form in the sense of [K, (7.3.2), p. 41]. Both maps induce group homomorphisms $tr_{k'/k} : W(k') \to W(k)$, $tr_{k'/k} : W(X') \to W(X)$ and $T : W^\varepsilon(A \otimes_k k') \to W^\varepsilon(A)$, $T : W^\varepsilon(A \otimes_k \mathcal{O}_{X'}) \to W^\varepsilon(A \otimes_k \mathcal{O}_X)$. As in [K, p. 62], we can show that

$$T(U_\tau(\mathcal{M}, h) \otimes (\mathcal{F}, \gamma)) \sim (\mathcal{M}, h) \otimes tr_{k'/k}(\mathcal{F}, \gamma)$$

or, equivalently,

$$T(((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})) \otimes (\mathcal{F}, \gamma)) \sim (\mathcal{M}, h) \otimes tr_{k'/k}(\mathcal{F}, \gamma)$$

for all ε-hermitian spaces $(\mathcal{M}, h)$ over $\mathcal{A} = A \otimes_k \mathcal{O}_X$ and symmetric bilinear spaces $(\mathcal{F}, \gamma)$ over X'. Analogously,

$$T(((M, h) \otimes_A (A \otimes_k k')) \otimes (F, \gamma)) \sim (M, h) \otimes tr_{k'/k}(F, \gamma)$$

for all ε-hermitian spaces (M, h) over A and nonsingular symmetric bilinear spaces (F, γ) over k'. Since $[k' : k]$ is odd, we get

$$tr_{k'/k}(\langle 1 \rangle_{\mathcal{O}_{X'}}) \sim \langle 1 \rangle_{\mathcal{O}_X},$$

$$T((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})) \sim (\mathcal{M}, h) \text{ and } T((M, h) \otimes_A (A \otimes_k k')) \sim (M, h)$$

as in [K, p. 62]. For an ε-hermitian space $(\mathcal{M}, h)$ over $\mathcal{A}$ it follows that

$$(\mathcal{M}, h) \cong (\mathcal{M}, h) \otimes \langle 1 \rangle_{\mathcal{O}_X} \sim (\mathcal{M}, h) \otimes tr_{k'/k}(\langle 1 \rangle_{\mathcal{O}_{X'}})$$

$$\sim T(((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})) \otimes \langle 1 \rangle_{\mathcal{O}_{X'}})$$

$$\sim T((M', h') \otimes_{A'} (A' \otimes_{k'} \mathcal{O}_{X'})) \sim T(M', h') \otimes_A (A \otimes \mathcal{O}_X),$$

for a suitable hermitian space (M', h') over A', where the second last equivalence holds by the assumption that $U_\tau : W^\varepsilon(A') \to W^\varepsilon(A \otimes_k \mathcal{O}_{X'})$ is surjective. □

Corollary 5. *Let X be a Brauer Severi variety of odd index. Let $\mathrm{Mat}_s(D)$ be the central simple algebra associated to X. Let k'/k be a finite separable field extension which is a maximal subfield of D, such that*

$$U_\tau : W^\varepsilon(A') \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_{X'})$$

is surjective ($X' \cong \mathbb{P}_k^{n-1}$), then

$$U_\tau : W^\varepsilon(A) \longrightarrow W^\varepsilon(A \otimes_k \mathcal{O}_X)$$

is surjective.

3. Extension theorem for hermitian spaces

Let X be a scheme such that $2 \in H^0(X, \mathcal{O}_X^\times)$ and $\mathcal{A}$ an algebra over X with an $\mathcal{O}_X$-linear involution σ. An ϵ-hermitian space $(\mathcal{M}, h)$ with $\epsilon = 1$ is called a *hermitian space*. For a hermitian space $(\mathcal{M}, h)$, a subbundle $\mathcal{N} \subset \mathcal{M}$ is called *totally isotropic* if $\mathcal{N} \subset \mathcal{N}^\perp$. For a totally isotropic subbundle $\mathcal{N} \subset \mathcal{M}$, we obtain an induced hermitian space $(\overline{\mathcal{M}}, \overline{h})$ by setting $\overline{\mathcal{M}} = \mathcal{N}^\perp/\mathcal{N}$ and writing $\iota : \mathcal{N}^\perp \hookrightarrow \mathcal{M}$ for the inclusion, $\pi : \mathcal{N}^\perp \to \mathcal{M}$ for the projection. Then $\overline{h}$ is uniquely determined by $\iota^* \circ h \circ \iota = \pi^* \circ \overline{h} \circ \pi$. We get a short exact sequence

$$0 \longrightarrow \mathcal{N}^\perp \xrightarrow{\ \kappa\ } \overline{\mathcal{M}} \oplus \mathcal{M} \xrightarrow{(\kappa^*, \overline{h} \oplus -h)} \mathcal{N}^{\perp\,*} \longrightarrow 0$$

with $\kappa = (\pi, \mathrm{id})$ implying that $(\overline{\mathcal{M}}, \overline{h}) \perp (\mathcal{M}, -h)$ is metabolic. Since $(\mathcal{M}, h) \perp (\mathcal{M}, -h)$ is metabolic as well, $(\mathcal{M}, h)$ and $(\overline{\mathcal{M}}, \overline{h})$ are Witt equivalent. We get a short exact sequence of locally free right $\mathcal{A}$-modules of constant finite rank

$$0 \longrightarrow \mathcal{N} \xrightarrow{\ \iota\ } \mathcal{N}^\perp \xrightarrow{\ \pi\ } \overline{\mathcal{M}} \longrightarrow 0\cdot$$

Analogously as observed in [Pu2, 4], we can reverse this construction as follows:

For a locally free right $\mathcal{A}$-module $\mathcal{M}$ of constant finite rank, let $\{\mathrm{Ext}^i(\mathcal{M}, \cdot)\}$ be the right derived functor of the group of $\mathcal{A}$-module homomorphisms $\mathrm{Hom}_\mathcal{A}(\mathcal{M}, \cdot)$, which is a universal contravariant δ-functor from locally free right $\mathcal{A}$-modules of constant finite rank to abelian groups.

Theorem 6. *Let $(\mathcal{G}, b)$ be a hermitian space over $\mathcal{A}$ and*

$$(1) \qquad 0 \longrightarrow \mathcal{N} \xrightarrow{\ \iota\ } \mathcal{B} \xrightarrow{\ \pi\ } \mathcal{G} \longrightarrow 0$$

a short exact sequence of locally free right $\mathcal{A}$-modules of constant finite rank. Suppose that

$$\mathrm{Ext}^1(\mathcal{N}^*, \mathcal{N}) = \mathrm{Ext}^2(\mathcal{N}^*, \mathcal{N}) = 0.$$

Then there exists a hermitian space $(\mathcal{M}, h)$ and identifications of $\mathcal{N}$, $\mathcal{B}$ in $\mathcal{M}$ such that $\mathcal{B} = \mathcal{N}^\perp$ in $(\mathcal{M}, h)$ and $(\mathcal{G}, b) \cong (\overline{M}, \overline{h})$. In particular, $(\mathcal{G}, b)$ and $(\mathcal{M}, h)$ are Witt equivalent.

Moreover, for $(\mathcal{M}, h)$ as in Theorem 6, we have $\mathcal{B}^\perp = \mathcal{N}$, hence the sequence

$$(2) \qquad 0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \xrightarrow{\ \iota^* h\ } \mathcal{B}^* \longrightarrow 0$$

is exact, with ι being the inclusion $\mathcal{B} \hookrightarrow \mathcal{M}$.

For the proof of this result, we need the following elementary results which we state here for the convenience of the reader:

Lemma 7. *(a) Let (P) and (Q) be two extensions of of locally free right $\mathcal{A}$-modules of constant finite rank such that*

$$\begin{array}{ccccccccc} (P) & 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ (Q) & 0 & \longrightarrow & \mathcal{M}' & \xrightarrow{p^*\iota} & \mathcal{M}_1 & \xrightarrow{\kappa^*} & \mathcal{M}_2'' & \longrightarrow & 0 \end{array}$$

with $\alpha'' : \mathcal{M}'' \longrightarrow \mathcal{M}_1''$ an $\mathcal{A}$-linear map. If $\xi \in \operatorname{Ext}^1(\mathcal{M}'', \mathcal{M}')$ (resp. $\xi_1 \in \operatorname{Ext}^1(\mathcal{M}_1'', \mathcal{M}')$) corresponds to the extension (P) (resp. (Q)), then the following statements are equivalent:
(i) There exists an $\mathcal{A}$-linear map $\alpha : \mathcal{M} \longrightarrow \mathcal{M}_1$ making the above diagram commutative.
(ii) $\operatorname{Ext}^1(\alpha'', \mathcal{M}')\xi_1 = \xi$.
(b) Let

$$\begin{array}{ccccccccc} (P) & 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M} & \xrightarrow{\pi} & \mathcal{M}'' & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ (Q) & 0 & \xrightarrow{\iota_1} & \mathcal{M}' & \xrightarrow{p^*\iota} & \mathcal{M}_1 & \xrightarrow{\kappa^*} & \mathcal{M}_2'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of of locally free right $\mathcal{A}$-modules of constant finite rank with exact rows. Then an $\mathcal{A}$-linear map $\beta : \mathcal{M} \longrightarrow \mathcal{M}_1$ makes the diagram commutative as well if and only if there exists an $\mathcal{A}$-linear map $\gamma : \mathcal{M}'' \longrightarrow \mathcal{M}_1$ such that

$$\beta = \alpha + \iota_1 \gamma \pi.$$

In this case γ is unique.

The proof of Theorem 6 is now analogous to the proof in [A], Erster Schritt:

Proof. We dualize (1) and replace $\mathcal{G}^*$ by $\mathcal{G}$ via b. This yields the short exact sequence

$$(3) \qquad 0 \longrightarrow \mathcal{G} \xrightarrow{\pi^* b} \mathcal{B}^* \xrightarrow{\iota^*} \mathcal{N}^* \longrightarrow 0.$$

By applying $\{\operatorname{Ext}^i(\cdot, \mathcal{N})\}$ to (3) we obtain a long exact sequence. In particular,

$$(4) \qquad 0 = \operatorname{Ext}^1(\mathcal{N}^*, \mathcal{N}) \longrightarrow \operatorname{Ext}^1(\mathcal{B}^*, \mathcal{N}) \xrightarrow{\operatorname{Ext}^1(\pi^* b, \mathcal{N})} \operatorname{Ext}^1(\mathcal{G}, \mathcal{N}) \longrightarrow \operatorname{Ext}^2(\mathcal{N}^*, \mathcal{N}) = 0.$$

Therefore

$$\operatorname{Ext}^1(\pi^* b, \mathcal{N}) : \operatorname{Ext}^1(\mathcal{B}^*, \mathcal{N}) \longrightarrow \operatorname{Ext}^1(\mathcal{G}, \mathcal{N})$$

is an isomorphism. Now let $\xi \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{N})$ correspond to the isomorphism class of extension (1). Then we thus find a unique $\xi_1 \in \operatorname{Ext}^1(\mathcal{B}^*, \mathcal{N})$ such that $\operatorname{Ext}^1(\pi^* b, \mathcal{N})(\xi_1) = \xi$. This yields an extension of locally free right $\mathcal{A}$-modules of constant finite rank

$$(5) \qquad 0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \xrightarrow{p} \mathcal{B}^* \longrightarrow 0$$

over X, see (2). Using (1) and (5) we obtain the following commutative diagram (Lemma 7 (a)):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\iota} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{G} & \longrightarrow & 0 \\
 & & {\scriptstyle id}\downarrow & & {\scriptstyle \kappa}\downarrow & & {\scriptstyle \pi^* b}\downarrow & & \\
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{M} & \xrightarrow{p} & \mathcal{B}^* & \longrightarrow & 0 \\
 & & & & {\scriptstyle \iota^* p}\downarrow & & {\scriptstyle \iota^*}\downarrow & & \\
 & & & & \mathcal{N}^* & \xrightarrow{id} & \mathcal{N}^* & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{6}
$$

Diagram chasing confirms that the middle column of the above diagram is also exact. Using that $b = b^*$ we dualize and obtain

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{G}^* & \xrightarrow{\pi^*} & \mathcal{B}^* & \xrightarrow{\iota^*} & \mathcal{N}^* & \longrightarrow & 0 \\
 & & {\scriptstyle b\pi}\uparrow & & {\scriptstyle \kappa^*}\uparrow & & {\scriptstyle id}\uparrow & & \\
0 & \longrightarrow & \mathcal{B} & \xrightarrow{p^*} & \mathcal{M}^* & \longrightarrow & \mathcal{N}^* & \longrightarrow & 0 \\
 & & {\scriptstyle \iota}\uparrow & & {\scriptstyle p^*\iota}\uparrow & & & & \\
 & & \mathcal{N} & \xrightarrow{id} & \mathcal{N} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By replacing $\mathcal{G}^*$ with $\mathcal{G}$ via b we replace $b\pi$ by π and π^* by $\pi^* b$. We obtain

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\iota} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{G} & \longrightarrow & 0 \\
 & & {\scriptstyle id}\downarrow & & {\scriptstyle p^*}\downarrow & & {\scriptstyle \pi^* b}\downarrow & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{p^*\iota} & \mathcal{M}^* & \xrightarrow{\kappa^*} & \mathcal{B}^* & \longrightarrow & 0 \\
 & & & & {\scriptstyle \iota^* p}\downarrow & & {\scriptstyle \iota^*}\downarrow & & \\
 & & & & \mathcal{N}^* & \xrightarrow{id} & \mathcal{N}^* & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & 0 & &
\end{array}
\tag{7}
$$

Let $\xi_1^* \in \mathrm{Ext}^1(\mathcal{B}^*, \mathcal{N})$ correspond to the extension

$$(8) \qquad 0 \longrightarrow \mathcal{N} \xrightarrow{p^*\iota} \mathcal{M}^* \xrightarrow{\kappa^*} \mathcal{B}^* \longrightarrow \cdot$$

Then $\mathrm{Ext}^1(\pi^*b, \mathcal{N})\xi_1^* = \xi$ by Lemma 7 (a), thus $\xi_1^* = \xi$ and the extensions (5) and (8) are isomorphic. (This step does not generalize to ϵ-hermitian forms with $\epsilon \neq 1$.) Therefore there exists an $\mathcal{A}$-linear map $h : \mathcal{M} \longrightarrow \mathcal{M}^*$ which makes the following diagram commutative:

$$(9) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \xrightarrow{\kappa\iota} & \mathcal{M} & \xrightarrow{p} & \mathcal{B}^* & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle h}\downarrow & & \mathrm{id}\downarrow & & \\ 0 & \longrightarrow & \mathcal{N} & \xrightarrow{p^*\iota} & \mathcal{M}^* & \xrightarrow{\kappa^*} & \mathcal{B}^* & \longrightarrow & 0 \end{array}$$

h is an isomorphism and by Lemma 7 (b) unique up to summands of the form $p^*\iota\beta p$ with $\beta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \xrightarrow{\iota} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{G} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle p^*, h\kappa}\downarrow & & {\scriptstyle \pi^*b}\downarrow & & \\ 0 & \longrightarrow & \mathcal{N} & \xrightarrow{p^*\iota} & \mathcal{M}^* & \xrightarrow{\kappa^*} & \mathcal{B}^* & \longrightarrow & 0 \end{array}$$

is made commutative by both maps written next to the arrow in the middle, since we have $h\kappa\iota = p^*\iota$ by (9) and $\kappa^*p^* = \pi^*b\pi = p\kappa = \kappa^*h\kappa$ by (7), (6) and (9). Lemma 7 (b) implies that there exists a $\gamma \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{N})$ such that $h\kappa = p^* + p^*\iota\gamma\pi$. Since $\mathrm{Ext}^1(\mathcal{N}^*, \mathcal{N}) = 0$, (3) induces the exact sequence

$$(10) \qquad \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N}) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\iota^*, \mathcal{N})} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N}) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\pi^*b, \mathcal{N})} \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{N}) \longrightarrow 0\cdot$$

Therefore $\gamma = \beta\pi^*b$ for some $\beta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$ which yields

$$h\kappa = p^* + p^*\iota\beta\pi^*b\pi = p^* + p^*\iota\beta p\kappa$$

and so $(h - p^*\iota\beta p)\kappa = p^*$. Since h is unique up to certain summands, see above, we may assume that

$$(11) \qquad h\kappa = p^*.$$

Moreover, h is uniquely determined by this equation together with (9), up to summands of the form $p^*\iota\beta p$ with $\beta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$ such that $p^*\iota\beta p\kappa = 0$. We also have $p^*\iota\beta p\kappa = 0$ if and only if $p^*\iota\beta\pi^*b\pi = 0$ by (6), if and only if $\beta\pi^*b = 0$ ($p^*\iota$ is injective, π surjective), if and only if $\beta = \alpha\iota^*$ by (10). Therefore h is uniquely determined up to summands of the form $p^*\iota\alpha\iota^*p$ with $\alpha \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$.

Now $h^* : \mathcal{M} \longrightarrow \mathcal{M}^*$ satisfies $h^*\kappa = (\kappa^*h)^* = p^*$ by (9), hence (11), and $h^*\kappa\iota = p^*\iota$, $\kappa^*\mathcal{H}^* = (h\kappa)^* = p$ by (11), hence (9). Therefore the fact that h is uniquely determined up to summands of the form $p^*\iota\alpha\iota^*p$ with $\alpha \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$ even yields a unique $\alpha \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$ satisfying $h^* = h + p^*\iota\alpha\iota^*p$ and dualizing implies that $\alpha = -\alpha^*$. Replacing h by $h + \frac{1}{2}p^*\iota\alpha\iota^*p$ if necessary, we may assume in addition that

$$h = h^*.$$

We have thus obtained a hermitian space $(\mathcal{M}, h)$ containing $\mathcal{N}$ as a subbundle via $\kappa\iota$ by (6), such that

$$\mathcal{N}^{\perp} = \ker(\iota^*\kappa^* h) = \ker(\iota^*(h\kappa)^*) = \ker(\iota^* p) = \mathrm{im}(\kappa)$$

and $\mathcal{N} = \mathrm{im}(\kappa\iota) \subset \mathrm{im}(\kappa)$. We conclude that $\mathcal{N}$ is totally isotropic and that $\mathcal{B}$, viewed as a subbundle of $\mathcal{M}$ via κ, can be identified with $\mathcal{N}^{\perp}$. Under these identifications, the diagram

$$\begin{array}{ccccccc} \mathcal{B} & \xrightarrow{\kappa} & \mathcal{M} & \xrightarrow{h} & \mathcal{M}^* & \xrightarrow{\kappa^*} & \mathcal{B}^* \\ {\scriptstyle\pi}\downarrow & & & & & & \downarrow{\scriptstyle\pi^*} \\ \mathcal{G} & & & \xrightarrow{b} & & & \mathcal{G}^* \end{array}$$

corresponds to the equation displayed in the first paragraph of 3.1, it commutes because of (7). Hence $(\mathcal{G}, b)$ and $(\overline{\mathcal{M}}, \overline{h})$ are isometric, as claimed. The last assertion now follows easily. □

Note that our assumption that $2 \in H^0(X, \mathcal{O}_X^{\times})$ is needed in the proof and cannot be omitted.

4. A generalization of Horrock's Theorem

Let k be a field and D be a division algebra over k. Let $X = \mathbb{P}_k^n$ and $\tau : \mathbb{P}_k^n \to \operatorname{Spec} k$ be the structure morphism. Let $\mathcal{D} = \tau^* D \cong D \otimes_k \mathcal{O}_X$.

Given a locally free right $\mathcal{D}$-module $\mathcal{E}$, let $\mathcal{E}(m) = \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{E}$ for any integer m. For a locally free right $\mathcal{D}$-module $\mathcal{E}$ define

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{E}(*)) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}^i(\mathcal{D}, \mathcal{E}(j))$$

for integers $j \geq 0$.

We generalize Horrock's Theorem [B-H, Sect. 5, Lemma 1] which was an important ingredient in the proofs of [A] and [Pu2]:

Theorem 8. *A locally free right $\mathcal{D}$-module $\mathcal{E}$ satisfies*

$$\mathcal{E} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$$

if and only if

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{E}(*)) = 0 \quad (i \in \mathbb{Z}, 0 < i < n). \tag{12}$$

Proof. By the cohomology of projective space, the condition (12) is necessary.

We prove that it is sufficient by induction on n. For $n = 1$ every locally free right $\mathcal{D}$-module $\mathcal{E}$ is of the form

$$\mathcal{E} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$$

[K, p. 407, VII.(3.1.1)], so there is nothing to prove. So suppose $n > 1$ and assume that the assertion holds for $n-1$ in place of n. $Z = \mathbb{P}_k^{n-1}$ is a closed subscheme of X with inclusion $i : Z \hookrightarrow X$. Via identification with the hyperplane $x_n = 0$, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0.$$

Let $\mathcal{E}$ be a locally free right $\mathcal{D}$-module satisfying (12), then tensor the above sequence with $\mathcal{E}(j)$ to obtain

$$0 \longrightarrow \mathcal{E}(j-1) \longrightarrow \mathcal{E}(j) \longrightarrow i_*[(\mathcal{E}|_Z)(j)] \longrightarrow 0. \tag{13}$$

By (12) this yields

$$\mathrm{Ext}^i(\mathcal{D}, (\mathcal{E}|_Z)(*)) = 0 \text{ for } 0 < i < n-1,$$

so, by the induction hypothesis, $\mathcal{E}|_Z$ is a direct sum of locally free right $\mathcal{D}|_Z$-modules of rank one of the kind $\mathcal{D}|_Z(m)$: there are integers $s_1, \dots, s_t$, $\mathcal{F} = \mathcal{D}(s_1) \oplus \dots \mathcal{D}(s_t)$ and an isomorphism

$$\Psi : \mathcal{F}|_Z \longrightarrow \mathcal{E}|_Z$$

of locally free right $\mathcal{D}|_Z$-modules. Put $j = 0$ in (13) then

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{E} \longrightarrow i_*(\mathcal{E}|_Z) \longrightarrow 0.$$

is a short exact sequence of locally free right $\mathcal{D}$-modules, where $\mathcal{E} \longrightarrow i_*(\mathcal{E}|_Z)$ is the canonical restriction map. Applying $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \cdot)$ to this yields the exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{E}) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, i_*(\mathcal{E}|_Z)) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}|_Z, \mathcal{E}|_Z) \longrightarrow \mathrm{Ext}^1(\mathcal{F}, \mathcal{E}(-1))$$

and since we assume that $\mathcal{E}$ satisfies (12),

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{E}(-1)) \cong \bigoplus_{j=1}^{t} \mathrm{Ext}^1(\mathcal{D}(s_j), \mathcal{E}(-1)) \cong \bigoplus_{j=1}^{t} \mathrm{Ext}^1(\mathcal{D}, \mathcal{E}(-s_j - 1)) = 0.$$

Therefore the natural map $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{E}) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}|_Z, \mathcal{E}|_Z)$ is surjective, so that Ψ extends to a $\mathcal{D}$-linear homomorphism $\varphi : \mathcal{F} \to \mathcal{E}$. Now view φ as an $\mathcal{O}_X$-linear map between vector bundles $\mathcal{F}$ and $\mathcal{E}$ over X: then

$$\det \varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\det \mathcal{F}, \det \mathcal{E}) \cong H^0(X, (\det \mathcal{F})^\vee \otimes (\det \mathcal{E})) \cong H^0(X, \mathcal{O}_X(m))$$

for some integer m. Restricting this to Z shows that $m = 0$ and thus $\det \varphi \in k^\times$. Hence φ is an isomorphism. $\square$

5. Killing extension groups for $X = \mathbb{P}_k^{n-1}$

The proof of surjectivity of the base change morphism $\tau^* : W(k) \to W(X)$ between the Witt rings of k and a Brauer Severi variety X in [A], [Pu 2, 3], used the killing of cohomology groups. In our setup, this corresponds to the following observations we phrase in terms of extension groups. We phrase the proofs in a general setting in order to see if and where they could be used in a more general setup.

Let k be a field of characteristic not 2 and D be a division algebra over k. Let $X = \mathbb{P}_k^{n-1}$, $\mathcal{D} = \tau^* D \cong D \otimes_k \mathcal{O}_X$ and $\mathcal{F}(m) = \mathcal{O}_X(m) \otimes \mathcal{F}$ for any integer m and any locally free right $\mathcal{D}$-module $\mathcal{F}$.

Every right D-module W may be viewed as a right module over $\mathrm{Spec}\, D$, so for a right $\mathcal{D}$-module $\mathcal{F}$, the notation $\mathcal{F} \otimes_D W = \mathcal{F} \otimes_{\mathcal{O}_{\mathrm{Spec}\, D}} W$ used in the following makes sense and is a $\mathcal{D}$-module.

Let $\Omega = \Omega_{X/k}$ be the sheaf of relative differentials of X over k and $\Omega^l = \Lambda^l \Omega$ the sheaf of l-forms over k. Define $\Omega_D^l = \Omega^l \otimes_{\mathcal{O}_X} \mathcal{D}$.

5.1. Let X be a scheme and $\mathcal{A}$ an algebra over X. Let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be an $\mathcal{A}$-linear map of right $\mathcal{A}$-modules. For any right $\mathcal{A}$-module $\mathcal{G}$ and $i \in \mathbb{N}_0$ we get an induced homomorphism

$$\mathrm{Ext}^i(\alpha, \mathcal{G}) : \mathrm{Ext}^i(\mathcal{F}_2, \mathcal{G}) \longrightarrow \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{G})$$

and

$$\{\mathrm{Ext}^i(\alpha, \cdot)\} : \{\mathrm{Ext}^i(\mathcal{F}_2, \cdot)\} \longrightarrow \{\mathrm{Ext}^i(\mathcal{F}_1, \cdot)\}$$

is a homomorphism of δ-functors.

Lemma 9. *Assume that D is a field extension of k. Let $l \in \mathbb{Z}$ with $0 \leq l < n-2$ and $\mathcal{F}$ a locally free $\mathcal{D}$-module. Then there exists a finite dimensional D-vector space W as well as an extension*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{P} \longrightarrow (\Omega^r \otimes_k D) \otimes_D W \longrightarrow 0.$$

of locally free $\mathcal{D}$-modules such that the connecting homomorphism

$$\delta : \mathrm{Hom}_{\mathcal{D}}(\Omega^l_D, \Omega^l_D \otimes_D W) \longrightarrow \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})$$

is an isomorphism.

Proof. Let W be an arbitrary free D-vector space of finite dimension. Multiplication by $x \in W$ yields a $\mathcal{D}$-linear map

$$\tau_x : \Omega^l_D \longrightarrow \Omega^l_D \otimes_D W, \quad s \to s \otimes x.$$

For a locally free $\mathcal{D}$-module $\mathcal{F}$, the map

$$\theta = \theta_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{D}}(\Omega^l_D \otimes_D W, \mathcal{F}) \longrightarrow \mathrm{Hom}_D(W, \mathrm{Hom}(\Omega^l_D, \mathcal{F}))$$

defined by

$$[\theta(\varphi)](x) = \varphi \circ \tau_x$$

for $\varphi \in \mathrm{Hom}_{\mathcal{D}}(\Omega^l_D \otimes_D W, \mathcal{F})$, $x \in W$ is an isomorphism with inverse satisfying

$$[\theta^{-1}(\Psi)](s \otimes x) = \Psi(x)s$$

for $\Psi \in \mathrm{Hom}_{\mathcal{D}}(W, \mathrm{Hom}(\Omega^l_D, \mathcal{F}))$, $s \in \Omega^l_D$, $x \in W$. θ is functorial in $\mathcal{F}$ and since the functor $\mathrm{Hom}_D(W, \cdot)$ is exact for a fixed D-vector space W, we get an induced isomorphism

$$\{\theta^i\} : \{\mathrm{Ext}^i(\Omega^l_D \otimes_D W, \cdot)\} \longrightarrow \{\mathrm{Hom}_{\mathcal{D}}(W, \mathrm{Ext}^i(\Omega^l_D, \cdot))\}$$

of universal δ-functors.

Now

$$\{\mathrm{Ext}^i(\tau_x, \cdot)\} : \{\mathrm{Ext}^i(\Omega^l_D \otimes_D W, \cdot)\} \longrightarrow \{\mathrm{Ext}^i(\Omega^l_D, \cdot)\}$$

is a homomorphism of δ-functors by 5.1. Let V be a finite dimensional D-vector space, then the evaluation map

$$\epsilon_x = \epsilon_{x,Z} : \mathrm{Hom}_D(W, V) \longrightarrow V, \quad \alpha \to \alpha(x)$$

is functorial in V. We obtain a diagram of δ-functors

$$\begin{array}{ccc} \{\mathrm{Ext}^i(\Omega^l_D \otimes_D W, \cdot)\} & \xrightarrow{\{\theta^i\}} & \{\mathrm{Hom}_{\mathcal{D}}(W, \mathrm{Ext}^i(\Omega^l_D, \cdot))\} \\ {\scriptstyle \{\mathrm{Ext}^i(\tau_x,\cdot)\}}\downarrow & & \downarrow{\scriptstyle \{\epsilon_{x,\mathrm{Ext}^i(\Omega^l_D,\cdot)}\}} \\ \{\mathrm{Ext}^i(\Omega^l_D, \cdot)\} = & & = \{\mathrm{Ext}^i(\Omega^l_D, \cdot)\} \end{array}$$

which commutes since it commutes in degree zero:

$$\epsilon_{x,\mathrm{Hom}(\Omega^l_D,\mathcal{F})}\theta(\varphi) = [\theta(\varphi)](x) = \varphi \circ \tau_x = \mathrm{Hom}(\tau_x, \mathcal{F})(\varphi).$$

This implies that

$$(A) \qquad [\theta^i(\zeta)](x) = \epsilon_x \theta^i(\zeta) = \mathrm{Ext}^i(\tau_x, \epsilon)(\zeta)$$

for all $i \in \mathbb{N}_0$, $\zeta \in \mathrm{Ext}^i(\Omega^l_D \otimes_D W, \mathcal{F})$ and $x \in W$.

Let $\lambda : W \longrightarrow \mathrm{Ext}^i(\Omega^l_D, \mathcal{F})$ be a $\mathcal{D}$-linear map, put $\zeta = (\theta^1)^{-1}(\lambda) \in \mathrm{Ext}^1(\Omega^l_D \otimes_D W, \mathcal{F})$ and write

$$(B) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{P} \longrightarrow (\Omega^l_D) \otimes_D W \longrightarrow 0.$$

for the extension of $(\Omega^l_D) \otimes_D W$ by $\mathcal{F}$ corresponding to ζ [H, III, Ex. 6.1]. Use the canonical maps

$$\mu : \mathrm{Hom}(\Omega^l_D, \Omega^l_D) \otimes_D W \longrightarrow \mathrm{Hom}(\Omega^l_D, \Omega^l_D \otimes_D W), \varphi \otimes x \to \tau_x \circ \varphi$$

and

$$\kappa : W \longrightarrow \mathrm{Hom}(\Omega^l_D, \Omega^l_D) \otimes_D W, x \to id \otimes x,$$

the diagram

$$\begin{array}{ccc} W & \xrightarrow{\lambda} & \mathrm{Ext}^1(\Omega^l_D, \mathcal{F}) \\ {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle \delta} \\ \mathrm{Hom}(\Omega^l_D, \Omega^l_D) \otimes_D W & \xrightarrow{\mu} & \mathrm{Hom}(\Omega^l_D, \Omega^l_D \otimes_D W) \end{array}$$

commutes, where δ is the connecting homomorphism arising from (B). Indeed, given $x \in W$,

$$\{\mathrm{Ext}^i(\tau_x, \cdot)\} : \{\mathrm{Ext}^i(\Omega^l_D \otimes_D W, \cdot)\} \longrightarrow \{\mathrm{Ext}^i(\Omega^l_D, \cdot)\}$$

is a homomorphism of δ-functors by 5.1. Thus the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\Omega^l_D \otimes_D W, \Omega^l_D \otimes_D W) & \xrightarrow{\delta} & \mathrm{Ext}^1(\Omega^l_D \otimes_D W, \mathcal{F}) \\ {\scriptstyle \mathrm{Hom}(\tau_x,\Omega^l_D\otimes_D W)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^1(\tau_x,\mathcal{F})} \\ \mathrm{Hom}(\Omega^l_D, \Omega^l_D \otimes_D W) & \xrightarrow{\delta} & \mathrm{Ext}^1(\Omega^l_D, \mathcal{F}) \end{array}$$

commutes, yielding

$$\delta\mu\kappa(x) = \delta\mu(id_{\Omega^l_D} \otimes x) = \delta(\tau_x) = \delta\mathrm{Hom}(\tau_x, \Omega^l_D \otimes_D W) =$$

$$\mathrm{Ext}^1(\tau_x, \mathcal{F})\, \delta\, (id_{\Omega^l_D \otimes W}) = \mathrm{Ext}^1(\tau_x, \mathcal{F})(\zeta) = [\theta^1(\zeta)](x) = \lambda(x)$$

by (A). κ is an isomorphism since $\mathrm{Hom}(\Omega^l, \Omega^l) \cong k$ (see for instance [Pu, 3.3(b)]), thus $\mathrm{Hom}(\Omega^l_D, \Omega^l_D) \cong D$. μ is functorial in W and commutes with direct sums. Hence it is an isomorphism as well since it obviously is so for $V = D$. Taking $W = \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})$ and $\lambda = id_W$, the map δ behaves a desired. □

Note that in this last step of the proof, we need $W = \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})$ to be a free D-module which is guaranteed if D is a field extension. It is not clear how to generalize this proof if D is not a field extension, even if we assume that every finitely generated projective right D-module is free. The result is needed to prove both Lemma 10 and Proposition 11.

5.2. In the ensuing lemma we will use the following property of Ext-functors: Let

$$0 \longrightarrow \mathcal{M}_j \longrightarrow \mathcal{F}_j \longrightarrow \mathcal{N}_j \longrightarrow 0$$

for $j = 1, 2$ be two short exact sequences of locally free right $\mathcal{D}$-modules. Then the diagram

$$\begin{array}{ccc} \mathrm{Ext}^i(\mathcal{M}_2, \mathcal{N}_1) & \stackrel{\delta}{\longrightarrow} & \mathrm{Ext}^{i+1}(\mathcal{M}_2, \mathcal{M}_1) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle -\delta} \\ \mathrm{Ext}^{i+1}(\mathcal{N}_2, \mathcal{N}_1) & \stackrel{\delta}{\longrightarrow} & \mathrm{Ext}^{i+2}(\mathcal{N}_2, \mathcal{M}_1) \end{array}$$

commutes for all $i \geq 0$ ([H-S, IV.9.9] or just adapt [N, Satz 3.6]).

Lemma 10. *Assume that D is a field extension of k. Let $l \in \mathbb{Z}$ with $0 \leq l < n-2$, $m = 0$ and $\mathcal{F}$ a locally free $\mathcal{D}$-module such that*

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{F}(*)) = 0 \qquad (0 < i < l+1),$$

$$\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) = 0 \qquad (j > m).$$

Then, in the situation of Lemma 9, the connecting homomorphism

$$\delta : \mathrm{Ext}^l(\mathcal{D}, \Omega^l_D \otimes W, \mathcal{F}) \longrightarrow \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F})$$

is an isomorphism.

Proof. For $l = 0$ this is shown in Lemma 9, thus we assume $l > 0$. Let $\mathcal{E}$ be a $\mathcal{D}$-module, $i, j, p \in \mathbb{Z}$ such that $i \geq 0$, $1 \leq p \leq l$ and

$$0 \longrightarrow \Omega^l \longrightarrow \mathcal{O}_X(-l)^{\binom{n}{l}} \longrightarrow \Omega^{l-1} \longrightarrow 0$$

be the extended Euler sequence of X [Pu2, (3.1)]. Tensoring by $\mathcal{D}$ yields the short exact sequence

$$0 \longrightarrow \Omega^l_D \longrightarrow \mathcal{D}(-l)^{\binom{n}{l}} \longrightarrow \Omega^{l-1}_D \longrightarrow 0$$

of $\mathcal{D}$-modules and twisting it by $\mathcal{O}_X(j)$ the short exact sequence

$$0 \longrightarrow \Omega^l_D(j) \longrightarrow \mathcal{D}(-l+j)^{\binom{n}{l}} \longrightarrow \Omega^{l-1}_D(j) \longrightarrow 0$$

of $\mathcal{D}$-modules.

This induces a long exact Ext-sequence, part of it looking as follows:

$$\mathrm{Ext}^{i+l-p}(\mathcal{D}(-p+j), \mathcal{E})^{\binom{n}{p}} \longrightarrow \mathrm{Ext}^{i+l-p}(\Omega^p_D(j), \mathcal{E})^{\binom{n}{p}}$$
$$\stackrel{\delta_p}{\longrightarrow} \mathrm{Ext}^{i+l-(p-1)}(\Omega^{p-1}_D(j), \mathcal{E}) \longrightarrow \mathrm{Ext}^{i+l-p+1}(\Omega^p_D(j), \mathcal{E}).$$

Combining for $p = 1, \ldots, l$ we get a homomorphism

$$\overline{\delta} : \delta_1 \ldots \delta_l : \mathrm{Ext}^i(\Omega^l_D(j), \mathcal{E}) \longrightarrow \mathrm{Ext}^{i+l}(\mathcal{D}(j), \mathcal{E})$$

which is injective, resp. surjective, if each δ_p is injective, resp. surjective. This is the case if

$$\mathrm{Ext}^{i+l-p}(\mathcal{D}(-p+j), \mathcal{E}) = \mathrm{Ext}^{i+l-p}(\mathcal{D}, \mathcal{E}(-p+j)) = 0$$

resp.

$$\mathrm{Ext}^{i+l-p+1}(\mathcal{D}(-p+j),\mathcal{E}) = \mathrm{Ext}^{i+l-p+1}(\mathcal{D},\mathcal{E}(-p+j)) = 0$$

for $1 \leq p \leq l$.

Applying this to the special cases $i = j = 0$, $\mathcal{E} = \Omega^l_D \otimes_D W$ and $i = 1$, $j = 0$, $\mathcal{E} = \mathcal{F}$, we obtain that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\Omega^l_D, \Omega^l_D \otimes_D W) & \xrightarrow{\overline{\delta}} & \mathrm{Ext}^l(\mathcal{D}, \Omega^l_D \otimes_D W) \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\ \mathrm{Ext}^1(\Omega^l_D, \mathcal{F}) & \xrightarrow{\overline{\delta}} & \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}) \end{array}$$

commutes up to a sign by 5.2. For $1 \leq p \leq l$, we have

$$\mathrm{Ext}^{l-p+1}(\mathcal{D}, \Omega^l_D \otimes_D W \otimes \mathcal{O}_X(p)) = \mathrm{Ext}^{l-p+1}(\mathcal{D}, \Omega^l_D \otimes \mathcal{O}_X(p))^{\mathrm{rank}\, W} = 0$$

since $\mathrm{Ext}^{l-p+1}(\mathcal{O}_X, \Omega^l(p)) = 0$ by [Pu2, 3.3 (c)] and also

$$\mathrm{Ext}^{l-p+2}(\mathcal{D}, \Omega^l_D \otimes_D W \otimes \mathcal{O}_Z(p)) = \mathrm{Ext}^{l-p+2}(\mathcal{D}, \Omega^l_D \otimes \mathcal{O}_X(p))^{\mathrm{rank}\, W} = 0$$

since $\mathrm{Ext}^{l-p+2}(\mathcal{O}_X, \Omega^l(p)) = 0$ by [Pu2, 3.3 (d)], so the upper map $\overline{\delta}$ is injective.

For $1 \leq p \leq l$, we have

$$\mathrm{Ext}^{l-p+1}(\mathcal{D}, \mathcal{F}(p)) = 0$$

and also

$$\mathrm{Ext}^{l-p+2}(\mathcal{D}, \mathcal{F}(p)) = 0$$

by the hypothesis on $\mathcal{F}$, so the lower map $\overline{\delta}$ is an isomorphism. The left-hand map δ is an isomorphism by Lemma 9, therefore also the right-hand side map δ must be an isomorphism, as desired. □

We can now prove:

Proposition 11. *Assume that D is a field extension of k.*

Let $l, m \in \mathbb{Z}$ with $0 \leq l \leq n-2$ and let $\mathcal{F}$ be a locally free $\mathcal{D}$-module such that

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{F}(*)) = 0 \qquad (0 < i < l+1),$$

$$\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) = 0 \qquad (j > m).$$

Then there is a finite dimensional D-vector space W and an extension

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{P} \longrightarrow \Omega^l_D(-m) \otimes_D W \longrightarrow 0.$$

such that

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{P}(j)) = 0 \qquad (i, j \in \mathbb{Z}, 0 < i < l+1),$$

$$\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{P}(j)) = 0 \qquad (j \in \mathbb{Z}, j \geq m),$$

$$\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{P}(j)) \cong \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) \qquad (j \in \mathbb{Z}, j < m).$$

Proof. By twisting by $\mathcal{O}_X(m)$ we may assume that $m=0$.

Twisting the short exact sequence of Lemma 9 by $\mathcal{O}_X(j)$ yields the short exact sequence

$$0 \longrightarrow \mathcal{F}(j) \longrightarrow \mathcal{P}(j) \longrightarrow (\Omega^r_D \otimes_D W)(j) \longrightarrow 0.$$

By hypotheses on $\mathcal{F}$, $\mathrm{Ext}^i(\mathcal{D},\mathcal{F}(*))=0$ for $0<i<l+1$. Moreover, $\mathrm{Ext}^i(\mathcal{D},\Omega^l_D(j)\otimes_D W)=0$, unless $i=l$ and $j=0$ [Pu2, 3.3(d)], forcing $\mathrm{Ext}^i(\mathcal{D},\mathcal{P}(j))=0$ for $0<i<l+1$, $i\neq l$ or $j\neq 0$. In case $i=l$ and $j=0$, the exact sequence

$$0 \longrightarrow \mathrm{Ext}^l(\mathcal{D},\mathcal{P}) \longrightarrow \mathrm{Ext}^l(\mathcal{D},\Omega^r_D\otimes_D W) \xrightarrow{\delta} \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{F})$$

together with Lemma 10 shows that $\mathrm{Ext}^l(\mathcal{D},\mathcal{P})=0$. Summing up, $\mathrm{Ext}^i(\mathcal{D},\mathcal{P}(j))=0$ for $0<i<l+1$.

Now consider the exact sequence

$$\begin{aligned}\mathrm{Ext}^l(\mathcal{D},\Omega^r_D\otimes_D W(j)) &\xrightarrow{\delta} \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{F}(j))\\ &\longrightarrow \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{P}(j)) \longrightarrow \mathrm{Ext}^{l+1}(\mathcal{D},(\Omega^l_D\otimes_D W)(j))=0,\end{aligned}$$

the last extension group being zero by [Pu2, 3.3 (d)]. If $j\neq 0$, also $\mathrm{Ext}^l(\mathcal{D},(\Omega^r_D\otimes_D W)(j))=0$ which implies $\mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{P}(j))\cong\mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{F}(j))=0$ for $j>0$. If $j=0$, δ is an isomorphism by Lemma 10 which shows $\mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{P})=0$ and completes the proof. □

Lemma 12. *Let D be a field extension of k. Let $l,m\in\mathbb{Z}$ with $0\leq l\leq \frac{n-1}{2}-1$ and $m\geq -l-1$ for $l=\frac{n-1}{2}-1$. Given any finite dimensional D-vector space W, put $\mathcal{R}=\Omega^l_D(-m)\otimes_D W$. Then*

$$\mathrm{Ext}^1(\mathcal{R},\mathcal{R}^*)=\mathrm{Ext}^2(\mathcal{R},\mathcal{R}^*)=0$$

Proof. Ext^i is additive in both variables, so we may assume that $W=D$ and have to show that

$$\mathrm{Ext}^i(\Omega^l_D(-m),(\Omega^l_D(-m))^*)=0$$

for $i=1,2$. Put $j=-m$, $\mathcal{F}=\Omega^l_D(-m)^*$, then as in the proof of Lemma 10 we obtain a homomorphism

$$\overline{\delta}:\mathrm{Ext}^i(\Omega^l_D(-m),(\Omega^l_D(-m))^*)\longrightarrow \mathrm{Ext}^{i+l}(\mathcal{D},(\Omega^l_D(-2m))^*).$$

Since $n-1\geq 2l+1$ we have $l<n-1-i-l+p$ for all $p=1,\dots,l$. By Serre-duality, we know that $H^{i+l-p}(X,\Omega^l(-m)^\vee(p+m))=0$, therefore also $\mathrm{Ext}^{i+l-p}(\mathcal{D},\Omega^l_D(-m)^*(p+m))=0$, which together with [Pu2, 3.3 (d)] proves injectivity of $\overline{\delta}$ as in the proof of Lemma 10. Using Serre-duality and [Pu2, 3.3 (d)], we can check that $H^{i+l}(X,\Omega^l(-2m)^\vee)=0$ therefore $\mathrm{Ext}^{i+l}(\mathcal{D},\Omega^l_D(-2m)^*)=0$ for $i=1,2$. Hence $\overline{\delta}$ is surjective as well. □

6. Hermitian forms over field extensions of k

6.1. Let k be a field of characteristic not 2 and X a Brauer Severi variety over k. Let l/k be a separable field extension with a k-linear involution σ. Put $Y=X\times_k l$ and $X_s=X\times_k k_s\cong\mathbb{P}^{n-1}_{k_s}$ for k_s a separable closure of k. Recall from [Pu2, 5.2] that every line bundle $\mathcal{O}_{X_s}(m)$ has a G-invariant isomorphism class, where $G=\mathrm{Gal}\,(k_s/k)$ is the Galois group of k_s/k. That means $\mathcal{O}_{X_s}(m)\cong\mathcal{O}_{X_s}(m)^\tau$ for all $\tau\in G$. We look at (σ-)hermitian spaces $(\mathcal{M}, h)$ over $\mathcal{D} = l \otimes_k \mathcal{O}_X$, pointing out that $l \otimes_k \mathcal{O}_X = \pi_* \mathcal{O}_Y$, see 1.5. In other words, Y is affine over X and defined by the sheaf of $\mathcal{O}_X$-algebras $l \otimes_k \mathcal{O}_X$:

$$Y = \underline{\mathrm{Spec}}_X(l \otimes_k \mathcal{O}_X)$$

[H, II. Ex. 5.17].

Let $\mathcal{M}$ be a right $\mathcal{D}$-module which is locally free of finite rank. Then $\mathcal{M}$ canonically is an $\mathcal{O}_X$-module and we denote the associated $\mathcal{O}_Y$-module by $\widetilde{\mathcal{M}}$ as in [H, II. Ex. 5.17].

Proposition 13. *Suppose $Y = \mathbb{P}_l^{n-1}$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\widetilde{\mathcal{M}}$ splits into a direct sum of line bundles is Witt equivalent to a hermitian space extended from l.*

Proof. If $\widetilde{\mathcal{M}}$ splits into the direct sum of line bundles, then

$$\widetilde{\mathcal{M}} \cong \bigoplus_{i=1}^{t} \mathcal{O}_Y(s_i)$$

as $\mathcal{O}_Y$-module. We have $\mathcal{O}_Y(m) \cong \mathcal{O}_Y(m)^*$ if and only if $m = 0$ [Pu2, 5.2]. Hence there is no non-trivial line bundle over Y which is selfdual with respect to *. By the Krull-Schmidt Theorem for hermitian spaces [K, (6.3.1), p. 98] and by [K, (6.4.1), p. 99],

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_l (l \otimes \mathcal{O}_Y) \perp \text{ a hyperbolic space.}$$

□

Theorem 14. *Suppose $X = \mathbb{P}_k^{n-1}$. Then*

$$U_\tau : W^1(l) \longrightarrow W^1(l \otimes_k \mathcal{O}_X)$$

is surjective.

The proof is similar to the one given in [A] or in [Pu2, 5.1]:

Proof. If $n = 2$, for every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ the vector bundle $\mathcal{M}$ splits into a direct sum of line bundles, hence surjectivity follows from Proposition 13 and we may assume $n \geq 3$.

We show by induction on $a \geq 0$: If $(\mathcal{M}, h)$ is a hermitian space over $\mathcal{D} = l \otimes_k \mathcal{O}_X$ such that

$$a = \max\{i \in \mathbb{Z} \mid 0 \leq i < n-1, \mathrm{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) \neq 0\}$$

then $(\mathcal{M}, h)$, up to Witt equivalence, is a hermitian space extended from l. Note that the set on the right hand side is not empty here.

If $a = 0$ then $\mathrm{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) = 0$ for $0 < i < n-1$, so by the generalization of Horrocks' Theorem, $\mathcal{M} \cong \mathcal{D}(s_1) \oplus \cdots \oplus \mathcal{D}(s_t)$ for some s_i's and, by Proposition 13, $(\mathcal{M}, h)$ is Witt equivalent to some hermitian space extended from l. This settles the induction beginning.

In the induction step, let $a > 0$ and suppose the induction hypothesis holds for all nonnegative integers $a' < a$.

There is no harm in assuming $a < n-1$, so if $l = n - 2 - a$, then

$$(14) \quad 0 \leq l < n-2.$$

It suffices to show that a hermitian space $(\mathcal{M}, h)$ with

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{M}(*)) = 0 \text{ for } 0 < i < l+1,$$

up to Witt equivalence is extended from l. This will be done by induction on

$$s = \dim \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)).$$

If $s = 0$ then $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)) = \mathrm{Ext}^{n-a-1}(\mathcal{D}, \mathcal{M}(*)) = 0$, therefore

$$\max\{i \in \mathbb{Z} \,|\, 0 \leq i < n-1, \mathrm{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) \neq 0\} < a$$

and we are done by induction hypothesis on a. If $s > 0$,

$$(15) \quad l \text{ is the least nonnegative integer such that } \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)) \neq 0.$$

By [H, III, Ex. 6.10], $\mathrm{Ext}^q(\mathcal{D}, \mathcal{M}(j)) = \mathrm{Ext}^q(\mathcal{O}_Y, \widetilde{\mathcal{M}}(j)) = H^q(Y, \widetilde{\mathcal{M}}(j))$. Thus (15) is equivalent to saying that

$$(15') \quad l \text{ is the least nonnegative integer such that } H^{l+1}(Y, \widetilde{\mathcal{M}}(*)) \neq 0.$$

Using that $\mathcal{M} \cong \mathcal{M}^*$ and Serre duality we obtain

$$H^i(Y, \widetilde{\mathcal{M}}(j)) \cong H^i(Y, \widetilde{\mathcal{M}}^\vee(j)) \cong H^{n-1-i}(Y, \widetilde{\mathcal{M}}(-n-2-j))^\vee$$

and may conclude that

$$(16) \quad l + 1 \leq \frac{n-1}{2}.$$

Picking $m \in \mathbb{Z}$ maximal such that $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(m)) \neq 0$, we obtain

$$(17) \quad m \geq -l - 1 \text{ if } l + 1 = \frac{n-1}{2}.$$

(This is because for $l + 1 = \frac{n-1}{2}$, $m \geq -n - 2 - m$, thus $2m \geq -n - 2 = -2l - 4$ implying $m \geq -l - 1$.) In particular,

$$(18) \quad \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(j)) = 0 \text{ for } j > m.$$

Because of (14, 15, 18), $\mathcal{M}$ satisfies the hypothesis of Proposition 11 and there exists a locally free $\mathcal{D}$-module $\mathcal{E}$ and an extension

$$(19) \qquad 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow \mathcal{E} \longrightarrow 0.$$

such that $\mathcal{P}$ satisfies the conditions listed in Proposition 11. Hence

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}^*) = \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}^*) = 0.$$

Thus we may apply the Extension Theorem 6 to the dual of (19) with $\mathcal{M}$ replaced by $\mathcal{M}^*$ via h, i.e., to

$$(20) \qquad 0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{P}^* \longrightarrow \mathcal{M} \longrightarrow 0.$$

This way we obtain a hermitian space $(\mathcal{S}, b)$ Witt equivalent to $(\mathcal{M}, h)$ and an exact sequence of $\mathcal{D}$-modules

$$(21) \qquad 0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{S} \longrightarrow \mathcal{P} \longrightarrow 0.$$

Since $\mathrm{Ext}^i(\mathcal{D}, \Omega^l_D(-m)^*(j)) = 0$ for $0 < i \leq l+1$ this together with (20) yields

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{S}(*)) = 0 \text{ for } 0 < i < l+1$$

and that $\mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{S}(j)) \to \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{P}(j))$ is injective for all $j \in \mathbb{Z}$. The latter shows by using Proposition 11 that

$$\mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{S}(j)) = 0 \text{ for } j \geq m',$$

$$\dim \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{S}(j)) \leq \dim \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{M}(j)) \text{ for } j < m'.$$

Together this yields

$$\dim \mathrm{Ext}^{l+1}(\mathcal{D},\mathcal{S}(*)) \leq s.$$

Applying the induction hypothesis yields the assertion that $(\mathcal{S}, b)$ and hence also $(\mathcal{M}, h)$ is up to Witt equivalence extended from l. □

By Theorems 1, 2 and 4, this settles the case where X is associated to a central simple algebra of odd index:

Corollary 15. *(i) Let $X = \mathbb{P}^{n-1}_k$. Then*

$$U_\tau : W^1(l) \longrightarrow W^1(l \otimes_k \mathcal{O}_X)$$

is bijective.
(ii) Let X be a Brauer Severi variety associated to a central simple algebra of odd index. Then

$$U_\tau : W^1(l) \longrightarrow W^1(l \otimes_k \mathcal{O}_X)$$

is bijective.

6.2. Let X be a Brauer Severi variety over k with associated central simple algebra $\mathrm{Mat}_s(E)$, E a division algebra over k.

Proposition 16. *(i) Suppose there is a separable maximal subfield k' of E containing l. Let $X' = X \times_k k'$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\widetilde{\mathcal{M}} \otimes \mathcal{O}_{X'}$ splits into the direct sum of line bundles is Witt equivalent to a hermitian space (M_0, h_0) over l.*
(ii) Suppose there is a separable maximal subfield k' of E, such that l and k' are linearly disjoint. Let $Y' = X \times_k l'$ with $l' = l \otimes_k k'$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\widetilde{\mathcal{M}} \otimes \mathcal{O}_{Y'}$ splits into the direct sum of line bundles is Witt equivalent to a hermitian space (M_0, h_0) over l.

Proof. (i) Obviously, $X' \cong \mathbb{P}^{n-1}_{k'}$. If $\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X'}$ splits into the direct sum of line bundles

$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X'} \cong \bigoplus_{i=1}^{t} \mathcal{O}_{X'}(s_i),$$

then, by the theory developed in [AEJ],

$$\widetilde{\mathcal{M}} \cong \bigoplus_{i=1}^{t} \mathcal{L}(s_i) \oplus \bigoplus_{j=1}^{h} tr_{l_j/l}(\mathcal{N}_j)$$

as $\mathcal{O}_Y$-module with line bundles $\mathcal{N}_j$ over $Y_j = Y \times_l l_j$ which are not already defined over Y, l_j/l proper field extensions, and with the $tr_{l_j/l}(\mathcal{N}_j)$'s indecomposable. We have $\mathcal{O}_{X'}(m) \cong \mathcal{O}_{X'}(m)^*$ if and only if $m = 0$ [Pu2, 5.2]. Hence there is no non-trivial line bundle over Y which is selfdual with respect to *. Now consider an indecomposable vector bundle

$tr_{l_j/l}(\mathcal{N}_j)$. Then $tr_{l_j/l}(\mathcal{N}_j) \cong tr_{l_j/l}(\mathcal{N}_j)^*$ implies that $\mathcal{O}_{X'}(m) \oplus \cdots \oplus \mathcal{O}_{X'}(m) \cong \mathcal{O}_{X'}(-m) \oplus \cdots \oplus \mathcal{O}_{X'}(-m)$ [Pu2, 5.2], hence $m = 0$ and so there are no indecomposable $\mathcal{O}_Y$-modules of rank > 1 which are selfdual with respect to *. By the Krull-Schmidt Theorem for hermitian spaces [K, (6.3.1), p. 98],

$$(\widetilde{\mathcal{M}}, \widetilde{h}) \cong (M, h) \otimes_l (l \otimes \mathcal{O}_Y) \oplus \text{ a hyperbolic space}$$

and thus also $(\mathcal{M}, h)$ is Witt equivalent to a hermitian space (M_0, h_0) over l (we canonically identify hermitian forms over $l \otimes_k \mathcal{O}_X$ with hermitian forms over $\mathcal{D}$).
(ii) is proved analogously. □

Theorem 17. *Let E have even index. Then*

$$U_\tau : W^1(l) \longrightarrow W^1(l \otimes_k \mathcal{O}_X)$$

is surjective.

Proof. If l is a finite field extension of k with an involution σ and invariant field l^σ, for a Brauer-Severi variety X over k we may identify l^σ-algebras and modules over X with $\mathcal{O}_{X_{l^\sigma}}$-algebras and modules over X_{l^σ}, analogous to the $\mathcal{M} \to \widetilde{\mathcal{M}}$ construction. In particular, we may identify hermitian forms over $l \otimes_k \mathcal{O}_X$ with hermitian forms over $l \otimes_{l^\sigma} \mathcal{O}_{X_{l^\sigma}}$. This way we may restrict us without loss of generality to the case that $[l : k] = 2$. In this case, either there is a separable maximal subfield k' of E containing l, or there is a maximal separable subfield k' of E such that k' is linearly disjoint with l over k.
(i) Suppose that there is a separable maximal subfield k' of E containing l. For $n = 2$, $Y' \cong \mathbb{P}^1_{l'}$ and the assertion is proved in Proposition 16 (i). So we may assume $n \geq 3$. Let $\mathcal{M}' = \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$.

We show by induction on $a \geq 0$: If $(\mathcal{M}, h)$ is a hermitian space such that

$$a = \max\{i \in \mathbb{Z} \,|\, 0 \leq i \leq n-1, H^{n-i-1}(Y', \mathcal{M}') \neq 0\}$$

then $(\mathcal{M}, h)$, up to Witt equivalence, is a hermitian space (M_0, h_0) over l.

If $a = 0$ then $H^{n-i-1}(Y', \mathcal{M}'(j)) = 0$ for all j and for $0 \leq i \leq n-1$, so by Horrocks [B-H, Sect. 5, Lemma 1], $\mathcal{M}'$ splits into the direct sum of line bundles and, by Proposition 16 (i), $(\mathcal{M}, h)$ is Witt equivalent to some hermitian space (M_0, h_0) over l. This settles the induction beginning. In the induction step, let $a > 0$ and suppose the induction hypothesis holds for all nonnegative integers $a' < a$. Then the assertion is proved analogously as Theorem 14 using [Pu3, Proposition 4.1] (there use X_l instead of X), [Pu3, Lemma 4.4], Lemma 10 and the Extension Theorem 6.
(ii) Suppose there is a maximal separable subfield k' of E such that k' is linearly disjoint with l over k. Then the proof is analogous to the one in (i), but working over $\mathbb{P}^{n-1}_{l'}$ instead. □

Remark 18. Let $X = \mathbb{P}^{n-1}_{\mathbb{R}}$ and let σ be the standard involution on $\mathbb{C}$. Then $W^\varepsilon(\mathbb{C}) \cong W^\varepsilon(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_X)$. Since $W^{-1}(\mathbb{C}) \cong W^1(\mathbb{C}) \cong \mathbb{Z}$ [K, p. 63], this implies

$$W^{\pm 1}(\mathbb{C}) \cong W^{\pm 1}(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_X) \cong \mathbb{Z}.$$

7. Hermitian spaces over division algebras

7.1. Let k be a field of characteristic not 2 and D be a division algebra over k with a k-linear involution σ. Let $X = \mathbb{P}^1_k$, $\mathcal{D} = \tau^* D \cong D \otimes_k \mathcal{O}_X$ and $\mathcal{D}(m) = \mathcal{O}_X(m) \otimes \mathcal{D}$ for any integer m.

The Theorem of Krull-Schmidt holds for locally free right $\mathcal{D}$-modules [K, p. 96] and for ϵ-hermitian spaces over $\mathcal{D}$ if we restrict to $\epsilon = \pm 1$, by [K, p. 99, (6.5.1)].

Proposition 19. *Let $X = \mathbb{P}^1_k$ and $\epsilon = \pm 1$. Every ϵ-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ such that*

$$\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$$

is Witt equivalent to an ϵ-hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$, where (M_0, h_0) is an ϵ-hermitian space over D.

Note that for the possible extension of these results described in Section 7.2 we would need this proposition also for $X = \mathbb{P}^n_k$ (and $\epsilon = 1$).

Proof. We have $\mathcal{D}(m) \cong \mathcal{D}(m)^*$ if and only if $m = 0$ [K, p. 96, (5.4.1)]. Hence $\mathcal{D}$ itself is the only locally free right $\mathcal{D}$-module of rank 1 which is selfdual with respect to *. Any ϵ-hermitian space with underlying vector bundle of type $\{\mathcal{D}(m), \mathcal{D}(m)^*\}$ with $m \neq 0$ is isometric to a hyperbolic space [K, p. 99, (6.4.2)]. By the Krull-Schmidt Theorem for ϵ-hermitian spaces ([S, p. 272] or [K, p. 96, p. 99]), $\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$ implies that

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X) \perp \perp_{j=1}^{h} (\mathcal{D}(m_j) \oplus \mathcal{D}(-m_j), \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix})$$

for a suitable ϵ-hermitian space over D and suitable $m_j \neq 0$. We conclude that

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X) \perp \text{ a hyperbolic space.}$$

□

Corollary 20. *Let $X = \mathbb{P}^1_k$ and $\epsilon = \pm 1$. Every ϵ-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ is Witt equivalent to an ϵ-hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$ with (M_0, h_0) an ϵ-hermitian space over D. In particular,*

$$U_\tau : W^\epsilon(D) \longrightarrow W^\epsilon(D \otimes \mathcal{O}_X)$$

is bijective.

Proof. For $X = \mathbb{P}^1_k$, every ϵ-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ satisfies $\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$ [K, p. 407, VII.(3.1.1)]. □

7.2. Let $X = \mathbb{P}^n_k$ and $\operatorname{char} k \neq 2$. It would be desirable to prove that for a division algebra D with a k-linear involution σ the group homomorphism

$$U_\tau : W^1(D) \longrightarrow W^1(D \otimes \mathcal{O}_X)$$

is surjective. However, we will leave this open for now and only briefly discuss the problems arising in a possible proof.

First of all, we need to assume that

$$\operatorname{Ext}^{l+1}(\mathcal{D},\mathcal{M}(*)).$$

is of *finite rank* as a right D-module and that $\operatorname{Ext}^{n-1}(\mathcal{D},\mathcal{M}(*))\neq 0$.

For $n=1$ the assertion has been proved in Corollary 20, so let $n\geq 2$. Imitating the proofs in [Pu2] or [A] we proceed as follows: it would suffice to show by induction on $a\geq 0$ that if $(\mathcal{M},h)$ is a hermitian space satisfying

$$a=\max\{\, i\in\mathbb{Z}\,|\,0\leq i<n, \operatorname{Ext}^{n-i-1}(\mathcal{D},\mathcal{M}(*))\neq 0\},$$

then $(\mathcal{M},h)$ is Witt equivalent to a hermitian space $(M_0,h_0)\otimes_D (D\otimes\mathcal{O}_X)$.

If $a=0$ then $\operatorname{Ext}^{n-i}(\mathcal{D},\mathcal{M}(j))=0$ for all j, $0<i<n$ and $\mathcal{M}\cong\mathcal{D}(m_1)\oplus\cdots\oplus\mathcal{D}(m_t)$ by Theorem 8 (the generalized Horrocks Theorem). If Proposition 19 can be generalized to $\mathbb{P}^n_k$, this would imply that $(\mathcal{M},h)$ is Witt equivalent to a hermitian space $(M_0,h_0)\otimes_D (D\otimes\mathcal{O}_X)$ and settle the induction beginning.

In the induction step, let $a>0$ and suppose that the induction hypothesis holds for all nonnegative integers $a'<a$. There is no harm in assuming $a<n$, thus we have $0\leq l=n-1-a<n-1$. It suffices to show that a hermitian space $(\mathcal{M},h)$ with $\operatorname{Ext}^i(\mathcal{D},\mathcal{M}(*))=0$ for $0<i<l+1$ is Witt equivalent to a hermitian space which is extended from D. This is done by induction on

$$s=\dim\operatorname{Ext}^{l+1}(\mathcal{D},\mathcal{M}(*)).$$

If $s=0$ then we are done by the induction hypothesis on a. If $s>0$ then l is the least nonnegative integer such that $\operatorname{Ext}^{l+1}(X,\mathcal{M}(*))\neq 0$. We next would have to be able to conclude that $l+1\leq\frac{n}{2}$. It is not clear if we can show it by using Serre-duality. Let $m\in\mathbb{Z}$ be maximal such that $\operatorname{Ext}^{l+1}(\mathcal{D},\mathcal{M}(m))\neq 0$. If $l+1=\frac{n}{2}$, $m\geq -m-1-m$, hence $2m\geq -n-1=-2l-3$, forcing $m\geq -l-1$ if $l+1=\frac{n}{2}$.

We would now need to apply a similar result as Proposition 11 in our setting here if we want to proceed with our proof along the same lines as in Theorem 14. However, it is not clear how to prove a statement like this, see Section 5. At this point all we can say is that $(\mathcal{M},q_h)$, the quadratic space over X induced by $(\mathcal{M},h)$, is Witt equivalent to a quadratic space defined over k. We leave it open if it is possible to fix these gaps in the proof.

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