NOTE ON THE COHOMOLOGICAL INVARIANT OF PFISTER FORMS

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ABSTRACT. The cohomological invariant ring of the n-Pfister forms is isomorphic to the invariant ring in that of an elementary abelian 2-group of rank n under a $GL_n(\mathbb{Z}/2)$-action.

1. INTRODUCTION

Let G be an algebraic group over k with $ch(k) \neq 2$. The cohomological invariant $Inv^*(G;\mathbb{Z}/2)$ is (roughly speaking) the ring of natural functors $H^1(F;G) \to H^*(F;\mathbb{Z}/2)$ for the category of finitely generated field F over k. (For details, see the excellent book [Ga-Me-Se]). Moreover, we can define the cohomological invariant $Inv^*(Pfister_n;\mathbb{Z}/2)$ of n-Pfister forms, while there does not exist the corresponding group G for $n \geq 4$. In Theorem 18.1 in [Ga-Me-Se], this invariant has been computed.

In this note, we show that this cohomological invariant is isomorphic to the invariant ring in that of an elementary abelian 2-group of rank n under a $GL_n(\mathbb{Z}/2)$-action, namely,

$$Inv^*(Pfister_n;\mathbb{Z}/2) \cong Inv^*((\mathbb{Z}/2)^n;\mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)}.$$

2. MOTIVIC COHOMOLOGY AND COHOMOLOGICAL INVARIANT

We recall the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2)$ for a smooth scheme X over k with $ch(k) \neq 2$. By the Milnor conjecture (which is now solved by Voevosky), we know $H^{*,*'}(X;\mathbb{Z}/2) \cong H^*_{et}(X;\mathbb{Z}/2)$ for $* \leq *'$. Take $\tau \in H^{0,1}(Spec;\mathbb{Z}/2) \cong \mathbb{Z}/2$ as a nonzero element. It is known that $H^{*,*'}(X;\mathbb{Z}/2) = 0$ for $(* - *') > dim(X)$. Hence we have

$$H^{*,*'}(Spec(k);\mathbb{Z}/2) \cong H^*_{et}(Spec(k);\mathbb{Z}/2)[\tau].$$

Let us write $H^{*,*'} = H^{*,*'}(Spec(k);\mathbb{Z}/2)$ and $H^* = H^*_{et}(Spec(k);\mathbb{Z}/2)$ so that $H^{*,*'} \cong H^*[\tau]$.

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Let BG be a classifying space of G ([To], [Vo2]). Let $H^*(X;H^{*'}_{\mathbb{Z}/2})$ be the sheaf cohomology where $H^n_{\mathbb{Z}/2}$ is the Zarisky sheaf induced from the presheaf $H^n_{et}(V;\mathbb{Z}/p)$ for open subset V of X. Then Totaro proved that

$$Inv^*(BG;\mathbb{Z}/2)\cong H^0(BG;H^*_{\mathbb{Z}/2})$$

in a letter to Serre [Ga-Me-Se]. The Milnor conjecture implies the Beilinson and Lichtenbaum conjecture (see [Vo2,3]). This fact implies the following long exact sequences of motivic and sheaf cohomology theories (Lemma 3.1 in [Or-Vi-Vo], [Vo3])

$$\to H^{m,n-1}(X;\mathbb{Z}/2)\overset{\times\tau}{\to} H^{m,n}(X;\mathbb{Z}/2)$$
$$\to H^{m-n}(X;H^n_{\mathbb{Z}/2})\to H^{m+1,n-1}(X;\mathbb{Z}/2)\overset{\times\tau}{\to}.$$

Thus we have

Theorem 2.1. *There is an additive isomorphism*

$$Inv^*(G;\mathbb{Z}/2)\cong H^{*,*}(BG;\mathbb{Z}/2)/(\tau)\oplus Ker(\tau)|H^{*+1,*-1}(BG;\mathbb{Z}/2).$$

As an application, we first consider the case $G=\mathbb{Z}/2$. The $mod(2)$ motivic cohomology is computed in [Vo1,2].

$$H^{*,*'}(B\mathbb{Z}/2;\mathbb{Z}/2)\cong H^{*,*'}[y]\otimes\Delta(x)$$

with $\beta(x)=y$, hence $deg(y)=(2,1)$ and $deg(x)=(1,1)$. Here Voevodsky shows ([Vo1,2])

$$x^2=\tau y+\rho x\quad with\ \rho=(-1)\in H^1=k^*/(k^*)^2.$$

Next consider their product $G=(\mathbb{Z}/2)^n$. The cohomology $H^{*,*'}(B\mathbb{Z}/2;\mathbb{Z}/2)$ has the Kunneth formula (also by Voevodsky). Hence the motivic cohomology is given

$$H^{*,*'}(BG;\mathbb{Z}/2)\cong H^{*,*'}[y_1,...,y_n]\otimes\Delta(x_1,..,x_n)$$

where $\beta(x_i)=y_i$ and $x_i^2=\tau y_i+\rho x_i$. Hence from Theorem 2.1, we get (as stated in [Ga-Me-Se])

Lemma 2.2. *Let G be an elementary abelian 2-group of $rank(G)=n$. Then $Inv^*(G;\mathbb{Z}/2)\cong H^*\otimes\Delta(x_1,...,x_n)$ with $x_i^2=\rho x_i$.*

3. Dickson invariants

Recall that the mod 2 (topological) cohomology

$$H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)\cong\mathbb{Z}/2[x_1,...,x_n]\quad |x_i|=1.$$

It is well known that the invariant ring under the $GL_n(\mathbb{Z}/2)$-action is the Dickson algebra

$$H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)}\cong\mathbb{Z}/2[d_{n,0},...,d_{n,n-1}]$$

where each generator $d_{n,i}$ is given by

$$w_t(x) = \Pi_{\epsilon_i=0 \ or \ 1}(t + \epsilon_1 x_1 + ...\epsilon_n x_n)$$
$$= t^{2^n} + d_{n,n-1}t^{2^{n-1}} + d_{n,n-2}t^{2^{n-2}} + ... + d_{n,0}t.$$

Examples. Let us write by w_i the elementary symmetric function for x_j in $H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)$. Then

$$\begin{cases} d_{2,1} = x_1^2 + x_1x_2 + x_2^2 = w_1^2 + w_2 \\ d_{2,0} = x_1^2x_2 + x_1x_2^2 = w_1w_2 \end{cases}.$$

We want to know the Dickson invariant ring in the cohomological invariant. Let us consider

$$U = Inv^*_{k=\mathbb{R}}((\mathbb{Z}/2)^n;\mathbb{Z}/2) \cong H^{*,*'}(B(\mathbb{Z}/2)^n|_{\mathbb{R}};\mathbb{Z}/2)/(\tau)$$
$$\cong \mathbb{Z}/2[\rho] \otimes \Delta(x_1,...,x_n), \quad x_i^2 = \rho x_i.$$

For example, in U, we see $d_{2,0} = \rho x_1x_2 + x_1\rho x_2 = 0$, and

$$d_{2,1} = \rho x_1 + x_1x_2 + \rho x_2 = \rho w_1 + w_2.$$

Lemma 3.1. *In U, we have $d_{n,i} = 0$ for $i < n-1$ and*

$$d_{n,n-1} = \sum_{i\geq 1}^{n} w_i \rho^{2^{n-1}-i} = (\rho + x_1)...(\rho + x_n)\rho^{2^{n-1}-n} + \rho^{2^{n-1}}.$$

Proof. Decompose that

$$w_t(x) = \Pi(t + \epsilon_1x_1 + ...\epsilon_{n-1}x_{n-1}) \times \Pi(t + x_n + \epsilon_1x_1 + ...\epsilon_{n-1}x_{n-1}).$$

By induction on n, we assume this element is

$$(t^{2^{n-1}} + d_{n-1,n-2}t^{2^{n-2}})((t + x_n)^{2^{n-1}} + d_{n-1,n-2}(t + x_n)^{2^{n-2}}).$$

Letting $d_{n-1,n-2} = d$, $t^{2^{n-2}} = T$ and $x^{2^{n-2}} = X$, we see that the above formula is

$$= (T^2 + dT)(T^2 + X^2 + dT + dX)$$
$$= T^4 + (d^2 + dX + X^2)T^2 + (d^2X + dX^2)T.$$

Here note $X^2 = \rho^* X = \rho^{*'} x_n$, $d^2 = \rho^* d$ (since $(\rho + x)^2 = \rho(\rho + x)$). So $(d^2X + dX^2) = 0$. Let $a = a_{n-1} = (\rho + x_1)...(\rho + x_{n-1})$. Then we have

$$d^2 + dX + X^2 = \rho^* d + \rho^{*-1} dx_n + \rho^{*'} x_n$$
$$= \rho^{\sharp}(a + \rho^{*''}) + \rho^{\sharp-1}(a + \rho^{*''})x_n + \rho^{*'} x_n$$
$$= \rho^{\sharp-1}a(\rho + x_n) + \rho^{\sharp''} = \rho^{\sharp-1}a_n + \rho^{\sharp''}$$

as desired. □

Corollary 3.2. *Let us write*

$$e_n = \rho^{-2^{n-1}+n} d_{n,1} = \sum_{i\geq 1}^{n} w_i \rho^{n-i} = (\rho + x_1)...(\rho + x_n) + \rho^n.$$

Then we have

$$Inv^*((\mathbb{Z}/2)^n; \mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)} \cong H^*\{1, e_n\}.$$

Proof. By $Ideal(\rho)$, we consider the associated graded algebra

$$gr(H^* \otimes \Delta(x_1, ..., x_n)) \cong gr(H^*) \otimes \Lambda(x_1, ..., x_n) \quad (x_i^2 = 0).$$

Note $e_n = w_n$ in the above graded algebra. We can see

$$\Lambda(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)} \cong \mathbb{Z}/2\{1, w_n\}$$

as following arguments.

Since $x_i^2 = 0$, the only possibility of invariant is w_s. Suppose $s < n$. Write

$$w_s = x_1(\sum_{1\neq i_k} x_{i_1}...x_{i_{s-1}}) + (\sum_{1\neq i_k} x_{i_1}...x_{i_s}).$$

Consider the action $x_{12} : x_1 \mapsto x_1 + x_2$ but $x_{12} : x_i \mapsto x_i$ for $i > 1$. Then

$$(x_{12}-1)w_s = x_2(\sum_{1\neq i_k} x_{i_1}...x_{i_{s-1}}) = x_2x_3...x_{s+1} + ... \neq 0 \quad in\ \Lambda(x_1, ..., x_n).$$

All elements in H^* and e_n are really invariants in $H^* \otimes \Delta(x_1, ..., x_n)$. Thus we have the corollary. □

Let $n = 2$ and $G = SO_3$ or $n = 3$ and $G = G_2$ the exceptional group. Then G has only one conjugacy class A_n of maximal elementary abelian 2 groups of rank n. The Weyl group $W_G(A_n) \cong GL_n(\mathbb{Z}/2)$. Hence we have the restriction map

$$Inv^*(G; \mathbb{Z}/2) \to Inv^*(A_n; \mathbb{Z}/2)^{W_G(A_n)} \cong H^*\{1, e_n\}.$$

The result in [Ga-Me-Se] shows this map is an isomorphism.

4. Pfister forms

The most important quadratic forms are Pfister forms. Given $a = (a_1, ..., a_n) \in (k^*/(k^*)^2)^{\times n}$, the n-th Pfister form P_a is the quadratic form defined as

$$P_a = \langle\langle a_1, ..., a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes ... \otimes \langle 1, -a_n\rangle$$
$$= \oplus_{1\leq i_1<...<i_s\leq n} \langle(-a_{i_1})...(-a_{i_s})\rangle.$$

Given a quadratic form $q_a = \langle a_1, ..., a_n \rangle$, the total Stiefel-Whitney class is given by

$$w_t(q_a) = \Pi(t + a_i).$$

Hence we know

$$w_t(P_a) = \Pi_{\epsilon_i = 0 \ or \ 1}(t + \epsilon_1(\rho + x_1) + ... + \epsilon_n(\rho + x_n))$$

identifying $x_i = (a_i)$. Hence the following proposition follows the preceding lemma. (Substitute $x_i + \rho$ for x_i in the right hand side of the euation in Lemma 3.1.)

Proposition 4.1. *Let $x_i = (a_i) \in k^*/(k^*)^2$ and $w_n = x_1...x_n$. Then*

$$w_t(P_a) = t^{2^n} + (w_n + \rho^n)\rho^{2^{n-1}-n}t^{2^{n-1}}.$$

Next consider the map from $(k^*/(k^*)^2)^{\times n}$ to the set of n-th Pfister forms $Pfist_n$ defined by

$$p : a = (a_1, ..., a_n) \mapsto P_{-a} = \langle\langle -a_1, ..., -a_n \rangle\rangle = \langle 1, a_1 \rangle \otimes ... \otimes \langle 1, a_n \rangle.$$

This map induces the map of cohomological invariants

$$p^* : Inv^*(Pfister_n; \mathbb{Z}/2) \to Inv^*((k^*/(k^*)^2)^{\times n}; \mathbb{Z}/2).$$

Here the last invariant is isomorphic to

$$Inv^*((\mathbb{Z}/2)^n; \mathbb{Z}/2) \cong H^* \otimes \Delta(x_1, ..., x_n).$$

On $H^* \otimes \Delta(x_1, ..., x_n)$, we can define the usual $GL_n(\mathbb{Z}/2)$-action. This action is also written as follows. Consider the Bruhat decomposition

$$GL_n(\mathbb{Z}/2) = \coprod_{w \in S_n} BwB$$

where B is the Borel group generated by upper triangular matrices, and S_n is the n-th symmetric group generated by transition matrices. The group B is generated by $x_{ij} = 1 + e_{ij}$; the elementary matrix with (i, j) entries 1 with the relations

$$x_{ij}^2 = 1, \quad [x_{ij}, x_{kl}] = \begin{cases} x_{il} & if \ j = k \\ 0 & otherwise. \end{cases}$$

Define $w(x_i) = x_{w(i)}$ for $w \in S_n$ and

$$x_{ij}(x_i) = x_i + x_j, \quad x_{ij}(x_k) = x_k \ \ for \ i \neq k.$$

Then the $GL_n(\mathbb{Z}/2)$-action is decided on $Inv^*((\mathbb{Z}/2)^n; \mathbb{Z}/2)$.

Theorem 4.2. *The above map p^* induces the isomorphism*

$$Inv^*(Pfister_n; \mathbb{Z}/2) \cong H^* \otimes \Delta(x_1, ..., x_n)^{GL_n(\mathbb{Z}/2)} \cong H^*\{1, e_n\}.$$

Proof. On $(k^*/(k^*)^2)^{\times n}$, we can define the $GL_n(\mathbb{Z}/2)$-action by

$$x_{ij}(a_1,...a_n)=(a_1,...,a_{i-1},a_ia_j,a_{i+1},...,a_n),$$
$$w(a_1,...,a_n)=(a_{w(1)},...,a_{w(n)}).$$

This induces the action on $Inv^*((\mathbb{Z}/2)^n;\mathbb{Z}/2)$ by $w(x_i)=x_{w(i)}$ and

$$x_{ij}(x_i)=x_i+x_j, \quad x_{ij}(x_k)=x_k \quad for\ i\neq k.$$

Define a $GL_n(\mathbb{Z}/2)$ action on $Pfister_n$ by $x_{ij}p(a)=p(x_{ij}(a))$. Then this action is invariant, indeed,

$$x_{12}\langle\langle a_1,a_2\rangle\rangle = px_{12}(-a_1,-a_2)=p(a_1a_2,-a_2)$$
$$=\langle\langle -a_1a_2,a_2\rangle\rangle=\langle 1,a_1a_2,-a_2,-a_1a_2^2\rangle$$
$$=\langle 1,a_1a_2,-a_2,-a_1\rangle=\langle\langle a_1,a_2\rangle\rangle.$$

Hence we have the map

$$q^*: Inv^*(Pfister_n;\mathbb{Z}/2)\to H^*\otimes\Delta(x_1,...,x_n)^{GL_n(\mathbb{Z}/2)}.$$

Since $P_{-a}\mapsto e_n=w_n(P_{-a})+\rho^n$ is a cohomology invariant and hence the above map is epic.

For each finitely generated field F over k, the map $p:(K^*/(K^*)^2)^n\to Pfister_n|_K$ is of course an epimorphism. Hence the induced map

$$p^*(x):(K^*/(K^*)^2)^n\to Pfister_n\xrightarrow{x} H^n(K;\mathbb{Z}/2)$$

is always injective. □

If we consider the map

$$q: a=(a_1,...,a_n)\mapsto P_a=\langle\langle a_1,...,a_n\rangle\rangle,$$

then the map q^* also induces the isomorphism

$$Inv^*(Pfister_n;\mathbb{Z}/2)\cong H^*\otimes\Delta(x_1,...,x_n)^{GL_n(\mathbb{Z}/2)_{II}}\cong H^*\{1,w_n\}$$

where $GL(\mathbb{Z}/2)_{II}$ is the unusual action defined by $x_{ij}(x_i)=\rho+x_i+x_j$.

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