ON THE ALGEBRAIC COBORDISM SPECTRA MSL AND MSp

IVAN PANIN AND CHARLES WALTER

Abstract. We construct algebraic cobordism spectra **MSL** and **MSp**. They are commutative monoids in the category of symmetric $T^{\wedge 2}$-spectra. The spectrum **MSp** comes with a natural symplectic orientation given either by a tautological Thom class $th^{\mathbf{MSp}} \in \mathbf{MSp}^{4,2}(\mathbf{MSp}_2)$, a tautological Pontryagin class $p_1^{\mathbf{MSp}} \in \mathbf{MSp}^{4,2}(HP^\infty)$ or any of six other equivalent structures. For a commutative monoid E in the category $SH(S)$ we prove that assignment $\varphi \mapsto \varphi(th^{\mathbf{MSp}})$ identifies the set of homomorphisms of monoids $\varphi \colon \mathbf{MSp} \to E$ in the motivic stable homotopy category $SH(S)$ with the set of tautological Thom elements of symplectic orientations of E. A weaker universality result is obtained for **MSL** and special linear orientations.

1. Introduction

A dozen years ago Voevodsky [15] constructed the algebraic cobordism spectrum **MGL** in the motivic stable homotopy category $SH(S)$. This gave a new cohomology theory $\mathbf{MGL}^{*,*}$ on smooth schemes and on motivic spaces. Later Vezzosi [14] put a commutative monoid structure on **MGL**. This gave a product to $\mathbf{MGL}^{*,*}$. The commutative monoid structure can even be constructed in the symmetric monoidal model category of symmetric T-spectra, with $T = \mathbf{A}^1/(\mathbf{A}^1 - 0)$ the Morel-Voevodsky object (Panin, Pimenov and Röndigs [10]).

In this paper we construct the algebraic special linear and symplectic cobordism spectra **MSL** and **MSp**. The construction of **MSL** is straightforward although there is one slightly subtle point. We equip each space BSL_n and $\mathbf{MSL}_n$ with an action of GL_n which is compatible with the monoid structure $BSL_m \times BSL_n \to BSL_{m+n}$ induced by the direct sum of subbundles. This gives an action of the subgroup $\Sigma_n \subset GL_n$ of permutation matrices. But to define the unit of the monoid structure we need the action on BSL_n to have fixed points. The natural action of SL_n has fixed points, but the natural action of GL_n does not. So we use an embedding $\Sigma_n \subset Sp_{2n} \subset SL_{2n}$. This means that our **MSL** is a commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra. The categories of symmetric T-spectra and of symmetric $T^{\wedge 2}$-spectra are both symmetrical monoidal, and their homotopy categories are equivalent symmetric monoidal categories (Theorem 3.2). So a symmetric $T^{\wedge 2}$-spectrum structure is quite satisfactory, and it seems to be a natural structure for this spectrum.

Cobordism spectra and the cohomology theories they define are expected to have some universal properties among certain classes of cohomology theories. For instance Voevodsky's and Levine and Morel's algebraic cobordism theories are universal among oriented cohomology theories [6, 10, 14]. We should therefore expect **MSL** to have some degree of universality for special linearly oriented theories. Recall that a special linear bundle (E, λ) over X is a pair consisting of a vector bundle E and an isomorphism of line bundles $\lambda \colon \mathcal{O}_X \cong \det E$.

Date: October 28, 2010.

The work was supported by Universite de Nice - Sophia-Antipolis , the Presidium of RAS Program "Fundamental Research in modern mathematics", the joint DFG-RFBR grant 09-01-91333 NNIO-a, and the RFBR-grant 10-01-00551.

A special linear orientation on a cohomology theory $A^{*,*}$ is an assignment to every special linear bundle of a Thom class $th(E,\lambda) \in A^{2n,n}(E, E-X) = A^{2n,n}_X(E)$ with $n = \operatorname{rk} E$ which is functorial, multiplicative, and such that the multiplication maps $- \cup th(E,\lambda)\colon A^{*,*}(X) \to A^{*+2n,*+n}(E, E-X)$ are isomorphisms. In the motivic context we generally also require that the Thom class of the trivial line bundle over a point be $\Sigma_T 1_A \in A^{2,1}(T) = A^{2,1}(\mathbf{A}^1, \mathbf{A}^1 - 0)$. Hermitian K-theory and Balmer's derived Witt groups are examples of special linearly oriented theories which are not oriented.

The universality properties we show for **MSL** are as follows. A morphism of commutative monoids $\varphi\colon (\mathbf{MSL}, \mu^{SL}, e^{SL}) \to (A, \mu, e)$ in $SH(S)$ determines naturally a special linear orientation on $A^{*,*}$ with Thom classes written $th^{\varphi}(E,\lambda)$. The compatibility of φ with the monoid structure ensures the multiplicativity of the Thom classes (Theorem 5.5).

Conversely, a special linear orientation on $A^{*,*}$ with Thom classes $th(E,\lambda)$ determines a morphism $\varphi\colon \mathbf{MSL} \to A$ in $SH(S)$ with $th^{\varphi}(E,\lambda) = th(E,\lambda)$ for all (E,λ). This φ is unique modulo a certain subgroup $\varprojlim^1 A^{2n-1,n}(MSL_n^{\langle n\rangle}) \subset Hom_{SH(S)}(\mathbf{MSL}, A)$. The obstruction $\varphi \circ \mu^{SL} - \mu_A \circ (\varphi \wedge \varphi)$ to having a morphism of monoids lies in a similarly defined subgroup of $Hom_{SH(S)}(\mathbf{MSL} \wedge \mathbf{MSL}, A)$ (Theorem 5.9).

It would be interesting to know if these obstruction subgroups vanish for Witt groups and hermitian K-theory. The necessary calculations are likely very close to Balmer and Calmès's computation of Witt groups of Grassmannians [1].

Our **MSp** is defined similarly with an action of Sp_{2n} on the spaces BSp_{2n} and $\mathbf{MSp}_{2n}$. The actions of the subgroups $\Sigma_n \subset Sp_{2n}$ make **MSp** a commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra. For **MSp** we can do much more than for **MSL** because we have the quaternionic projective bundle theorem [13, Theorem 8.2] for symplectically oriented cohomology theories. Therefore for any symplectically oriented cohomology theory $A^{*,*}$ we have Pontryagin classes for symplectic bundles, and we can compute the cohomology of quaternionic Grassmannians [13, §11] and of the spaces BSp_{2r} and $\mathbf{MSp}_{2r}$ (§§8–9). Our main result is the following theorem.

Theorem 1.1. *Let (A,μ,e) be a commutative monoid in $SH(S)$. Then the following sets are in canonical bijection*:

(*a*) *symplectic Thom structures on the bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ such that for the trivial rank 2 bundle $\mathbf{A}^2 \to pt$ we have $th(\mathbf{A}^2, \omega_2) = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$,*

(*b*) *Pontryagin structures on $(A^{*,*}, \partial, \times, 1_A)$ for which $p_1(\mathfrak{U}_{HP^1}, \phi_{HP^1}) \in A^{4,2}(HP^1, h_\infty) \subset A^{4,2}(HP^1)$ corresponds to $-\Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$ under the canonical motivic homotopy equivalence $(HP^1, h_\infty) \simeq T^{\wedge 2}$,*

(*c*) *Pontryagin classes theories on $(A^{*,*}, \partial, \times, 1_A)$ with the same normalization condition on $p_1(\mathfrak{U}_{HP^1}, \phi_{HP^1})$ as in* (*b*),

(*d*) *symplectic Thom classes theories on $(A^{*,*}, \partial, \times, 1_A)$ such that for the trivial rank 2 bundle $\mathbf{A}^2 \to pt$ we have $th(\mathbf{A}^2, \omega_2) = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$,*

(α) *classes $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$ with $\vartheta|_{T^{\wedge 2}} = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$,*

(β) *classes $\varrho \in A^{4,2}(HP^\infty, h_\infty)$ with $\varrho|_{HP^1} \in A^{4,2}(HP^1, h_\infty)$ corresponding to $-\Sigma_T^2 1_A \in A^{4,2}(T^{\wedge 2})$ under the canonical motivic homotopy equivalence $(HP^1, h_\infty) \cong T^{\wedge 2}$,*

(δ) *sequences of classes $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3, \dots)$ with $\vartheta_r \in A^{4r,2r}(\mathbf{MSp}_{2r})$ for each r satisfying $\mu_{rs}^* \vartheta_{r+s} = \vartheta_r \times \vartheta_s$ for all r, s, and $\vartheta_1|_{T^{\wedge 2}} = \Sigma_T^2 1_A$,*

(ε) *morphisms $\varphi\colon (\mathbf{MSp}, \mu^{Sp}, e^{Sp}) \to (A, \mu, e)$ of commutative monoids in $SH(S)$.*

The bijections are explicit and are given in a series of theorems in the last part of the paper. The presence of (ε) among them is the universality of **MSp** as a symplectically oriented theory.

The equivalence of (a), (b), (c) and (d) was already shown in [13] in a different axiomatic context. The ability of the motivic language used here to handle tautological classes such as the (α), (β) and (δ) is very useful by itself. But our main new observation is that in the motivic unstable homotopy category $H_\bullet(S)$ we have a commutative diagram (Theorem 7.7).

$$\begin{tikzcd}
 & BSp_{2r} \arrow[dl, bend right, "\text{structure map}"'] \arrow[dr, bend left, "\text{quotient}"] & \\
\mathbf{MSp}_{2r} \arrow[rr, leftrightarrow, "\cong", "\mathbf{A}^N\text{-bundles and excision}"'] & & BSp_{2r}/BSp_{2r-2}.
\end{tikzcd}$$

What is surprising about this diagram is that it is the homotopy colimit of diagrams (7.5) of finite-dimensional schemes and their quotient spaces which have a fourth side which is an inclusion of quaternionic Grassmannians of different dimensions which is in no way a motivic equivalence. But in the infinite-dimensional colimit the fourth side becomes $\mathbf{A}^1$-homotopic to the identity map of BSp_{2r}, and the picture simplifies significantly.

The fact that this diagram is three-sided instead of four-sided helps us to see more conceptual proofs of two of the trickier points of [13]. One is the construction of the higher-rank symplectic Thom classes and the proof of their multiplicativity. The commutativity of the diagram and the computations of the cohomology of quaternionic Grassmannians imply that given a symplectic Thom structure on $A^{*,*}$, the pullback along the structure map gives an injection $A^{*,*}(\mathbf{MSp}_{2r}) \to A^{*,*}(BSp_{2r})$, and the isomorphism $A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom} \cong A^{*,*}(BSp_{2r})$ defined by the symplectic Thom structure identifies the image of $A^{*,*}(\mathbf{MSp}_{2r})$ with the principal two-sided ideal generated by p_r. This makes it easy to define the higher-rank tautological symplectic Thom classes (the ϑ_r of (δ)) with the classes of $A^{*,*}(\mathbf{MSp}_{2r})$ identified with $(-1)^r p_r \in A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom}$. Their multiplicativity is also easily established.

The other tricky point of [13] for which the diagram helps is the reconstitution of the symplectic Thom structure from the Pontryagin structure. The tautological rank 2 Thom class is a $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$, and it is tempting to identify it (up to sign) with the tautological rank 2 Pontryagin class $\varrho \in A^{4,2}(BSp_2/BSp_0) = A^{4,2}(HP^\infty, h_\infty)$ using the horizontal motivic homotopy equivalence. But the Pontryagin is actually (up to sign) the pullback of ϑ along the structure map of the Thom space. For the three-sided diagram this is no problem: in $H_\bullet(S)$ the structure map $HP^\infty \to \mathbf{MSp}_2$ is the composition of the horizontal isomorphism $(HP^\infty, h_\infty) \cong \mathbf{MSp}_2$ with the pointing map.

It is not difficult to define spectra **MO** and **MSO** which resemble formally **MGL** and our **MSL** and **MSp**. However, our proof of even our most basic result about **MSL** (Theorem 5.5) uses the fact that special linear bundles are locally trivial in the Zariski topology. So we omit **MO** and **MSO**.

2. Preliminaries

Let S be a noetherian scheme of finite Krull dimension, and let $\mathcal{S}m/S$ be the category of smooth quasi-projective schemes over S. We will assume that S admits an ample family of line bundles so that for any X in $\mathcal{S}m/S$ there exists an affine bundle $Y \to X$ with Y an affine scheme. This condition is used a number of times in this paper, and it was also used in the proof of the symplectic splitting principle in [13, Theorem 10.2].

The category $\mathcal{S}m\mathcal{O}p/S$ has objects (X, U) where X is in $\mathcal{S}m/S$ and $U \subset X$ is an open subscheme. A morphism $f\colon (X, U) \to (X', U')$ in $\mathcal{S}m\mathcal{O}p/S$ is a morphism $f\colon X \to X'$ of S-schemes with $f(U) \subset U'$. We often write X in place of $(X, \varnothing)$.

A *bigraded ring cohomology theory* $(A^{*,*}, \partial, \times, 1)$ on $\mathcal{S}m\mathcal{O}p/S$ is a contravariant functor $A^{*,*}$ from $\mathcal{S}m\mathcal{O}p/S$ to the category of bigraded abelian groups which satisfies étale excision and $\mathbf{A}^1$-homotopy invariance and which has localization long exact sequences

$$\cdots \to A^{*,*}(X, U) \to A^{*,*}(X) \to A^{*,*}(U) \xrightarrow{\partial} A^{*+1,*}(X, U) \to \cdots .$$

The $\times$ product is assumed to be functorial, bilinear, associative, and compatible with the bigrading with a two-sided unit 1.

In this paper we work mainly with the motivic unstable and stable homotopy categories $H_\bullet(S)$ and $SH(S)$. The former is the homotopy category of a model category $\mathbf{M}_\bullet(S)$ of pointed motivic spaces over S with motivic weak equivalences. There are several versions of this model category with different underlying categories and with different choices of fibrations and cofibrations. See [2, 4, 5, 8, 11, 15, 16] among other papers. It is not essential which of these model category structures is used. However, we give geometric constructions symmetric T- and $T^{\wedge 2}$-spectra using the the Morel-Voevodsky object $T = \mathbf{A}^1/(\mathbf{A}^1 - 0)$ itself. So the best adapted model structures is perhaps the flasque motivic model category of [4] which is known to be cellular (so we can apply Hovey's results [3] in the proof of Theorem 3.2) but for which T is cofibrant.

The category $\mathbf{M}_\bullet(S)$ is equipped with a symmetric monoidal structure $(\mathbf{M}_\bullet(S), \wedge, S^0)$. The smash-product is taken sectionwise, and S^0 is the constant simplicial zero-sphere. This smash-product induces a smash-product on $H_\bullet(S)$ such that the natural functor $\mathbf{M}_\bullet(S) \to H_\bullet(S)$ becomes a strict symmetric monoidal functor $(\mathbf{M}_\bullet(S), \wedge, S^0) \to (H_\bullet(S), \wedge, S^0)$.

We set $T = \mathbf{A}^1/(\mathbf{A}^1 - 0)$. A *$T$-spectrum* E is a sequence $(E_0, E_1, \dots)$ of pointed motivic spaces equipped with a sequence of structure maps $\sigma_n\colon E_n \wedge T \to E_{n+1}$ of pointed motivic spaces. A morphism of T-spectra $E \to E'$ is a sequence of maps $f_n\colon E_n \to E'_n$ of pointed motivic spaces which commute with the structure maps. The category of T-spectra $Sp(\mathbf{M}_\bullet(S), T)$ can be equipped with a motivic stable model structure as in [5, 15, 16]. Its homotopy category is $SH(S)$. This category can be equipped with a structure of a symmetric monoidal category $(SH(S), \wedge, \mathbf{1})$, satisfying the conclusions of [15, Theorem 5.6]. The unit $\mathbf{1}$ of that monoidal structure is the T-sphere spectrum $\mathbb{S} = (S^0, T, T \wedge T, \dots)$.

Every T-spectrum $E = (E_0, E_1, \dots)$ represents a cohomology theory on the category of pointed motivic spaces $\mathbf{M}_\bullet(S)$. Namely, let S^n_s and S^n_t be as in [15, (16)]. Let $S^{p,q} = S^{p-q}_s \wedge S^q_t$. We write

$$E^{p,q}(A) = Hom_{SH(S)}(\Sigma^\infty_T A, E \wedge S^{p,q})$$

as in [15, §6]. There is a canonical element in $E^{2n,n}(E_n)$, denoted as

$$\Sigma^\infty_T E_n(-n) \xrightarrow{u_n} E.$$

It is represented by the canonical map $(*, \dots, *, E_n, E_n \wedge T, \dots) \to (E_0, E_1, \dots, E_n, \dots)$ of T-spectra.

A *T-ring spectrum* is a monoid (E, μ, e) in $(SH(S), \wedge, \mathbf{1})$. The cohomology theory $E^{*,*}$ defined by a T-ring spectrum is a ring cohomology theory on $\mathbf{M}_\bullet(S)$. To see recall the standard isomorphism $S^{i,j} \wedge S^{k,l} \cong S^{i+k,j+l}$ given by the composition

$$(S^{i-j}_s \wedge S^j_t) \wedge (S^{k-l}_s \wedge S^l_t) \cong (S^{i-j}_s \wedge S^{k-l}_s) \wedge (S^j_t \wedge S^l_t) \cong S^{i-j+k-l}_s \wedge S^{j+l}_t.$$

For $X, Y \in \mathbf{M}_\bullet(k)$ let $\alpha\colon \Sigma^\infty(X) \to S^{i,j} \wedge E$ and $\beta\colon \Sigma^\infty(Y) \to S^{k,l} \wedge E$ be elements of $E^{i,j}(X)$ and $E^{k,l}(Y)$ respectively. Following Voevodsky [15] define $\alpha \times \beta \in E^{i+k,j+l}(X \wedge Y)$ as the composition

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty(X) \wedge \Sigma^\infty(Y) \xrightarrow{\alpha\wedge\beta} E \wedge S^{i,j} \wedge E \wedge S^{k,l} \cong E \wedge E \wedge S^{i+k,j+l} \xrightarrow{id\wedge\mu} E \wedge S^{i+k,j+l}.$$

This gives a functorial product which is associative, has a two-sided unit, and takes cofibration sequences to long exact sequences.

A *commutative* T-ring spectrum is a commutative monoid (E, μ, e) in $(SH(S), \wedge, \mathbf{1})$. To describe the properties of the associated cohomology theory we make some definitions.

Definition 2.1. Let $in_T\colon T \to T$ be a morphism of pointed motivic spaces induced by the morphism $\mathbf{A}^1 \to \mathbf{A}^1$ sending $t \mapsto -t$. One has the equality

$$Hom_{SH(S)}(pt_+, pt_+) = Hom_{SH(S)}(T, T).$$

We write in for in_T regarded as an element of $Hom_{SH(S)}(pt_+, pt_+)$. For a commutative monoid (A, m, e) set $\epsilon = in^*(e) \in A^{0,0}(pt_+)$.

Remark 2.2. The morphism $\mathbf{A}^2 \to \mathbf{A}^2$ which sends $(t_1, t_2) \mapsto (-t_1, -t_2)$ is $\mathbf{A}^1$-homotopic to the identity morphism because $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ is a product of elementary matrices. Whence we have $\epsilon \times \epsilon = e \in A^{0,0}(pt_+)$.

Definition 2.3. A ring cohomology theory on $\mathbf{M}_\bullet(S)$ is *ϵ-commutative* if for $\alpha \in E^{i,j}(X)$, $\beta \in E^{k,l}(Y)$ one has $\sigma^*_{X,Y}(\alpha \times \beta) = (-1)^{ik}\epsilon^{jl}(\beta \times \alpha) \in E^{i+k,j+l}(X \wedge Y)$ where $\sigma_{X,Y}\colon X \wedge Y \to Y \wedge X$ switches the factors.

Thus ϵ-commutativity is a specific form of bigraded commutativity.

Theorem 2.4 (Morel)**.** *Let (E, μ, e) be a commutative monoid in $SH(S)$. Then the data $(E^{*,*}, \partial, \times, e)$ is an associative and ϵ-commutative ring cohomology theory on $\mathbf{M}_\bullet(k)$; $e \in E^{0,0}(S^0)$ is the two-sided unit of the ring structure.*

Note, that if $(i, j) = (2m, 2n)$ or $(k, l) = (2m, 2n)$, then $\sigma^*_{A,B}(\alpha \times \beta) = \beta \times \alpha$.

3. Commutative T- and $T^{\wedge 2}$-monoids

We compare the categories of symmetric T-spectra and symmetric $T^{\wedge 2}$-spectra. Recall the definition for $K = T$ or $K = T^{\wedge 2}$.

Definition 3.1. A *symmetric K-spectrum E* is a sequence of pointed spaces $(E_0, E_1, E_2, \dots)$ with each E_n equipped with an action of the symmetric group $\Sigma_n \times E_n \to E_n$ and with a morphism $\sigma_n\colon E_n \wedge K \to E_{n+1}$ such that the induced maps $E_n \wedge K^{\wedge m} \to E_{n+m}$ are $(\Sigma_n \times \Sigma_m)$-equivariant for all n and m.

The categories of symmetric T- and $T^{\wedge 2}$-spectra both have a symmetric monoidal product $\wedge$. They are symmetric monoidal model categories for the stable model structure [3, 5].

Theorem 3.2. *The homotopy categories of $Sp^\Sigma(\mathbf{M}_\bullet(S), T)$ and of $Sp^\Sigma(\mathbf{M}_\bullet(S), T^{\wedge 2})$ are equivalent symmetric monoidal categories.*

Proof. The proof of this theorem is essentially the same as that given for topological S^1- and S^2-spectra in [11, Theorem A.44]. The inclusion $Sp^\Sigma(\mathbf{M}_\bullet(S), T) \to Sp^\Sigma(Sp^\Sigma(\mathbf{M}_\bullet(S), T), T^{\wedge 2})$ is a Quillen equivalence by [3, Theorem 9.1] because $- \wedge T^{\wedge 2}$ is a Quillen self-equivalence of

$Sp^\Sigma(\mathbf{M}_\bullet(S), T)$. Similarly the inclusion $Sp^\Sigma(\mathbf{M}_\bullet(S), T^{\wedge 2}) \to Sp^\Sigma(Sp^\Sigma(\mathbf{M}_\bullet(S), T^{\wedge 2}), T)$ is a Quillen equivalence. The two categories of symmetric bispectra are isomorphic with the identical stable model structure by arguments like those used in the proof of [3, Theorem 10.1]. Hovey's work requires that the model structure have certain properties, but in the flasque model structure [4] these properties hold, and T and $T^{\wedge 2}$ are cofibrant.

The symmetric monoidal structures are the same because (i) the inclusions of the categories of symmetric spectra in the categories of symmetric bispectra are symmetric monoidal functors like any inclusion $\Sigma_K^\infty \colon \mathcal{C} \to Sp^\Sigma(\mathcal{C}, K)$, and (ii) the symmetric monoidal structures on the two isomorphic categories of symmetric bispectra are the same. □

For natural numbers m, n we denote by $c_{m,n} \in \Sigma_{m+n}$ the (m, n)-*shuffle permutation.* It acts by $c_{m,n}(i) = i + n$ for $1 \le i \le m$ and $c_{m,n}(i) = i - m$ for $m + 1 \le i \le m + n$.

Definition 3.3. A *commutative K-monoid* E in $\mathbf{M}_\bullet(S)$ is a sequence of pointed motivic spaces $(E_0, E_1, E_2, \dots)$ with each space equipped with an action $\Sigma_n \times E_n \to E_n$ of the symmetric group, plus morphisms

$$\begin{aligned} e_0 &\colon \mathbf{1}_{\mathbf{M}_\bullet(S)} \to E_0, \\ e_1 &\colon K \to E_1 \\ \mu_{mn} &\colon E_m \wedge E_n \to E_{m+n} \end{aligned} \tag{3.1}$$

in $\mathbf{M}_\bullet(S)$ such that each μ_{mn} is $(\Sigma_m \times \Sigma_n)$-equivariant and such that the compositions

$$\begin{aligned} E_n &\xrightarrow{\cong} E_n \wedge \mathbf{1}_{M_\bullet(S)} \xrightarrow{1 \wedge e_0} E_n \wedge E_0 \xrightarrow{\mu_{n0}} E_n \\ E_n &\xrightarrow{\cong} \mathbf{1}_{M_\bullet(S)} \wedge E_n \xrightarrow{e_0 \wedge 1} E_0 \wedge E_n \xrightarrow{\mu_{0n}} E_n \end{aligned} \tag{3.2}$$

are the identity maps, and the diagrams

$$\begin{array}{ccc} E_\ell \wedge E_m \wedge E_n & \xrightarrow{\mu_{\ell m} \wedge 1} & E_{\ell+m} \wedge E_n \\ {\scriptstyle 1 \wedge \mu_{mn}} \downarrow & & \downarrow {\scriptstyle \mu_{\ell+m,n}} \\ E_\ell \wedge E_{m+n} & \xrightarrow{\mu_{\ell,m+n}} & E_{\ell+m+n} \end{array} \qquad \begin{array}{ccc} E_m \wedge E_n & \xrightarrow{\mu_{mn}} & E_{m+n} \\ {\scriptstyle \text{switch}} \downarrow & & \cong \downarrow {\scriptstyle c_{m,n}} \\ E_n \wedge E_m & \xrightarrow{\mu_{nm}} & E_{n+m} \end{array} \tag{3.3}$$

commute in $\mathbf{M}_\bullet(S)$ with $c_{m,n}$ the isomorphism given by the action of the (m, n)-shuffle permutation.

Theorem 3.4. *Let E be a commutative K-monoid. Define maps σ_n as the compositions*

$$\sigma_n \colon E_n \wedge K \xrightarrow{1 \wedge e_1} E_n \wedge E_1 \xrightarrow{\mu_{n1}} E_{n+1}. \tag{3.4}$$

Then the spaces $(E_0, E_1, E_2, \dots)$ equipped with the actions $\Sigma_n \times E_n \to E_n$ and the bonding maps σ_n form a symmetric K-spectrum E. Moreover, the morphisms $\mu \colon E \wedge E \to E$ induced by the μ_{mn} and $e \colon \Sigma_K^\infty \mathbf{1}_{\mathbf{M}_\bullet(S)} \to E$ composed of the maps $e_n \colon K^{\wedge n} \to E_n$ induced by e_0, e_1 and the μ_{mn} make (E, μ, e) a commutative monoid in $Sp^\Sigma(\mathbf{M}_\bullet(S), K)$.

Sketch of proof. To show that E is a symmetric K-spectrum one has to verify that the induced maps $E_n \wedge K^{\wedge j} \to E_{n+j}$ are $(\Sigma_n \times \Sigma_j)$-equivariant. To show that the maps μ_{mn} define a morphism $E \wedge E \to E$ one has to verify that they are K-linear and K-bilinear in the sense of [5, (4.6)–(4.7)]. One has to verify that each e_n is Σ_n-equivariant. Finally, one has to verify the commutative monoid axioms. All the verifications are formal, straightforward and left to the reader. □

4. The symmetric $T^{\wedge 2}$-spectrum MSL

We construct a commutative T-monoid $\mathbf{MSL}$. Each space $\mathbf{MSL}_n$ comes equipped with an action of GL_n such that the multiplication maps μ_{mn} of the monoid structure are $(GL_m \times GL_n)$-equivariant. We then get actions of the Σ_n from the embeddings $\Sigma_n \to GL_n$ given by permutation matrices. The need for an action of GL_n with fixed points — necessary for the proper definition of the unit maps — rather than merely of SL_n is the delicate part of the construction.

We begin by reviewing the construction of $\mathbf{MGL}$ originally done in [15, §6.3] and of its monoid structure given in [10, 14].

For each integer $n \geq 0$ let $\Gamma_n = \mathcal{O}_S^{\oplus n}$ be the trivial rank-n vector bundle. For each integer $p \geq 1$ let $Gr(n, np) = Gr(n, \Gamma_n^{\oplus p})$. Let $\mathcal{T}GL_{n,np} \to Gr(n, np)$ be the tautological subbundle. The inclusions $(1, 0)\colon \Gamma_n^{\oplus p} \to \Gamma_n^{\oplus p} \oplus \Gamma_n = \Gamma_n^{\oplus p+1}$ induce closed embeddings $Gr(n, np) \hookrightarrow Gr(n, np + n)$ and monomorphisms $\operatorname{Th} \mathcal{T}GL_{n,np} \to \operatorname{Th} \mathcal{T}GL_{n,np+n}$ of Thom spaces. We set

$$\begin{aligned} BGL_n &= \operatorname{colim}_{p\in\mathbb{N}} Gr(n, np),\\ \mathcal{T}GL_{n,n\infty} &= \operatorname{colim}_{p\in\mathbb{N}} \mathcal{T}GL_{n,np},\\ \mathbf{MGL}_n &= \operatorname{colim}_{p\in\mathbb{N}} \operatorname{Th} \mathcal{T}GL_{n,np}. \end{aligned}$$

The diagonal action of $GL_n = GL(\Gamma_n)$ on each $Gr(n, np) = Gr(n, \Gamma_n^{\oplus p})$ is compatible with the inclusions over increasing p. Moreover, the $\mathcal{T}GL_{n,np}$ are GL_n-equivariant vector bundles. This induces actions

$$\Sigma_n \times \mathbf{MGL}_n \subset GL_n \times \mathbf{MGL}_n \to \mathbf{MGL}_n\,.$$

Concatenation of bases induces isomorphisms $\Gamma_m \oplus \Gamma_n = \Gamma_{m+n}$ which induce $(GL_m \times GL_n)$-equivariant maps

$$\oplus\colon Gr(m, mp) \times Gr(n, np) \to Gr(m + n, mp + np)$$

and therefore $(GL_m \times GL_n)$-equivariant maps

$$\begin{aligned} \oplus\colon BGL_m \times BGL_n &\to BGL_{m+n},\\ \mu^{GL}_{mn}\colon \mathbf{MGL}_m \wedge \mathbf{MGL}_n &\to \mathbf{MGL}_{m+n}\,. \end{aligned}$$

Finally each $Gr(n, np)$ is pointed by the point corresponding to the trivial rank-n subbundle

$$\Gamma_n \xrightarrow{(1,0,\ldots,0)} \Gamma_n^{\oplus p}.$$

In the colimit this gives a S-valued point x_n of BGL_n, which is fixed by the action of GL_n. The Thom space of the fiber of $\mathcal{T}(n, n\infty)$ over x_n is $\Gamma_n \cong \mathbf{A}^n$, and the inclusion $x_n \hookrightarrow BGL_n$ induces a map of Thom spaces

$$e^{GL}_n\colon T^{\wedge n} \to \mathbf{MGL}_n\,.$$

Definition 4.1. The *algebraic cobordism spectrum* $\mathbf{MGL}$ is the commutative monoid in the category of symmetric T-spectra associated to the commutative T-monoid composed of the spaces $\mathbf{MGL}_n$, the actions $\Sigma_n \times \mathbf{MGL}_n \to \mathbf{MGL}_n$, the maps $e^{GL}_0\colon pt_+ \to \mathbf{MGL}_0$ and $e^{GL}_1\colon T \to \mathbf{MGL}_1$ and the maps $\mu^{GL}_{mn}\colon \mathbf{MGL}_m \wedge \mathbf{MGL}_n \to \mathbf{MGL}_{m+n}$.

We now move on to defining $\mathbf{MSL}$. We begin with the spaces. For $n = 0$ we have $SL_0 = GL_0 = \{1\}$. So we set $BSL_0 = pt$. The Thom space of a zero vector bundle over a scheme X is the externally pointed space X_+. So we set $\mathbf{MSL}_0 = pt_+$.

Now suppose $n > 0$. Over each $Gr(n, np)$ there is the line bundle $\mathcal{O}_{Gr(n,np)}(-1) = \det \mathcal{T}GL_{n,np}$. Removing the zero section gives a smooth scheme

$$SGr(n, np) = \mathcal{O}_{Gr(n,np)}(-1) - Gr(n, np).$$

The projection

$$\pi = \pi_{n,np} \colon SGr(n, np) \to Gr(n, np)$$

is a principal $\mathbb{G}_m$-bundle. Write

$$\mathcal{T}SL_{n,np} = \pi^* \mathcal{T}GL_{n,np}.$$

The inclusion $SGr(n, np) \hookrightarrow \mathcal{O}_{Gr(n,np)}(-1)$ and the cartesian diagram

$$\begin{tikzcd} \pi^*\mathcal{O}_{Gr(n,np)}(-1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{O}_{Gr(n,np)}(-1) \arrow[d] \\ SGr(n,np) \arrow[r, "\pi"'] & Gr(n,np) \end{tikzcd}$$

gives a nowhere vanishing section of $\pi^*\mathcal{O}_{Gr(n,np)}(-1) = \det \mathcal{T}SL_{n,np}$. The corresponding isomorphism $\lambda_{n,np} \colon \mathcal{O}_{SGr(n,np)} \cong \det \mathcal{T}SL_{n,np}$ makes $(\mathcal{T}SL_{n,np}, \lambda_{n,np})$ the *tautological special linear bundle* over $SGr(n, np)$.

We set

$$\begin{aligned} BSL_n &= \operatorname{colim}_{p\in\mathbb{N}} SGr(n, np), \\ \mathcal{T}SL_{n,n\infty} &= \operatorname{colim}_{p\in\mathbb{N}} \mathcal{T}SL_{n,np}, \\ \mathbf{MSL}_n &= \operatorname{colim}_{p\in\mathbb{N}} \operatorname{Th} \mathcal{T}SL_{n,np}. \end{aligned}$$

We next define the multiplication maps. Morphisms of S-schemes $X \to SGr(n, np)$ are in bijection with pairs (f, λ) with $f \colon X \to Gr(n, np)$ a morphism and $\lambda \colon \mathcal{O}_X \cong \det f^*\mathcal{T}GL_{n,np}$ an isomorphism. There are unique maps

$$(\oplus, \otimes) \colon SGr(m, mp) \times SGr(n, np) \to SGr(m + n, mp + np)$$

corresponding to the morphisms of representable functors

$$\begin{array}{ccc} Hom(X, SGr(m, mp)) \times Hom(X, SGr(n, np)) & \longrightarrow & \operatorname{Hom}(X, SGr(m + n, mp + np)) \\ ((f, \lambda), (g, \lambda_1)) & \longmapsto & (f \oplus g, \lambda \otimes \lambda_1). \end{array}$$

They induce maps

$$\begin{aligned} (\oplus, \otimes) &\colon BSL_m \times BSL_n \to BSL_{m+n} \\ \mu^{SL}_{mn} &\colon \mathbf{MSL}_m \wedge \mathbf{MSL}_n \to \mathbf{MSL}_{m+n}. \end{aligned}$$

We now discuss the group actions. Since $\mathcal{T}GL_{n,np}$ is a GL_n-equivariant bundle over $Gr(n, np)$, there is an induced action of GL_n on the complement of the zero section of the determinant line bundle. This is an action $GL_n \times SGr(n, np) \to SGr(n, np)$. In the colimit this gives an action $GL_n \times BSL_n \to BSL_n$. But there is a problem.

The unit maps $e^{GL}_n \colon T^{\wedge n} \to \mathbf{MGL}_n$ were defined using points $x_n \colon pt \to BGL_n$ which were fixed under the action of GL_n. To define unit maps for a T-monoid $\mathbf{MSL}$ we need fixed points for the action of at least Σ_n on BSL_n, preferably lying over x_n. We have a cartesian diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BSL_n \arrow[d] \\ pt \arrow[r, "x_n"'] & BGL_n. \end{tikzcd}$$

The action of $GL_n = GL(\Gamma_n)$ on the fiber Γ_n of $\mathcal{T}GL_{n,n\infty}$ over the fixed point x_n is the standard representation of GL_n. So the induced action on the fiber $\mathbb{G}_m$ over x_n is $g \cdot t = \det(g)t$. Thus there are fixed points for the action of the alternating group $\mathfrak{A}_n \subset SL_n$ on BSL_n lying over the fixed point $x_n \in BGL_n(pt)$ used to define the unit maps on $\mathbf{MGL}_n$ but not for the action of $\Sigma_n \subset GL_n$ (except in characteristic 2).

So we use the embedding $\Sigma_n \subset Sp_{2n} \subset SL_{2n}$ which sends $\sigma \in \Sigma_n$ to the permutation matrix associated to $\bar\sigma \in \Sigma_{2n}$ where we have $\bar\sigma(2i-1) = 2\sigma(i) - 1$ and $\bar\sigma(2i) = 2\sigma(i)$. This gives us an action $\Sigma_n \times BSL_{2n} \to BSL_{2n}$ which fixes pointwise the fiber over x_n.

Therefore we define the spaces of the commutative $T^{\wedge 2}$-monoid $\mathbf{MSL}$ to be the $\mathbf{MSL}_{2n}$. Each is equipped with the action of $\Sigma_n \times \mathbf{MSL}_{2n} \to \mathbf{MSL}_{2n}$ induced by the action of SL_{2n}.

We now define the unit maps. Points $pt \to BSL_n$ lifting the point $x_n \colon pt \to BGL_n$ are in bijection with isomorphisms $\lambda \colon \mathcal{O}_S \cong \det x_n^* \mathcal{T}GL_{n,n\infty} = \Lambda^n \Gamma_n$. Let $f_1, \dots, f_n$ be the standard basis of $\Gamma_n = \mathcal{O}_S^{\oplus n}$. We let $y_n \colon pt \to BSL_n$ be the lifting of x_n corresponding to $\lambda = f_1 \wedge \dots \wedge f_n$. The fiber of $\mathcal{T}SL_{n,n\infty}$ over y_n is $\Gamma_n \cong \mathbf{A}^n$, and we let

$$e_n^{SL} \colon T^{\wedge n} \to \mathbf{MSL}_n$$

be the map of Thom spaces induced by y_n. It is SL_n-equivariant. Note that $e_0^{SL} \colon pt_+ \to \mathbf{MSL}_0 = pt_+$ is the identity.

Having identified the components of the structure $\mathbf{MSL}$, we have to assemble them. It appears as if $\mathbf{MSL}$ is a commutative monoid in the category of alternating T-spectra. We do not know how to work in that category. But there is underlying structure.

Definition 4.2. The *algebraic special linear cobordism spectrum* $\mathbf{MSL}$ refers to three related objects.

(a) The commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra associated to the commutative $T^{\wedge 2}$-monoid composed of the spaces $\mathbf{MSL}_{2n}$, the actions $\Sigma_n \times \mathbf{MSL}_{2n} \to \mathbf{MSL}_{2n}$, the maps $e_0^{SL} \colon pt_+ \to \mathbf{MSL}_0$ and $e_2^{SL} \colon T^{\wedge 2} \to \mathbf{MSL}_2$ and the maps $\mu_{2m,2n}^{SL} \colon \mathbf{MSL}_{2m} \wedge \mathbf{MSL}_{2n} \to \mathbf{MSL}_{2m+2n}$.

(b) The T-spectrum with spaces $\mathbf{MSL}_n$, bonding maps $\mathbf{MSL}_n \wedge T \to \mathbf{MSL}_n \wedge \mathbf{MSL}_1 \to \mathbf{MSL}_{n+1}$ induced by e_1^{SL} and $\mu_{n,1}^{SL}$, equipped with the morphism of T-spectra $e \colon \Sigma_T^\infty pt_+ \to \mathbf{MSL}$ and the structural maps μ_{mn}^{SL}.

(c) Their common underlying $T^{\wedge 2}$-spectrum.

The properties of the commutative monoid structure that we require are given in the following theorem.

Theorem 4.3. *The* $(\mathbf{MSL}, \mu^{SL}, e^{SL})$ *is a commutative monoid in* $SH(S)$*, and the canonical maps* $u_n \colon \Sigma_T^\infty \mathbf{MSL}_n(-n) \to \mathbf{MSL}$ *and the* μ_{mn}^{SL} *make the following diagram commute for all* *m and n*

$$\begin{tikzcd} \Sigma_T^\infty \mathbf{MSL}_m(-m) \wedge \Sigma_T^\infty \mathbf{MSL}_n(-n) \arrow[r, "\Sigma_T^\infty \mu^{SL}_{mn}"] \arrow[d, "u_m \wedge u_n"'] & \Sigma_T^\infty \mathbf{MSL}_{m+n}(-m-n) \arrow[d, "u_{m+n}"] \\ \mathbf{MSL} \wedge \mathbf{MSL} \arrow[r, "\mu^{SL}"] & \mathbf{MSL}. \end{tikzcd} \tag{4.1}$$

Proof. A commutative monoid in $Sp^\Sigma(\mathbf{M}_\bullet(S), T^{\wedge 2})$ gives a commutative monoid in $SH(S)$ by Theorem 3.2. When m and n are even, the diagram in $Sp^\Sigma(\mathbf{M}_\bullet(S), T^{\wedge 2})$ corresponding to (4.1) commutes by formal arguments. When say m is even and n is odd, the diagram

$$\begin{tikzcd} \Sigma_T^\infty \mathbf{MSL}_m(-m) \wedge \Sigma_T^\infty \mathbf{MSL}_n(-n) \wedge \Sigma_T^\infty T \arrow[r, "\Sigma_T^\infty \mu^{SL}_{mn} \wedge 1"] \arrow[d, "1 \wedge \Sigma_T^\infty \sigma_n"'] & \Sigma_T^\infty \mathbf{MSL}_{m+n}(-m-n) \wedge \Sigma_T^\infty T \arrow[d, "\Sigma_T^\infty \sigma_{m+1}"] \\ \Sigma_T^\infty \mathbf{MSL}_m(-m) \wedge \Sigma_T^\infty \mathbf{MSL}_{n+1}(-n) \arrow[r, "\Sigma_T^\infty \mu^{SL}_{m,n+1}"] \arrow[d, "u_m \wedge u_{n+1}"'] & \Sigma_T^\infty \mathbf{MSL}_{m+n+1}(-m-n) \arrow[d, "u_{m+n+1}"] \\ \mathbf{MSL} \wedge \mathbf{MSL}(1) \arrow[r, "\mu^{SL}"] & \mathbf{MSL}(1) \end{tikzcd}$$

commutes because m and $n+1$ are even. One may desuspend. The other cases are similar. □

5. Special linear orientations

We now investigate the relationship between special linear orientations on a ring cohomology theory E, as defined in [12, Definition 3.1] and homomorphisms $\varphi\colon \mathbf{MSL} \to A$ of commutative monoids in $SH(S)$.

A *special linear vector bundle* over X is a pair (E, λ) with $E \to X$ a vector bundle and $\lambda\colon \mathcal{O}_X \cong \det E$ an isomorphism of line bundles. An *isomorphism* $\phi\colon (E,\lambda) \cong (E',\lambda')$ of special linear vector bundles is an isomorphism $\phi\colon E \cong E'$ of vector bundles such that $(\det\phi)\circ\lambda = \lambda'$.

Definition 5.1. A *special linear orientation* on a bigraded ϵ-commutative ring cohomology theory $A^{*,*}$ on $\mathcal{S}m\mathcal{O}p/S$ is a rule which assigns to every special linear vector bundle (E, λ) of rank n over an X in $\mathcal{S}m/S$ a class $th(E,\lambda) \in A^{2n,n}(E, E - X)$ satisfying the following conditions:

(1) For an isomorphism $f\colon (E,\lambda) \cong (E_1,\lambda_1)$ we have $th(E,\lambda) = f^*\, th(E_1,\lambda_1)$.
(2) For $u\colon Y \to X$ we have $u^*\, th(E,\lambda) = th(u^*(E,\lambda))$ in $A^{2n,n}(u^*E, u^*E - Y)$.
(3) The maps $-\cup th(E,\lambda)\colon A^{*,*}(X) \to A^{*+2n,*+n}(E, E-X)$ are isomorphisms.
(4) We have
$$th(E_1 \oplus E_2, \lambda_1 \otimes \lambda_2) = q_1^*\, th(E_1,\lambda_1) \cup q_2^*\, th(E_2,\lambda_2),$$
where q_1, q_2 are the projections from $E_1 \oplus E_2$ onto its summands. Moreover, for the zero bundle $\mathbf{0} \to pt$ we have $th(\mathbf{0}) = 1_A \in A^{0,0}(pt)$.

The class $th(E,\lambda)$ is the *Thom class* of the special linear bundle, and $e(E,\lambda) = z^*\, th(E,\lambda) \in A^{2n,n}(X)$ is its *Euler class*.

This definition is analogous to the Thom classes theory version of the definition of an orientation [9, Definition 3.32].

For any n the functor $- \wedge T^{\wedge n} \colon SH(S) \to SH(S)$ is a self-equivalence. So it induces isomorphisms

$$- \wedge T^{\wedge n} \colon Hom_{SH(S)}(X, A \wedge S^{p,q}) \xrightarrow{\cong} Hom_{SH(S)}(X \wedge T^{\wedge n}, A \wedge S^{p,q} \wedge T^{\wedge n})$$

for any X and (p,q) and any cohomology theory on $\mathcal{S}m/S$ defined by a commutative monoid (A, μ, e) in $SH(S)$. We also write these isomorphisms as

$$\Sigma_T^n \colon A^{p,q}(X) \xrightarrow{\cong} A^{p+2n,q+n}(X \times \mathbf{A}^n, X \times (\mathbf{A}^n - 0))$$

This isomorphism coincides with $- \times \Sigma_T^n 1_A$. Thus $A^{*,*}$ automatically has Thom classes for trivial bundles: the pullbacks of $\Sigma_T^n 1_A$.

Definition 5.2. A special linear orientation on a bigraded ring cohomology theory $A^{*,*}$ on $\mathcal{S}m\mathcal{O}p/S$ which is representable by a commutative monoid in $SH(S)$ is *normalized* if

(5) for the trivial line bundle $\mathbf{A}^1 \to pt$ we have $th(\mathbf{A}^1, 1) = \Sigma_T 1_A \in A^{2,1}(\mathbf{A}^1, \mathbf{A}^1 - 0)$.

From the multiplicativity and functoriality conditions (4) and (2) in the definition of a special linear orientation one deduces the following result.

Lemma 5.3. *Suppose $A^{*,*}$ is a bigraded ring cohomology theory on $\mathcal{S}m\mathcal{O}p/S$ representable by a commutative monoid in $SH(S)$ with a normalized special linear orientation. For $X \in \mathcal{S}m/S$ let $(\mathcal{O}_X^{\oplus n}, \lambda_n)$ be the trivial special linear bundle of rank n over X. Then $th(\mathcal{O}_X^{\oplus n}, \lambda_n)$ is the pullback to X of $\Sigma_T^n 1_A$, and*

$$- \cup th(\mathcal{O}_X^{\oplus n}, \lambda_n) \colon A^{*,*}(X) \xrightarrow{\cong} A^{*+2n,*+n}(X \times \mathbf{A}^n, X \times (\mathbf{A}^n - 0))$$

is an isomorphism.

Now suppose $\varphi \colon \mathbf{MSL} \to A$ is a morphism in $SH(S)$. We associate to φ and a special linear bundle (E, λ) of rank n over an X in $\mathcal{S}m/S$ a class $th^{\varphi}(E, \lambda)$ defined as follows. By assumption the scheme X admits an ample family of line bundles. So there exists an affine bundle $f \colon Y \to X$ with Y an affine scheme. Then for some p there exist global sections $s_1, \dots, s_{np}$ of $f^*E^\vee$ generating $f^*E^\vee$. The data $(f^*E, s_1, \dots, s_{np})$ determine a morphism $\psi \colon Y \to Gr(n, np)$, and the data $(\psi, f^*\lambda)$ determine a morphism $\widetilde{\psi} \colon Y \to SGr(n, np)$. We have $\widetilde{\psi}^* \mathcal{T}SL_{n,np} \cong f^*E$. We deduce maps

$$\operatorname{Th} E \xleftarrow[\sim \mathrm{mot}]{\overline{f}} \operatorname{Th} f^*E \cong \operatorname{Th} \widetilde{\psi}^* \mathcal{T}SL_{n,np} \xrightarrow{\overline{\psi}} \operatorname{Th} \mathcal{T}SL_{n,np} \tag{5.1}$$

of pointed motivic spaces, which can be composed with the maps

$$\operatorname{Th} \mathcal{T}SL_{n,np} \xrightarrow{\text{inclusion}} \mathbf{MSL}_n \xrightarrow{u_n} \mathbf{MSL} \wedge T^{\wedge n} \xrightarrow{1 \wedge \varphi} A \wedge T^{\wedge n}. \tag{5.2}$$

in $SH(S)$. The composition of (5.1) and (5.2) gives a class

$$th^{\varphi}(E, \lambda) \in Hom_{SH(S)}(\operatorname{Th} E, A \wedge T^{\wedge n}) = A^{2n,n}(E, E - X).$$

Lemma 5.4. *The classes $th^{\varphi}(E, \lambda)$ depend only on the special linear bundle (E, λ) and the morphism $\varphi \colon \mathbf{MSL} \to A$ in $SH(S)$.*

Proof. First suppose f fixed. Let $(s_1, \dots, s_{np})$ and $(t_1, \dots, t_{nq})$ be two families of sections generating $f^*E^\vee$ with $p \geq q$. There are $\mathbf{A}^1$-homotopies between the morphisms $\operatorname{Th} f^*E \to \mathbf{MSL}_n$ in $\mathbf{M}_\bullet(S)$ defined by the family $(s_1, \dots, s_{np})$, the family $(s_1, \dots, s_{np}, t_1, \dots, t_{nq})$, the family $(t_1, \dots, t_{nq}, 0, \dots, 0, t_1, \dots, t_{nq})$, and the family $(t_1, \dots, t_{nq})$. So we get the same morphism $\operatorname{Th} f^*E \to \mathbf{MSL}_n$ in $H_\bullet(S)$ and the same morphism $\operatorname{Th} E \to A \wedge T^{\wedge n}$ in $SH(S)$.

Now suppose given a second affine bundle $g\colon Z \to X$ with Z affine and sections $(u_1, \ldots, u_{nr})$ generating $g^*E^\vee$. Let $g'\colon Y \times_X Z \to Y$ and $f'\colon Y \times_X Z \to Z$ be the projections. The morphisms $\operatorname{Th} E \to \mathbf{MSL}_n$ in $H_\bullet(S)$ defined by f and $(s_1, \ldots, s_{np})$, by $g'f$ and $(g'^*s_1, \ldots, g'^*s_{np})$, by $f'g$ and $(f'^*u_1, \ldots, f'^*u_{nr})$ and by g and $(u_1, \ldots, u_{nr})$ are then the same. So we again get the same morphism $\operatorname{Th} E \to A \wedge T^{\wedge n}$ in $SH(S)$. $\square$

Theorem 5.5. *For a homomorphism $\varphi\colon \mathbf{MSL} \to A$ of commutative monoids in $SH(S)$, the classes $th^\varphi(E, \lambda)$ define a normalized special linear orientation on the bigraded ring cohomology theory $A^{*,*}$ on $\mathcal{S}m\mathcal{O}p(S)$.*

In particular the identity homomorphism induces a normalized special linear orientation on $\mathbf{MSL}^{*,*}$.

Proof. The functoriality conditions (1) and (2) follow easily from the construction of the classes $th^\varphi(E, \lambda)$. The multiplicativity condition (4) holds because of Theorem 4.3 and because φ is a homomorphism of monoids. The normalization condition (5) holds because $th^\varphi(\mathbf{A}^1, 1)$ and $\Sigma_T 1_A$ are both equal to the composition

$$T \xrightarrow{e_1^{SL}} \mathbf{MSL}_1 \xrightarrow{u_1} \mathbf{MSL} \wedge T \xrightarrow{\varphi \wedge 1} A \wedge T.$$

The isomorphism condition (3) holds for trivial special linear bundles because of the normalization condition and Lemma 5.3. It then holds for general special linear bundles by a Mayer-Vietoris argument because special linear bundles are locally trivial in the Zariski topology. $\square$

Now suppose that M and A are (symmetric) T-spectra. Then we have an inverse system of abelian groups

$$\cdots \to A^{2n+2,n+1}(M_{n+1}) \xrightarrow{\alpha_{n+1}} A^{2n,n}(M_n) \to \cdots \to A^{0,0}(M_0) \tag{5.3}$$

where the map α_n associates to the map $v\colon M_{n+1} \to A \wedge T^{\wedge n+1}$ in $SH(S)$ the composition

$$M_n \xrightarrow{\sigma_n^*} \Omega_T M_{n+1} \xrightarrow{v} \Omega_T(A \wedge T^{\wedge n+1}) \cong A \wedge T^{\wedge n}$$

in $SH(S)$. There is a similar inverse system

$$\cdots \to A^{4n+4,2n+2}(M_{n+1} \wedge M_{n+1}) \to A^{4n,2n}(M_n \wedge M_n) \to \cdots \to A^{0,0}(M_0 \wedge M_0). \tag{5.4}$$

For the following theorem see for example [11, Corollaries 3.4 and 3.5].

Theorem 5.6. *For any (symmetric) T- or $T^{\wedge 2}$-spectra M and A we have exact sequences of abelian groups*

$$0 \to \varprojlim{}^1 A^{2n-1,n}(M_n) \to Hom_{SH(S)}(M, A) \to \varprojlim A^{2n,n}(M_n) \to 0,$$
$$0 \to \varprojlim{}^1 A^{4n-1,2n}(M_n \wedge M_n) \to Hom_{SH(S)}(M \wedge M, A) \to \varprojlim A^{4n,2n}(M_n \wedge M_n) \to 0.$$

This theorem is actually a special case of the following result [11, Lemma 3.3].

Theorem 5.7. *Let $E = \operatorname{hocolim}_{i \in \mathbb{N}} E^{\langle i \rangle}$ be a sequential homotopy colimit of T-spectra. Then for any T-spectrum A and any (p, q) we have an exact sequence of abelian groups*

$$0 \to \varprojlim_{i \in \mathbb{N}}{}^1 A^{p-1,q}(E^{\langle i \rangle}) \to A^{p,q}(E) \to \varprojlim_{i \in \mathbb{N}} A^{p,q}(E^{\langle i \rangle}) \to 0.$$

We wish to apply Theorem 5.6 when A is a commutative monoid in $SH(S)$ with a normalized special linear orientation on $A^{*,*}$ and when M is a commutative monoid isomorphic to $\mathbf{MSL}$ in $SH(S)$. (Note that the exact sequences depend on the levelwise weak equivalence class of M, which is a finer invariant than its isomorphism class in $SH(S)$.) However, the special linear orientation provides Thom classes for special linear bundles over finite-dimensional smooth schemes and not over the infinite-dimensional ind-schemes BSL_n. So the orientation does not provide us with classes in the $A^{2n,n}(\mathbf{MSL}_n)$. But we can solve this problem as follows.

For each n and p write

$$MSL_n^{\langle p\rangle} = \mathrm{Th}\,\mathcal{T}SL_{n,np}.$$

For $n = 0$ this is $MSL_0^{\langle p\rangle} = pt_+$. The actions of Σ_n on $\mathbf{MSL}_n$ and the structural maps e_n^{SL} and μ_{mn}^{SL} constructed in the previous section are colimits of actions and structural maps

$$\Sigma_n \times MSL_n^{\langle p\rangle} \to MSL_n^{\langle p\rangle},$$
$$e_n^{\langle p\rangle} \colon T^{\wedge n} \to MSL_n^{\langle p\rangle},$$
$$\mu_{mn}^{\langle p\rangle} \colon MSL_m^{\langle p\rangle} \wedge MSL_n^{\langle p\rangle} \to MSL_{m+n}^{\langle p\rangle}.$$

We thus get a direct system of commutative T-monoids

$$MSL^{\langle 1\rangle} \to MSL^{\langle 2\rangle} \to \cdots \to MSL^{\langle p\rangle} \to \cdots$$

whose colimit is $\mathbf{MSL}$. We can now define a "diagonal" commutative T-monoid $\mathbf{MSL}^{fin}$ with spaces

$$\mathbf{MSL}_n^{fin} = MSL_n^{\langle n\rangle}$$

with the actions $\Sigma_n \times MSL_n^{\langle n\rangle} \to MSL_n^{\langle n\rangle}$ and unit maps $e_n^{\langle n\rangle}$ given above and with multiplication maps the compositions

$$\mu_{mn}^{fin} \colon MSL_m^{\langle m\rangle} \wedge MSL_n^{\langle n\rangle} \xrightarrow{\text{inclusion}} MSL_m^{\langle m+n\rangle} \wedge MSL_n^{\langle m+n\rangle} \xrightarrow{\mu_{mn}^{\langle m+n\rangle}} MSL_{m+n}^{\langle m+n\rangle}.$$

A cofinality argument now gives the nontrivial part of the following result.

Theorem 5.8. *The inclusion $\mathbf{MSL}^{fin} \hookrightarrow \mathbf{MSL}$ defines a homomorphism of commutative monoids in the category of symmetric T-spectra which is a motivic stable weak equivalence.*

Thus the inclusion becomes an isomorphism of commutative monoids in $SH(S)$. So Theorem 5.6 gives us an exact sequence

$$0 \to \varprojlim{}^1 A^{2n-1,n}(MSL_n^{\langle n\rangle}) \to Hom_{SH(S)}(\mathbf{MSL}, A) \to \varprojlim A^{2n,n}(MSL_n^{\langle n\rangle}) \to 0 \tag{5.5}$$

and a similar exact sequence for $Hom_{SH(S)}(\mathbf{MSL} \wedge \mathbf{MSL}, A)$.

Theorem 5.9. *Suppose (A, μ_A, e_A) is a commutative monoid in $SH(S)$ with a normalized special linear orientation on $A^{*,*}$ given by Thom classes $th(E,\lambda)$. Then there exists a morphism $\varphi \colon \mathbf{MSL} \to A$ in $SH(S)$ such that $th^{\varphi}(E,\lambda) = th(E,\lambda)$ for all special linear bundles over all X in $\mathcal{S}m/S$. This φ is unique modulo the subgroup*

$$\varprojlim{}^1 A^{2n-1,n}(MSL_n^{\langle n\rangle}) \subset Hom_{SH(S)}(\mathbf{MSL}, A).$$

It satisfies $\varphi(e_{\mathbf{MSL}}) = e_A$. The obstruction $\varphi \circ \mu_{\mathbf{MSL}} - \mu_A \circ (\varphi \wedge \varphi)$ to φ being a homomorphism of monoids lies in the subgroup

$$\varprojlim{}^1 A^{4n-1,2n}(MSL_n^{\langle n\rangle} \wedge MSL_n^{\langle n\rangle}) \subset Hom_{SH(S)}(\mathbf{MSL} \wedge \mathbf{MSL}, A).$$

Proof. For every n and p the tautological special linear bundle $(\mathcal{T}SL_{n,np}, \lambda_{n,np})$ over the scheme $SGr(n,np)$ has a Thom class, which we will abbreviate to $th_{n,np} \in A^{2n,n}(MSL_n^{\langle p\rangle})$. Pullback along the inclusion $MSL_n^{\langle p-1\rangle} \hookrightarrow MSL_n^{\langle p\rangle}$ sends

$$th_{n,np} \mapsto th_{n,n(p-1)} \,. \tag{5.6}$$

Pullback along the bonding map $MSL_{n-1}^{\langle p\rangle} \wedge T \to MSL_n^{\langle p\rangle}$ induced by $e_1^{\langle p\rangle}$ and $\mu_{n-1,1}^{\langle p\rangle}$ sends

$$th_{n,np} \mapsto th_{n-1,(n-1)p} \times th(\mathbf{A}^1, 1) = \Sigma_T\, th_{n-1,(n-1)p} \,. \tag{5.7}$$

So as n and p vary, we get an element

$$\bar\varphi = (th_{n,np})_{n,p} \in \varprojlim_{n,p} A^{2n,n}(MSL_n^{\langle p\rangle}) = \varprojlim_{n} A^{2n,n}(MSL_n^{\langle n\rangle}).$$

Let $\varphi \in Hom_{SH(S)}(\mathbf{MSL}, A)$ be an element mapping onto $\bar\varphi$ under the surjection in the exact sequence (5.5).

The image of φ under the composition

$$Hom_{SH(S)}(\mathbf{MSL}, A) \to \varprojlim A^{2n,n}(MSL_n^{\langle n\rangle}) \to A^{2n,n}(MSL_n^{\langle n\rangle})$$

is the composition

$$\mathrm{Th}\, \mathcal{T}SL_{n,n^2} = MSL_n^{\langle n\rangle} \xrightarrow{u_n} \mathbf{MSL}^{fin} \wedge T^{\wedge n} \xrightarrow{\sim} \mathbf{MSL} \wedge T^{\wedge n} \xrightarrow{\varphi\wedge 1} A \wedge T^{\wedge n}$$

which is the $th^\varphi(\mathcal{T}SL_{n,n^2}, \lambda_{n,n^2})$ defined by (5.1)–(5.2). Thus we have $th^\varphi(E,\lambda) = th(E,\lambda)$ for $(E,\lambda) = (\mathcal{T}SL_{n,n^2}, \lambda_{n,n^2})$. The Thom classes for the $(\mathcal{T}SL_{n,n^2}, \lambda_{n,n^2})$ determine the Thom classes for all $(\mathcal{T}SL_{n,np}, \lambda_{n,np})$ by formulas (5.6)–(5.7). These in turn determine the Thom classes for all (E,λ) by formulas (5.1)–(5.2). So we have $th^\varphi(E,\lambda) = th(E,\lambda)$ for all special linear bundles.

Similarly for $\psi\colon \mathbf{MSL} \to A$ we have $th^\psi(E,\lambda) = th^\varphi(E,\lambda)$ for all special linear bundles if and only if ψ and φ have the same image in $\varprojlim A^{2n,n}(MSL_n^{\langle n\rangle})$. This happens if and only if $\psi - \varphi$ is in the kernel, which is the first $\varprojlim^1$ of the statement of the theorem.

By construction $e_{\mathbf{MSL}}$ is the canonical map $\Sigma_T^\infty pt_+ = \Sigma_T^\infty \mathbf{MSL}_0 \to \mathbf{MSL}$. Therefore we have $\varphi(e_{\mathbf{MSL}}) = th_{0,0} = th(\mathbf{0}) = e_A \in A^{0,0}(pt)$ as declared.

By multiplicativity and functoriality we have an equality

$$th_{n,n^2} \times th_{n,n^2} = th(p_1^*\mathcal{T}SL_{n,n^2} \oplus p_2^*\mathcal{T}SL_{n,n^2}, p_1^*\lambda_{n,n^2} \otimes p_2^*\lambda_{n,n^2}) = \mu_{nn}^{fin*}\, th_{2n,4n^2}$$

of members of $A^{4n,2n}(MSL_n^{\langle n\rangle} \wedge MSL_n^{\langle n\rangle})$. This equality means that the outer perimeter of the diagram

$$\begin{tikzcd}
\Sigma^\infty_T MSL_n^{\langle n\rangle}(-n) \wedge \Sigma^\infty_T MSL_n^{\langle n\rangle}(-n) \arrow[r, "\mu^{fin}_{nn}"] \arrow[d, "u_n \wedge u_n"] \arrow[ddd, bend right=60, "th_{n,n^2} \wedge th_{n,n^2}"'] & \Sigma^\infty_T MSL_{2n}^{\langle 2n\rangle}(-2n) \arrow[d, "u_{2n}"] \arrow[ddd, bend left=60, "th_{2n,4n^2}"] \\
\mathbf{MSL}^{fin} \wedge \mathbf{MSL}^{fin} \arrow[r, "\mu^{fin}_{\mathbf{MSL}}"] \arrow[d, "\text{inclusion}"] & \mathbf{MSL}^{fin} \arrow[d, "\text{inclusion}"] \\
\mathbf{MSL} \wedge \mathbf{MSL} \arrow[r, "\mu_{\mathbf{MSL}}"] \arrow[d, "\varphi\wedge\varphi"] & \mathbf{MSL} \arrow[d, "\varphi"] \\
A \wedge A \arrow[r, "\mu_A"] & A
\end{tikzcd}$$

commutes. The half-circles commute by the previous calculations, and the top two squares commute. Therefore we have

$$\big(\varphi \circ \mu_{\mathbf{MSL}} - \mu_A \circ (\varphi \wedge \varphi)\big) \circ \text{inclusion} \circ (u_n \wedge u_n) = 0$$

for all n. So the image of the obstruction class $\varphi \circ \mu_{\mathbf{MSL}} - \mu_A \circ (\varphi \wedge \varphi)$ under the surjection

$$Hom_{SH(S)}(\mathbf{MSL} \wedge \mathbf{MSL}, A) \longrightarrow \varprojlim A^{4n,2n}(MSL_n^{\langle n\rangle} \wedge MSL_n^{\langle n\rangle}) \longrightarrow 0$$

vanishes. Therefore the obstruction class lies in the kernel, which is the second $\varprojlim^1$ of the statement of the theorem. $\square$

6. The symmetric $T^{\wedge 2}$-spectrum MSp

We now define the commutative $T^{\wedge 2}$-monoid and symmetric $T^{\wedge 2}$-spectrum **MSp**.

We write the standard symplectic form on the trivial vector bundle of rank $2n$ as

$$\omega_{2n} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

From the symplectic isometry $(\mathcal{O}_S^{\oplus 2n}, \omega_{2n}) \cong (\mathcal{O}_S^{\oplus 2}, \omega_2)^{\oplus n}$ we see that the action of Σ_n given by permutations of the n orthogonal direct summands $(\mathcal{O}_S^{\oplus 2}, \omega_2)$ gives an embedding $\Sigma_n \to Sp_{2n}$. Hence Sp_{2n}-actions restrict to Σ_n-actions.

In [13] we defined the quaternionic Grassmannian $HGr(r, n)$ as the open subscheme of the Grassmannian $Gr(2r, 2n)$ parametrizing rank $2r$ subspaces of $\mathcal{O}_S^{\oplus 2n}$ on which the restriction of ω_{2n} is nondegenerate. The restriction of the tautological subbundle over the Grassmannian is the tautological symplectic subbundle $\mathcal{T}Sp_{r,n}$. It is equipped with the symplectic form $\phi_{r,n} = \omega_{2n}|_{\mathcal{T}Sp_{r,n}}$. For $r = 1$ we write $HP^n = HGr(1, n+1)$ and $HP^\infty = \operatorname{colim}_n HP^n$.

To construct **MSp** we look at the particular schemes $HGr(n, np) = HGr(n, (\mathcal{O}_S^{\oplus 2n}, \omega_{2n})^{\oplus p})$. Each has a natural action of Sp_{2n} induced by the diagonal action of Sp_{2n} on the p summands of

$(\mathcal{O}_S^{\oplus 2n}, \omega_{2n})^{\oplus p}$. The vector bundles $\mathcal{T}Sp_{n,np} \to HGr(n,np)$ and the inclusions $HGr(n,np) \to HGr(n,np+n)$ are Sp_{2n}-equivariant. We set

$$\begin{aligned} BSp_{2n} &= \operatorname{colim}_{p\in\mathbb{N}} HGr(n,np), \\ \mathcal{T}Sp_{n,n\infty} &= \operatorname{colim}_{p\in\mathbb{N}} \mathcal{T}Sp_{n,np}, \\ \mathbf{MSp}_{2n} &= \operatorname{colim}_{p\in\mathbb{N}} \operatorname{Th} \mathcal{T}Sp_{n,np}. \end{aligned}$$

As with **MGL** and **MSL** the isomorphisms

$$(\mathcal{O}_S^{\oplus 2m}, \omega_{2m}) \oplus (\mathcal{O}_S^{\oplus 2n}, \omega_{2n}) \cong (\mathcal{O}_S^{\oplus 2m+2n}, \omega_{2m+2n})$$

and the direct sum induce $(Sp_{2m} \times Sp_{2n})$-equivariant maps

$$\begin{aligned} \oplus\colon BSp_{2m} \times BSp_{2n} &\to BSp_{2m+2n}, \\ \mu_{mn}^{Sp}\colon \mathbf{MSp}_{2m} \wedge \mathbf{MSp}_{2n} &\to \mathbf{MSp}_{2m+2n}. \end{aligned} \tag{6.1}$$

Each $HGr(n,np)$ is pointed by the point corresponding to the symplectic subbundle which is the first direct summand $(\mathcal{O}_S^{\oplus 2n}, \omega_{2n}) \oplus 0^{\oplus p-1} \subset (\mathcal{O}_S^{\oplus 2n}, \omega_{2n})^{\oplus p}$. In the colimit this yields points $z_{2n}\colon pt \to BSp_{2n}$. The point z_{2n} is fixed by the Sp_{2n}-action. The action of Sp_{2n} on the fiber of $\mathcal{T}Sp_{n,n\infty}$ over z_{2n} is the standard representation of Sp_{2n}. The inclusion of the fiber induces an inclusion of Thom spaces

$$e_{2n}^{Sp}\colon T^{\wedge 2n} \to \mathbf{MSp}_{2n} \tag{6.2}$$

which is Sp_{2n}-equivariant. The action of the subgroup $\Sigma_n \subset Sp_{2n}$ on $T^{\wedge 2n} = (T^{\wedge 2})^{\wedge n}$ permutes the n factors $T^{\wedge 2}$.

The spaces $\mathbf{MSp}_{2n}$ with the actions and structural maps verify the axioms of a commutative $T^{\wedge 2}$-monoid.

Definition 6.1. The *algebraic symplectic cobordism spectrum* **MSp** is the commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra associated to the commutative $T^{\wedge 2}$-monoid composed of the spaces $\mathbf{MSp}_{2n}$, the actions $\Sigma_n \times \mathbf{MSp}_{2n} \to \mathbf{MSp}_{2n}$ the maps $e_0^{Sp}\colon pt_+ \to \mathbf{MSp}_0$ and $e_2^{Sp}\colon T^{\wedge 2} \to \mathbf{MSp}_2$ and the maps $\mu_{mn}^{Sp}\colon \mathbf{MSp}_{2m} \wedge \mathbf{MSp}_{2n} \to \mathbf{MSp}_{2m+2n}$.

This **MSp** defines a commutative monoid in $SH(S)$ by Theorem 3.2.

7. Quaternionic Grassmannian bundles

We review the geometry of quaternionic projective bundles and Grassmannian bundles studied in [13, §§3–5]. We then translate some of the results into a more motivic language.

Given (E,ϕ) a symplectic bundle of rank $2n$ over a scheme X and an integer $0 \le r \le n$, there is a quaternionic Grassmannian bundle $p\colon HGr(r,E,\phi) \to X$ whose fiber over $x \in X$ is the quaternionic Grassmannian parametrizing $2r$-dimensional subspaces of E_x on which ϕ_x is nondegenerate. We write $\mathcal{U}_{r,E} \subset p^*E$ for the tautological rank $2r$ subbundle over $HGr(r,E,\phi)$. Morphisms $f\colon Y \to HGr(r,E,\phi)$ are in bijection with pairs (g,U) where $g\colon Y \to X$ is a morphism and $U \subset g^*(E,\phi)$ is a symplectic subbundle of rank $2r$ over Y.

Since $\mathcal{U}_{r,E}$ is a subbundle on which the symplectic form is fiberwise nondegenerate, it has an orthogonal complement such that $\mathcal{U}_{r,E} \oplus \mathcal{U}_{r,E}^{\perp} = p^*E$. The symplectic subbundle $\mathcal{U}_{r,E}^{\perp} \subset p^*E$ classifies an isomorphism

$$HGr(r,E,\phi) \xrightarrow{\cong} HGr(n-r,E,\phi). \tag{7.1}$$

Now let $(F,\psi) = (\mathcal{O}_X^{\oplus 2}, \omega_2) \oplus (E,\phi)$. We have a natural embedding

$$HGr(r,E,\phi) \hookrightarrow HGr(r,F,\psi) \tag{7.2}$$

classified by the symplectic subbundle $0 \oplus \mathcal{U}_{r,E} \subset \mathcal{O}_X^{\oplus 2} \oplus E = F$. The normal bundle of this embedding can be naturally identified with the vector bundle $N = \mathcal{H}om(\mathcal{U}_{r,E}, \mathcal{O}_X^{\oplus 2})$ over $HGr(r,E,\phi)$. This bundle is a direct sum decomposition $N = N^+ \oplus N^-$ where

$$N^+ = \mathcal{H}om(\mathcal{U}_{r,E}, \mathcal{O}_X \oplus 0), \qquad N^- = \mathcal{H}om(\mathcal{U}_{r,E}, 0 \oplus \mathcal{O}_X).$$

The basic result concerning the geometry of the closed embedding (7.2) is the following.

Theorem 7.1 ([13, Theorem 4.1])**.** (*a*) *The normal bundle of the embedding* (7.2) *has a canonical open embedding* $\nu\colon N \hookrightarrow Gr(2r,F)$. *The zero section is sent identically onto* $HGr(r,E,\phi)$.

(*b*) *We have* $\nu(N^+) = HGr(r,F,\psi) \cap Gr_S(2r, \mathcal{O}_X \oplus 0 \oplus E)$. *Consequently* $\nu(N^+) \subset HGr(r,F,\psi)$ *is a closed subscheme, as is* $\nu(N^-) \subset HGr(r,F,\psi)$.

(*c*) *There are natural isomorphisms of vector bundles* $N^+ \cong N^- \cong \mathcal{U}_{r,E}^\vee \cong \mathcal{U}_{r,E}$.

(*d*) *There is a natural section* s_+ *of* $\mathcal{U}_{r,F}$ *intersecting the zero section transversally in* N^+ *and similarly for* N^-.

(*e*) *Let* $\pi_+\colon N^+ \to HGr(r,E,\phi)$ *be the structural map. Then* $\pi_+^*(\mathcal{U}_{r,E}, \phi|_{\mathcal{U}_{r,E}})$ *is isometric to* $(\mathcal{U}_{r,F}, \psi|_{\mathcal{U}_{r,F}})|_{N^+}$ *and similarly for* N^-.

The second basic result about the geometry of symplectic Grassmannian bundles involves the following embeddings

$$HGr(r-1,E,\phi) \xhookrightarrow[\text{closed}]{\sigma} HGr(r,F,\psi) - \nu(N^+) \xhookrightarrow{\text{open}} HGr(r,F,\psi). \tag{7.3}$$

The composition is the closed embedding classified by the symplectic subbundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{U}_{r-1,E} \subset \mathcal{O}^{\oplus 2} \oplus E = F$.

Theorem 7.2 ([13, Theorems 5.1 and 5.2])**.** *There are morphisms over* X

$$HGr(r,F,\psi) - \nu(N^+) \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xrightarrow{q} HGr(r-1,E,\phi)$$

with g_1 *an* $\mathbf{A}^{2r-1}$*-bundle,* g_2 *an* $\mathbf{A}^{2r-2}$*-bundle, and* q *an* $\mathbf{A}^{4n+1}$*-bundle. Moreover, there is a section* s *of* q *such that the composition* $g_1g_2s\colon HGr(r-1,E,\phi) \to HGr(r,F,\psi) - \nu(N^+)$ *is the closed embedding* σ *of* (7.3).

The section s appears at the end of the proof of [13, Theorem 5.2]. The statement of the theorem only contains the consequence that σ induces isomorphisms of cohomology groups.

These two theorems have the following consequence.

Theorem 7.3. *Let* (E,ϕ) *be a symplectic bundle of rank* $2n$ *over a smooth* S*-scheme* X*, and let* $(F,\psi) = (\mathcal{O}_X^{\oplus 2}, \omega_2) \oplus (E,\phi)$. *For* $1 \le r \le n$ *let* $\mathcal{U}_{r,E}$ *be the tautological symplectic subbundle over* $HGr(r,E,\phi)$. *Let* $HGr(r-1,E,\phi) \hookrightarrow HGr(r,F,\psi)$ *be the closed embedding of* (7.3). *Then there is a canonical zigzag of motivic weak equivalences*

$$\mathrm{Th}\,\mathcal{U}_{r,E} \xrightarrow{\sim} HGr(r,F,\psi)/(HGr(r,F,\psi) - \nu(N^+)) \xleftarrow{\sim} HGr(r,F,\psi)/HGr(r-1,E,\phi)$$

inducing an isomorphism in the motivic unstable homotopy category $H_\bullet(S)$. *These isomorphisms commute with the maps induced by inclusions* $(E,\phi) \hookrightarrow (E,\phi) \oplus (E_1,\phi_1)$ *of symplectic bundles and with the maps induced by base changes* $Y \to X$.

Proof. In the geometry of Theorem 7.1 we identify N, N^+ and N^- with their images in $Gr(2r, F)$ under the open embedding ν. Then there are motivic weak equivalences

$$\begin{tikzcd}
N/(N - N^+) & (N \cap HGr(r,F,\psi))/(N \cap HGr(r,F,\psi) - N^+) \arrow[l, "\sim"', "\text{excision}"] \arrow[d, "\sim"', "\text{excision}"] \\
\operatorname{Th} \mathcal{U}_{r,E} \cong N^-/(N^- - HGr(r,E,\phi)) \arrow[u, "\text{section of vector bundle}", "\sim"'] \arrow[ur] \arrow[r, "\text{inclusion}"'] & HGr(r,F,\psi)/(HGr(r,F,\psi) - N^+)
\end{tikzcd}$$

The arrows not explicitly labeled $\sim$ are nevertheless motivic weak equivalences by the 2-out-of-3 axiom. The morphisms g_1, g_2 and q of Theorem 7.2 are affine bundles, so they, the section s, the composition σ, and the map

$$HGr(r,F,\psi)/HGr(r-1,E,\phi) \xrightarrow{\sim} HGr(r,F,\psi)/(HGr(r,F,\psi) - N^+)$$

of quotient spaces induced by σ are all motivic weak equivalences.

The functoriality is straightforward. □

Because of (7.1) the quotient appearing in Theorem 7.3 is also isomorphic to

$$HGr(n-r+1, F, \psi)/HGr(n-r+1, E, \phi).$$

The case $r = n$ of Theorem 7.3 therefore gives the following result. The corresponding result for ordinary Grassmannian bundles is [8, Proposition 3.2.17(3)].

Theorem 7.4. *Suppose that (E, ϕ) is a symplectic bundle of rank $2n$ over a smooth S-scheme X. Let $HP(E,\phi) \hookrightarrow HP(\mathcal{O}_X^{\oplus 2} \oplus E, \omega_2 \oplus \phi)$ be the natural closed embedding. Then we have isomorphisms in $H_\bullet(S)$*

$$\operatorname{Th} E \cong HGr(n, (\mathcal{O}_X^{\oplus 2}, \omega_2) \oplus (E,\phi))/HGr(n-1, E, \phi) = HP((\mathcal{O}_X^{\oplus 2}, \omega_2) \oplus (E,\phi))/HP(E,\phi).$$

For $X = pt$ and trivial E Theorem 7.3 gives the following result.

Theorem 7.5. *There are canonical isomorphisms*

$$\operatorname{Th} \mathcal{U}_{HGr(r,n)} \cong HGr(r, 1+n)/HGr(r-1, n)$$

in $H_\bullet(S)$. These isomorphisms are compatible with the inclusions of quaternionic projective spaces. Therefore we have commutative diagrams of inclusions and isomorphisms in $H_\bullet(S)$

$$\begin{tikzcd}
T^{\wedge 2r} \arrow[r, "e_r^{Sp}"] \arrow[d, "\cong"'] & \mathbf{MSp}_{2r} \arrow[d, "\cong"] \\
HP^r/HP^{r-1} \arrow[r, "\text{inclusion}"'] & BSp_{2r}/BSp_{2r-2}
\end{tikzcd} \tag{7.4}$$

The motivic weak equivalences of Theorem 7.5 fit into a commutative diagram

$$\begin{tikzcd}
HGr(r,n) \arrow[rr, "\text{inclusion}"] \arrow[d, "\text{section map}"'] & & HGr(r,1+n) \arrow[d, "\text{quotient}"] \\
\operatorname{Th} \mathcal{U}_{HGr(r,n)} \arrow[r, "\sim mot"] & Y_{r,n} & HGr(r,1+n)/HGr(r-1,n) \arrow[l, "\sim mot"']
\end{tikzcd} \tag{7.5}$$

The *section map* is the structure map of the Thom space induced by a section of the vector bundle. We have a colimit as $n \to \infty$:

$$\begin{tikzcd}
BSp_{2r} = HGr(r,\infty) \arrow[rr, "\text{shift}"] \arrow[d, "\text{section map}"'] & & BSp_{2r} = HGr(r,1+\infty) \arrow[d, "\text{quotient}"] \\
\mathbf{MSp}_{2r} \arrow[r, "\sim mot"] & Y_{r,\infty} & BSp_{2r}/BSp_{2r-2} \arrow[l, "\sim mot"']
\end{tikzcd} \tag{7.6}$$

The *shift map* is the map induced by the *shift endomorphism* of $(\mathcal{O}^{\oplus 2}, \omega_2)^{\oplus\infty}$ acting on sequences of sections of $\mathcal{O}$ by

$$(s_1, s_2, s_3, s_4, \dots) \mapsto (0, 0, s_1, s_2, s_3, s_4, \dots).$$

Lemma 7.6. *There is an $\mathbf{A}^1$-homotopy of symplectic endomorphisms of $(\mathcal{O}^{\oplus 2}, \omega_2)^{\oplus\infty}$ linking the shift endomorphism of $(\mathcal{O}^{\oplus 2}, \omega_2)^{\oplus\infty}$ to the identity.*

Proof. Any matrix in any $Sp_{2r}(\mathbb{Z})$ is a product of elementary symplectic matrices. So there exists an $M(t) \in Sp_4(\mathbb{Z}[t])$ with

$$M(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

for example

$$M(t) = \begin{pmatrix} 1-t^2 & 0 & -2t+13t^3-14t^5+4t^7 & 8t^2-12t^4+4t^6 \\ 0 & 1-t^2 & -2t^2+2t^4 & -t+2t^3 \\ t & 0 & 1-7t^2+10t^4-4t^6 & -4t+8t^3-4t^5 \\ 0 & 2t-t^3 & 2t-4t^3+2t^5 & 1-3t^2+2t^4 \end{pmatrix}.$$

Consider the sequence of infinite matrices

$$f_n(t) = \begin{pmatrix} I_{2n-2} & 0 & 0 \\ 0 & M(t) & 0 \\ 0 & 0 & I_\infty \end{pmatrix}.$$

Then $f_n(1)$ acts on $(\mathcal{O}^{\oplus 2}, \omega_2)^{\oplus\infty}$ by exchanging the n^{th} and $(n+1)^{\text{st}}$ summands $(\mathcal{O}^{\oplus 2}, \omega_2)$, while $f_n(0)$ is the identity. The infinite product

$$F(t) = f_1(t) f_2(t) f_3(t) \cdots$$

is well-defined because the first $2n$ columns of $f_1 f_2 \cdots f_N$ are independent of N for all $N \geq n$. Each column of $F(t)$ contains only a finite number of nonzero entries. (More precisely, writing $F(t) = (a_{ij}(t))$, we have $a_{i,2n-1}(t) = a_{i,2n}(t) = 0$ for $i > 2n+2$.) The endomorphism $F(t)$ is preserves the symplectic form $\omega_2^{\oplus\infty}$ because the automorphisms $f_n(t)$ all do. Clearly $F(0)$ is the identity. The finite product $f_1(1) f_2(1) \cdots f_N(1)$ permutes cyclically the first $N+1$ summands of $(\mathcal{O}^{\oplus 2}, \omega_2)^{\infty}$ and fixes the others. The infinite product $F(1)$ is the shift map. $\square$

Theorem 7.7. *In the motivic unstable homotopy category $H_\bullet(S)$ we have a commutative diagram*

$$\begin{tikzcd} & BSp_{2r} \arrow[dl, bend right, "\text{structure map}"'] \arrow[dr, bend left, "\text{quotient}"] & \\ \mathbf{MSp}_{2r} \arrow[rr, leftrightarrow, "\cong", "\mathbf{A}^N\text{-bundles and excision}"'] & & BSp_{2r}/BSp_{2r-2}. \end{tikzcd}$$

8. The quaternionic projective bundle theorem

The most basic form a symplectic orientation is a symplectic Thom structure [13, Definition 7.1]. The version of the definition for bigraded ϵ-commutative theories is as follows.

Definition 8.1. A *symplectic Thom structure* on a bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ on $\mathcal{S}m\mathcal{O}p/S$ is a rule which assigns to each rank 2 symplectic bundle (E, ϕ) over an X in $\mathcal{S}m/S$ an element $th(E, \phi) \in A^{4,2}(E, E - X)$ with the following properties:

(1) For an isomorphism $u \colon (E, \phi) \cong (E_1, \phi_1)$ one has $th(E, \phi) = u^*\, th(E_1, \phi_1)$.
(2) For a morphism $f \colon Y \to X$ with pullback map $f_E \colon f^*E \to E$ one has $f_E^*\, th(E, \phi) = th(f^*E, f^*\phi)$.
(3) For the trivial rank 2 bundle $\mathbf{A}^2 \to pt$ with the symplectic form $\omega_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the map
$$- \times th(\mathbf{A}^2, \omega_2) \colon A^{*,*}(X) \to A^{*+4,*+2}(X \times \mathbf{A}^2, X \times (\mathbf{A}^2 - 0))$$
is an isomorphism for all X.

The *Pontryagin class* of (E, ϕ) is $p_1(E, \phi) = -z^*\, th(E, \phi) \in A^{4,2}(X)$ where $z \colon X \to E$ is the zero section.

The quaternionic projective bundle theorem is proven in [13] using the symplectic Thom structure and not any other version of a symplectic orientation. It is proven first for trivial bundles.

Theorem 8.2 ([13, Theorem 8.1])**.** *Let $(A^{*,*}, \partial, \times, 1_A)$ be a bigraded ϵ-commutative ring cohomology theory with a symplectic Thom structure. Let $(\mathcal{U}_{HP^n}, \phi_{HP^n})$ be the tautological rank 2 symplectic subbundle over HP^n and $\zeta = p_1(\mathcal{U}_{HP^n}, \phi_{HP^n})$ its Pontryagin class. Then for any X in $\mathcal{S}m/S$ we have an isomorphism of bigraded rings*
$$A^{*,*}(HP^n \times X) \cong A^{*,*}(X)[\zeta]/(\zeta^{n+1}).$$

A Mayer-Vietoris argument gives the more general theorem [13, Theorem 8.2].

Theorem 8.3 (Quaternionic projective bundle theorem)**.** *Let $(A^{*,*}, \partial, \times, 1_A)$ be a bigraded ϵ-commutative ring cohomology theory with a symplectic Thom structure. Let (E, ϕ) be a symplectic bundle of rank $2n$ over X, let $(\mathcal{U}, \phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $HP(E, \phi)$, and let $\zeta = p_1(\mathcal{U}, \phi|_{\mathcal{U}})$ be its Pontryagin class. Then we have an isomorphism of bigraded $A^{*,*}(X)$-modules*
$$(1, \zeta, \dots, \zeta^{n-1}) \colon A^{*,*}(X) \oplus A^{*,*}(X) \oplus \cdots \oplus A^{*,*}(X) \to A^{*,*}(HP(E, \phi)).$$

Definition 8.4. Under the hypotheses of Theorem 8.3 there are unique elements $p_i(E, \phi) \in A^{4i,2i}(X)$ for $i = 1, 2, \dots, n$ such that
$$\zeta^n - p_1(E, \phi) \cup \zeta^{n-1} + p_2(E, \phi) \cup \zeta^{n-2} - \cdots + (-1)^n p_n(E, \phi) = 0.$$

The classes $p_i(E,\phi)$ are called the *Pontryagin classes* of (E,ϕ) with respect to the symplectic Thom structure of the cohomology theory (A,∂). For $i > n$ one sets $p_i(E,\phi) = 0$, and one sets $p_0(E,\phi) = 1$.

For a rank 2 symplectic bundle (E,ϕ) the classes $p_1(E,\phi)$ defined by Definitions 8.1 and 8.4 coincide.

Corollary 8.5. *The Pontryagin classes of a trivial symplectic bundle vanish.*

Among the consequences of the quaternionic projective bundle theorem is the symplectic splitting principle [13, Theorem 10.2]. We used it to prove the Cartan sum formula for Pontryagin classes [13, Theorem 10.5].

Theorem 8.6. *Let $(A^{*,*},\partial,\times,1_A)$ be a bigraded ϵ-commutative ring cohomology theory with a symplectic Thom structure. Suppose $(F,\psi) \cong (E_1,\phi_1) \oplus (E_2,\phi_2)$ is an orthogonal direct sum of symplectic bundles over an X in $\mathcal{S}m/S$. Then for all i we have*

$$p_i(F,\psi) = p_i(E_1,\phi_1) + \sum_{j=1}^{i-1} p_{i-j}(E_1,\phi_1)p_j(E_2,\phi_2) + p_i(E_2,\phi_2). \tag{8.1}$$

The quaternionic projective bundle theorem also allowed us to compute the cohomology of quaternionic Grassmannians. To explain our results we need to recall a number of facts about symmetric polynomials. They may be found in for example [7, Chap. 1, §§1–3].

Let $\Lambda_r \subset \mathbb{Z}[x_1,\ldots,x_r]$ be the ring of symmetric polynomials in r variables. Let e_i denote the i^{th} elementary symmetric polynomial, and h_i the i^{th} *complete symmetric polynomial*, the sum of all the monomials of degree i. Set $e_0 = h_0 = 1$ and $e_i = h_i = 0$ for $i < 0$ and also $e_i = 0$ for $i > r$. We have $\Lambda_r = \mathbb{Z}[e_1,\ldots,e_r]$. There is a recurrence relation $h_m + \sum_{i=1}^{r}(-1)^r e_i h_{m-i} = 0$.

Let

$$\Pi_r = \{\text{partitions } \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ of length } l(\lambda) \le r\}.$$

Write $\delta = (r-1, r-2, \ldots, 1, 0)$. For $\lambda \in \Pi_r$ let $a_{\lambda+\delta} = \det(x_i^{\lambda_j+r-j})_{1\le i,j\le r}$. Then $a_{\lambda+\delta}$ is a skew-symmetric polynomial and therefore divisible by the Vandermonde determinant a_δ. The quotient $s_\lambda = a_{\lambda+\delta}/a_\delta$ is the *Schur polynomial* for λ. It is symmetric of degree $|\lambda| = \sum \lambda_i$. One has $s_{(1^i)} = e_i$ and $s_{(i)} = h_i$. The $a_{\lambda+\delta}$ with $l(\lambda) \le r$ form a $\mathbb{Z}$-basis of the skew-symmetric polynomials in r variables, so the s_λ with $l(\lambda) \le r$ form a $\mathbb{Z}$-basis of Λ_r. Denote by λ' the partition dual to λ. We have formulas

$$s_\lambda = \det(e_{\lambda'_i-i+j})_{1\le i,j\le m} = \det(h_{\lambda_i-i+j})_{1\le i,j,\le r}, \tag{8.2}$$

for $m \ge l(\lambda')$ and $r \ge l(\lambda)$. Set

$$\Pi_{r,n-r} = \{\text{partitions } \lambda \text{ of length } l(\lambda) = \lambda'_1 \le r \text{ and with } \lambda_1 \le n-r\}$$

The set $\Pi_{r,n-r}$ has $\binom{n}{r}$ members. We will use the following results.

Proposition 8.7. *The quotient map*

$$\mathbb{Z}[e_1,\ldots,e_r] \to \mathbb{Z}[e_1,\ldots,e_r]/(h_{n-r+1},\ldots,h_n)$$

sends $\{s_\lambda \mid \lambda \in \Pi_r - \Pi_{r,n-r}\} \mapsto 0$, and it sends $\{s_\lambda \mid \lambda \in \Pi_{r,n-r}\}$ onto a homogeneous $\mathbb{Z}$-basis of the quotient ring.

Proof. For $\lambda \in \Pi_r - \Pi_{r,n-r}$ the first line of the determinant $s_\lambda = \det(h_{\lambda_i - i + j})$ consists of h_k with $k \geq n - r + 1$. These are all in $(h_{n-r+1}, \ldots, h_n)$ because of the recurrence relation satisfied by the h_k. So they are sent to 0 in the quotient.

The rank of the quotient as a $\mathbb{Z}$-module is $\prod \deg h_i / \prod \deg e_i = \binom{n}{r}$. Since this is the same as the cardinality of $\Pi_{r,n-r}$, and since the images of the s_λ with $\lambda \in \Pi_{r,n-r}$ generate the quotient as a $\mathbb{Z}$-module, they form a $\mathbb{Z}$-basis of the quotient. $\square$

Let the $\overline{e}_i$ and the $\overline{h}_i$ be, respectively, the elementary and complete symmetric polynomials in $r-1$ variables. The natural quotient map sends $e_i \mapsto \overline{e}_i$ for $i < r$ and $e_r \mapsto 0$, while it sends $h_i \mapsto \overline{h}_i$ for all i.

Proposition 8.8. *The kernel of the quotient map*

$$\mathbb{Z}[e_1, \ldots, e_r]/(h_{n-r+1}, \ldots, h_n) \to \mathbb{Z}[\overline{e}_1, \ldots, \overline{e}_{r-1}]/(\overline{h}_{n-r+1}, \ldots, \overline{h}_{n-1}) \to 0$$

is the image of the injection

$$0 \to \mathbb{Z}[e_1, \ldots, e_r]/(h_{n-r}, \ldots, h_{n-1}) \xrightarrow{e_r \times -} \mathbb{Z}[e_1, \ldots, e_r]/(h_{n-r+1}, \ldots, h_n).$$

Proof. The kernel is the free $\mathbb{Z}$-module with basis $\{s_\lambda \mid \lambda \in \Pi_{r,n-r} - \Pi_{r-1,n-r}\}$. These are the λ with $\lambda_r \geq 1$ and thus $\lambda'_1 = r$. The formula $s_\lambda = \det(e_{\lambda'_i - i + j})$ shows that for such λ one has $s_\lambda = e_r s_\mu$ with $\mu = (\lambda_1 - 1, \ldots, \lambda_r - 1) \in \Pi_{r,n-r-1}$. These s_μ form a basis of the ring on the left of the second displayed line. $\square$

Theorem 8.9 ([13, Theorem 11.2])**.** *Let $(\mathcal{U}_{r,n}, \phi_{r,n})$ be the tautological symplectic bundle of rank $2r$ on $HGr(r,n)$. Then for any bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ with a symplectic Thom structure and any X in $\mathcal{S}m/S$ the map*

$$A^{*,*}(X)[e_1, \ldots, e_r]/(h_{n-r+1}, \ldots, h_n) \xrightarrow{\cong} A^{*,*}(HGr(r,n) \times X) \tag{8.3}$$

sending $e_i \mapsto p_i(\mathcal{U}_{r,n}, \phi_{r,n})$ for all i is an isomorphism of bigraded rings.

Theorem 8.10 ([13, Theorem 11.4])**.** *Let $\alpha_{r,n} \colon HGr(r,n) \hookrightarrow HGr(r,n+1)$ be the usual inclusion. For any bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ with a symplectic Thom structure and any X in $\mathcal{S}m/S$ the map*

$$(\alpha_{r,n} \times 1)^* \colon A^{*,*}(HGr(r,n+1) \times X) \to A^{*,*}(HGr(r,n) \times X)$$

is a surjection which the isomorphisms (8.3) *identify with the natural surjection*

$$A^{*,*}(X)[e_1, \ldots, e_r]/(h_{n-r+2}, \ldots, h_n, h_{n+1}) \to A^{*,*}(X)[e_1, \ldots, e_r]/(h_{n-r+1}, h_{n-r+2}, \ldots, h_n).$$

Theorem 8.11 ([13, Theorem 11.4])**.** *Let $\beta_{r,n} \colon HGr(r,n) \to HGr(1+r, 1+n)$ be the usual inclusion. For any bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ with a symplectic Thom structure and any X in $\mathcal{S}m/S$ the map*

$$(\beta_{r,n} \times 1)^* \colon A^{*,*}(HGr(1+r, 1+n) \times X) \to A^{*,*}(HGr(r,n) \times X)$$

is a surjection which the isomorphisms (8.3) *identify with the surjection*

$$A^{*,*}(pt)[e_1, \ldots, e_r, e_{r+1}]/(h_{n-r+1}, \ldots, h_n, h_{n+1}) \to A^{*,*}(pt)[e_1, \ldots, e_r]/(h_{n-r+1}, \ldots, h_n)$$

of $A^{*,*}(pt)[e_1, \ldots, e_r]$-algebras sending $e_{r+1} \mapsto 0$.

9. The cohomology of BSp_{2r} and $\mathbf{MSp}_{2r}$

Theorem 9.1. *Let (A, μ, e) be a commutative T-ring spectrum with a symplectic Thom structure on $A^{*,*}$. Then the isomorphism $BSp_{2r} = \operatorname{colim}_n HGr(r,n)$ induces isomorphisms*

$$A^{*,*}(BSp_{2r}) \xrightarrow{\cong} \varprojlim_{n\to\infty} A^{*,*}(HGr(r,n)) \xleftarrow{\cong} A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom}$$

of bigraded rings. The second isomorphism sends the variable p_i to the inverse system of i^{th} Pontryagin classes $(p_i(\mathcal{U}_{r,n}))_{n\geq r}$.

Here $A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom}$ is the bigraded ring of homogeneous power series. The $\varprojlim$ is taken in the category of bigraded rings.

Proof. We have $BSp_{2r} = \operatorname{colim}_n HGr(r,n)$, so by Theorem 5.7 we have an exact sequence

$$0 \to \varprojlim_{n\in\mathbb{N}}{}^1 A^{*-1,*}(HGr(r,n)) \to A^{*,*}(BSp_{2r}) \to \varprojlim_{n\in\mathbb{N}} A^{*,*}(HGr(r,n)) \to 0.$$

The connecting maps are the $\alpha^*_{r,n}$ of Theorem 8.10, which are surjective. So the $\varprojlim^1$ vanishes, and the first map of the statement of the theorem is an isomorphism.

Let $I_d \subset A^{*,*}(pt)[p_1,\dots,p_r]$ be the two-sided ideal generated by the monomials $p_1^{a_1}p_2^{a_2}\dots p_r^{a_r}$ with $\sum ia_i \geq d$. We have inclusions $I_{r(n-r)+1} \subset (h_{n-r+1},\dots,h_n) \subset I_{n-r+1}$ because all partitions $\lambda \in \Pi_{r,n-r}$ have $|\lambda| \leq r(n-r)$. So we have

$$\begin{aligned}\varprojlim_n A^{*,*}(HGr(r,n)) &\cong \varprojlim_n A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+1},\dots,h_n)\\ &\cong \varprojlim_d A^{*,*}(pt)[p_1,\dots,p_r]/I_d = A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom}. \qquad \square\end{aligned}$$

Theorem 9.2. *Let (A, μ, e) be a commutative T-ring spectrum with a symplectic Thom structure on $A^{*,*}$. Then for any r and n the motivic homotopy equivalence*

$$\operatorname{Th}\mathcal{U}_{HGr(r,n)} \cong HGr(r,1+n)/HGr(r-1,n)$$

of Theorem 7.5 induces isomorphisms

$$A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+1},\dots,h_n) \xrightarrow[\cong]{\cup p_r} A^{*+4r,*+2r}(\operatorname{Th}\mathcal{U}_{HGr(r,n)}) \tag{9.1}$$

of two-sided bigraded modules over $A^{*,*}(HGr(r,n)) \cong A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+1},\dots,h_n)$. Moreover, the isomorphism $\mathbf{MSp}_{2r} = \operatorname{colim}_n \operatorname{Th}\mathcal{U}_{HGr(r,n)}$ induces isomorphisms of bigraded modules over $A^{*,*}(BSp_{2r}) \cong A^{*,*}(pt)[[p_1,\dots,p_r]]^{hom}$

$$A^{*+4r,*+2r}(\mathbf{MSp}_{2r}) \xrightarrow{\cong} \varprojlim_{n\to\infty} A^{*+4r,*+2r}(\operatorname{Th}\mathcal{U}_{HGr(r,n)}) \xleftarrow{\cong} p_rA^{*,*}(pt)[[p_1,\dots,p_r]]^{hom} \tag{9.2}$$

Proof. Applying $A^{*,*}$ to the cofiber sequence

$$HGr(r-1,n) \xrightarrow{\beta_{r-1,n}} HGr(r,1+n) \longrightarrow \operatorname{Th}\mathcal{U}_{HGr(r,n)}$$

gives a long exact sequence of cohomology. The map $\beta_{r-1,n}$ induces a surjection of cohomology groups by Theorem 8.11, so we have $A^{*,*}(\operatorname{Th}\mathcal{U}_{HGr(r,n)}) \cong \ker\beta^*_{r-1,n}$. This kernel is identified by Proposition 8.8, giving the isomorphism (9.1). In principle this is an isomorphism of two-sided modules over $A^{*,*}(HGr(r,1+n)) \cong A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+2},\dots,h_{n+1})$. But the modules are annihilated by h_{n-r+1}, so they are also two-sided modules over the quotient ring $A^{*,*}(HGr(r,n))$.

The inclusion $\mathrm{Th}\,\mathcal{U}_{HGr(r,n)} \hookrightarrow \mathrm{Th}\,\mathcal{U}_{HGr(r,n+1)}$ induces a commutative diagram

$$\begin{tikzcd}
A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+2},\dots,h_n,h_{n+1}) \arrow[r, "\cup p_r", "\cong"'] \arrow[d, two heads] & A^{*+4r,*+2r}(\mathrm{Th}\,\mathcal{U}_{HGr(r,n+1)}) \arrow[d, two heads] \\
A^{*,*}(pt)[p_1,\dots,p_r]/(h_{n-r+1},h_{n-r+1}\dots,h_n) \arrow[r, "\cup p_r", "\cong"'] & A^{*+4r,*+2r}(\mathrm{Th}\,\mathcal{U}_{HGr(r,n)}).
\end{tikzcd}$$

The inverse limit gives the righthand isomorphism of (9.2). The vertical map on the left is surjective, so the vertical map on the right is as well. Therefore the $\varprojlim^1$ vanishes in the exact sequence

$$0 \to \varprojlim_{n\in\mathbb{N}}{}^1 A^{*-1,*}(\mathrm{Th}\,\mathcal{U}_{HGr(r,n)}) \to A^{*,*}(\mathbf{MSp}_{2r}) \to \varprojlim_{n\in\mathbb{N}} A^{*,*}(\mathrm{Th}\,\mathcal{U}_{HGr(r,n)}) \to 0.$$

obtained from Theorem 5.7. □

For any r let $z_{2r}\colon BSp_{2r} \to \mathrm{Th}\,\mathcal{U}_{BSp_{2r}} = \mathbf{MSp}_{2r}$ be the structure map induced by the zero section of the tautological symplectic bundle $\mathcal{U}_{BSp_{2r}} \to BSp_{2r}$.

Theorem 9.3. *Let (A,μ,e) be a commutative T-ring spectrum with a symplectic Thom structure on $A^{*,*}$. The map $z_{2r}^*\colon A^{*,*}(\mathbf{MSp}_{2r}) \to A^{*,*}(BSp_{2r})$ of two-sided $A^{*,*}(BSp_{2r})$-modules is injective and identifies $A^{*,*}(\mathbf{MSp}_{2r})$ with the two-sided principal ideal generated by $p_r \in A^{4r,2r}(BSp_{2r})$.*

Proof. By Theorem 7.7 we have cofiber sequence $BSp_{2r-2} \xrightarrow{i_{2r}} BSp_{2r} \xrightarrow{z_{2r}} \mathbf{MSp}_{2r}$ yielding a long exact sequence of cohomology groups. By the previous theorems these are isomorphic to (in simplified notation)

$$\cdots \to A^{*-4r,*-2r}[[p_1,\dots,p_r]] \xrightarrow{z_{2r}^*} A^{*,*}[[p_1,\dots,p_r]] \xrightarrow{i_{2r}^*} A^{*,*}[[p_1,\dots,p_{r-1}]] \to \cdots \tag{9.3}$$

Since i_{2r} is the colimit of the inclusion maps $\beta_{r-1,n}\colon A(r-1,n) \to A(r,1+n)$ of Theorem 8.11, i_{2r}^* is the quotient by the ideal generated by p_r. It is surjective in all bidegrees. It follows that z_{2r}^* is injective in all bidegees and is the inclusion of that ideal. □

The direct sum of symplectic bundle induces compatible monoid structures on the BSp_{2r} and the $\mathbf{MSp}_{2r}$. So the following diagram commutes for all r and s

$$\begin{tikzcd}
BSp_{2r} \times BSp_{2s} \arrow[r, "m_{rs}"] \arrow[d, "z_{2r}\times z_{2s}"'] & BSp_{2r+2s} \arrow[d, "z_{2r+2s}"] \\
\mathbf{MSp}_{2r} \wedge \mathbf{MSp}_{2s} \arrow[r, "\mu_{rs}"] & \mathbf{MSp}_{2r+2s}.
\end{tikzcd} \tag{9.4}$$

Theorem 9.4. *Let (A,μ,e) be a commutative T-ring spectrum with a symplectic Thom structure on $A^{*,*}$. Then the isomorphisms*

$$BSp_{2r} \times BSp_{2s} = \operatorname{colim}_n\big(HGr(r,rn) \times HGr(s,sn)\big)$$
$$\mathbf{MSp}_{2r} \wedge \mathbf{MSp}_{2s} = \operatorname{colim}_n(\mathrm{Th}\,\mathcal{U}_{HGr(r,rn)} \wedge \mathrm{Th}\,\mathcal{U}_{HGr(s,sn)})$$

induces a commutative diagram of isomorphisms and monomorphisms of two-sided graded $A^{*,*}(BSp_{2r} \times BSp_{2s})$-modules

$$\begin{tikzcd}
A^{*,*}(\mathbf{MSp}_{2r} \wedge \mathbf{MSp}_{2s}) \arrow[r, "\cong"] \arrow[d, hook, "(z_{2r}\times z_{2s})^*"'] & \varprojlim_{n\to\infty} A^{*,*}(\operatorname{Th}\mathcal{U}_{r,rn} \wedge \operatorname{Th}\mathcal{U}_{s,sn}) \arrow[d, hook] & p'_r p''_s A^{*,*}[[p'_1,\dots,p'_r,p''_1,\dots,p''_s]] \arrow[l, "\cong"'] \arrow[d, hook, "\text{inclusion}"] \\
A^{*,*}(BSp_{2r} \times BSp_{2s}) \arrow[r, "\cong"] & \varprojlim_{n\to\infty} A^{*,*}(HGr(r,rn) \times HGr(s,sn)) & A^{*,*}[[p'_1,\dots,p'_r,p''_1,\dots,p''_s]]. \arrow[l, "\cong"']
\end{tikzcd}$$

Moreover, these isomorphisms identify the diagram obtained by applying $A^{*,*}$ to (9.4) with the diagram of rings and ideals

$$\begin{tikzcd}
p_{r+s}A^{*,*}(pt)[[p_1,\dots,p_{r+s}]]^{hom} \arrow[r] \arrow[d, hook] & p'_r p''_s A^{*,*}(pt)[[p'_1,\dots,p'_r,p''_1,\dots,p''_s]]^{hom} \arrow[d, hook] \\
A^{*,*}(pt)[[p_1,\dots,p_{r+s}]]^{hom} \arrow[r] & A^{*,*}(pt)[[p'_1,\dots,p'_r,p''_1,\dots,p''_s]]^{hom}
\end{tikzcd}$$

where the horizontal maps send $p_i \mapsto p'_i + \sum_{j=1}^{i-1} p'_{i-j}p''_j + p''_i$. Moreover, the horizontal maps of the last diagram are also injective.

Proof. The construction of the first diagram is much the same as in the previous theorems. The second diagram follows. For the last statement of the theorem, let $t_1,\dots,t_{r+s}$ be independent indeterminates of bidegree $(2,1)$. Then the composition of the bottom horizontal map with the map

$$A^{*,*}(pt)[[p'_1,\dots,p'_r,p''_1,\dots,p''_s]]^{hom} \to A^{*,*}(pt)[[t_1,\dots,t_{r+s}]]^{hom}$$

sending $p'_i \mapsto e_i(t_1,\dots,t_r)$ and $p''_j \mapsto e_j(t_{r+1},\dots,t_{r+s})$ is the inclusion of the ring of symmetric homogeneous power series in the ring of homogeneous power series. That is injective. □

The final calculation in this section ought to be that of $A^{*,*}(\mathbf{MSp})$. However, we will put this off until Theorem 13.1 because we wish to make the calculation using a symplectic Thom classes theory and not just a symplectic Thom structure.

10. Tautological Thom elements

Suppose that (A,μ,e) is a commutative T-ring spectrum. Let $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$.

We associate to ϑ and a symplectic bundle (E,ϕ) of rank 2 over an X in $\mathcal{S}m/S$ a class $th^{\vartheta}(E,\phi)$ defined as follows. By assumption the scheme X admits an ample family of line bundles. So there exists an affine bundle $f\colon Y \to X$ with Y an affine scheme. Then for some p there exist global sections $s_1,\dots,s_p$ of $f^*E^\vee$ generating $f^*E^\vee$. There then exist global functions a_{ij} on Y such that $f^*\phi = \sum_{1\le i<j\le p} a_{ij}s_i \wedge s_j$. We set $t_i = \sum_{j=i+1}^{p} a_{ij}s_j$ so that we have $\sum_i s_i \wedge t_i = f^*\phi$. The map $(s_1,t_1,\dots,s_p,t_p)\colon f^*E \to \mathcal{O}_Y^{\oplus 2p}$ embeds $(f^*E, f^*\phi)$ as a symplectic subbundle of $(\mathcal{O}_Y^{\oplus 2p}, \omega_{2p})$. So it is classified by a map $\psi\colon Y \to HGr(1,p) = HP^{p-1}$ such that $\psi^*(\mathcal{T}Sp_{1,p},\phi_{1,p}) = f^*(E,\phi)$. This gives us maps of (ind)-schemes

$$X \xleftarrow[\mathbf{A}^N\text{-bundle}]{f} Y \xrightarrow{\psi} HP^{p-1} \xrightarrow{\text{inclusion}} BSp_2 = HP^\infty \tag{10.1}$$

and of pointed motivic spaces

$$\operatorname{Th} E \xleftarrow[\sim\text{mot}]{\overline{f}} \operatorname{Th} f^*E \cong \operatorname{Th} \widetilde{\psi}^* \mathcal{T}Sp_{1,p} \xrightarrow{\overline{\psi}} \operatorname{Th} \mathcal{T}Sp_{1,p} \tag{10.2}$$

of pointed motivic spaces, which can be composed with the maps

$$\operatorname{Th} \mathcal{T}Sp_{1,p} \xrightarrow{\text{inclusion}} \mathbf{MSp}_2 \xrightarrow{\vartheta} A \wedge T^{\wedge 2}. \tag{10.3}$$

in $SH(S)$. The composition of (10.2) and (10.3) gives a class

$$th^{\vartheta}(E,\phi) \in Hom_{SH(S)}(\operatorname{Th} E, A \wedge T^{\wedge 2}) = A^{4,2}(E, E - X).$$

The following lemma is proven in the same way as Lemma 5.4.

Lemma 10.1. *The classes $th^{\vartheta}(E,\phi)$ depend only on the rank 2 symplectic bundle (E,ϕ) and the morphism $\vartheta\colon \Sigma_T^\infty \mathbf{MSp}_2(-2) \to A$ in $SH(S)$.*

Recall the inclusion $e_2^{Sp}\colon T^{\wedge 2} \to \mathbf{MSp}_2$ of (6.2).

Theorem 10.2. *Let (A,μ,e) be a commutative T-ring spectrum. Then the map which assigns to a class ϑ as above the family of classes $th^{\vartheta}(E,\phi)$ is a bijection between the sets of*

(α) classes $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$ with $\vartheta|_{T^{\wedge 2}} = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$, and

(a) symplectic Thom structures on the bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ such that for the trivial rank 2 bundle $\mathbf{A}^2 \to pt$ we have $th(\mathbf{A}^2, \omega_2) = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$.

Proof. A proof similar to that of Theorem 5.5 shows that for a ϑ as in (α), the family of classes $th^{\vartheta}(E,\phi)$ form a symplectic Thom structure with the stated normalization condition. Note that this uses the fact that all symplectic bundles are locally trivial in the Zariski topology.

Now suppose we have a symplectic Thom structure with the stated normalization condition. For every n the tautological rank 2 symplectic bundle bundle over HP^{n-1} has a Thom class which we will abbreviate as

$$th_n = th(\mathcal{U}_{HP^{n-1}}, \phi_{HP^{n-1}}) \in A^{4,2}(\operatorname{Th} \mathcal{U}_{HP^{n-1}}).$$

Pullback along the inclusion $\operatorname{Th} \mathcal{U}_{HP^{n-1}} \to \operatorname{Th} \mathcal{U}_{HP^n}$ sends $th_{n+1} \mapsto th_n$. So as n varies, we get an element

$$\bar{\vartheta} = (th_n)_{n \in \mathbb{N}} \in \varprojlim A^{4,2}(\operatorname{Th} \mathcal{U}_{HP^{n-1}}).$$

We have $\mathbf{MSp}_2 = \operatorname{colim} \operatorname{Th} \mathcal{U}_{HP^{n-1}}$, and by Theorem 9.2 the natural map

$$A^{*,*}(\mathbf{MSp}_2) \xrightarrow{\cong} \varprojlim A^{*,*}(\operatorname{Th} \mathcal{U}_{HP^{n-1}}).$$

is an isomorphism. Let $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$ be the unique class lifting $\bar{\vartheta}$. As in the proof of Theorem 5.9 we have $th^{\vartheta}(E,\phi) = th(E,\phi)$ for all rank 2 symplectic bundles. Moreover, for ϑ and ξ in $A^{4,2}(\mathbf{MSp}_2)$ we have $th^{\vartheta}(E,\phi) = th^{\xi}(E,\phi)$ for all symplectic bundles if and only if ϑ and ξ have the same image in the inverse limit. But that happens only for $\vartheta = \xi$. □

Definition 10.3. The class $\vartheta \in A^{4,2}(\mathbf{MSp}_2)$ is the *tautological Thom element* of the symplectic orientation on $A^{*,*}$ whose rank 2 symplectic Thom classes are the $th^{\vartheta}(E,\phi)$.

The canonical morphism $u_2\colon \Sigma_T^\infty \mathbf{MSp}_2(-2) \to \mathbf{MSp}$ which is part of the counit of the adjunction between $\Sigma_T^\infty(-2)$ and its right adjoint the forgetful functor Ev_2 defines an element $\vartheta_{\mathbf{MSp}} \in \mathbf{MSp}^{4,2}(\mathbf{MSp}_2)$. It satisfies $\vartheta_{\mathbf{MSp}}|_{T^{\wedge 2}} = \Sigma_T^2 1_{\mathbf{MSp}}$ because both elements correspond to the composition $u_2 \circ e_2^{Sp}\colon T^{\wedge 2} \to \mathbf{MSp} \wedge T^{\wedge 2}$.

Definition 10.4. The *standard symplectic Thom structure* on $\mathbf{MSp}^{*,*}$ is the one whose universal Thom element is the $\vartheta_{\mathbf{MSp}}$ we have just described.

11. Tautological Pontryagin elements

The bigraded version of the definition of a Pontryagin structure [13, Definition 12.1] is as follows.

Definition 11.1. A *Pontryagin structure* on a bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ on $\mathcal{S}m\mathcal{O}p/S$ is a rule which assigns to each rank 2 symplectic bundle (E, ϕ) over an X in $\mathcal{S}m/S$ an element $p_1(E, \phi) \in A^{4,2}(X)$ with the following properties:

1. For $(E_1, \phi_1) \cong (E_2, \phi_2)$ we have $p_1(E_1, \phi_1) = p_1(E_2, \phi_2)$.
2. For a morphism $f\colon Y \to X$ we have $f^* p_1(E, \phi)) = p_1(f^*E, f^*\phi)$.
3. For the tautological rank 2 symplectic subbundle $(\mathcal{U}_{HP^1}, \phi_{HP^1})$ on HP^1 the map
$$(1, p_1(\mathcal{U}_{HP^1}, \phi_{HP^1}))\colon A^{*,*}(X) \oplus A^{*-4,*-2}(X) \to A^{*,*}(HP^1 \times X)$$
is an isomorphism for all X in $\mathcal{S}m/S$.
4. For the trivial rank 2 symplectic bundle $(\mathbf{A}^2, \omega_2)$ over pt we have $p_1(\mathbf{A}^2, \omega_2) = 0$ in $A^{4,2}(pt)$.

The Pontryagin classes associated to a symplectic Thom structure by the formula $p_1(E, \phi) = -z^*\, th(E, \phi) \in A^{4,2}(X)$ of Definition 8.1 form a Pontryagin structure because of the functoriality of the Thom classes, the quaternionic projective bundle theorem 8.2 and Corollary 8.5.

For $r = 1$ the diagram (7.4) of morphisms in $H_\bullet(S)$ becomes

$$\begin{tikzcd} T^{\wedge 2} \arrow[r, "e_1^{Sp}"] \arrow[d, "\cong"'] & \mathbf{MSp}_2 \arrow[d, "\cong"] \\ (HP^1, h_\infty) \arrow[r, "\text{inclusion}"'] & (HP^\infty, h_\infty) \end{tikzcd} \tag{11.1}$$

with $h_\infty = pt \to HP^1$ a point such that $h_\infty^*(\mathcal{U}_{HP^1}, \phi_{HP^1})$ is a trivial symplectic bundle. We will call the two vertical arrows the *canonical motivic homotopy equivalences.*

Suppose that (A, μ, e) is a commutative T-ring spectrum. Let $\varrho \in A^{4,2}(HP^\infty, h_\infty)$. For a rank 2 symplectic bundle (E, ϕ) over X the construction of (10.1) composed with the quotient by the pointing and with ϱ gives us a zigzag

$$X \xleftarrow[\sim]{\mathbf{A}^N\text{-bundle}} Y \to HP^\infty \to (HP^\infty, h_\infty) \xrightarrow{\varrho} A \wedge T^{\wedge 2} \tag{11.2}$$

in which the pullbacks to Y of (E, ϕ) and of $(\mathcal{U}_{HP^\infty}, \phi_{HP^\infty})$ are isomorphic. The composition is a class $p_1^\varrho(E, \phi) \in A^{4,2}(X)$. This class depends only on (E, ϕ) and ϱ by arguments similar to those of Lemmas 5.4 and 10.1.

Theorem 11.2. *Let (A, μ, e) be a commutative T-ring spectrum. Then the map which assigns to a class ϱ as above the family of classes $p_1^\varrho(E, \phi)$ is a bijection between the sets of*

(β) *classes $\varrho \in A^{4,2}(HP^\infty, h_\infty)$ with $\varrho|_{HP^1} \in A^{4,2}(HP^1, h_\infty)$ corresponding to $-\Sigma_T^2 1_A \in A^{4,2}(T^{\wedge 2})$ under the canonical motivic homotopy equivalence $(HP^1, h_\infty) \cong T^{\wedge 2}$, and*

(b) *Pontryagin structures on $(A^{*,*}, \partial, \times, 1_A)$ for which $p_1(\mathcal{U}_{HP^1}, \phi_{HP^1}) \in A^{4,2}(HP^1, h_\infty) \subset A^{4,2}(HP^1)$ corresponds to $-\Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$ under the canonical motivic homotopy equivalence $(HP^1, h_\infty) \simeq T^{\wedge 2}$.*

The proof is like that of Theorem 10.2. The classes $p_1^{\varrho}(E,\phi)$ satisfy condition (3) of Definition 11.1 because of an argument like Lemma 5.3 and the isomorphism $T^{\wedge 2} \cong (HP^1, h_\infty)$. They satisfy condition (4) because $p_1^{\varrho}(\mathbf{A}^2, \omega_2) = h_\infty^* \varrho = 0$. The proof that there is a unique ϱ corresponding to each Pontryagin structure invokes the isomorphism $A^{*,*}(HP^\infty) \cong \varprojlim A^{*,*}(HP^n)$ which is the case $r = 1$ of Theorem 9.1.

Definition 11.3. The class $\varrho \in A^{4,2}(HP^\infty, h_\infty)$ is the *tautological Pontryagin element* of the symplectic orientation on $A^{*,*}$ whose rank 2 Pontryagin classes are the $p_1^{\varrho}(E,\phi)$.

Theorem 11.4. *Let (A, μ, e) be a commutative T-ring spectrum. Then the canonical motivic homotopy equivalence $\mathbf{MSp}_2 \cong (HP^\infty, h_\infty)$ plus change-of-sign gives a bijection between the sets of*

(α) the tautological Thom elements ϑ of Theorem 10.2 and

(β) the tautological Pontryagin elements ϱ of Theorem 11.2.

The composition (a) $\leftrightarrow$ (α) $\leftrightarrow$ (β) $\leftrightarrow$ (b) with the bijections of Theorems 10.2 and 11.2 is the same as the rule which assigns to a symplectic Thom structure with classes $th(E,\phi)$ the Pontryagin structure with classes $p_1(E,\phi) = -z^*\, th(E,\phi)$ for $z\colon X \to \operatorname{Th} E$ the structural map of the Thom space.

Proof. The first statement follows from the existence and compatibility of the canonical motivic homotopy equivalences of (11.1). For the second, given a rank 2 symplectic bundle (E,ϕ) on X we have a diagram

$$\begin{tikzcd}
X \arrow[d, "z"'] & Y \arrow[l, "f"', "\sim"] \arrow[r] \arrow[d] & HP^\infty \arrow[r] \arrow[d] & (HP^\infty, h_\infty) \arrow[d, "\varrho"] \arrow[dl, "\cong"'] \\
\operatorname{Th} E & \operatorname{Th} f^*E \arrow[l, "\sim"'] \arrow[r] & \mathbf{MSp}_2 \arrow[r, "-\vartheta"] & A \wedge T^{\wedge 2}
\end{tikzcd}$$

in which the squares commute by compatibility of the structural maps of Thom spaces with pullbacks, the upper triangle commutes by Theorem 7.7, and the lower triangle commutes because of the rule giving the bijection (α) $\leftrightarrow$ (β). We deduce the equality $p_1^{\varrho}(E,\phi) = -z^*\, th^{\vartheta}(E,\phi)$ in $Hom_{SH(S)}(X, A \wedge T^{\wedge 2}) = A^{4,2}(X)$. $\square$

The bigraded version of the definition of a Pontryagin classes theory [13, Definition 14.1] is as follows.

Definition 11.5. A *Pontryagin classes theory* on a bigraded ϵ-commutative ring cohomology theory $(A^{*,*}, \partial, \times, 1_A)$ on $\mathcal{S}m\mathcal{O}p/S$ is a rule assigning to every symplectic bundle (F,ψ) over every X in $\mathcal{S}m/S$ elements $p_i(F,\psi) \in A^{4i,2i}(X)$ for all $i \geq 1$ satisfying

(1) For $(F_1,\psi_1) \cong (F_2,\psi_2)$ we have $p_i(F_1,\psi_1) = p_i(F_2,\psi_2)$ for all i.
(2) For a morphism $f\colon Y \to S$ we have $f^* p_i(F,\psi) = p_i(f^*E, f^*\psi)$ for all i.
(3) For the tautological rank 2 symplectic subbundle $(\mathcal{U}_{HP^1}, \phi_{HP^1})$ on HP^1 the maps
$$(1, p_1(\mathcal{U}_{HP^1}, \phi_{HP^1}))\colon A^{*,*}(X) \oplus A^{*-4,*-2}(X) \to A^{*,*}(HP^1 \times X)$$
are isomorphisms for all X.
(4) For the trivial rank 2 symplectic bundle $(\mathbf{A}^2, \omega_2)$ over pt we have $p_1(\mathbf{A}^2, \omega_2) = 0$ in $A^{4,2}(pt)$.
(5) For an orthogonal direct sum of symplectic bundles $(F,\psi) \cong (F_1,\psi_1) \oplus (F_2,\psi_2)$ we have $p_i(F,\psi) = p_i(F_1,\psi_1) + \sum_{j=1}^{i-1} p_{i-j}(F_1,\psi_1) p_j(F_2,\psi_2) + p_i(F_2,\psi_2)$ for all i.

(6) For (F,ψ) of rank $2r$ we have $p_i(F,\psi)=0$ for $i>r$.

One may also set $p_0(F,\psi)=1$ and even $p_i(F,\psi)=0$ for $i<0$.

Definition 8.4 associates Pontryagin classes to a symplectic Thom structure on $(A^{*,*},\partial,\times,1_A)$. They form a Pontryagin classes theory because the quaternionic projective bundle Theorems 8.2 and 8.3, Corollary 8.5 and the Cartan sum formula (Theorem 8.6).

Theorem 11.6. *Let (A,μ,e) be a commutative T-ring spectrum. Then the forgetful map gives a bijection between the sets of*

(c) Pontryagin classes theories on $(A^{*,*},\partial,\times,1_A)$ with the normalization condition on $p_1(\mathcal{U}_{HP^1},\phi_{HP^1})$ of Theorem 11.2 and

(b) Pontryagin structures on $(A^{*,*},\partial,\times,1_A)$ with the same normalization condition.

The inverse bijection is given by assigning to a Pontryagin structure first the symplectic Thom structure associated to it by Theorem 11.4 and then the the Pontryagin classes theory associated to the symplectic Thom structure by Definition 8.4.

Proof. The chain of associations $(b)\to(a)\to(c)\to(b)$ gives the identity because for rank 2 symplectic bundles the classes $p_1(E,\phi)$ given in Definitions 8.1 and 8.4 coincide.

The chain of associations $(c)\to(b)\to(a)\to(c)$ gives the identity because for a symplectic bundle (F,ψ) of rank $2r$ on X if we let $\pi\colon HP(F,\psi)\to X$ be the associated quaternionic projective bundle with rank 2 tautological subbundle $(\mathcal{U},\phi)$, then from the orthogonal direct sum $\pi^*(F,\psi)=(\mathcal{U},\phi)\oplus(\mathcal{U},\phi)^\perp$ and the axioms we get

$$0=(-1)^r p_r\big((\mathcal{U},\phi)^\perp\big)=p_1(\mathcal{U},\phi)^r-\pi^*p_1(F,\psi)\cup p_1(\mathcal{U},\phi)^{r-1}+\cdots+(-1)^r\pi^*p_r(F,\psi).$$

Hence the Pontryagin classes defined by $(c)\to(b)\to(a)\to(c)$ coincide with the original ones. □

12. Higher rank symplectic Thom classes

The bigraded version of the definition of a symplectic Thom classes theory [13, Definition 14.2] is as follows.

Definition 12.1. A *symplectic Thom classes theory* on a bigraded ϵ-commutative ring cohomology theory $(A^{*,*},\partial,\times,1_A)$ on $\mathcal{S}m\mathcal{O}p/S$ is a rule assigning to every symplectic bundle (F,ψ) over every scheme X in $\mathcal{S}m/S$ an element $th(F,\psi)\in A^{4r,2r}(F,F-X)$ with $2r=\operatorname{rk}F$ with the following properties:

(1) For an isomorphism $u\colon(F,\psi)\cong(F_1,\psi_1)$ we have $th(F,\psi)=u^*\,th(F_1,\psi_1)$.
(2) For $f\colon Y\to X$, writing $f_F\colon f^*F\to F$ for the pullback, we have $f_F^*\,th(F,\psi)=th(f^*F,f^*\psi))\in A^{4r,2r}(f^*F,f^*F-Y)$.
(3) The maps $\cup\,th(F,\psi)\colon A^{*,*}(X)\to A^{*+4r,*+2r}(F,F-X)$ are isomorphisms.
(4) We have $th\big((F_1,\psi_1)\oplus(F_2,\psi_2)\big)=q_1^*\,th(F_1,\psi_1)\cup q_2^*\,th(F_2,\psi_2)$, where q_1,q_2 are the projections from $F_1\oplus F_2$ onto its factors. Moreover, for the zero bundle $\mathbf{0}\to pt$ we have $th(\mathbf{0})=1_A\in A^{0,0}(pt)$.

The classes $th(F,\psi)$ are *symplectic Thom classes*.

Let (A,μ,e) be a commutative T-ring spectrum. Suppose we have a sequence of classes $\boldsymbol{\vartheta}=(\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ with $\vartheta_r\in A^{4r,2r}(\mathbf{MSp}_{2r})$ for each r. Then for any symplectic bundle (F,ψ) of rank $2r$ over X one can use ϑ_r to define a class $th^{\boldsymbol{\vartheta}}(F,\psi)$ by the construction already described in (10.1)–(10.3) for rank 2. For a rank 0 bundle $\mathbf{0}_X\to X$ we set $th(\mathbf{0}_X)=1_X\in A^{0,0}(X)$. These classes are well-defined by the same argument as in Lemma 5.4 and 10.1.

Recall the inclusion $e_2^{Sp}\colon T^{\wedge 2} \to \mathbf{MSp}_2$ of (6.2) and the monoid maps $\mu_{rs}\colon \mathbf{MSp}_{2r} \wedge \mathbf{MSp}_{2s} \to \mathbf{MSp}_{2r+2s}$ of (6.1).

Theorem 12.2. *Let (A,μ,e) be a commutative T-ring spectrum. Then the map which assigns to a sequence of classes $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ as above the family of classes $th^{\boldsymbol{\vartheta}}(F,\psi)$ is a bijection between the sets of*

(δ) sequences of classes $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ with $\vartheta_r \in A^{4r,2r}(\mathbf{MSp}_{2r})$ for each r satisfying $\mu_{rs}^*\vartheta_{r+s} = \vartheta_r \times \vartheta_s$ for all r,s, and $\vartheta_1|_{T^{\wedge 2}} = \Sigma_T^2 1_A$, and

(d) symplectic Thom classes theories on $(A^{*,*},\partial,\times,1_A)$ such that for the trivial rank 2 bundle $\mathbf{A}^2 \to pt$ we have $th(\mathbf{A}^2,\omega_2) = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$.

The proof is essentially the same as that of Theorem 10.2. The class ϑ_r is the *tautological symplectic Thom element* of rank $2r$.

Recall that for a commutative T-ring spectrum (A,μ,e) with a symplectic Thom structure on $(A^{*,*},\partial,\times,1_A)$ the Thom space structural map $z_r\colon BSp_{2r} \to \mathbf{MSp}_{2r}$ has the property that $z_r^*\colon A^{*,*}(\mathbf{MSp}_{2r}) \to A^{*,*}(BSp_{2r})$ is injective, and that the isomorphism

$$A^{*,*}(BSp_{2r}) \xleftarrow{\cong} A^{*,*}(pt)[[p_1,\dots,p_r]] \tag{12.1}$$

derived from the symplectic Thom structure identifies the image of z_r^* with the two-sided ideal generated by p_r (Theorems 9.1, 9.2 and 9.3).

Theorem 12.3. *Let (A,μ,e) be a commutative T-ring spectrum. Then the assignment $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots) \mapsto \vartheta_1$ gives a bijection between the sets of*

(δ) sequences of classes $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ satisfying the conditions of Theorem 12.2 and

(α) the tautological rank 2 Thom elements ϑ of Theorem 10.2.

The inverse bijection sends $\vartheta \mapsto \boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ where $z_r^*\vartheta_r \in A^{4r,2r}(BSp_{2r})$ is the element corresponding to $(-1)^r p_r$ under the isomorphism (12.1) *derived from the symplectic Thom structure associated to ϑ by Theorem 10.2.*

Proof. Clearly the mapping $(\delta) \to (\alpha)$ is well-defined.

We will show that $(\alpha) \to (\delta)$ is well-defined. Suppose ϑ satisfies the conditions of (α). The classes $(\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ verify the condition $\mu_{rs}^*\vartheta_{r+s} = \vartheta_r \times \vartheta_s$ because in Theorem 9.4 the classes in the second diagram verify $p_{r+s} \mapsto p_r' p_s''$. The class ϑ_1 is obtained by the construction corresponding to the assignments $(\alpha) \to (\beta) \to (\alpha)$ of Theorem 11.4. So by that theorem we have $\vartheta_1 = \vartheta$. So we have $\vartheta_1|_{T^{\wedge 2}} = \Sigma_T^2 1_A$. Therefore $(\alpha) \to (\delta)$ is well-defined. In addition this shows that $(\alpha) \to (\delta) \to (\alpha)$ is the identity.

Now suppose given $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ satisfying (δ), and let $\boldsymbol{\vartheta}' = (\vartheta_1',\vartheta_2',\vartheta_3',\dots)$ be the result obtained by applying $(\delta) \to (\alpha) \to (\delta)$. We have already seen that we have $\vartheta_1 = \vartheta_1'$. The equalities $\vartheta_r = \vartheta_r'$ follow by induction using the injectivity of the maps m_{rs}^* of Theorem 9.4. □

For a symplectic bundle (F,ψ) of rank $2r$ over X the Pontryagin classes (γ) and the symplectic Thom classes (δ) are related by

$$p_r(F,\psi) = (-1)^r z^*\, th(F,\psi). \tag{12.2}$$

13. The universality of MSp

Theorem 13.1. *Let (A,μ,e) be a commutative T-ring spectrum with a symplectic Thom classes theory on $A^{*,*}$. Then we have isomorphisms of bigraded rings*

$$A^{*,*}(\mathbf{MSp}) \xrightarrow{\cong} \varprojlim A^{*+4r,*+2r}(\mathbf{MSp}_{2r}) \xleftarrow{\cong} A^{*,*}(pt)[[p_1,p_2,p_3,\dots]]^{hom},$$

$$A^{*,*}(\mathbf{MSp}\wedge\mathbf{MSp}) \xrightarrow{\cong} \varprojlim A^{*+8r,*+4r}(\mathbf{MSp}_{2r}\wedge\mathbf{MSp}_{2r}) \xleftarrow{\cong} A^{*,*}(pt)[[p_1',p_2',\dots,p_1'',p_2'',\dots]]^{hom}.$$

Proof. By Theorem 12.2 the symplectic Thom classes theory has associated to it a sequence $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ of tautological symplectic Thom classes with the property that $\vartheta_r \in A^{4r,2r}(\mathbf{MSp}_{2r})$ is the Thom class of the tautological symplectic subbundle $(\mathcal{U}_{BSp_{2r}},\phi_{BSp_{2r}})$ over $BSp_{2r} = HGr(r,\infty)$ and with $\vartheta_1|_{T^{\wedge 2}} = \Sigma_T^2 1_A$. Set also $\vartheta_0 = 1_A \in A^{0,0}(pt) = A^{0,0}(\mathbf{MSp}_0)$.

By Theorem 5.6 the group $A^{*,*}(\mathbf{MSp})$ fits into the short exact sequence

$$0 \to \varprojlim{}^1 A^{*+4r-1,*+2r}(\mathbf{MSp}_{2r}) \to A^{*,*}(\mathbf{MSp}) \to \varprojlim A^{*+4r,*+2r}(\mathbf{MSp}_{2r}) \to 0$$

where the connecting maps in the tower are given by the top line of the commutative diagram

$$\begin{tikzcd}
A^{*+4r-4,*+2r-2}(\mathbf{MSp}_{2r-2}) & A^{*+4r,*+2r}(\mathbf{MSp}_{2r-2}\wedge T^{\wedge 2}) \arrow[l, "\Sigma_T^{-2}"'] & A^{*+4r,*+2r}(\mathbf{MSp}_{2r}) \arrow[l, "\sigma^*"'] \\
A^{*,*}(BSp_{2r-2}) \arrow[u, "\cong", "-\cup\vartheta_{r-1}"'] & A^{*,*}(BSp_{2r-2}) \arrow[l, "\mathrm{id}"'] \arrow[u, "\cong", "-\cup\sigma^*\vartheta_r"'] & A^{*,*}(BSp_{2r}). \arrow[l, "i_{2r}^*"'] \arrow[u, "\cong", "-\cup\vartheta_r"']
\end{tikzcd}$$

The map σ^* is the pullback along the bonding map

$$\mathbf{MSp}_{2r-2}\wedge T^{\wedge 2} \xrightarrow{1\times e_1} \mathbf{MSp}_{2r-2}\wedge\mathbf{MSp}_2 \xrightarrow{\mu_{r-1,1}} \mathbf{MSp}_{2r}$$

of the symmetric $T^{\wedge 2}$-spectrum. Thus we have $\sigma^*\vartheta_r = \vartheta_{r-1}\times\Sigma_T^2 1_A$. The diagram therefore commutes. The vertical maps are isomorphisms by condition (3) of the definition of a symplectic Thom classes theory. The map i_{2r}^* is the surjection $A^{*,*}[[p_1,\dots,p_r]] \twoheadrightarrow A^{*,*}[[p_1,\dots,p_{r-1}]]$ of (9.3). This gives us the second isomorphism of the theorem, while the surjectivity of the connecting maps in the inverse system gives the vanishing of the $\varprojlim^1$ and the first isomorphism.

The calculations for $A^{*,*}(\mathbf{MSp}\wedge\mathbf{MSp})$ are similar. □

Let $\varphi\colon \mathbf{MSp}\to A$ be a morphism in $SH(S)$. For each $r\ge 1$ let $\vartheta_r^\varphi \in A^{4r,2r}(\mathbf{MSp}_{2r})$ be the composition

$$\Sigma_T^\infty \mathbf{MSp}_{2r}(-2r) \xrightarrow{u_{2r}} \mathbf{MSp} \xrightarrow{\varphi} A,$$

and let $\boldsymbol{\vartheta}^\varphi = (\vartheta_1^\varphi,\vartheta_2^\varphi,\vartheta_3^\varphi,\dots)$.

Theorem 13.2. *Suppose (A,μ,e) is a commutative monoid in $(SH(S),\wedge,\mathbf{1})$. Then the assignment $\varphi\mapsto\boldsymbol{\vartheta}^\varphi = (\vartheta_1^\varphi,\vartheta_2^\varphi,\vartheta_3^\varphi,\dots)$ gives a bijection between the sets of*

(ε) *morphisms $\varphi\colon (\mathbf{MSp},\mu^{Sp},e^{Sp})\to(A,\mu,e)$ of commutative monoids in $SH(S)$, and*

(δ) *sequences of classes $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3,\dots)$ satisfying the conditions of Theorem 12.2.*

The proof of this theorem is substantially the same as that of Theorem 5.9. The differences are, first, that the $\boldsymbol{\vartheta}$ comes from a unique $\varphi\colon \mathbf{MSp}\to A$ because the map $A^{0,0}(\mathbf{MSp}) \to \varprojlim A^{4r,2r}(\mathbf{MSp}_{2r})$ of Theorem 13.1 is an isomorphism. Second, the obstruction to φ being a morphism of monoids vanishes because $A^{0,0}(\mathbf{MSp}\wedge\mathbf{MSp}) \to \varprojlim A^{8r,4r}(\mathbf{MSp}_{2r}\wedge\mathbf{MSp}_{2r})$ is also an isomorphism.

References

[1] P. Balmer and B. Calmès, *Witt groups of Grassmann varieties*, (2008). Preprint, 2008.

[2] B. I. Dundas, O. Röndigs, and P. A. Østvær, *Motivic functors*, Doc. Math., 8 (2003), pp. 489–525 (electronic).

[3] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra, 165 (2001), pp. 63–127.

[4] D. C. Isaksen, *Flasque model structures for simplicial presheaves*, K-Theory, 36 (2005), pp. 371–395.

[5] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math., 5 (2000), pp. 445–552 (electronic).

[6] M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[7] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 2nd ed., 1995.

[8] F. Morel and V. Voevodsky, *$\mathbf{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math., (1999), pp. 45–143 (2001).

[9] I. Panin, *Oriented cohomology theories of algebraic varieties*, K-Theory, 30 (2003), pp. 265–314. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[10] I. Panin, K. Pimenov, and O. Röndigs, *A universality theorem for Voevodsky's algebraic cobordism spectrum*, Homology, Homotopy Appl., 10 (2008), pp. 211–226.

[11] ———, *On Voevodsky's algebraic K-theory spectrum*, in Algebraic topology, vol. 4 of Abel Symp., Springer, Berlin, 2009, pp. 279–330.

[12] I. Panin and C. Walter, *On the motivic commutative ring spectrum* $\mathbf{BO}$. Preprint, 2010.

[13] ———, *Quaternionic Grassmannians and Pontryagin classes in algebraic geometry*. Preprint, 2010.

[14] G. Vezzosi, *Brown-Peterson spectra in stable $\mathbb{A}^1$-homotopy theory*, Rend. Sem. Mat. Univ. Padova, 106 (2001), pp. 47–64.

[15] V. Voevodsky, *$\mathbf{A}^1$-homotopy theory*, Doc. Math., Extra Vol. I (1998), pp. 579–604 (electronic).

[16] V. Voevodsky, O. Röndigs, and P. A. Østvær, *Voevodsky's Nordfjordeid lectures: Motivic homotopy theory*, in Motivic homotopy theory, Universitext, Springer, Berlin, 2007, pp. 147–221.

Steklov Institute of Mathematics at St. Petersburg, Russia

Laboratoire J.-A. Dieudonné (UMR 6621 du CNRS), Département de mathématiques, Université de Nice – Sophia Antipolis, 06108 Nice Cedex 02, France