ISOTROPY OVER FUNCTION FIELDS OF PFISTER FORMS

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Abstract

The question of which quadratic forms become isotropic when extended to the function field of a given form is studied. A formula for the minimum dimension of the minimal isotropic forms associated to such extensions is given, and some consequences thereof are outlined. Especial attention is devoted to function fields of Pfister forms. Here, the relationship between excellence concepts and the isotropy question is explored. Moreover, in the case where the ground field is formally real and has finite Hasse number, the isotropy question is answered for forms of sufficiently large dimension.

1 Introduction

Certain field invariants in quadratic form theory (for example the u-invariant, the Hasse number, the Pythagoras number) are defined as the suprema of the dimensions of anisotropic quadratic forms of a given type. A fruitful method of establishing that such an invariant attains a particular value was introduced by Merkurjev (see [M]). It serves as one source of motivation for the following question.

Question 1.1. Given a quadratic form φ over a field F, which anisotropic quadratic forms over F become isotropic when extended to the function field of φ over F?

While this question appears to be extremely difficult to resolve, some noteworthy progress has been made in this direction (see [L, Ch.X]). More is known regarding the following related question.

Question 1.2. Given a quadratic form φ over a field F, which anisotropic forms over F become hyperbolic when extended to the function field of φ over F?

The Cassels-Pfister Subform Theorem [L, Ch.X, Theorem 4.5] gives a partial answer to this question by providing necessary conditions in terms of subform containment. One would obtain a complete answer to Question 1.1, again in terms of subform containment, if one could classify the isotropic forms that are minimal with respect to subform containment. Towards this end, we study the dimensions of such forms in the second section of this article.

In the case where φ is a Pfister form, a complete answer is known to Question 1.2 [L, Ch.X, Theorem 4.9]. Thus, it is justified to devote particular attention to Question 1.1 in the context of function fields of Pfister forms, particularly since the property of excellence can only arise for such function fields (see [K2, Theorem 7.13]). Consequently, sections three, four and five of this paper are primarily concerned with addressing Question 1.1 for function fields of Pfister forms. The third section explores excellence concepts and their relation to Question 1.1. Building on this, the fourth section answers Question 1.1 for forms of certain dimensions, and provides bounds on the range of these dimensions. The final section tackles Question 1.1 for function fields of Pfister forms over formally real fields of finite Hasse number, and offers an answer for forms whose dimension is greater than the Hasse number.

Throughout, we highlight cases where our investigations allow for short or simple recoveries of established results.

Henceforth, we will let F denote a field of characteristic different from two. The term "form" will refer to a regular quadratic form. Every form over F can be diagonalised. Given $a_1, \ldots, a_n \in F^\times$, one denotes by $\langle a_1, \ldots, a_n \rangle$ the n-dimensional quadratic form $a_1X_1^2 + \ldots + a_nX_n^2$. If φ and ψ are forms over F, we denote by $\varphi \perp \psi$ their orthogonal sum and by $\varphi \otimes \psi$ their tensor product. For $m \in \mathbb{N}$, we will denote the orthogonal sum of m copies of φ by $m \times \varphi$. We use $a\varphi$ to denote $\langle a \rangle \otimes \varphi$ for $a \in F^\times$. We write $\varphi \simeq \psi$ to indicate that φ and ψ are isometric. Two forms φ and ψ over F are *similar* if $\varphi \simeq a\psi$ for some $a \in F^\times$. For φ a form over F and K/F a field extension, we write φ_K when we view φ as a form over K. A form over F is *isotropic* if it represents zero non-trivially, and *anisotropic* otherwise. Every form φ has a decomposition $\varphi \simeq \psi \perp i \times \langle 1, -1 \rangle$ where the anisotropic form ψ and the integer i are uniquely determined, with ψ being referred to as the *anisotropic part of* φ, denoted φ_{an}, and i being labelled the *Witt index of* φ, denoted $i_W(\varphi)$. A form φ is *hyperbolic* if its anisotropic part is trivial, whereby $i_W(\varphi) = \frac{1}{2}\dim\varphi$. A form τ is a *subform of* φ if $\varphi \simeq \tau \perp \gamma$ for some form γ, in which case we will write $\tau \subset \varphi$. The following basic fact (see [L, Ch.I, Ex. 16]) will be employed frequently.

Lemma 1.3. *If $\tau \subset \varphi$ with $\dim\tau \geqslant \dim\varphi - i_W(\varphi) + 1$, then τ is isotropic.*

Given $n \in \mathbb{N}$, an *n-fold Pfister form* is a form isometric to $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$ for some $a_1, \ldots, a_n \in F^\times$. We let PF denote the class of Pfister forms over F, and P_nF the class of n-fold Pfister forms over F. For $\pi \in PF$, a form τ over F is a *generalised Pfister neighbour of* π if there exists a form γ over F such that $\tau \subset \pi \otimes \gamma$ and $\dim\tau > \frac{1}{2}\dim(\pi \otimes \gamma)$. In particular, if $\dim\gamma = 1$ then τ is said to be a *Pfister neighbour of* π. Since isotropic Pfister forms are hyperbolic ([L, Ch.X, Theorem 1.7]), Lemma 1.3 demonstrates that the isotropy of a Pfister form implies the isotropy of its generalised Pfister neighbours.

For a form φ over F with $\dim\varphi = n \geqslant 2$ and $\varphi \not\simeq \langle 1, -1 \rangle$, the *function field $F(\varphi)$ of* φ is the quotient field of the integral domain $F[X_1, \ldots, X_n]/(\varphi(X_1, \ldots, X_n))$ (this is the function field of the affine quadric $\varphi(X) = 0$ over F). As per [L, Ch.X, Theorem 4.1], $F(\varphi)/F$ is a purely transcendental extension if and only if φ is isotropic over F. To avoid case distinctions, we set $F(\varphi) = F$ if $\dim\varphi \leqslant 1$ or $\varphi \simeq \langle 1, -1 \rangle$. The positive integer $i_W(\varphi_{F(\varphi)})$ is called the *first Witt index of* φ, and is denoted by $i_1(\varphi)$. For all extensions K/F such that φ_K is isotropic, $i_1(\varphi) \leqslant i_W(\varphi_K)$ (see [K1, Proposition 3.1 and Theorem 3.3]). We will often invoke [H3, Theorem 1]:

Theorem 1.4. (*Hoffmann*) *Let ψ be anisotropic over F. If $\dim\psi \leqslant 2^n < \dim\varphi$ for some $n \in \mathbb{N}$, then $\psi_{F(\varphi)}$ is anisotropic.*

Combining Lemma 1.3 with Theorem 1.4, one obtains [H3, Corollary 1], namely that if φ is an anisotropic form with $\dim\varphi = 2^n + k$ where $0 < k \leqslant 2^n$, then $i_1(\varphi) \leqslant k$. If $i_1(\varphi) = k$ in this situation, whereby $\dim\varphi - i_1(\varphi)$ is a power of two, φ is said to have *maximal splitting*. Two anisotropic forms φ and ψ over F are *isotropy equivalent* if for every K/F we have that φ_K is isotropic if and only if ψ_K is isotropic. For φ and ψ anisotropic forms over F, if φ is isotropic over $F(\psi)$ then φ is isotropic over any field extension K/F such that ψ_K is isotropic (since

$K(\psi)/K$ is a purely transcendental extension). Thus, one recovers the observation in [K1] that φ is isotropy equivalent to ψ if and only if $\varphi_{F(\psi)}$ is isotropic and $\psi_{F(\varphi)}$ is isotropic.

For further details regarding the above, we refer the reader to [L].

2 Minimal isotropy and essential dimension

Given an extension K/F, an anisotropic form ψ over F is *minimal K-isotropic* if ψ_K is isotropic and, for every proper subform φ of ψ, the form φ_K is anisotropic. As a consequence of Lemma 1.3, we note that every minimal K-isotropic form ψ satisfies $i_W(\psi_K) = 1$. Since every form over F that becomes isotropic over K contains a minimal K-isotropic form, a determination of the minimal K-isotropic forms would provide an answer to the question of which anisotropic forms over F become isotropic when extended to K. Towards this end, we introduce the following set:

$$\mathcal{M}(K/F) = \{\dim\psi \mid \psi \text{ is a minimal } K\text{-isotropic form over } F\}.$$

The invariants $\min\mathcal{M}(K/F)$ and $\sup\mathcal{M}(K/F)$ were introduced in [H1], wherein they are denoted by $t_{\min}(K/F)$ and $t_{\max}(K/F)$, and have since been studied in the case where $K = F(\varphi)$ for φ a form over F. In particular, it was shown in [HVG] that $\sup\mathcal{M}(F(\varphi)/F)$ can be infinite when $\dim\varphi = 3$.

For φ an anisotropic form over F, we denote by $\mathrm{edim}(\varphi)$ the number $\dim\varphi - i_1(\varphi) + 1$, defined as the essential dimension of the form by Izhboldin (see [IKKV] for more details). In this section, we will regularly employ the isotropy criteria provided by [KM, Theorem 4.1]:

Theorem 2.1. (*Karpenko, Merkurjev*) *Let φ and ψ be anisotropic forms over F.*

(*i*) *If $\psi_{F(\varphi)}$ is isotropic, then* $\mathrm{edim}(\psi) \geqslant \mathrm{edim}(\varphi)$,

(*ii*) *If* $\mathrm{edim}(\psi) = \mathrm{edim}(\varphi)$, *then $\psi_{F(\varphi)}$ is isotropic if and only if $\varphi_{F(\psi)}$ is isotropic.*

Corollary 2.2. *If φ and ψ are isotropy-equivalent anisotropic forms over F, then* $\mathrm{edim}(\varphi) = \mathrm{edim}(\psi)$.

Proof. This follows immediately from Theorem 2.1(i). □

Corollary 2.3. *Let φ and ψ be anisotropic forms over F. If $\psi_{F(\varphi)}$ is isotropic, then* $\dim\psi \geqslant \mathrm{edim}(\varphi)$.

Proof. This follows immediately from Theorem 2.1(i). □

Remark 2.4. Theorem 2.1(i) was established through the usage of advanced algebro-geometric machinery. Given its importance as an isotropy criterion over function fields of quadratic forms, it would be desirable to obtain a proof of this result solely by means of classical quadratic form theory. To this end, the following argument demonstrates that it suffices to find such a proof of Corollary 2.3.

Assuming Corollary 2.3, we have that $\min\mathcal{M}(F(\varphi)/F)$ is greater than or equal to $\mathrm{edim}(\varphi)$, whereby Lemma 1.3 implies that $\min\mathcal{M}(F(\varphi)/F)$ equals $\mathrm{edim}(\varphi)$. Moreover, since every subform of ψ of dimension $\dim\psi - i_W(\psi_{F(\varphi)}) + 1$ is isotropic over $F(\varphi)$ by Lemma 1.3, this equality implies that $\dim\psi - i_W(\psi_{F(\varphi)}) + 1$ is greater than or equal to $\mathrm{edim}(\varphi)$, whereby Theorem 2.1(i) follows.

Arguing as above, we obtain our opening result, which relates the essential dimension of φ to the problem of determining the minimal $F(\varphi)$-isotropic forms.

Theorem 2.5. *Any anisotropic form φ over F satisfies* $\min \mathcal{M}(F(\varphi)/F) = \mathrm{edim}(\varphi)$.

Proof. By Lemma 1.3, every subform τ of φ with $\dim \tau = \dim \varphi - i_1(\varphi) + 1$ is isotropic over $F(\varphi)$. Hence, $\min \mathcal{M}(F(\varphi)/F) \leqslant \dim \varphi - i_1(\varphi) + 1$. On the other hand, if ψ is a form over F with $\dim \psi \leqslant \dim \varphi - i_1(\varphi)$, then $\dim \psi - i_1(\psi) < \dim \varphi - i_1(\varphi)$, whereby $\psi_{F(\varphi)}$ is anisotropic by Theorem 2.1. Hence $\min \mathcal{M}(F(\varphi)/F) \geqslant \dim \varphi - i_1(\varphi) + 1$, establishing the equality. □

[AO, Example 1.5] demonstrates that there exist 5-dimensional isotropy-equivalent forms which are non-similar. In particular, since [H3, Corollary 1] implies that a 5-dimensional anisotropic form φ over F satisfies $i_1(\varphi) = 1$, whereby $\mathrm{edim}(\varphi) = 5$, this shows that minimal $F(\varphi)$-isotropic forms of minimum dimension need not be similar to subforms of φ.

Corollary 2.6. *Let ψ and φ be isotropy-equivalent anisotropic forms over F. Every subform of ψ of dimension $\dim \psi - i_W(\psi_{F(\varphi)}) + 1$ is a minimal $F(\varphi)$-isotropic form.*

Proof. Lemma 1.3 implies that such subforms of ψ are isotropic over $F(\varphi)$, whereby $\dim \psi - i_W(\psi_{F(\varphi)}) + 1 \geqslant \min \mathcal{M}(F(\varphi)/F)$. Moreover, $\dim \psi - i_W(\psi_{F(\varphi)}) + 1 \leqslant \mathrm{edim}(\psi) = \mathrm{edim}(\varphi)$, whereby $\dim \psi - i_W(\psi_{F(\varphi)}) + 1 \leqslant \min \mathcal{M}(F(\varphi)/F)$ by Theorem 2.5. □

Corollary 2.7. *An anisotropic form φ over F satisfies $i_1(\varphi) = 1$ if and only if φ is a minimal $F(\varphi)$-isotropic form.*

Proof. This follows directly from Corollary 2.6, Theorem 2.5 or Theorem 2.1. □

Corollary 2.6 does not hold for arbitrary pairs of anisotropic forms. As per [H1, Section 3.3], there exists an example of a field F and anisotropic forms γ and ψ over F, where $\pi \in P_2F$ and $\dim \gamma = 6$, such that $i_W(\gamma_{F(\pi)}) = 1$ but γ is not a minimal $F(\pi)$-isotropic form (indeed, it is shown that γ contains two non-similar 5-dimensional minimal $F(\pi)$-isotropic forms). In Section 5 we will provide a complementary example, Example 5.8, which demonstrates that for φ and γ anisotropic forms over a field F such that $i_W(\gamma_{F(\varphi)}) = m$, where $m \in \mathbb{N}$, the form γ need not contain any minimal $F(\varphi)$-isotropic forms of dimension $\dim \gamma - m + 1$.

Theorem 2.5 allows us to describe those forms which have maximal splitting.

Corollary 2.8. *An anisotropic form ψ over F has maximal splitting if and only if there exists a form φ over F with $\dim \varphi - i_W(\varphi_{F(\psi)}) = 2^{n-1}$, where n is such that $2^{n-1} \leqslant \dim \psi < 2^n$.*

Proof. Letting $\varphi = \psi$ gives the left-to-right implication. Conversely, if φ is such that $\dim \varphi - i_W(\varphi_{F(\psi)}) = 2^{n-1}$, then Lemma 1.3 implies that every subform of φ of dimension $2^{n-1} + 1$ is isotropic over $F(\psi)$. Hence, $\min \mathcal{M}(F(\psi)/F) = 2^{n-1} + 1$ by Theorem 1.4, whereby Theorem 2.5 implies that ψ has maximal splitting. □

Returning to Theorem 2.1 itself, our next two results highlight the extreme cases, where equality of the respective essential dimensions is forced.

Corollary 2.9. *Let φ and ψ be anisotropic forms over F such that $2^{n-1} < \dim \psi \leqslant 2^n$ and $2^{n-1} < \dim \varphi \leqslant 2^n$. Assume that ψ has maximal splitting. Then $\psi_{F(\varphi)}$ is isotropic if and only if φ and ψ are isotropy equivalent.*

Proof. The right-to-left implication is clear. Conversely, Theorem 2.1 implies that $\operatorname{edim}(\psi) = \operatorname{edim}(\varphi)$, whereby we obtain that φ and ψ are isotropy equivalent. □

Corollary 2.10. *Let φ and ψ be anisotropic forms over F such that $\dim\varphi = \dim\psi$ and $i_1(\varphi) = 1$. The following are equivalent:*

(*a*) *$\psi_{F(\varphi)}$ is isotropic,*

(*b*) *φ and ψ are isotropy equivalent,*

(*c*) *ψ is a minimal $F(\varphi)$-isotropic form.*

Proof. Assuming (*a*), Theorem 2.1 shows that $\operatorname{edim}(\psi) = \operatorname{edim}(\varphi)$, whereby (*b*) follows. Assuming (*b*), since $\dim\psi = \operatorname{edim}(\varphi)$, Theorem 2.5 implies that ψ is a minimal $F(\varphi)$-isotropic form, establishing (*c*). Finally, (*c*) clearly implies (*a*). □

Pfister neighbours have maximal splitting, as per [H3, Proposition 3] (one can also see this via Corollary 2.2, since a Pfister neighbour is isotropy equivalent to its associated Pfister form). In general, forms with maximal splitting need not be Pfister neighbours, as is demonstrated in [H3, Example 2]. However, this correspondence does hold for forms of certain dimension, as we will see in the following short proof of [H3, Theorem 3].

Theorem 2.11. (*Hoffmann*) *Let ψ be an anisotropic form over F with $\dim\psi = 2^{n-1}+1$ for some $n \geqslant 4$. Let γ be an anisotropic form over F with $2^n - 3 \leqslant \dim\gamma$. Then $\psi_{F(\gamma)}$ is isotropic if and only if there exists an n-fold Pfister form π such that ψ and γ are Pfister neighbours of π.*

Proof. The right-to-left implication is clear. Conversely, since $\dim\psi = 2^{n-1}+1$, we have that $i_1(\psi) = 1$. If $\psi_{F(\gamma)}$ is isotropic, then ψ and γ are isotropy equivalent by Corollary 2.9. Hence γ has maximal splitting, whereby [K1, Theorem 5.8] and [K2, Corollary 8.2] imply that γ is a Pfister neighbour of some $\pi \in P_nF$. Hence γ is isotropy equivalent to π, whereby ψ is isotropy equivalent to π. Since $\pi_{F(\gamma)}$ is isotropic, [L, Ch.X, Theorem 4.5] implies that $\gamma \subset a\pi$ for some $a \in F^{\times}$. Moreover, since $\gamma_{F(\pi)}$ is isotropic, Theorem 1.4 implies that $\dim\gamma > \frac{1}{2}\dim\pi$, whereby we can conclude that ψ is a Pfister neighbour of π. □

We end this section with some characterisations of anisotropic Pfister neighbours.

Proposition 2.12. *Let π be an anisotropic n-fold Pfister form over F and γ an anisotropic form over F. The following are equivalent:*

(*a*) *γ is a Pfister neighbour of π,*

(*b*) *γ and π are isotropy equivalent,*

(*c*) *γ has maximal splitting, $\dim\gamma \leqslant \dim\pi$ and $\gamma_{F(\pi)}$ is isotropic.*

Proof. Since Pfister neighbours have maximal splitting, (*a*) clearly implies (*c*).

Assuming (*c*), since $\gamma_{F(\pi)}$ is isotropic, we have that $\operatorname{edim}(\gamma) \geqslant \operatorname{edim}(\pi)$ by Theorem 2.1. Furthermore, since $\dim\gamma \leqslant \dim\pi$ and γ has maximal splitting, we have that $\operatorname{edim}(\gamma) \leqslant \operatorname{edim}(\pi)$. Hence $\operatorname{edim}(\gamma) = \operatorname{edim}(\pi)$, whereby Theorem 2.1 shows that (*b*) holds.

As per the proof of Theorem 2.11, assuming (*b*), [L, Ch.X, Theorem 4.5] and Theorem 1.4 respectively show that $\gamma \subset a\pi$ for some $a \in F^{\times}$ and $\dim\gamma > \frac{1}{2}\dim\pi$, whereby (*a*) follows. □

Corollary 2.13. *Let π be an anisotropic n-fold Pfister form over F and γ an anisotropic form over F such that* $\dim\gamma = 2^{n-1}+1$. *If $\gamma_{F(\pi)}$ is isotropic, then γ is a Pfister neighbour of π.*

Proof. This follows directly from Proposition 2.12, since $i_1(\gamma)=1$. □

3 Excellence

A field extension K/F is said to be *excellent* if, for every form ϑ over F, the anisotropic part of ϑ_K is defined over F, that is, $(\vartheta_K)_{\mathrm{an}} \simeq \gamma_K$ for some form γ over F. For $m \in \mathbb{N}$, we say that K/F is m-*excellent* if, for every form ϑ over F with $\dim\vartheta \leqslant m$, there exists a form γ over F such that $(\vartheta_K)_{\mathrm{an}} \simeq \gamma_K$.

Combining [K2, Theorem 7.13] and [H3, Proposition 3], the following is known:

Proposition 3.1. (*Knebusch* ($\Rightarrow$) *and Hoffmann* ($\Leftarrow$)) *Let φ be an anisotropic form over F. Then $(\varphi_{F(\varphi)})_{\mathrm{an}}$ is defined over F if and only if φ is a Pfister neighbour.*

Thus, the only anisotropic quadratic forms whose function fields can be excellent are Pfister neighbours. Indeed, for $\pi \in P_nF$ anisotropic, the extension $F(\pi)/F$ is excellent when $n \leqslant 2$, and is not excellent in general when $n \geqslant 3$ (see [EKM, Ch.IV, Section 29]).

As a result of Proposition 3.1, if φ is not a Pfister neighbour, then $F(\varphi)/F$ is not $(\dim\varphi)$-excellent. However, one may justifiably examine m-excellence for arbitrary function fields $F(\varphi)/F$ when m is less than $\dim\varphi$, and the opening comments of this section address this topic.

Proposition 3.2. *If φ is an anisotropic form over F with* $\dim\varphi > 2^n$, *then $F(\varphi)/F$ is 2^n-excellent.*

Proof. This follows directly from Theorem 1.4, since an anisotropic form ψ over F with $\dim\psi \leqslant 2^n$ is such that $\psi_{F(\varphi)}$ is anisotropic. □

Proposition 3.3. *Let φ and ψ be isotropy-equivalent anisotropic forms over F and γ an anisotropic form over F. Then $i_W(\gamma_{F(\varphi)}) = i_W(\gamma_{F(\psi)})$. Moreover, $(\gamma_{F(\varphi)})_{\mathrm{an}}$ is defined over F if and only if $(\gamma_{F(\psi)})_{\mathrm{an}}$ is defined over F.*

Proof. By [L, Ch.X, Theorem 4.1], $F(\varphi,\psi)$ is a purely-transcendental extension of $F(\varphi)$ and of $F(\psi)$. Thus $i_W(\gamma_{F(\varphi)}) = i_W(\gamma_{F(\varphi,\psi)}) = i_W(\gamma_{F(\psi)})$. Consequently, letting $(\gamma_{F(\varphi)})_{\mathrm{an}} \simeq \delta_{F(\varphi)}$ for some form δ over F, we can conclude that $\gamma \perp -\delta$ becomes hyperbolic over $F(\psi)$, whereby $(\gamma_{F(\psi)})_{\mathrm{an}} \simeq (\delta_{F(\psi)})_{\mathrm{an}} \simeq \delta_{F(\psi)}$. □

Our next observation is that Proposition 3.2 cannot be improved in general.

Proposition 3.4. *Let ψ and φ be anisotropic forms over F such that* $2^n+1 = \dim\psi \leqslant \dim\varphi$ *for some $n \in \mathbb{N}$ and $\psi_{F(\varphi)}$ is isotropic. Then $(\psi_{F(\varphi)})_{\mathrm{an}}$ is defined over F if and only if ψ is a Pfister neighbour.*

Proof. Since $\psi_{F(\varphi)}$ is isotropic, Theorem 2.1 implies that $\mathrm{edim}(\psi) = \mathrm{edim}(\varphi)$, whereby ψ and φ are isotropy equivalent. Thus, $(\psi_{F(\varphi)})_{\mathrm{an}}$ is defined over F if and only if $(\psi_{F(\psi)})_{\mathrm{an}}$ is defined over F by Proposition 3.3, which occurs if and only if ψ is a Pfister neighbour by Proposition 3.1. □

For the rest of this section, we will consider how the aforementioned excellence concepts relate to $F(\pi)/F$ when π is an anisotropic Pfister form. We begin by examining which forms φ over F are such that $(\varphi_{F(\pi)})_{\mathrm{an}}$ is defined over F. Proposition 3.1 establishes that Pfister neighbours of π have this property. Indeed, if $a \in F^\times$ and α, μ are forms over F such that $\alpha \perp \mu \simeq a\pi$ with $\dim \alpha > \dim \mu$, then $(\alpha_{F(\pi)})_{\mathrm{an}} \simeq -\mu_{F(\pi)}$. We next show that certain forms containing Pfister neighbours of π also possess this property.

Proposition 3.5. *Suppose that an anisotropic form φ over F contains a Pfister neighbour of an n-fold Pfister form π. If $i_W(\varphi_{F(\pi)}) = 1$ or if $\dim \varphi \leqslant 2^{n-1} + 3$, then $(\varphi_{F(\pi)})_{\mathrm{an}}$ is defined over F.*

Proof. Let $\varphi \simeq \tau \perp \varphi'$, where τ is a Pfister neighbour of π such that $\tau \perp \gamma \simeq a\pi$. Hence, $\varphi_{F(\pi)} \simeq \varphi'_{F(\pi)} \perp i_W(\tau_{F(\pi)}) \times \langle 1, -1\rangle \perp -\gamma_{F(\pi)}$ and $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq ((\varphi' \perp -\gamma)_{F(\pi)})_{\mathrm{an}}$. If $i_W(\varphi_{F(\pi)}) = 1$, then $\varphi' \perp -\gamma$ is anisotropic over $F(\pi)$. If $\dim \varphi \leqslant 2^{n-1} + 3$, then $\dim \varphi' \leqslant 2$, whereby $\dim(\varphi' \perp -\gamma) \leqslant 2^{n-1} + 1$. If $\varphi' \perp -\gamma$ is isotropic over $F(\pi)$, then it is a Pfister neighbour of π by Corollary 2.13. Hence $(\varphi_{F(\pi)})_{\mathrm{an}}$ is defined over F. ∎

[H2, Corollary 4.2] establishes that $F(\pi)/F$ is 6-excellent in the case where π is a 3-fold Pfister form. Owing to this fact, we can obtain a slight improvement of the preceding result in this case.

Proposition 3.6. *Let φ be an anisotropic form over F with $\dim \varphi \leqslant 8$ and π an anisotropic 3-fold Pfister form over F. If φ contains a Pfister neighbour of π, then $(\varphi_{F(\pi)})_{an}$ is defined over F.*

Proof. As above, letting $\varphi \simeq \tau \perp \varphi'$, where τ is a Pfister neighbour of π such that $\tau \perp \gamma \simeq a\pi$, we have that $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq ((\varphi' \perp -\gamma)_{F(\pi)})_{\mathrm{an}}$. Since $\dim(\varphi' \perp -\gamma) \leqslant 6$, [H2, Corollary 4.2] implies that $((\varphi' \perp -\gamma)_{F(\pi)})_{\mathrm{an}}$ is defined over F. ∎

For certain generalised Pfister neighbours α of π, we can prove that $(\alpha_{F(\pi)})_{\mathrm{an}}$ is defined over F.

Proposition 3.7. *Let α be an anisotropic generalised Pfister neighbour of an n-fold Pfister form π, with μ and ϑ forms over F such that $\alpha \perp \mu \simeq \pi \otimes \vartheta$ with $\dim \alpha > \dim \mu$. If $i_W(\alpha_{F(\pi)}) = 1$, then $(\alpha_{F(\pi)})_{\mathrm{an}} \simeq -\mu_{F(\pi)}$.*

Proof. Since $\alpha \perp \mu \simeq \pi \otimes \vartheta$, we have that $(\alpha_{F(\pi)})_{\mathrm{an}} \simeq -(\mu_{F(\pi)})_{\mathrm{an}}$. As $i_W(\alpha_{F(\pi)}) = 1$ by assumption, we have that $\dim(\alpha_{F(\pi)})_{\mathrm{an}} \geqslant \dim \mu$, whereby we must have that $-\mu_{F(\pi)} \simeq (\alpha_{F(\pi)})_{\mathrm{an}}$. ∎

We note that the above result does not follow from Proposition 3.5, as generalised Pfister neighbours need not contain Pfister neighbours:

Example 3.8. As per [HVG, Section 4], for a certain field F and a particular $\pi \in P_2F$, there exists a minimal $F(\pi)$-isotropic form ψ_m of dimension $2m + 1$ for every $m \in \mathbb{N}$. Since $F(\pi)/F$ is an excellent extension (see [EKM, Section 29]), [H1, Lemma 3.1.2] implies that ψ_m is a generalised Pfister neighbour of π for every $m \in \mathbb{N}$. By minimality, ψ_m does not contain a Pfister neighbour of π when $m \geqslant 2$.

The "only if" part of the following result is [H1, Lemma 3.1.2].

Proposition 3.9. (*Hoffmann* ($\Rightarrow$)) *Let ψ be a minimal $F(\pi)$-isotropic form for some Pfister form π. Then $(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F if and only if ψ is a generalised Pfister neighbour of π.*

Proof. If $(\psi_{F(\pi)})_{\mathrm{an}} \simeq \gamma_{F(\pi)}$ for some form γ over F, then $\psi \perp -\gamma$ becomes hyperbolic over $F(\pi)$. Moreover, $\psi \perp -\gamma$ is anisotropic over F, as otherwise we would have that $\psi \simeq \langle d\rangle \perp \psi'$ and $\gamma \simeq \langle d\rangle \perp \gamma'$ for some $d \in F\times$, whereby $\psi' \perp -\gamma'$ would also be hyperbolic over $F(\pi)$. However, since $\dim\psi > \dim\gamma$, this cannot occur, as otherwise $\psi'_{F(\pi)}$ would be isotropic, contradicting the minimality of ψ. Thus, [L, Ch.X, Theorem 4.9] implies that $\psi \perp -\gamma \simeq \pi \otimes \vartheta$ for some form ϑ, whereby ψ is a generalised Pfister neighbour of π.

Conversely, since ψ is a minimal $F(\pi)$-isotropic form, we have that $i_W(\psi_{F(\pi)}) = 1$. Hence, if ψ is a generalised Pfister neighbour of π, Proposition 3.7 implies that $(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F. □

We conclude this section with some characterisations of excellence and m-excellence for function fields of Pfister forms.

Theorem 3.10. *Let π be an anisotropic Pfister form over F. For $m \in \mathbb{N}$, the following are equivalent:*

(a) *$F(\pi)/F$ is m-excellent,*

(b) *$(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F for every minimal $F(\pi)$-isotropic form ψ over F of dimension m or less,*

(c) *Every minimal $F(\pi)$-isotropic form over F of dimension m or less is a generalised Pfister neighbour of π.*

Proof. (a) clearly implies (b). Moreover, Proposition 3.9 establishes the equivalence of (b) and (c). To conclude, we will show that (b) implies (a).

Let $\varphi_{F(\pi)}$ be isotropic, where $\dim\varphi \leqslant m$. Then $\varphi \simeq \psi \perp \gamma$, where ψ is a minimal $F(\pi)$-isotropic form. Since $(\psi_{F(\pi)})_{\mathrm{an}} \simeq \delta_{F(\pi)}$ for some form δ over F, we have that $\varphi_{F(\pi)} \simeq \langle 1,-1\rangle_{F(\pi)} \perp \delta_{F(\pi)} \perp \gamma_{F(\pi)}$. Now consider $\varphi_1 := \delta \perp \gamma$. If φ_1 is anisotropic over $F(\pi)$, then $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq (\varphi_1)_{F(\pi)}$ and we are done. Otherwise, φ_1 is isotropic over $F(\pi)$, whereby $\varphi_1 \simeq \psi_1 \perp \gamma_1$ for ψ_1 a minimal $F(\pi)$-isotropic form. Iterating our argument, we obtain that $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq (\varphi_n)_{F(\pi)}$ for some $n \in \mathbb{N}$. □

Corollary 3.11. (*Hoffmann* (a) $\iff$ (c)) *Let π be an anisotropic Pfister form over F. The following are equivalent:*

(a) *$F(\pi)/F$ is excellent,*

(b) *$(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F for every minimal $F(\pi)$-isotropic form ψ,*

(c) *Every minimal $F(\pi)$-isotropic form is a generalised Pfister neighbour of π.*

The equivalence of (a) and (c) in the above is [H1, Theorem 3.1.3]. This provides an answer to Question 1.1 for function fields of Pfister forms that are excellent extensions. As per the proof of Theorem 3.10, the equivalences of (a) and (b) in Theorem 3.10 and Corollary 3.11 hold for arbitrary extensions K/F.

4 Bounds on the dimensions where excellence holds

Whereas $F(\pi)/F$ is excellent for all $\pi \in P_nF$ when $n \leqslant 2$, Izhboldin [I, Proposition 1.2] proved that for every $n \geqslant 3$ and any anisotropic $\pi \in P_nF$, there exists a field K/F such that $K(\pi)/K$ is not excellent. In particular, there exists a form φ over K with $\dim \varphi = \dim \pi$ such that $(\varphi_{K(\pi)})_{\mathrm{an}}$ is not defined over K by [I, Lemma 2.4].

As remarked above, Corollary 3.11 provides an answer to Question 1.1 for function fields of Pfister forms that are excellent extensions. In light of Izhboldin's results, one cannot use Corollary 3.11 to obtain information regarding isotropy over $F(\pi)$ when $n \geqslant 3$ without first placing restrictions on either F or $\pi \in P_nF$. However, Theorem 3.10 provides isotropy criteria over all function fields of Pfister forms. Let $\Phi : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ be given by

$$\Phi(n) = \sup\{m \in \mathbb{N} \mid F(\pi)/F \text{ is } m\text{-excellent for every field } F \text{ and } \pi \in P_nF\}.$$

Proposition 4.1. *Let $n \geqslant 3$ and let π be an n-fold Pfister form over F. An anisotropic form φ with $\dim \varphi \leqslant \Phi(n)$ is isotropic over $F(\pi)$ if and only if it contains a Pfister neighbour of π.*

Proof. The right-to-left implication is clear. Conversely, φ contains a minimal $F(\pi)$-isotropic form ψ. Since $F(\pi)/F$ is $\Phi(n)$-excellent and $\dim \psi \leqslant \Phi(n)$, Theorem 3.10 implies that ψ is a generalised Pfister neighbour of π. As per Izhboldin's examples [I], we have that $\Phi(n) < \dim \pi$, whereby ψ is a Pfister neighbour of π. □

Question 4.2. For each $n \geqslant 3$, what is the value of $\Phi(n)$?

Proposition 4.3. *$\Phi(n) \geqslant 2^{n-1} + 1$ for every $n \in \mathbb{N}$.*

Proof. Let $\pi \in P_nF$ and ψ be anisotropic forms over F such that $\dim \psi \leqslant 2^{n-1} + 1$ and $\psi_{F(\pi)}$ is isotropic. Theorem 1.4 implies that $\dim \psi = 2^{n-1} + 1$. Hence, Corollary 2.13 implies that ψ is a Pfister neighbour of π, whereby $(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F. Hence, $F(\pi)/F$ is $(2^{n-1} + 1)$-excellent. □

We note that $\Phi(3) \geqslant 6$ by [H2, Corollary 4.2].

Corollary 4.4. *Let π be an anisotropic 3-fold Pfister form over F and ψ a 6-dimensional anisotropic form over F. Then $\psi_{F(\pi)}$ is isotropic if and only if ψ contains a Pfister neighbour of π. In particular, there are no 6-dimensional minimal $F(\pi)$-isotropic forms.*

Proof. Since $\Phi(3) \geqslant 6$, Proposition 4.1 gives the equivalence. Consequently, every 6-dimensional anisotropic form over F that becomes isotropic over $F(\pi)$ necessarily contains a 5-dimensional Pfister neighbour of π, and hence cannot be a minimal $F(\pi)$-isotropic form. □

In order to establish upper bounds on the values of $\Phi(n)$ when $n \geqslant 3$, we will require the following result:

Proposition 4.5. *Let ψ be a minimal $F(\pi)$-isotropic form for some n-fold Pfister form π over F. If $(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F, then $\dim \psi = m2^{n-1} + 1$ for some $m \in \mathbb{N}$.*

Proof. By Proposition 3.9, ψ is a generalised Pfister neighbour of π, whereby there exists a form γ over F such that $\psi \subset \pi \otimes \gamma$ and $\dim \psi > \frac{1}{2} \dim(\pi \otimes \gamma)$. Letting $\dim \gamma = m$ for some $m \in \mathbb{N}$, the minimality of ψ implies the result. □

Corollary 4.6. (*Hoffmann*) *For π an anisotropic 2-fold Pfister form over F, every minimal $F(\pi)$-isotropic form has odd dimension.*

Proof. Since $F(\pi)/F$ is excellent, Proposition 4.5 implies that every minimal $F(\pi)$-isotropic form is of dimension $2m+1$ for some $m \in \mathbb{N}$. □

Corollary 4.7. *Let π be an n-fold Pfister form over F and ψ a minimal $F(\pi)$-isotropic form. If $2^{n-1}+2 \leqslant \dim\psi \leqslant 2^n$, then $(\psi_{F(\pi)})_{\mathrm{an}}$ is not defined over F.*

Proof. This is an immediate corollary of Proposition 4.5. □

In [I, Lemma 2.4], Izhboldin established the existence of 2^n-dimensional minimal $F(\pi)$-isotropic forms ψ for all $n \geqslant 3$, where π is an n-fold Pfister form over a certain field F. Additionally, he proved that these forms are such that $(\psi_{F(\pi)})_{\mathrm{an}}$ is not defined over F, a result we can recover directly by invoking Corollary 4.7. These examples belong to the class of twisted Pfister forms, $P_{n,m}F$, which Hoffmann studied in [H4]. For $1 \leqslant m < n$, a form φ over F is contained in $P_{n,m}F$ if $\dim\varphi = 2^n$ and $\varphi \simeq (\pi_1 \perp -\pi_2)_{\mathrm{an}}$ where π_1 (respectively π_2) is an n-fold (respectively m-fold) Pfister form over F. For all $n \geqslant 3$ and for all m satisfying $1 \leqslant m \leqslant n-2$, Hoffmann provided examples in [H4, Section 8] of fields F, forms $\varphi \in P_{n,m}F$ and n-fold Pfister forms π over F such that $(\varphi_{F(\pi)})_{\mathrm{an}}$ is not defined over F. The following result concerns the minimal $F(\pi)$-isotropic forms contained within these examples.

Proposition 4.8. *For $n \geqslant 3$, let a field F, forms $\varphi \in P_{n,m}F$ and an n-fold Pfister form π over F be as in [H4, Example 8.1] or [H4, Example 8.3]. The minimal $F(\pi)$-isotropic forms $\psi \subset \varphi$ are such that $(\psi_{F(\pi)})_{\mathrm{an}}$ is not defined over F.*

Proof. We note that such $\varphi \in P_{n,m}F$ satisfy the criteria of [H4, Proposition 7.6]. As a consequence, for all m such that $1 \leqslant m \leqslant n-2$, the minimal $F(\pi)$-isotropic forms $\psi \subset \varphi$ satisfy $2^{n-1}+2 \leqslant \dim\psi \leqslant 2^n - 2^{m-1}+1$. Thus, Corollary 4.7 implies that $(\psi_{F(\pi)})_{\mathrm{an}}$ is not defined over F. □

Corollary 4.9. $\Phi(n) \leqslant 2^n - 2^{n-3}$ *for every $n \geqslant 3$.*

Proof. For certain fields F and certain n-fold Pfister forms π over F, Proposition 4.8 implies the existence of minimal $F(\pi)$-isotropic forms ψ such that $(\psi_{F(\pi)})_{\mathrm{an}}$ is not defined over F. As per the proof of Proposition 4.8, for all m such that $1 \leqslant m \leqslant n-2$, these forms ψ satisfy $2^{n-1}+2 \leqslant \dim\psi \leqslant 2^n - 2^{m-1}+1$. The result follows by letting $m = n-2$. □

5 Fields of finite Hasse number

We will let X_F denote the space of orderings of F, with ${r_P}^+(\varphi)$ (respectively ${r_P}^-(\varphi)$) denoting the number of positive (respectively negative) coefficients in a diagonalisation of φ with respect to $P \in X_F$. A field F is *formally real* if -1 is not a sum of squares in F, a condition which is equivalent to $X_F \neq \emptyset$ (see [L, Ch.VIII, Theorem 1.10]). Elman, Lam and Wadsworth ([ELW, Theorem 3.5 and Remark 3.6]) proved that $P \in X_F$ extends to an ordering of $F(\varphi)$ if and only if φ is *indefinite at* P, that is, ${r_P}^+(\varphi) > 0$ and ${r_P}^-(\varphi) > 0$. Clearly, $X_F = X_{F(\varphi)}$ if and only if φ is *totally indefinite*, that is, indefinite at every $P \in X_F$. The *Hasse number of* F is defined to be

$$\tilde{u}(F) := \sup\{\dim\varphi \mid \varphi \text{ is anisotropic and totally indefinite over } F\}.$$

In this section, we study a special case of Question 1.1 for function fields of Pfister forms, namely that where F is formally real and $\tilde{u}(F)$ is finite. The next result provides an answer to this question for those forms over F of dimension greater than $\tilde{u}(F)$, by offering a classification of the minimal $F(\pi)$-isotropic forms contained therein, where π is a Pfister form over F.

Theorem 5.1. *Let π be an anisotropic n-fold Pfister form over F, and φ an anisotropic form over F such that $\dim\varphi > \tilde{u}(F)$. Then $\varphi_{F(\pi)}$ is isotropic if and only if φ contains a Pfister neighbour of π.*

Proof. The right-to-left implication is clear. Conversely, suppose that $\varphi_{F(\pi)}$ is isotropic. Since $\tilde{u}(F) < \dim\varphi$ and φ is anisotropic over F, we can conclude that F is real (as otherwise φ would be trivially totally indefinite and hence isotropic over F) and that there exists $Q \in X_F$ such that φ is definite at Q. If $P \in Y$, then [H1, Lemma 4.4.3] implies that $0 < {r_P}^{+}(\varphi), {r_P}^{-}(\varphi) \leqslant 2^{n-1}$ does not hold. Moreover, since $\varphi_{F(\pi)}$ is isotropic, Theorem 1.4 implies that $\dim\varphi \geqslant 2^{n-1}+1$, whereby if $0 < {r_P}^{+}(\varphi), {r_P}^{-}(\varphi) \leqslant 2^{n-1}$ does not hold, we can conclude that either ${r_P}^{+}(\varphi) > 2^{n-1}$ or ${r_P}^{-}(\varphi) > 2^{n-1}$. Furthermore, since isotropic forms must necessarily be totally indefinite, if $P \in X_F$ extends to an ordering of $F(\pi)$ (that is, $P \notin Y$ by [ELW, Theorem 3.5 and Remark 3.6]), then φ must be indefinite with respect to P. Hence, φ fulfills all of the conditions in [H1, Lemma 4.4.5], whereby one may conclude that it contains a Pfister neighbour of π. □

We offer the following improvement of [H1, Theorem 4.4.6] as a corollary of Theorem 5.1:

Theorem 5.2. *For $\tilde{u}(F) \leqslant 2^{n-1}+1$ and π an anisotropic n-fold Pfister form over F, the minimal $F(\pi)$-isotropic forms are exactly the Pfister neighbours of π of dimension $2^{n-1}+1$. In particular, $F(\pi)/F$ is excellent.*

Proof. All forms which become isotropic over $F(\pi)$ are necessarily of dimension at least $2^{n-1}+1$ by Theorem 1.4. Since $\tilde{u}(F) \leqslant 2^{n-1}+1$, Theorem 5.1 implies that all the minimal $F(\pi)$-isotropic forms are of dimension $2^{n-1}+1$. Moreover, all such forms are Pfister neighbours of π by Corollary 2.13. Hence, $F(\pi)/F$ is excellent by Corollary 3.11. □

We will proceed to list some further corollaries of Theorem 5.1, beginning by making explicit the consequence thereof employed in the above proof. We note that the following result is contained in [H5, Theorem 5.3], where an analogous statement is presented for iterated function fields of Pfister forms, and thus, it may be viewed as a short recovery of this result for function fields of a single Pfister form:

Corollary 5.3. (*Hoffmann*) $2^{n-1}+1 \leqslant \sup\mathcal{M}(F(\pi)/F) \leqslant \max\{2^{n-1}+1, \tilde{u}(F)\}$, *for π an anisotropic n-fold Pfister form over F.*

Proof. Theorem 1.4 gives the lower bound. Letting $\psi_{F(\pi)}$ be isotropic, if $\dim\psi > \tilde{u}(F)$, then Theorem 5.1 implies that ψ contains a Pfister neighbour of dimension $2^{n-1}+1$. Hence if ψ is a minimal $F(\pi)$-isotropic form, $\dim\psi \leqslant \max\{2^{n-1}+1, \tilde{u}(F)\}$. □

As a corollary of the above, we can give a short proof of [HVG, Proposition 2.6], a result of concerning function fields of conics (or equivalently, function fields of 2-fold Pfister forms):

Corollary 5.4. (*Hoffmann, Van Geel*) *Let F be formally real with $\tilde{u}(F) \leqslant 2n$ for $n \in \mathbb{N}$, and ρ an anisotropic conic over F. Then* $\sup \mathcal{M}(F(\rho)/F) \leqslant \max\{3, 2n-1\}$.

Proof. Since $\dim \rho = 3$, $\sup \mathcal{M}(F(\rho)/F) = \sup \mathcal{M}(F(\pi)/F)$ for some $\pi \in P_2F$. Hence, $\sup \mathcal{M}(F(\rho)/F) \leqslant \max\{3, \tilde{u}(F)\}$ by Corollary 5.3. The statement follows, since Corollary 4.6 implies that $F(\rho)$-minimal forms are of odd dimension. □

Proposition 5.5. *Let φ and a Pfister form π be anisotropic forms over F. If φ is such that* $\dim((\varphi_{F(\pi)})_{\mathrm{an}}) \geqslant \tilde{u}(F) - 1$, *then $(\varphi_{F(\pi)})_{\mathrm{an}}$ is defined over F.*

Proof. Without loss of generality, we may assume that $\varphi_{F(\pi)}$ is isotropic, whereby $\dim \varphi \geqslant \tilde{u}(F) + 1$. Theorem 5.1 implies that $\varphi \simeq \varphi' \perp \tau$, where τ is a Pfister neighbour of π with $\tau \perp \gamma \simeq a\pi$. Hence $\varphi_{F(\pi)} \simeq \varphi'_{F(\pi)} \perp i_W(\tau_{F(\pi)}) \times \langle 1, -1\rangle \perp -\gamma_{F(\pi)}$. If $\varphi_1 \simeq \varphi' \perp -\gamma$ is anisotropic over $F(\pi)$, we are done. Otherwise, since $(\varphi_{F(\pi)})_{\mathrm{an}} \simeq (\varphi_{1\,F(\pi)})_{\mathrm{an}}$, we may iterate the process, whereby for some n we will obtain that $\varphi_{n\,F(\pi)} \simeq (\varphi_{F(\pi)})_{\mathrm{an}}$. □

Proposition 5.6. *Let π be an anisotropic Pfister form over F. Then $F(\pi)/F$ is excellent if and only if $F(\pi)/F$ is $\tilde{u}(F)$-excellent.*

Proof. The left-to-right implication is clear, as is the right-to-left one in the case where $\tilde{u}(F) = \infty$, so we will assume that $F(\pi)/F$ is $\tilde{u}(F)$-excellent where $\tilde{u}(F) < \infty$. If $\tilde{u}(F) \leqslant 2^{n-1} + 1$, Corollary 5.2 gives the result. Hence, we may assume that $\tilde{u}(F) > 2^{n-1} + 1$, whereby Corollary 5.3 implies that there are no minimal $F(\pi)$-isotropic forms of dimension $> \tilde{u}(F)$. Since $F(\pi)/F$ is $\tilde{u}(F)$-excellent, every minimal $F(\pi)$-isotropic form ψ is such that $(\psi_{F(\pi)})_{\mathrm{an}}$ is defined over F, whereby Corollary 3.11 implies that $F(\pi)/F$ is excellent. □

As a result of [H2, Corollary 4.2], one can conclude that $F(\pi)/F$ is 6-excellent for all Pfister forms π over F. Hence, as a corollary of Proposition 5.6, we can recover the following component of [H5, Corollary 4.8]:

Corollary 5.7. (*Hoffmann*) *Let F be a field such that $\tilde{u}(F) \leqslant 6$. Then $F(\pi)/F$ is excellent for every anisotropic Pfister form π over F.*

Proof. Since $\tilde{u}(F) \leqslant 6$, $F(\pi)/F$ is $\tilde{u}(F)$-excellent for every $\pi \in PF$, whereby Proposition 5.6 establishes the result. □

We conclude by invoking Theorem 5.1 to establish the example previously referred to in Section 2.

Example 5.8. Let F be a formally real field with $\tilde{u}(F) = 4$. Let $n \geqslant 1$ and let γ be an anisotropic form over F of dimension $5 + n$ that becomes isotropic over $F(\pi)$, where $\pi \in P_2F$. Since [L, Ch.X, Theorem 4.9] implies that γ cannot become hyperbolic over $F(\pi)$ in the case where $n = 1$, we have that $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1 > 4$ for all n. Theorem 5.1 implies that every subform of γ of dimension $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1$ necessarily contains a Pfister neighbour of π. Thus γ contains no minimal $F(\pi)$-isotropic forms of dimension $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1$.

Acknowledgements. I am greatly indebted to Karim Johannes Becher for his input and feedback regarding a preliminary version of this article. Additionally, I would like to thank Thomas Unger, Lukas Prinzen and Andrew Dolphin for their helpful comments. I gratefully acknowledge the support I received through an IRCSET-Marie Curie International Mobility Fellowship from the Irish Research

Council for Science, Engineering and Technology, co-funded by Marie Curie Actions under FP7. I am very appreciative of the hospitality shown to me by the Fachbereich Mathematik und Statistik and the Zukunftskolleg at Universtät Konstanz.

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