Indecomposable and Noncrossed Product Division Algebras over Curves over Complete Discrete Valuation Rings

Feng Chen[1]

Dept. of Math & CS, Emory University, Atlanta GA 30322

Abstract

Let T be a complete discrete valuation ring and $\hat{X}$ a smooth projective curve over $S = \mathrm{Spec}(T)$ with closed fibre X. Denote by F the function field of $\hat{X}$ and by $\hat{F}$ the completion of F with respect to the discrete valuation defined by X, the closed fibre. In this paper, we construct indecomposable and noncrossed product division algebras over F. This is done by defining an index preserving group homomorphism $s : \mathrm{Br}(\hat{F})' \to \mathrm{Br}(F)'$, and using it to lift indecomposable and noncrossed product division algebras over $\hat{F}$.

1. Introduction

Let $\hat{X}$ be a smooth projective curve over $S = \mathrm{Spec}(T)$, where T is a complete discrete valuation ring with uniformizer t. Let $F = K(\hat{X})$ be the function field, and let $\hat{F} = \hat{K}(\hat{X})$ be the completion with respect to the discrete valuation defined by the closed fibre X. We define an index-preserving homomorphism

$$\mathrm{Br}(\hat{F})' \to \mathrm{Br}(F)'$$

that splits the restriction map res : $\mathrm{Br}(F)' \to \mathrm{Br}(\hat{F})'$. Here $\mathrm{Br}(-)$ denotes the Brauer group of $-$ and the "prime" denotes the union of the n-torsion part of $\mathrm{Br}(F)$, where n is prime to the characteristic of k, the residue field of T. Using the method of Brussel [5] and Brussel [4], we can construct indecomposable and noncrossed product division algebras over $\hat{F}$, and lift these constructions to F using our homomorphism, generalizing the constructions in Brussel et al. [6], where indecomposable and noncrossed product division algebras over function fields of p-adic curves are constructed.

Recall that if K is a field, a K-*division algebra* D is a division ring that is finite-dimensional and central over K. The *period* or *exponent* of D is the order of the class $[D]$ in $\mathrm{Br}(K)$, and the *index* of D is the square root of D's K-dimension. A *noncrossed product* is a K-division algebra whose structure is not given by a Galois 2-cocycle. Noncrossed products were first constructed by Amitsur [2], settling a longstanding open problem. Since then there have been several other constructions, including Saltman [22], Jacob and Wadsworth [18], Brussel [5].

A K-division algebra is *indecomposable* if it cannot be expressed as the tensor product of two nontrivial K-division algebras. It is easy to see that all division algebras of period not a prime power are decomposable, so the problem of producing an indecomposable division algebra is only

Email address: `fchen@emory.edu` (Feng Chen)

interesting when the period and index are unequal prime powers. Therefore we will only consider division algebras of prime power period and index in this paper. Then it is not hard to see that all division algebras of equal (prime power) period and index are trivially indecomposable. Albert constructed decomposable division algebras in the 1930's, but indecomposable division algebras of unequal (2-power) period and index did not appear until Saltman [23]and Amitsur et al. [3]. Since then there have been several constructions, including Tignol [27], Jacob and Wadsworth [17], Jacob [16], Schofield and Van den Bergh [25], Brussel [4] and McKinnie [21].

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2. Patching over Fields

Our construction is based on the method of *patching over fields* introduced in Harbater and Hartmann [14]. In this section, we will recall this method. Throughout this section, T will be a complete discrete valuation ring with uniformizer t, fraction field K and residue field k. Let $\hat{X}$ be a smooth projective T-curve with function field F such that the reduced irreducible components of its closed fibre X is regular. (Given F, such an $\hat{X}$ always exists by resolution of singularities; cf. Abhyankar [1] or Lipman [20]). Let $f : \hat{X} \to \mathbb{P}^1_T$ be a finite morphism such that the inverse image S of $\infty \in \mathbb{P}^1_k$ contains all the points of X at which distinct irreducible components meet. (Such a morphism exists by Harbater and Hartmann [14, Proposition 6.6]). We will call $(\hat{X}, S)$ a *regular T-model* of F.

We follow Harbater and Hartmann [14, Section 6] to introduce the notation. Given an irreducible component X_0 of X with generic point η, consider the local ring of $\hat{X}$ at η. For a (possibly empty) proper subset U of X_0, we let R_U denote the subring of this local ring consisting of rational functions that are regular at each point of U. In particular, R_ϕ is the local ring of $\hat{X}$ at the generic point of the component X_0. The t-adic completion of R_U is denoted by $\hat{R}_U$. If P is a closed point of X, we write R_P for the local ring of $\hat{X}$ at P, and $\hat{R}_P$ for its completion at its maximal ideal. A height 1 prime ideal $\mathfrak{p}$ that contains t determines a *branch* of X at P, i.e., an irreducible component of the pullback of X to $\mathrm{Spec}(\hat{R}_P)$. Similarly the contraction of $\mathfrak{p}$ to the local ring of $\hat{X}$ at P determines an irreducible component X_0 of X, and we say that $\mathfrak{p}$ *lies on* X_0. Note that a branch $\mathfrak{p}$ uniquely determines a closed point P and an irreducible component X_0. In general, there can be several branches $\mathfrak{p}$ on X_0 at a point P; but if X_0 is smooth at P then there is a unique branch $\mathfrak{p}$ on X_0 at P. We write $\hat{R}_\mathfrak{p}$ for the completion of the localization of $\hat{R}_P$ at $\mathfrak{p}$; thus $\hat{R}_P$ is contained in $\hat{R}_\mathfrak{p}$, which is a complete discrete valuation ring.

Since $\hat{X}$ is normal, the local ring R_P is integrally closed and hence unibranched; and since T is a complete discrete valuation ring, R_P is excellent and hence $\hat{R}_P$ is a domain (cf. Grothendieck and Dieudonné [13, Scholie 7.8.3(ii,iii,vii)]). For nonempty U as above and $Q \in U$, $\hat{R}_U/t^n\hat{R}_U \to \hat{R}_Q/t^n\hat{R}_Q$ is injective for all n and hence $\hat{R}_U \to \hat{R}_Q$ is also injective. Thus $\hat{R}_U$ is also a domain. Note that the same is true if U is empty. The fraction field s of the domains $\hat{R}_U, \hat{R}_P$ and $\hat{R}_\mathfrak{p}$ will be denoted by $F_U, F_P$and $F_\mathfrak{p}$.

If $\mathfrak{p}$ is a branch at P lying on the closure of $U \subset X_0$, then there are natural inclusions of $\hat{R}_P$ and $\hat{R}_U$ into $\hat{R}_\mathfrak{p}$, and hence of F_P and F_U into $F_\mathfrak{p}$. The inclusion of $\hat{R}_P$ was observed above; for $\hat{R}_U$, note that the localization of R_U and of R_p at the generic point of X_0 are the same; and this localization is naturally contained in the t-adically complete ring $\hat{R}_{\mathfrak{p}}$. Thus so is R_U and hence its t-adic completion $\hat{R}_U$.

In the above context, assume $f : \hat{X} \to \mathbb{P}^1_T$ is a finite morphism such that $\mathfrak{P} = f^{-1}(\infty)$ contains all points at which distinct irreducible components of the closed fibre $X \subset \hat{X}$ meet (Such an f always exists by Harbater and Hartmann [14, Proposition 6.6]). We let $\mathfrak{U}$ be the collection of irreducible components U of $f^{-1}(\mathbb{A}^1_k)$, and let $\mathfrak{B}$ be the collection of all branches $\mathfrak{p}$ at all points of $\mathfrak{P}$.

The inclusions of $\hat{R}_U$ and of $\hat{R}_Q$ into $\hat{R}_{\mathfrak{p}}$, for $\mathfrak{p} = (U, Q)$, induce inclusions of the corresponding fraction fields F_U and F_Q into the fraction field $F_{\mathfrak{p}}$ of $\hat{R}_{\mathfrak{p}}$. Let I be the index set consisting of all $U, Q, \mathfrak{p}$ described above. Via the above inclusions, the collection of all F_ξ, for $\xi \in I$, then forms an inverse system with respect to the ordering given by setting $U \succ \mathfrak{p}$ and $Q \succ \mathfrak{p}$ if $\mathfrak{p} = (U, Q)$.

Under the above hypotheses, suppose that for every field extension L of F, we are given a category $\mathfrak{A}(L)$ of algebraic structures over L(i.e. finite dimensional L-vector spaces with additional structure, e.g. associative L-algebras), along with base-change functors $\mathfrak{A}(L) \to \mathfrak{A}(L')$ when $L \subseteq L'$. An $\mathfrak{A}$-*patching problem* for $(\hat{X}$,$S)$ consists of an object V_ξ in $\mathfrak{A}(F_\xi)$ for each $\xi \in I$, together with isomorphisms $\phi_{U,\mathfrak{p}} : V_U \otimes_{F_U} F_{\mathfrak{p}} \to V_{\mathfrak{p}}$ and $\phi_{Q,\mathfrak{p}} : V_Q \otimes_{F_Q} F_{\mathfrak{p}} \to V_{\mathfrak{p}}$ in $\mathfrak{A}(F_{\mathfrak{p}})$. These patching problems form a category, denoted by $\mathrm{PP}_{\mathfrak{A}}(\hat{X}, S)$, and there is a base change functor $\mathfrak{A}(F) \to \mathrm{PP}_{\mathfrak{A}}(\hat{X}, S)$.

If an object $V \in \mathfrak{A}(F)$ induces a given patching problem up to isomorphism, we will say that V is a *solution* to that patching problem, or that it is *obtained by patching* the objects V_ξ. We similarly speak of obtaining a morphism over F by patching morphisms in $\mathrm{PP}_{\mathfrak{A}}(\hat{X}, S)$. The next result is given by Harbater and Hartmann [14, Theorem 7.2].

Theorem 1. *Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth connected projective T-curve with closed fibre X. Let $U_1, U_2 \subseteq X$, let $U_0 = U_1 \cap U_2$, and let $F_i := F_{U_i} (i = 0, 1, 2)$. Let $U = U_1 \cup U_2$ and form the fibre product of groups $\mathrm{Br}(F_1) \times_{\mathrm{Br}(F_0)} \mathrm{Br}(F_2)$ with respect to the maps $\mathrm{Br}(F_i) \to \mathrm{Br}(F_0)$ induced by $F_i \hookrightarrow F_0$. Then the base change map $\beta : \mathrm{Br}(F_U) \to \mathrm{Br}(F_1) \times_{\mathrm{Br}(F_0)} \mathrm{Br}(F_2)$ is a group isomorphism.*

The above Theorem says that giving a Brauer class over a function field F is equivalent to giving compatible division algebras over the patches. The nice thing about patching Brauer classes over a function field F is that we have good control of the index, which is stated in Harbater et al. [15, Theorem 5.1].

Theorem 2. *Under the above notation, let A be a central simple F-algebra. Then* $\mathrm{ind}(A) = \mathrm{lcm}_{\xi \in \mathfrak{P} \cup \mathfrak{U}}(\mathrm{ind}(A_{F_\xi}))$.

To conclude this section, we record a variant of Hensel's Lemma from Harbater et al. [15, Lemma 4.5] that will be used over and over again in the index computation.

Lemma 3. *Let R be a ring and I an ideal such that R is I-adically complete. Let X be an affine R-scheme with structure morphism $\phi : X \to \mathrm{Spec} R$. Let $n \geq 0$. If $s_n : \mathrm{Spec}(R/I^n) \to X \times_R (R/I^n)$ is a section of $\phi_n : \phi \times_R (R/I^n)$ and its image lies in the smooth locus of ϕ, then s_n may be extended to a section of ϕ.*

3. Splitting Map

Let T be a complete discrete valuation ring with uniformizer t and residue field k. By a smooth curve $\hat{X}$ over T, we will mean a scheme $\hat{X}$ which is projective and smooth of relative dimension

1 over Spec(T). In particular, $\hat{X}$ is flat and of finite presentation over Spec(T). Let $F = K(\hat{X})$ be the function field of $\hat{X}$. Note that since $\hat{X}$ is smooth, the closed fibre X is smooth, integral, connected and of codimension 1, hence determines a discrete valuation ring on F. Let $\hat{F} = \hat{K}(\hat{X})$ be the completion of F with respect to this discrete valuation. Throughout the paper, n will denote an integer which is prime to the characteristic of k.

We will be using the following notation for cohomology groups in the sequel: For an integer r, we let

$$\mu_n^r = \begin{cases} \mu_n^{\otimes r} & \text{for } r \geq 0, \\ \hom(\mu_n^{\otimes -r}, \mu_n) & \text{for } r < 0. \end{cases}$$

For a fixed integer n, and for any field K, we will let $\mathrm{H}^q(K, r) = \mathrm{H}^q(K, \mu_n^{\otimes r})$ and $\mathrm{H}^q(K) = \mathrm{H}^q(K, q-1) = \mathrm{H}^q(K, \mu_n^{\otimes q-1})$. In particular, $\mathrm{H}^2(K) = {}_n\mathrm{Br}(K)$ will be the n-torsion part of the Brauer group of K; and $\mathrm{H}^1(K)$ will be the n-torsion part of the character group of K.

Adopting the above notation, in this section we will define a map $s : \mathrm{H}^2(\hat{F}) \to \mathrm{H}^2(F)$ and show that s has the following properties:

- s is a group homomorphism;
- s splits the restriction;
- s preserves index of Brauer classes.

Once such a map s is defined, we could use it to construct indecomposable division algebras and noncrossed product division algebras over F, as in section 5.

3.1. Construction over an Open Affine Subset

Given an element $\hat{\gamma} \in \mathrm{H}^2(\hat{F})$, we will define a lift γ_U to F_U of $\hat{\gamma}$. Note that since $\hat{F}$ is a complete discretely valued field with t a unifomizer, and with $k(X)$ the residue field. We have an exact Witt Sequence as in Garibaldi et al. [10, II.7.10 and II.7.11],

$$0 \to \mathrm{H}^2(k(X)) \to \mathrm{H}^2(\hat{F}) \to \mathrm{H}^1(k(X)) \to 0 \tag{1}$$

split (non-canonically) by the cup product with $(t) \in \mathrm{H}^1(k(X))$. Hence each element $\hat{\gamma} \in \mathrm{H}^2(\hat{F})$ can be written as a sum $\gamma_0 + (\chi_0, t)$, with $\gamma_0 \in \mathrm{H}^2(k(X))$ and $\chi_0 \in \mathrm{H}^1(k(X))$ (Note that here we are identifying $\mathrm{H}^r(k(X))$ as a subgroup of $\mathrm{H}^r(\hat{F})$, for $r = 1, 2$, as in Garibaldi et al. [10, II.7.10 and II.7.11]). Here we use the notation (χ_0, t) to denote the cup product $\chi_0 \cup (t)$, and we will use this notation throughout the paper without further explanation.

Let U be an open affine subset of X so that neither γ_0 nor χ_0 ramifies at any closed point of U. This implies that $\gamma_0 \in \mathrm{H}^2(k[U])$ and $\chi_0 \in \mathrm{H}^1(k[U])$ by purity(cf, Colliot-Thélène [8]), where $k[U]$ denotes the ring of regular functions of the affine scheme U.

By Cipolla [7], there exists a canonical isomorphism $\mathrm{H}^2(\hat{R}_U) \to \mathrm{H}^2(k[U])$ since $\hat{R}_U$ is t-adically complete and $k[U] \cong \hat{R}_U/(t)$; therefore there is a unique lift of γ_0 to $\mathrm{H}^2(\hat{R}_U)$. At the same time, Grothendieck and Raynaud [12, Théorèm 8.3] implies that there is a unique lift of χ_0 to $\mathrm{H}^1(\hat{R}_U)$ as well. Taking $\tilde{\gamma}_0$ and $\tilde{\chi}_0$ as the lifts of γ_0 and χ_0 to $\hat{R}_U$, we will let

$$\gamma_U = \tilde{\gamma}_0 + (\tilde{\chi}_0, t) \tag{2}$$

be the lift of $\hat{\gamma}$ to $\mathrm{H}^2(F_U)$.

3.2. Construction over Closed Points

Fix an open affine subset U of X and let $\mathfrak{P} = X\backslash U$. In order to apply the patching result we recalled in 2, we need to define a γ_P for each $P \in \mathfrak{P}$ in such a way that when $\mathfrak{p} = (U, P)$ is the unique branch of U at P, the restriction to $F_{\mathfrak{p}}$ of γ_P and γ_U agree with each other, i.e., $\mathrm{res}_{F_{\mathfrak{p}}}(\gamma_P) = \mathrm{res}_{F_{\mathfrak{p}}}(\gamma_U)$ (Recall there are field embeddings $F_P \hookrightarrow F_{\mathfrak{p}}$ and $F_U \hookrightarrow F_{\mathfrak{p}}$ for $\mathfrak{p} = (U, P)$, as in Section 2, hence there are restrictions $\mathrm{res} : \mathrm{H}^2(F_U) \to \mathrm{H}^2(F_{\mathfrak{p}})$ and $\mathrm{res} : \mathrm{H}^2(F_P) \to \mathrm{H}^2(F_{\mathfrak{p}})$. For more details on these restriction maps, see Serre [26]).

Note that since $\hat{X}$ is regular and the closed fibre X is smooth, the maximal ideal of the local ring R_P is generated by two generators, t and π. So is $\hat{R}_P$.

We define γ_P in the following way: There is a field embedding $F_U \to F_{\mathfrak{p}}$, hence a canonical restriction $\mathrm{res} : \mathrm{H}^2(F_U) \to \mathrm{H}^2(F_{\mathfrak{p}})$. Let $\gamma_{\mathfrak{p}}$ be the image of γ_U under this restriction. Observe that $F_{\mathfrak{p}}$ is a complete discretely valued field with residue field $\kappa(\mathfrak{p})$; furthermore, $\kappa(\mathfrak{p})$ is also a complete discretely valued field with residue field $\kappa(P)$. Therefore, applying Garibaldi et al. [10, II.7.10 and II.7.11] twice, we get the following decomposition of $\mathrm{H}^2(F_{\mathfrak{p}})$:

$$\mathrm{H}^2(F_{\mathfrak{p}}) \cong \mathrm{H}^2(\kappa(P)) \oplus \mathrm{H}^1(\kappa(P)) \oplus \mathrm{H}^1(\kappa(P)) \oplus \mathrm{H}^0(\kappa(P)). \tag{3}$$

In other words, each element $\gamma_{\mathfrak{p}} \in \mathrm{H}^2(F_{\mathfrak{p}})$ can be written as $\gamma_{\mathfrak{p}} = \gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi^r), t)$, where $\gamma_{0,0} \in \mathrm{H}^2(\kappa(P)), \chi_1, \chi_2 \in \mathrm{H}^1(\kappa(P)), r \in \mathrm{H}^0(\kappa(P)) \cong \mathbb{Z}/n\mathbb{Z}$ and (π^r) denote the image in $\mathrm{H}^1(\kappa(P))$ of π^r under the Kummer map. Note that by our notation, $\mathrm{H}^0(\kappa(P)) = \mathrm{H}^0(\kappa(P), \mu_n^{-1}) = \mathbb{Z}/n\mathbb{Z}$.

In order to define a lift for $\gamma_{\mathfrak{p}}$ to F_P, we first show that all characters in $\mathrm{H}^1(\kappa(\mathfrak{p}))$ can be lifted by proving the following lemma.

Lemma 4. *Let $\chi \in \mathrm{H}^1(\kappa(\mathfrak{p}))$ be a character. Then there is a unique $\tilde{\chi} \in \mathrm{H}^1(F_P)$ that lifts χ.*

Proof. Since $\kappa(\mathfrak{p})$ is a complete discretely valued field with residue field $\kappa(P)$, we have the classical Witt's decomposition for χ,

$$\chi = \chi_0 + (\pi^r),$$

where $\chi_0 \in \mathrm{H}^1(\kappa(P))$ and $r \in \mathrm{H}^0(\kappa(P))$. Note that χ_0 can be lifted without any difficulty by Grothendieck and Raynaud [12, Théorèm 8.3]; the only trouble comes from (π^r).

Let $L, L_0/\kappa(\mathfrak{p})$ be the field extension determined by χ, χ_0 respectively. Then L_0 is the maximal unramified subextension of $\kappa(\mathfrak{p})$ inside L and L/L_0 is a totally ramified extension determined by the character (π^r). Now Fesenko and Vostokov [9, Theorem II.3.5] implies that (π^r) can be lifted to $\mathrm{H}^1(F_P)$ in a unique fashion as well, since $\kappa(\mathfrak{p})$ is a complete discretely valued field. □

Now we are ready to define a lift for $\hat{\gamma}$ in $\mathrm{H}^2(F_P)$. Again Cipolla [7] implies that $\mathrm{H}^2(\kappa(P)) \cong \mathrm{H}^2(\hat{R}_P)$ and Lemma 4 implies that $\chi_1, \chi_2 + (\pi^r)$ can be lifted to $\mathrm{H}^1(\hat{R}_P)$ uniquely. Hence each component of $\mathrm{H}^2(F_{\mathfrak{p}})$ can be lifted to $\hat{R}_P$, and thus we will set

$$\gamma_P = \tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi) + (\tilde{\chi}_2 + (\pi^r), t). \tag{4}$$

where $\tilde{\gamma}, \tilde{\chi}_1, \tilde{\chi}_2$ are the lifts of $\gamma_{0,0,}\chi_1, \chi_2$ to $\hat{R}_P$ (and hence to F_P), respectively. Therefore this γ_P is a unique lift of $\gamma_{\mathfrak{p}}$ to F_P. The assignment of $s_P(\gamma_{\mathfrak{p}}) = s_P$ will yield a map $s_P : \mathrm{H}^2(F_{\mathfrak{p}}) \to \mathrm{H}^2(F_P)$. It is not hard to see that s_P is a group homomorphism, since it is a group homomorphism on each of the components.

3.3. The Map is Well Defined

In this section we show that γ_U and γ_P that we constructed in Section 3.1 and Section3.2 are compatible in the sense of patching, that is $\mathrm{res}_{F_\mathfrak{p}}(\gamma_U) = \mathrm{res}_{F_\mathfrak{p}}(\gamma_P)$ for each $P \in \mathfrak{P} = X \backslash U$ when $\mathfrak{p} = (U, P)$ is the unique branch of U at P.

We claim that the compatibility will be proved if we can show that s_P splits the restriction map $\mathrm{res}_{F_\mathfrak{p}} : \mathrm{H}^2(F_P) \to \mathrm{H}^2(F_\mathfrak{p})$, or equivalently, $\mathrm{res}_{F_\mathfrak{p}} \circ s_P$ is the identity map. This is true because $\gamma_P = s_P(\gamma_\mathfrak{p}) = s_P \circ \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$, hence we would have that $\mathrm{res}_{F_\mathfrak{p}}(\gamma_P) = \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$ if $\mathrm{res}_{F_\mathfrak{p}} \circ s_P$ is the identity map. So it suffices to prove the following

Proposition 5. *s_P as defined in 3.2 splits the restriction* $\mathrm{res} : \mathrm{H}^2(F_P) \to \mathrm{H}^2(F_\mathfrak{p})$*, that is,* $\mathrm{res} \circ s_P$ *is the identity map.*

Proof. Take an arbitrary element $\gamma_\mathfrak{p} \in \mathrm{H}^2(F_\mathfrak{p})$. As in section 3.2, we write $\gamma_\mathfrak{p} = \gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi^r), t)$. Therefore it is easily checked that

$$\begin{aligned} \mathrm{res} \circ s_P(\gamma_\mathfrak{p}) &= \mathrm{res} \circ s_P(\gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi^r), t)) \\ &= \mathrm{res}(\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi) + (\tilde{\chi}_2 + (\pi^r), t)) \\ &= \gamma_{0,0} + (\chi_1, \pi) + (\chi_2 + (\pi^r), t) \\ &= \gamma_\mathfrak{p}. \end{aligned}$$

□

Thus γ_U, γ_P will patch and yield $\gamma \in \mathrm{H}^2(F)$, by Harbater and Hartmann [14, Theorem 7.2]. But there is one more thing we have to check before we can say we have a map $s : \mathrm{H}^2(\hat{F}) \to \mathrm{H}^2(F)$: we need to show that γ is independent of the choice of the open affine subset U of X. In order to do this, we prove the following

Lemma 6. *Let T be a complete discrete valuation ring with residue field k; let $\hat{X}$ be a smooth projective T-curve with function field F and closed fibre X. Let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by X, and denote by $k(X)$ the corresponding residue field. Take an element $\hat{\gamma} = \gamma_0 + (\chi_0, t) \in \mathrm{H}^2(\hat{F})$, where $\gamma_0 \in \mathrm{H}^2(k(X))$ and $\chi_0 \in \mathrm{H}^1(k(X))$. Assume that U_1, U_2 are two open affine subsets of X so that neither γ_0, χ_0 is ramified on any point of $U_1 \cup U_2$. Let $\mathfrak{P}_1, \mathfrak{P}_2$ be the complements of U_1, U_2 respectively. We construct two Brauer classes $\gamma, \gamma' \in \mathrm{H}^2(F)$ by patching as we did above, while using U_1 and U_2 as the open affine subset in the construction, respectively. Then γ, γ' denote the same Brauer class in $\mathrm{H}^2(F)$.*

Proof. We first deal with the case where U_1 is contained in U_2. In this case we have a field embedding $F_{U_2} \hookrightarrow F_{U_1}$. Let γ_i be the lift of γ_0 to $\mathrm{H}^2(F_{U_i})$, we must have $\gamma_1 = \mathrm{res}_{F_{U_1}}(\gamma_2)$, since both γ_1 and γ_2 are the image of γ_0; in other words, $\mathrm{res}_{F_{U_2}}(\gamma) = \mathrm{res}_{F_{U_2}}(\gamma')$. By the construction in Section 3.2, it follows that for every $P \in \mathfrak{P}_2$, $\mathrm{res}_{F_P}(\gamma) = \mathrm{res}_{F_P}(\gamma')$. Therefore it follows that $\gamma = \gamma'$, by Harbater and Hartmann [14, Theorem 7.2]. This proves the Lemma in the case where U_1 is contained in U_2.

In the general case, let U_3 be an open affine subset of $U_1 \cap U_2$. Clearly γ_0 and χ_0 are both unramified at every point of U_3. Let $\gamma'' \in \mathrm{H}^2(F)$ be the Brauer class constructed by patching as above, using U_3 as the open affine subset in the construction. It follows that $\gamma'' = \gamma$ and $\gamma'' = \gamma'$ since U_3 is contained in both U_1 and U_2, by what we just proved for the case where one open affine subset is contained in the other. Hence $\gamma = \gamma' = \gamma'' \in \mathrm{H}^2(F)$, which proves the Lemma in the general case. □

3.4. s Splits the Restriction Map

Recall the notation: let T be a complete discrete valuation ring with residue field k and uniformizer t. Let $\hat{X}$ be a smooth projective T-curve with function field F and closed fibre X. Let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by X. Let $s : \mathrm{H}^2(\hat{F}) \to \mathrm{H}^2(F)$ be the map defined by patching as in section 3.1 and section 3.2. We will show that s splits the restriction map $\mathrm{res} : \mathrm{H}^2(F) \to \mathrm{H}^2(\hat{F})$. Hence index of Brauer classes cannot go up under the map s, because restriction can never raise index. In particular, we prove the following Proposition.

Proposition 7. *The map s is a section to the restriction map $\mathrm{res}_{\hat{F}} : \mathrm{H}^2(F) \to \mathrm{H}^2(\hat{F})$.*

Proof. It suffices to show that $\mathrm{res} \circ s$ is the identity map on $\mathrm{H}^2(\hat{F})$. Since $\mathrm{H}^2(\hat{F}) \cong \mathrm{H}^2(k(X)) \oplus \mathrm{H}^1(k(X))$, it suffices to show that $\mathrm{res}_{\hat{F}} \circ s$ is the identity map on both components; that is, given $\hat{\gamma} = \gamma_0 + (\chi_0, t)$ where $\gamma_0 \in \mathrm{H}^2(k(X))$ and $\chi_0 \in \mathrm{H}^1(k(X))$, the Proposition will follow if we can show that $\mathrm{res}_{\hat{F}} \circ s(\gamma_0) = \gamma_0$ and $\mathrm{res}_{\hat{F}} \circ s((\chi_0, t)) = (\chi_0, t)$.

Take an open affine subset U of X so that γ_0, χ_0 are both unramified on every point of U; that is, we have $\gamma_0 \in \mathrm{H}^2(k[U])$ and $\chi_0 \in \mathrm{H}^1(k[U])$. Note that we have the following commutative diagram (For a field E, $\mathrm{H}^2_{\mathrm{nr}}(E)$ denotes the unramified part of $\mathrm{H}^2_{\mathrm{nr}}(E)$, or equivalently, $\mathrm{H}^2_{\mathrm{nr}}(E) = \cap_v \mathrm{H}^2(E_v)$, where v runs through all discrete valuations on E, and E_v denotes the completion of E at v. See Colliot-Thélène [8] for more details on the unramified cohomology.):

$$\begin{tikzcd}
\mathrm{H}^2(k(X)) \arrow[rr, "\sim", "f"'] & & \mathrm{H}^2_{\mathrm{nr}}(\hat{F}) \arrow[d, "s"] \\
\mathrm{H}^2(\hat{R}_U) \arrow[u, hook, "g"] \arrow[d, hook, "h"] & & \mathrm{H}^2(F) \arrow[dll, "\mathrm{res}_{F_U}"] \\
\mathrm{H}^2(F_U) & &
\end{tikzcd}$$

The commutativity of the above diagram follows simply from the construction of over open affine subset we outline in Section 3.1. Therefore $\mathrm{res}_{\hat{F}}$ on $s(\gamma_0)$ is the same as $f \circ g \circ h^{-1} \circ \mathrm{res}_{F_U}$, and thus $\mathrm{res}_{\hat{F}} \circ s(\gamma_0) = f \circ g \circ h^{-1} \circ \mathrm{res}_{F_U} \circ s(\gamma_0) = \gamma_0$. (Note in fact h has no inverse; however we can find an inverse image under h for $\mathrm{res}_{F_U} \circ s(\gamma_0)$, so we write h^{-1} only merely as a shorthand notation here.)

To show that $\mathrm{res}_{\hat{F}} \circ s((\chi_0, t)) = (\chi_0, t)$, it suffices to show that $\mathrm{ram}(\mathrm{res}_{\hat{F}} \circ s((\chi_0, t))) = \chi_0$, where $\mathrm{ram} : \mathrm{H}^2(\hat{F}) \to \mathrm{H}^1(k(X))$ denotes the ramification map on $\mathrm{H}^2(\hat{F})$ with respect to the valuation determined by the closed fibre X. Since $\chi_0 \in \mathrm{H}^1(k[U])$, we have $\mathrm{ram}(\mathrm{res}_{\hat{F}} \circ s((\chi_0, t))) = \mathrm{ram}((\tilde{\chi}_0, t))$ where $\tilde{\chi}_0$ denotes the lift of χ_0 to $\mathrm{H}^1(\hat{R}_U)$, as we did in Section 3.1 (Since $\mathrm{H}^1(\hat{R}_U) \cong \mathrm{H}^1(k[U])$, $\tilde{\chi}_0$ can be viewed as as element of $\mathrm{H}^1(k[U])$, and hence element of $\mathrm{H}^1(k(X))$ via the injection $\mathrm{H}^1(k[U]) \hookrightarrow \mathrm{H}^1(k(X))$, and finally element of $\mathrm{H}^1(\hat{F})$ via the injection $\mathrm{H}^1(k(X)) \hookrightarrow \mathrm{H}^1(\hat{F})$). Therefore the image in $\mathrm{H}^1(\hat{F})$ of $\tilde{\chi}_0$ under the composition of these maps is in fact χ_0, since all these maps are injective. Then it is easy to see that $\mathrm{ram}((\tilde{\chi}_0, t)) = \tilde{\chi}_0 = \chi_0 \in \mathrm{H}^1(k[X])$, as desired. □

The following corollary is immediate:

Corollary 8. *Index of Brauer classes cannot go down under the map s.*

Proof. Take $\hat{\gamma} \in \mathrm{H}^2(\hat{F})$ and let $\gamma = s(\hat{\gamma})$. By Proposition 7 we must have that $\hat{\gamma} = \mathrm{res}_{\hat{F}}(\gamma)$, therefore $\mathrm{ind}(\hat{\gamma}) | \mathrm{ind}(\gamma)$. This proves that s can never lower index of Brauer classes. □

4. s Preserves Index of Brauer Classes

In this section, we will show that the splitting map s that we defined in section 3 has one more property that is crucial to the construction of indecomposable and noncrossed product division algebras over p-adic curves, that is, s preserves index of Brauer classes. In other words, $\mathrm{ind}(\hat{\gamma}) = \mathrm{ind}(\gamma) = \mathrm{ind}(s(\hat{\gamma}))$. We make the following elementary observation, which is true for Brauer classes over an arbitrary field.

Proposition 9. *Let k be an arbitrary field. Let $\gamma \in \mathrm{H}^2(k)$ be a Brauer class with the following decomposition: $\gamma = \gamma_0 + (\chi, t)$, where $\gamma_0 \in \mathrm{H}^2(k)$, $\chi \in \mathrm{H}^1(k)$ and t is an arbitrary element of k. Then $\mathrm{ind}(\gamma)|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$, where $\gamma_{0,l}$ denotes the base extension of γ_0 to l/k, where l is the field extension determined by χ.*

Proof. Let E/l be a minimal extension that splits $\gamma_{0,l}$. Then $[E : l] = \mathrm{ind}(\gamma_{0,l})$. Also there is some E'/k with $[E' : k] = \exp(\chi)$ which splits χ and hence (χ, t); therefore EE' will split γ, furthermore it is not hard to see that $[EE' : k]|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$ and hence $\mathrm{ind}(\gamma)|\mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi)$. □

We will apply Harbater et al. [15, Theorem 5.1], which states that $\mathrm{ind}(\gamma) = \mathrm{lcm}(\mathrm{ind}(\gamma_U), \mathrm{ind}(\gamma_P))$ for each $P \in \mathfrak{P}$. Since we already showed that s can never lower index of Brauer classes as in section 3.4, we will be done if we could show that $\mathrm{ind}(\gamma)|\mathrm{ind}(\hat{\gamma})$; therefore it suffices to show that $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\hat{\gamma})$ and $\mathrm{ind}(\gamma_P)|\mathrm{ind}(\hat{\gamma})$ for each $P \in \mathfrak{P}$, respectively. We will deal with them in order.

We start by recalling the notion of *Azumaya algebras* and their generalized Severi-Brauer varieties. The notion of a central simple algebra over a field can be generalized to the notion of an *Azumaya algebra* over a domain R (cf. Saltman [24, Chapter 2], or Grothendieck [11, Part I, Section 1]). The degree of an Azumaya algebra A over R is the degree of $A \otimes_R F$ as a central simple algebra over the fraction field F over R. The *Brauer group* of a domain R is defined as the set of equivalence classes of Azumaya algebras with the analogous operations, where one replaces the vector spaces V_i with projective modules in the definition of Brauer equivalences. If A is an Azumaya algebra of degree n over a domain R, and $1 \leq i < n$, there is a functorially associated smooth projective R-scheme $\mathrm{SB}_i(A)$, called the i-th *generalized Severi-Brauer variety of* A (cf. Van den Bergh [28, p. 334]). For each R-algebra S, the S-points of $\mathrm{SB}_i(A)$ are in bijection with the right ideals of $A_S = A \otimes_R S$ that are direct summands of the S-module A_S having dimension (i.e. S-rank) ni. If R is a field F, so that A is a central simple F-algebra, and if E/F is a field extension, then $\mathrm{SB}_i(A)(E) \neq \phi$ if and only if $\mathrm{ind}(A_E)$ divides i (cf. Knus et al. [19, Proposition 1.17]). Here $A_E \cong \mathrm{Mat}_m(D_E)$ for some E-division algebra D_E and some $m \geq 1$, and the right ideals of E-dimension ni are in natural bijection with the subspaces of D_E^m of D_E-dimension $i/\mathrm{ind}(A_E)$ (cf. Knus et al. [19, Proposition 1.12, Definition 1.9]). Thus the F-linear algebraic group $\mathrm{GL}_1(A) = \mathrm{GL}_m(D_F)$ acts transitively on the points of the F-scheme $\mathrm{SB}_i(A)$. We record Knus et al. [19, Proposition 1.17] here since we will be using it over and over again in the sequel.

Proposition 10. *Let A be a central simple algebra over a field F. The Severi-Brauer variety $\mathrm{SB}_r(A)$ has a rational point over an extension K/F if and only if the index $\mathrm{ind}(A_K)$ divides r. In particular, $\mathrm{SB}(A)$ has a rational point over K if and only if K splits A.*

4.1. Index Computation Over Affine Open Set

We compute $\mathrm{ind}(\gamma_U)$ in this section; in particular, we show that $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\hat{\gamma})$. Thanks to Lemma 6, it suffices to show that there exists an open affine subset $V \subset X$ so that $\mathrm{ind}(\gamma_V)|\mathrm{ind}(\hat{\gamma})$ since we could replace U by V if necessary in the construction we outlined in section 3.1 and this would not change $\gamma \in \mathrm{H}^2(K(\hat{X})$ by Lemma 6. Therefore we will prove the following proposition, which shows that there exists such an open affine subset V.

Proposition 11. *Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth projective T-curve with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation determined by X. Then for every $\hat{\gamma} \in \mathrm{H}^2(\hat{F})$, there exists an affine open subset $V \subset X$ such that* $\mathrm{ind}(\gamma_V)|\mathrm{ind}(\hat{\gamma})$*, where γ_V is the lift of $\hat{\gamma}$ to F_V as defined in section 3.1.*

Proof. Recall that $\hat{\gamma} = \gamma_0 + (\chi_{0,}t) \in \mathrm{H}^2(\hat{F})$ where $\gamma_0 \in \mathrm{H}^2(k(X))$ and $\chi_0 \in \mathrm{H}^1(k(X))$. Therefore $\mathrm{ind}(\hat{\gamma}) = \mathrm{ind}(\gamma_{0,l}) \cdot \exp(\chi_0)$, where $l/k(X)$ is the field extension determined by χ_0, by Jacob and Wadsworth [17, Theorem 5.15], since $\hat{F}$ is a complete discretely valued field.

Let U be an open affine subset of X such that neither γ_0 nor χ_0 ramifies on any point of U. Recall that $\gamma_U = \tilde{\gamma}_0 + (\tilde{\chi}_0, t)$ where $\tilde{\gamma}_0 \in \mathrm{H}^2(\hat{R}_U)$ and $\tilde{\chi}_0 \in \mathrm{H}^1(\hat{R}_U)$. Note that $\exp(\tilde{\chi}_0) = \exp(\chi_0)$ since $\mathrm{H}^1(\hat{R}_U) \cong \mathrm{H}^1(k(X))$. By Proposition 9, we have $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\tilde{\gamma}_{0,S}) \cdot \exp(\tilde{\chi}_0)$, where $S/\hat{R}_U$ denotes the Galois cyclic extension determined by $\tilde{\chi}_0$. Note when $V \subseteq U$, we have $\mathrm{H}^r(k[U]) \subseteq \mathrm{H}^r(k[V])$ by purity, and hence $\mathrm{H}^r(\hat{R}_U) \subseteq \mathrm{H}^r(\hat{R}_V)$; so we have $\tilde{\gamma}_0 \in \mathrm{H}^2(\hat{R}_V)$ and $\tilde{\chi}_0 \in \mathrm{H}^1(\hat{R}_V)$. Therefore it suffices to find some affine open subset $V \subset U$ such that $\mathrm{ind}(\tilde{\gamma}_{0,S'})|\mathrm{ind}(\gamma_{0,l})$, where $S'/\hat{R}_V$ denotes the Galois cyclic extension determined by $\tilde{\chi}_0$.

Let $i = \mathrm{ind}(\gamma_{0,l})$ be the index of the restriction of γ_0 to l. Then Proposition 10 implies that $\mathrm{SB}_i(\gamma_0)(l) \neq \phi$; in other words, there is an l-rational point in the i-th generalized Severi-Brauer variety of γ_0. Hence the $\mathrm{Spec}(k(X))$-morphism $\pi : \mathrm{SB}_i(\gamma_0) \times_{k(X)} l \to \mathrm{Spec}(l)$ has a section $\mathrm{Spec}(l) \to \mathrm{SB}_i(\gamma_0) \times_{k(X)} l$ over $\mathrm{Spec}(k(X))$, the generic point of the closed fibre U of $\mathrm{Spec}(\hat{R}_U)$. Choose a Zariski dense open subset $V \subseteq U$ such that this section over $\mathrm{Spec}(k(X))$ extends to a section over V, and such that the image of this latter section lies in an open subset of $\mathrm{SB}_i(\gamma_0) \times_{k(X)} l$ that is affine over $\hat{R}_U$. Then by Lemma 3, the section over V lifts to a section over $\mathrm{Spec}(\hat{R}_V)$, thus we obtain an L-rational point of $\mathrm{SB}_i(\tilde{\gamma}_0) \times_{\hat{R}_V} S'$, where L/F_V is the Galois cyclic extension determined by $\tilde{\chi}_0$; or equivalently, L is the fraction field of S'. This implies that $\mathrm{ind}(\tilde{\gamma}_{0,S'})|i = \mathrm{ind}(\gamma_{0,l})$ by Proposition 10 again. ☐

4.2. Index Computation Over Closed Points

It remains to show $\mathrm{ind}(\gamma_P)|\mathrm{ind}(\hat{\gamma})$. This is what we are going to do in this section. Note that γ_P is defined as $s_P \circ \mathrm{res}_{F_\mathfrak{p}}(\gamma_U)$, where $\mathrm{res}_{F_\mathfrak{p}}$ can only lower index of γ_U. Since we have already shown that $\mathrm{ind}(\gamma_U)|\mathrm{ind}(\hat{\gamma})$, we have that $\mathrm{ind}(\gamma)$ will be completely determined by $\mathrm{ind}(\gamma_U)$ if we could show that $\mathrm{ind}(\gamma_\mathfrak{p})$ does not go up under the map s_P. Therefore we just need to show that s_P cannot increase index of Brauer classes, or, $\mathrm{ind}(\gamma_P) = \mathrm{ind}(s_P(\gamma_\mathfrak{p}))|\mathrm{ind}(\gamma_\mathfrak{p})$.

We compute $\mathrm{ind}(\gamma_\mathfrak{p})$ first. Since $F_\mathfrak{p}$ is a complete discretely valued field, we have $\mathrm{ind}(\gamma_\mathfrak{p}) = \mathrm{ind}((\gamma_{0,0} + (\chi_1, \pi))_M) \cdot \exp(\chi_2 + (\pi^r))$, where $M/\kappa(\mathfrak{p})$ is the Galois cyclic extension determined by $\chi_2 + (\pi^r) \in \mathrm{H}^1(\kappa(\mathfrak{p}))$ by Jacob and Wadsworth [17, Theorem 5.15]. It is not hard to compute $\mathrm{ind}((\gamma_{0,0} + (\chi_1, \pi))_M)$: Since M is a finite extension of $\kappa(\mathfrak{p})$, which is a complete discretely valued field, we have that M is a complete discretely valued field as well. Let e be the ramification index of $M/\kappa(\mathfrak{p})$ and $\bar{M}$ the residue field of M. Then by Serre [26, Exercise XII.3.2], $(\gamma_{0,0} + (\chi_1,\pi))_M = (\gamma_{0,0})_{\bar{M}} + (e \cdot \chi_1, \pi')$, where π' is some uniformizer of M. Let $L/\kappa(\mathfrak{p})$ be the field extension determined by $e \cdot \chi_1$ and $\bar{L}$ the residue field of L. Then $\mathrm{ind}((\gamma_{0,0} + (\chi_1,\pi))_M) = \mathrm{ind}((\gamma_{0,0})_{\bar{M}} + (e \cdot \chi_1, \pi')) = \mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}}) \cdot \exp(e \cdot \chi_1)$.

Now that we have an index formula for Brauer classes over $F_\mathfrak{p}$, we are ready to show the following

Proposition 12. *Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth projective T-curve with closed fibre X. Suppose that U is an open affine subset of X and $P \in X \backslash U$ is a closed point. Let $\mathfrak{p} = (U, P)$ be the unique branch of U at P and let γ_P and $\gamma_{\mathfrak{p}}$ be defined as above. Then we have* $\mathrm{ind}(\gamma_P)|\mathrm{ind}(\gamma_{\mathfrak{p}})$.

Proof. By Proposition 9 we have that $\mathrm{ind}(\gamma_P)|\mathrm{ind}((\tilde{\gamma}_{0,0}+(\tilde{\chi}_1,\pi))_{\tilde{M}})\cdot\exp(\tilde{\chi}_2+(\pi^r))$, where $\tilde{M}/F_P$ is the Galois cyclic extension determined by $\tilde{\chi}_2+(\pi^r)$. We claim that $\exp(\tilde{\chi}_2+(\pi^r)) = \exp(\chi_2)+(\pi^r)$: we have that $\exp(\tilde{\chi}_2+(\pi^r)) = \mathrm{lcm}(\exp(\tilde{\chi}_2), \exp((\pi^r)))$ and $\exp(\chi_2+(\pi^r)) = \mathrm{lcm}(\exp(\chi_2), \exp((\pi^r)))$. Since $\exp(\tilde{\chi}_2) = \exp(\chi_2)$, we have proved that $\exp(\tilde{\chi}_2 + (\pi^r)) = \exp(\chi_2 + (\pi^r))$. Therefore this proposition will follow if we can show that $\mathrm{ind}((\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}})|\mathrm{ind}((\gamma_{0,0})_{\bar{M}} + (e \cdot \chi_1, \pi'))$.

Next we compute

$$\begin{aligned}
(\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}} &= (\tilde{\gamma}_{0,0})_{\tilde{M}} + (\tilde{\chi}_1, \pi)_{\tilde{M}} \\
&= (\tilde{\gamma}_{0,0})_{\tilde{M}} + ((\tilde{\chi}_1)_{\tilde{M}}, \pi) \\
&= (\tilde{\gamma}_{0,0})_{\tilde{M}} + ((\tilde{\chi}_1)_{\tilde{M}}, (\pi')^e) \\
&= (\tilde{\gamma}_{0,0})_{\tilde{M}} + (e \cdot (\tilde{\chi}_1)_{\tilde{M}}, \pi')
\end{aligned}$$

By Proposition 9 again we immediately see that $\mathrm{ind}((\tilde{\gamma}_{0,0} + (\tilde{\chi}_1, \pi))_{\tilde{M}})|\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}}) \cdot \exp(e \cdot (\tilde{\chi}_1)_{\tilde{M}})$, where $\tilde{L}/F_P$ denotes the Galois cyclic extension determined by $e \cdot \tilde{\chi}_1$. Clearly $\exp(e \cdot (\tilde{\chi}_1)_{\tilde{M}})| \exp(e \cdot (\chi_1))$, so we will be done if we can show that $\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}})|\mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$, which we will do in the following Lemma 13. ☐

Lemma 13. *In line with the notation in 12, we have that* $\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}})|\mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$.

Proof. Let $\tilde{M}'/F_P$ be the Galois cyclic extension determined by χ_2. Clearly it suffices to prove that $\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}'\tilde{L}})|\mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$ since $\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}\tilde{L}})|\mathrm{ind}((\tilde{\gamma})_{\tilde{M}'\tilde{L}})$. Let $i = \mathrm{ind}((\gamma_{0,0})_{\bar{M}\bar{L}})$. By Proposition 10, we have that $\mathrm{SB}_i(\gamma_{0,0})(\bar{M}\bar{L}) \neq \phi$, or equivalently, the morphism $\mathrm{SB}_i(\gamma_{0,0}) \times_{\kappa(P)} \bar{M}\bar{L}$ has a section $\mathrm{Spec}(\bar{M}\bar{L}) \to \mathrm{SB}_i(\gamma_{0,0}) \times_{\kappa(P)} \bar{M}\bar{L}$. By Lemma 3, this section lifts to a section over $\mathrm{Spec}(\hat{R}_P)$; thus we obtain a $\tilde{M}'\tilde{L}$-rational point of $\mathrm{SB}_i(\tilde{\gamma}_{0,0}) \times_{\hat{R}_P} S$(note that $\gamma_{0,0} \in \mathrm{H}^2(\hat{R}_P))$, where S is the integral closure of $\hat{R}_P$ in $\tilde{M}'\tilde{L}$; or equivalently, a $\tilde{M}'\tilde{L}$-rational point of $\mathrm{SB}_i(\tilde{\gamma}_{0,0}) \times_{F_P} \tilde{M}'\tilde{L}$. Therefore $\mathrm{ind}((\tilde{\gamma}_{0,0})_{\tilde{M}'\tilde{L}})|i$ again by Proposition 10, which proves this lemma. ☐

The following Corollary is immediate:

Corollary 14. *The homomorphism* $s : \mathrm{H}^2(\hat{F}) \to \mathrm{H}^2(F)$ *preserves index of Brauer classes.*

Proof. This is simply Corollary 8 plus Proposition 12. ☐

5. Indecomposable and noncrossed product Division Algebras over Curves over complete Discrete Valuation Rings

Let T be a complete discrete valuation ring. Let $\hat{X}$ be a smooth projective T-curve with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation determined by X. We construct indecomposable division algebras and noncrossed product division algebras over F of prime power index for all primes q where q is different from the characteristic of the residue field of T. Note that the existence of such algebras are already known when residue field of T is a finite filed, cf. Brussel et al. [6]. Our construction here is almost identical to Brussel et al. [6, Section 4], we list it here for the reader's convenience.

5.1. Indecomposable Division Algebras over F

First we recall the construction of indecomposable division algebras over $\hat{F}$, this is done in Brussel et al. [6, Proposition 4.2].

Proposition 15. *Let T be a complete discrete valuation ring and let $\hat{X}$ be a smooth projective curve over $\mathrm{Spec}(T)$ with closed fibre X. Let F be the function field of $\hat{X}$ and $\hat{F}$ the completion of F with respect to the discrete valuation induced by X. Let e, i be integers satisfying $l \leq e \leq 2e-1$. For any prime $q \neq \mathrm{char}(k)$, there exists a Brauer class $\hat{\gamma} \in \mathrm{H}^2(\hat{F})$ satisfying $\mathrm{ind}(\hat{\gamma}) = q^i, \exp(\hat{\gamma}) = q^e$ and whose underlying division algebra is indecomposable.*

Then we lift $\hat{\gamma}$ to F by using the splitting map s we defined in section 3, and show that the lift is in fact indecomposable.

Theorem 16. *In the notation of Theorem 15. Then there exists an indecomposable division algebra D over F such that $\mathrm{ind}(D) = q^i$ and $\exp(D) = q^e$.*

Proof. By Proposition 15, there exists $\hat{\gamma} \in \mathrm{Br}(\hat{F})$ with $\mathrm{ind}(\hat{\gamma}) = q^i$ and $\exp(\hat{\gamma}) = q^e$ and whose underlying division algebra is indecomposable. By Corollary 14, $\gamma = s(\hat{\gamma})$ has index q^i too. Since s splits the restriction map, we have $\exp(\gamma) = q^e$. We show the division algebra underlying γ is indecomposable.

We proceed by contradiction. Assume $\gamma = \beta_1 + \beta_2$ represents a nontrivial decomposition, then $\hat{\gamma} = \mathrm{res}_{\hat{F}}(\beta_1) + \mathrm{res}_{\hat{F}}(\beta_2)$. Since the index can only go down under restriction, we have that $\mathrm{ind}(\hat{\gamma}) = \mathrm{ind}(\mathrm{res}_{\hat{F}}(\beta_1)) \cdot \mathrm{ind}(\mathrm{res}_{\hat{F}}(\beta_2))$, which represents a nontrivial decomposition of the division algebra underlying $\hat{\gamma}$, a contradiction. ☐

5.2. Noncrossed Products over F

Again we will construct noncrossed product division algebras over $\hat{F}$ and use the splitting map s to lift it to F and show that the lift represents a noncrossed product division algebra over F.

The construction over $\hat{F}$ is in line with Brussel [5] where noncrossed products over $Q(t)$ and $Q((t))$ are constructed. In order to mimic the construction in Brussel [5], we need only note that both Chebotarev density theorem and the Gruwald-Wang theorem hold for global fields which are characteristic p function fields. Then the arguments in Brussel [5] apply directly to yield noncrossed products over $\hat{K}(\hat{X})$ of index and exponent given below:

The following is Brussel et al. [6, Theorem 4.7].

Theorem 17. *Let T be a complete discrete valuation ring with residue field k and let $\hat{X}$ be a smooth projective curve over $\mathrm{Spec}(T)$. Let F be the function field of $\hat{X}$ and let $\hat{F}$ be the completion of F with respect to the discrete valuation induced by the closed fibre. For any positive integer a, let ϵ_a be a primitive a-th root of unity. Set r and s to be maximum integers such that $\mu_{q^r} \subset k(X)^\times$ and $\mu_{q^s} \subset k(X)(\epsilon_{q^{r+1}})$. Let n, m be integers such that $n \geq 1, n \geq m$ and $n, m \in r \cup [s, \infty)$. Let a, l be integers such that $l \geq n + m + 1$ and $0 \leq a \leq 1 - n$. (See Brussel [5, Page 384-385] for more information regarding these constraints.) Let $q \neq \mathrm{char}(k)$ be a prime number. Then there exists noncrossed product division algebras over $\hat{F}$ with index q^{l+1} and exponent q^l.*

Corollary 18. *Let $R, k, \hat{X}, X, F, \hat{F}, q, a, l$ be as in Theorem 17. Then there exists noncrossed product division algebras over F of index q^{l+a} and exponent q^l.*

Proof. Let $\hat{\gamma}$ be the Brauer class representing a noncrossed product over $\hat{F}$ of index q^{l+a} and exponent q^l. Let D be the division algebra underlying the Brauer class $s(\hat{\gamma})$. By Corollary 14, we know that $\mathrm{ind}(D) = \mathrm{ind}(\hat{\gamma})$.

Assume that D is a crossed product with maximal Galois subfield M/F. Then $M\hat{F}$ splits $\hat{\gamma}$, is of degree $\mathrm{ind}(\hat{\gamma})$ and is Galois. This contradicts the fact that $\hat{\gamma}$ is a noncrossed product. $\square$

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