# OUTER AUTOMORPHISMS OF ALGEBRAIC GROUPS AND DETERMINING GROUPS BY THEIR MAXIMAL TORI

SKIP GARIBALDI

Abstract. We give a cohomological criterion for existence of outer automorphisms of a semisimple algebraic group over an arbitrary field. This criterion is then applied to the special case of groups of type $D_{2n}$ over a global field, which completes some of the main results from the paper "Weakly commensurable arithmetic groups and isospectral locally symmetric spaces" (Pub. Math. IHES, 2009) by Prasad and Rapinchuk and gives a new proof of a result from another paper by the same authors.

One goal of this paper is the (rather technical) Theorem 16 below, which completes some of the main results in the remarkable paper [PrR 09] by Gopal Prasad and Andrei Rapinchuk. For example, combining their Theorem 7.5 with our Theorem 16 gives:

**Theorem 1.** *Let $G_1$ and $G_2$ be connected absolutely simple algebraic groups over a number field $K$ that have the same $K$-isomorphism classes of maximal $K$-tori. Then:*

1. *$G_1$ and $G_2$ have the same Killing-Cartan type (and even the same quasi-split inner form) or one has type $B_n$ and the other has type $C_n$.*
2. *If $G_1$ and $G_2$ have the same Killing-Cartan type and that type is not $A_n$ for $n \geq 2$, $D_{2n+1}$, or $E_6$, then $G_1$ and $G_2$ are $K$-isomorphic.*

This result is essentially proved by Prasad-Rapinchuk in [PrR 09], except that paper omits types $D_{2n}$ for $2n \geq 4$ in (2). Our Theorem 16 gives a new proof of the $2n \geq 6$ case (treated by Prasad-Rapinchuk in a later paper [PrR 10, §9]) and settles the last remaining case of groups of type $D_4$. Note that in the final form stated above, Theorem 1 is complete, in that types $A_n$, $D_{2n+1}$, and $E_6$ are genuine exceptions by [PrR 09, 7.6].

Similarly, combining our Theorem 16 with the arguments in [PrR 09] implies that their Theorems 4, 8.16, and 10.4 remain true if you delete "$D_4$" from their statements—that is, the conclusions of those theorems regarding weak commensurability, locally symmetric spaces, etc., also hold for groups of type $D_4$.

The other goal of this paper is Theorem 8, which addresses the more general setting of a semisimple algebraic group $G$ over an arbitrary field $k$. That theorem gives a cohomological criterion for the existence of outer automorphisms of $G$, i.e., for the existence of $k$-points on non-identity components of $\mathrm{Aut}(G)$. This criterion and the examples we give of when it holds make up the bulk of the proof of Theorem 16, which concerns groups over global fields.

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**Notation.** A *global field* is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ for some prime $p$. A (non-archimedean) *local field* is the completion of a global field with respect to a discrete valuation, i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime $p$.

We write $H^d(k, G)$ for the $d$-th flat (fppf) cohomology set $H^d(\operatorname{Spec} k, G)$ when $G$ is an algebraic affine group scheme over a field $k$. In case $G$ is smooth, it is the same as the Galois cohomology set $H^d(\mathrm{Gal}(k), G(k_{\mathrm{sep}}))$ where $k_{\mathrm{sep}}$ denotes a separable closure of $k$ and $\mathrm{Gal}(k)$ denotes the group of $k$-automorphisms of $k_{\mathrm{sep}}$.

We refer to [PlR], [Sp 98], and [KMRT] for general background on semisimple algebraic groups. Such a group $G$ is an *inner form* of $G'$ if there is a class $\gamma \in H^1(k, \overline{G})$, for $\overline{G}$ the adjoint group of $G$, such that $G'$ is isomorphic to $G$ twisted by $\gamma$. We write $G_\gamma$ for the group $G$ twisted by the cocycle $\gamma$, following the TEX-friendly notation of [KMRT, p. 387] instead of Serre's more logical ${}_\gamma G$. We say simply that $G$ is *inner* or *of inner type* if it is an inner form of a split group; if $G$ is not inner then it is *outer*.

For a group scheme $D$ of multiplicative type, we put $D^*$ for its dual $\mathrm{Hom}(D, \mathbb{G}_m)$.

## 1. Background: the Tits algebras determine the Tits class

Fix a semisimple algebraic group $G$ over a field $k$. Its simply connected cover $\widetilde{G}$ and adjoint group $\overline{G}$ fit into an exact sequence

$$1 \longrightarrow Z \longrightarrow \widetilde{G} \longrightarrow \overline{G} \longrightarrow 1 \tag{2}$$

where $Z$ denotes the (scheme-theoretic) center of $\widetilde{G}$. Write $\delta \colon H^1(k, \overline{G}) \to H^2(k, Z)$ for the corresponding coboundary map.

There is a unique element $\nu_G \in H^1(k, \overline{G})$ such that the twisted group $\overline{G}_{\nu_G}$ is quasi-split [KMRT, 31.6], and the *Tits class* $t_G$ of $G$ is defined to be $t_G := -\delta(\nu_G) \in H^2(k, Z)$. The element $t_G$ depends only on the isogeny class of $G$.

For $\gamma \in H^1(k, \overline{G})$, the center of the twisted group $\widetilde{G}_\gamma$ is naturally identified with (and not merely isomorphic to) $Z$, and a standard twisting argument shows that

$$t_{G_\gamma} = t_G + \delta(\gamma). \tag{3}$$

**Example 4.** If $G$ itself is quasi-split, then $t_G = 0$ and for every $\gamma \in H^1(k, \overline{G})$ we have $t_{G_\gamma} = \delta(\gamma)$.

**Definition 5** (Tits algebras)**.** A *Tits algebra* of $G$ is an element

$$\chi(t_G) \in H^2(k(\chi), \mathbb{G}_m) \quad \text{for } \chi \in Z^*,$$

where $k(\chi)$ denotes the subfield of $k_{\mathrm{sep}}$ of elements fixed by the stabilizer of $\chi$ in $\mathrm{Gal}(k)$, i.e., $k(\chi)$ is the smallest separable extension of $k$ so that $\chi$ is fixed by $\mathrm{Gal}(k(\chi))$.

We can modify this definition to replace elements of $Z^*$ with weights. Fix a pinning for $\widetilde{G}$ over $k_{\mathrm{sep}}$ involving a maximal $k$-torus $\widetilde{T}$. As $Z$ is contained in $\widetilde{T}$, every weight $\lambda$—i.e., every $\lambda \in \widetilde{T}^*$—induces by restriction an element of $Z^*$ and we define $\lambda(t_G)$ to be $\lambda|_Z(t_G)$.

Regarding history, the Tits algebras of $G$ were defined in [T]. The class $\lambda(t_G)$ measures the failure of the irreducible representation of $\widetilde{G}$ with highest weight $\lambda$—which is defined over $k_{\mathrm{sep}}$—to be defined over $k$. Roughly speaking, a typical example of a Tits algebra is provided by the even Clifford algebra of the special orthogonal group of a quadratic form, see for example [KMRT, §27].

Obviously, the Tits class $t_G$ determines the Tits algebras $\chi(t_G)$. The converse also holds:

**Proposition 6.** *If $G$ is absolutely almost simple, then the natural map*

$$\prod \lambda \colon H^2(k, Z) \to \prod H^2(k(\lambda|_Z), \mathbb{G}_m)$$

*is injective, where the products range over minuscule weights $\lambda$.*

This proposition can probably be viewed as folklore. I learned it from Alexander Merkurjev and Anne Quéguiner. Below, we we will use the following restatement: Given classes $\gamma_1, \gamma_2 \in H^1(k, \overline{G})$, *if* $\lambda(\delta(\gamma_1)) = \lambda(\delta(\gamma_2))$ *for every minuscule weight $\lambda$ of $G$, then $\delta(\gamma_1) = \delta(\gamma_2)$.*

*Proof of Proposition 6.* As $\widetilde{G}$ is simply connected, the weight lattice $P$ in the pinning is identified with $\widetilde{T}$ and the root lattice $Q$ is the kernel of the restriction $\widetilde{T}^* \to Z^*$. Since $G$ is assumed absolutely almost simple, its root system is irreducible, and this surjection identifies the minuscule (dominant) weights with the nonzero elements of $Z^*$ [B, §VI.2, Exercise 5a]. Therefore, the claim is equivalent to showing that the map

$$\prod_{\chi \in Z^*} \chi \colon H^2(k, Z) \to \prod_{\chi \in Z^*} H^2(k(\chi), Z) \tag{7}$$

is injective. This claim depends only on $Z$, so we may replace $G$ with $G_{\nu_G}$ and so assume that $G$ is quasi-split.

We choose the maximal $k$-torus $\widetilde{T}$ and pinning in $\widetilde{G}$ so that the usual Galois action preserves the fundamental chamber, i.e., permutes the set of simple roots $\Delta$ and the fundamental dominant weights. In the short exact sequence $1 \to Z \to \widetilde{T} \to \widetilde{T}/Z \to 1$, the set $\Delta$ is a basis for the lattice $(\widetilde{T}/Z)^*$, so $H^1(k, \widetilde{T}/Z)$ is zero and the map $H^2(k, Z) \to H^2(k, \widetilde{T})$ is injective.

We fix a set $S$ of representatives of the $\mathrm{Gal}(k)$-orbits in $\Delta$ and write $\alpha_s$ (resp., $\lambda_s$) for the simple root (resp., fundamental dominant weight) corresponding to $s \in S$. Because $\widetilde{G}$ is simply connected,

$$\widetilde{T} \cong \prod_{s \in S} R_{k(\lambda_s|_Z)/k}(\operatorname{im} h_{\alpha_s})$$

where $h_{\alpha_s}$ denotes the homomorphism $\mathbb{G}_m \to \widetilde{T}$ corresponding to the coroot $\alpha_s^\vee$ [St, p. 44, Cor.]—note that the Weil restriction term makes sense because the stabilizers of $\alpha_s$, $\lambda_s$, and $\lambda_s|_Z$ all agree because of our particular choice of pinning. The inclusion of $Z$ in $\widetilde{T}$ amounts to the product $\prod_{s \in S} \lambda_s$, hence (7) is injective and the claim is proved. □

## 2. Outer automorphisms of semisimple groups

We maintain the notation of the previous section, so that $G$ is semisimple over a field $k$ and $\Delta$ is a set of simple roots, equivalently, the Dynkin diagram of $G$. The Galois action on $\Delta$ induces an action on $\mathrm{Aut}(\Delta)$, and in this way we view $\mathrm{Aut}(\Delta)$ as a finite étale (but not necessarily constant) group scheme. There is a map

$$\alpha \colon \mathrm{Aut}(G)(k) \to \mathrm{Aut}(\Delta)(k)$$

described for example in [Sp 98, §16.3], and one can ask if this map is surjective. That is, does every connected component of $\mathrm{Aut}(G) \times k_{\mathrm{sep}}$ that is defined over $k$ necessarily have a $k$-point?

One obstruction to $\alpha$ being surjective can come from the fundamental group, so we assume that $G$ is simply connected. (One could equivalently assume that $G$ is adjoint.) Another obstruction comes from the Tits class, as we now explain. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}(G) & \xrightarrow{\alpha} & \mathrm{Aut}(\Delta) \\ \downarrow & \swarrow & \\ \mathrm{Aut}(Z) & & \end{array}$$

where the diagonal arrow comes from the natural action of $\mathrm{Aut}(\Delta)$ on the coroot lattice. Hence $\mathrm{Aut}(\Delta)(k)$ acts on $H^2(k, Z)$ and we have:

**Theorem 8.** *Recall that $G$ is assumed semisimple and simply connected. Then there is an inclusion*

$$\mathrm{im}\,[\alpha\colon \mathrm{Aut}(G)(k) \to \mathrm{Aut}(\Delta)(k)] \subseteq \{\pi \in \mathrm{Aut}(\Delta)(k) \mid \pi(t_G) = t_G\}. \tag{9}$$

*Furthermore, the following are equivalent:*

(a) *Equality holds in* (9).
(b) *The sequence $H^1(k, Z) \to H^1(k, G) \to H^1(k, \mathrm{Aut}(G))$ is exact.*
(c) $\ker \delta \cap \ker \left[H^1(k, \overline{G}) \to H^1(k, \mathrm{Aut}(\overline{G}))\right] = 0$.

*Proof.* We consider the interlocking exact sequences

$$\begin{array}{ccccccc} & & & & H^1(k, Z) & & \\ & & & & \downarrow & & \\ & & & & H^1(k, G) & & \\ & & & & \downarrow q & & \\ \mathrm{Aut}(G)(k) & \xrightarrow{\alpha} & \mathrm{Aut}(\Delta)(k) & \xrightarrow{\beta} & H^1(k, \overline{G}) & \xrightarrow{\varepsilon} & H^1(k, \mathrm{Aut}(G)) \\ & & & & \downarrow \delta & & \\ & & & & H^2(k, Z) & & \end{array}$$

The crux is to prove that

$$\pi(t_{G_{\beta(\pi)}}) = t_G \quad \text{for } \pi \in \mathrm{Aut}(\Delta)(k). \tag{10}$$

Since $\overline{G}$ and $\mathrm{Aut}(G)$ are smooth, we may view their corresponding $H^1$'s as Galois cohomology. Put $\gamma := \beta(\pi)$, so $\gamma_\sigma = f^{-1}\,{}^{\sigma}\!f$ for some $f \in \mathrm{Aut}(G)(k_{\mathrm{sep}})$ and every $\sigma \in \mathrm{Gal}(k)$. The group $G_\gamma$ has the same $k_{\mathrm{sep}}$-points as $G$, but a different Galois action $\circ$ given by $\sigma \circ g = \gamma_\sigma \sigma g$ for $g \in G(k_{\mathrm{sep}})$, $\sigma \in \mathrm{Gal}(k)$, and where juxtaposition denotes the usual Galois action on $G$.

The map $f$ gives a $k$-isomorphism $G_\gamma \xrightarrow{\sim} G$. Sequence (2) gives a commutative diagram

$$\begin{array}{ccc} H^1(k, \overline{G}_\gamma) & \xrightarrow{\delta_\gamma} & H^2(k, Z) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ H^1(k, \overline{G}) & \xrightarrow{\delta} & H^2(k, Z). \end{array}$$

Let $\eta \in Z^1(k, \overline{G}_\gamma)$ be a 1-cocycle representing $\nu_{G_\gamma}$. Then $f(\eta)$ is a 1-cocycle in $Z^1(F, \overline{G})$ and $f$ is a $k$-isomorphism $f : (G_\gamma)_\eta \xrightarrow{\sim} G_{f(\eta)}$. Since $(G_\gamma)_\eta$ is $k$-quasi-split, we have $f(\nu_{G_\gamma}) = f(\eta) = \nu_G$. The commutativity of the diagram gives $f(t_{G_\gamma}) = t_G$, proving (10).

It follows that $\pi \in \mathrm{Aut}(\Delta)(k)$ satisfies $\pi(t_G) = t_G$ if and only if $t_{G_{\beta(\pi)}} = t_G$, if and only if $\delta(\beta(\pi)) = 0$. That is, in (9), the left side is $\ker \beta$ and the right side is $\ker \delta\beta$, which makes the inclusion in (9) and the equivalence of (a) and (c) obvious. Statement (b) says that $\ker \varepsilon q = \ker q$, i.e., $\ker \varepsilon \cap \mathrm{im}\, q = 0$, which is (c). $\square$

It is easy to find non-simple groups, even over $\mathbb{R}$, for which the inclusion (9) is proper, because the Tits index also provides an obstruction to equality. Here is an example to show that these are not the only obstructions, even over a number field.

**Example 11.** Fix a prime $p$ and write $x_1, x_2$ for the two square roots of $p$ in $k := \mathbb{Q}(\sqrt{p})$. For $i = 1, 2$, let $H_i$ be the group of type $G_2$ associated with the 3-Pfister quadratic form $\phi_i := \langle\langle -1, -1, x_i \rangle\rangle$. For $G = H_1 \times H_2$, the Tits index is

and $\mathrm{Aut}(\Delta)(k) = \mathbb{Z}/2\mathbb{Z}$, but $H_1$ is not isomorphic to $H_2$, so no $k$-automorphism of $G$ interchanges the two components.

Nonetheless, we now list "many" cases in which equality holds in (9).

**Example 12.** If $G$ is quasi-split, then $\alpha$ maps $\mathrm{Aut}(G)(k)$ onto $\mathrm{Aut}(\Delta)(k)$ by [SGA3, XXIV.3.10] or [KMRT, 31.4], so equality holds in (9).

**Example 13.** If $H^1(k, G) = 0$, then trivially Th. 8(b) holds. That is, (a)–(c) hold for every semisimple simply connected $G$ if $k$ is local (by Kneser-Bruhat-Tits), global with no real embeddings (Kneser-Harder-Chernousov), or the function field of a complex surface (de Jong-He-Starr-Gille), and conjecturally if the cohomological dimension of $k$ is at most 2 (Serre).

**Example 14.** Suppose $G$ is absolutely almost simple (and simply connected). Conditions (a)–(c) of the proposition hold trivially if $\mathrm{Aut}(\Delta)(k) = 1$, in particular if $G$ is not of type $A$, $D$, or $E_6$ or if $G$ has type ${}^6D_4$. Conditions (a)–(c) also hold:

(i) *if $G$ is of inner type.* If $G$ is of inner type $A$ ($n \geq 2$), then $\mathrm{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$ and the nontrivial element $\pi$ acts via $z \mapsto z^{-1}$ on $Z$, hence $\pi(t_G) = -t_G$. If $2t_G = 0$, then $G$ is $\mathrm{SL}_1(D)$ for $D$ a central simple algebra of degree $n+1$ such that there is an anti-automorphism $\sigma$ of $D$, hence $g \mapsto \sigma(g)^{-1}$ is a $k$-automorphism of $G$ mapping to $\pi$. (By a theorem of Albert [Sch, Th. 8.8.4] one can even arrange for $\sigma$ to have order 2, hence for this automorphism of $G$ to have order 2.)

Next let $G$ be of type ${}^1D_n$ for $n \geq 5$ and suppose that the nonidentity element $\pi \in \mathrm{Aut}(\Delta)(k)$ fixes the Tits class $t_G$. The group $G$ is isomorphic to

Spin$(A, \sigma, f)$ for some central simple $k$-algebra $A$ of degree $2n$ and quadratic pair $(\sigma, f)$ on $A$ such that the even Clifford algebra $C_0(A, \sigma, f)$ is isomorphic to a direct product $C_+ \times C_-$ of central simple algebras. Since $\pi$ fixes the Tits class, the algebras $C_+$ and $C_-$ are isomorphic. The equation $[A] + [C_+] - [C_-] = 0$ holds in the Brauer group of $k$ by [KMRT, 9.12] (alternatively, as a consequence of the fact that the cocenter is an abelian group of order 4). Therefore, $A$ is split. Let $\phi \in \mathrm{O}(A, \sigma, f)(k)$ be a hyperplane reflection as in [KMRT, 12.13]; it does not lie in the identity component of $\mathrm{O}(A, \sigma, f)$. The automorphism of $\mathrm{SO}(A, \sigma, f)$ given by $g \mapsto \phi g \phi^{-1}$ lifts to an automorphism of Spin$(A, \sigma, f)$ that is outer, i.e., that induces the automorphism $\pi$ on $\Delta$. (To recap: given a nonzero $\pi \in \mathrm{Aut}(\Delta)(k)$ that preserves the Tits class, we deduced that $A$ is split and therefore $(A, \sigma, f)$ has an improper isometry. Conversely, Lemma 1b from [Kn, p. 42] shows: if $\mathrm{char}\, k \neq 2$ and $(A, \sigma)$ has an improper isometry, then $A$ is split and obviously such a $\pi$ exists.)

For the remaining cases, we point out merely that an outer automorphism of order 3 in the $D_4$ case exists when $t_G = 0$ by triality [SpV, 3.6.3, 3.6.4] and an outer automorphism of order 2 in the $E_6$ case when $t_G = 0$ is provided by the "standard automorphism" of a $J$-structure [Sp 73, p. 150].

(ii) *if $G$ is the special unitary group of a hermitian form relative to a separable quadratic extension $K/k$*, i.e., $G$ is of type ${}^2A_n$ and $\mathrm{res}_{K/k}(t_G)$ is zero in $H^2(K, Z)$. We leave the details in this case as an exercise.

(iii) *if $k$ is real closed*. By the above cases, we may assume that $G$ has type ${}^2D_n$ (for $n \geq 4$) or ${}^2E_6$. In the first case, $\mathrm{Aut}(\Delta)(k) = \mathbb{Z}/2\mathbb{Z}$ (also for $n = 4$) and $G$ is the spin group of a quadratic form by [KMRT, 9.14], so a hyperplane reflection gives the desired $k$-automorphism.

In case $G$ has type ${}^2E_6$, combining pages 37, 38, 119, and 120 in [J] shows that the (outer) automorphism of the Lie algebra Jacobson denotes by $t$ is defined over $k$.

I don't know any examples of absolutely almost simple $G$ where conditions (a)–(c) fail. Furthermore, in all of the examples above, every $\pi$ from the right side of (9) is not only of the form $\alpha(f)$ for some $f \in \mathrm{Aut}(G)(k)$, but one can even pick $f$ to have the same order as $\pi$.

## 3. Groups of type $D_{\text{even}}$ over local fields

The main point of this section is to prove the following lemma.

**Lemma 15.** *Let $G$ be an adjoint semisimple group over a field $k$, and fix a maximal $k$-torus $T$ in $G$. If $z_1, z_2$ are in the image of the map $H^1(k, T) \to H^1(k, G)$ such that*

(1) *$G_{z_1}$ and $G_{z_2}$ are both quasi-split; or*
(2) *$T$ contains a maximal $k$-split torus in both $G_{z_1}$ and $G_{z_2}$ and*
    (a) *$k$ is real closed, or*
    (b) *$k$ is a (non-archimedean) local field and $G$ has type $D_{2n}$ for some $n \geq 2$,*

*then $z_1 = z_2$.*

*Proof.* For short, we write $G_i$ for $G_{z_i}$. In case (1), the uniqueness of the class $\nu_G \in H^1(k, G)$ such that $G_{\nu_G}$ is quasi-split (already used in §1) gives that $z_1 = \nu_G = z_2$. So suppose (2) holds. As $T$ is contained in both these groups, their Tits indexes are naturally identified over $k$. In particular, if one is quasi-split then so is the other, and we are done as in (1). So we assume that neither group is quasi-split.

In case (2a), where $k$ is real closed, one immediately reduces to the case where $G$ is absolutely simple. That case is trivial because the isomorphism class of an adjoint simple group is determined by its Tits index, so $G_1$ is isomorphic to $G_2$. The Tits index also determines the Tits algebras—see pages 211 and 212 of [T] for a recipe—so by Prop. 6, $\delta(z_1) = \delta(z_2)$. The claim now follows from Example 14(iii) and Theorem 8(c).

So assume for the remainder of the proof that (2b) holds. In particular, $\delta$ is injective. Number the simple roots of $G_1$ with respect to $T$ as in [B]. If $G_1$ has type ${}^2D_4$, we take $\alpha_1$ to be the root at the end of the Galois-fixed arm of the Tits index. Otherwise, we assign the numbering arbitrarily in case there is ambiguity (e.g., $\alpha_{2n-1}$ and $\alpha_{2n}$). Note that $G_1$ cannot have type ${}^3D_4$ or ${}^6D_4$, because it is not quasi-split.

As $2\omega_i$ is in the root lattice for every $i$, the Tits algebras $\omega_i(t_{G_1})$ for $i = 2n-1, 2n$ define up to $k$-isomorphism a quaternion (Azumaya) algebra $D$ over a quadratic étale $k$-algebra $\ell$. By the exceptional isomorphism $D_2 = A_1 \times A_1$ and a Tits algebra computation, $\mathrm{PGL}_1(D)$ is isomorphic to $\mathrm{PSO}(M_2(H), \sigma, f)$ for $H$ the quaternion algebra underlying $\omega_1(t_{G_1})$ and some quadratic pair $(\sigma, f)$ such that the even Clifford algebra $C_0(\sigma, f)$ is isomorphic to $D$, cf. [KMRT, 15.9]. Appending $2n-2$ hyperbolic planes to $(\sigma, f)$, we obtain a quadratic pair $(\sigma_0, f_0)$ such that $C_0(\sigma_0, f_0)$ is Brauer-equivalent to $D$. As $\mathrm{PSO}(M_{2n}(H), \sigma_0, f_0)$ has the same Tits algebras as $G_1$ (up to renumbering the simple roots of $G_1$), Prop. 6 and injectivity of $\delta$ implies that the two groups are isomorphic. (We have just given a characteristic-free proof of Tsukamoto's theorem [Sch, 10.3.6], relying on the Bruhat-Tits result that $\delta$ is injective.)

Now both $G_1$ and $G_2$ have the same Tits index and semisimple anisotropic kernels of Killing-Cartan type a product of $A_1$'s. As there is a unique quaternion division algebra over each finite extension of $k$, it follows that $G_1$ and $G_2$ have the same Tits class, i.e., $\delta(z_1) = \delta(z_2)$. $\square$

In the statement of (2b), we cannot replace "$D_{2n}$ for some $n \geq 2$" with "$D_\ell$ for some $\ell$" because the claim fails for groups of type $D_{\mathrm{odd}}$. This can been seen already for type $D_3 = A_3$: one can find $z_1, z_2 \in H^1(k, \mathrm{PGL}_4)$ so that $G_1$ and $G_2$ are both isomorphic to $\mathrm{Aut}(B)^\circ$ for a division algebra $B$ of degree 4, but $\delta(z_1) = -\delta(z_2)$ in $H^2(k, \mu_4) = \mathbb{Z}/4\mathbb{Z}$. Adding hyperbolic planes as in the proof of the lemma gives a counterexample for all odd $\ell$. This counterexample is visible in the proof: for groups $G_1, G_2$ of type $D_\ell$ with $\ell$ odd and $\geq 3$, the semisimple anisotropic kernels have Killing-Cartan type a product of $A_1$'s and an $A_3$ and the very last sentence of the proof fails.

## 4. Groups of type $D_{\mathrm{even}}$ over global fields

The following technical theorem concerning groups over a global field connects our Theorem 8 (about groups over an arbitrary field) with the results in [PrR 09]. It implies Theorem 9.1 of [PrR 10].

**Theorem 16.** *Let $G_1$ and $G_2$ be adjoint groups of type $D_{2n}$ for some $n \geq 2$ over a global field $K$, such that $G_1$ and $G_2$ have the same quasi-split inner form—i.e.,* *the smallest Galois extension of $K$ over which $G_1$ is of inner type is the same as for $G_2$. If there exists a maximal torus $T_i$ in $G_i$ for $i = 1$ and 2 such that*

(1) *there is a $K_{\mathrm{sep}}$-isomorphism $\phi : G_1 \to G_2$ whose restriction to $T_1$ is a $K$-isomorphism $T_1 \to T_2$; and*
(2) *there is a finite set $V$ of places of $K$ such that:*
  (a) *For all $v \notin V$, $G_1$ and $G_2$ are quasi-split over $K_v$.*
  (b) *For all $v \in V$, $(T_i)_{K_v}$ contains a maximal $K_v$-split torus of $(G_i)_{K_v}$;*

*then $G_1$ and $G_2$ are isomorphic over $K$.*

The hypotheses are what one obtains by assuming the existence of weakly commensurable arithmetic subgroups, see for example Theorems 1 and 6 and Remark 4.4 in [PrR 09]. Note that the groups appearing in the theorem can be trialitarian, i.e., of type ${}^3D_4$ or ${}^6D_4$. We remark that Bruce Allison gave an isomorphism criterion with very different hypotheses in [A, Th. 7.7].

*Proof.* Write $G$ for the unique adjoint quasi-split group that is an inner form of $G_1$ and $G_2$. By Steinberg [PlR, pp. 338, 339], there is a $K_{\mathrm{sep}}$-isomorphism $\psi_2 \colon G_2 \to G$ whose restriction to $T_2$ is defined over $K$. We put $\psi_1 := \psi_2\phi$ and $T := \psi_2(T_2) = \psi_1(T_1)$. Then $G_i$ is isomorphic to $G$ twisted by the 1-cocycle $\sigma \mapsto \psi_i({}^\sigma\psi_i)^{-1}$. But this 1-cocycle consists of elements of $\mathrm{Aut}(G)$ that fix $T$ elementwise, hence belong to $T$ itself. That is, for $i = 1, 2$, there is a cocycle $z_i$ in the image of $H^1(K, T) \to H^1(K, G)$ such that $G_i$ is isomorphic to $G$ twisted by $z_i$.[1]

Now Lemma 15 gives that $\mathrm{res}_{K_v/K}(z_1) = \mathrm{res}_{K_v/K}(z_2)$ for every $v$, hence $z_1 = z_2$ by the Kneser-Harder Hasse Principle [PlR, p. 336, Th. 6.22] and $G_1$ is isomorphic to $G_2$ over $K$. □

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[1]This argument does not use the fact that $K$ is a number field nor that $G_1$ and $G_2$ have type $D_{2n}$, so roughly speaking it applies generally to the situation where $G_1$ and $G_2$ share a maximal torus over the base field—more precisely, to the situation arising in Remark 4.4 of [PrR 09].

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Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

*E-mail address*: `skip@mathcs.emory.edu`