# UNITARY GRASSMANNIANS

NIKITA A. KARPENKO

Abstract. We study projective homogeneous varieties under an action of a projective unitary group (of outer type). We are especially interested in the case of (unitary) grassmannians of totally isotropic subspaces of a hermitian form over a field, the main result saying that these grassmannians are 2-incompressible if the hermitian form is generic. Applications to orthogonal grassmannians are provided.

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## 1. Introduction

In this paper, we study Chow rings, Chow motives, and incompressibility properties of projective homogeneous varieties under an action of an adjoint absolutely simple affine algebraic group of type $\mathcal{A}$ over an arbitrary field $F$. Such a group can be realized as the $F$-group of $K$-automorphism of an Azumaya $K$-algebra with an $F$-linear unitary involution, where $K$ is a quadratic étale $F$-algebra. The case where $K$ is split is pretty much studied in the literature (the varieties arising in that case are the generalized Severi-Brauer varieties and, more generally, varieties of flags of right ideals of a central simple $F$-algebra). In the present paper we are concentrated on the opposite case: $K$ is a field, but the central simple $K$-algebra is split, i.e., isomorphic to the algebra of endomorphisms of a finite-dimensional vector space $V$ over $K$. The involution on the algebra in this case is adjoint to some non-degenerate $K/F$-hermitian form on $V$.

Let $K/F$ be a separable quadratic field extension, $V$ a vector space over $K$ endowed with a non-degenerate $K/F$-hermitian form $h$. By a Jacobson theorem (cf. Corollary

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9.2), this classical object is completely determined by an even more classical one: the quadratic form $v \mapsto h(v, v)$ on $V$ considered this time as a vector space over $F$. Moreover, the quadratic forms arising this way from $K/F$-hermitian forms, are easily described as the tensor products of a non-degenerate (diagonal) bilinear form by the norm form of $K/F$ (which is a non-degenerate binary quadratic form, the quadratic form associated to the hermitian form $\langle 1\rangle$ on $K$).

Although this well-known and elementary observation shows that the study of hermitian forms is equivalent to the study of the "binary divisible" quadratic forms, it does not show that the hermitian forms are not interesting because they are just a more complicated replacement for a well-understood thing. Quite the contrary, first of all this observation shows that the binary divisible quadratic forms form an important class of quadratic forms. For instance, the subclass of quadratic forms divisible by an $n$-fold Pfister form (i.e., divisible by a product of $n$ binary forms: $n-1$ bilinear and 1 quadratic) for $n \geq 2$ is less important even if it is a natural generalization.

On the other hand, this observation provides a hope that the world around hermitian forms might give additional tools to study such quadratic forms. And indeed, it turns out that the projective homogeneous varieties related to hermitian forms (in particular, the unitary grassmannians) are more suitable (in particular, more "economical") geometric objects related to the binary divisible quadratic forms if compared with the orthogonal grassmannians – the geometric objects commonly attached to quadratic forms. The present paper demonstrates and exploits these phenomena.

The main result of the paper is Theorem 8.1 saying that the unitary grassmannians associated to a generic hermitian form are 2-incompressible [1] and that essential parts of their motives are indecomposable. The main application (which was the initial motivation) is Corollary 9.4, computing the $J$-invariant of the quadratic form attached to the generic hermitian form and therefore the maximal value of $J$-invariant of a quadratic form arising from a hermitian one (of a fixed dimension). Another striking application (in the opposite direction: from quadratic forms to hermitian ones) is the computation of the modulo 2 reduced Chow ring of a unitary grassmannians in terms of the Chow ring of the orthogonal grassmannians made in Corollary 9.10 (for odd-dimensional hermitian forms see Example 9.12). This computation is particularly interesting in the hyperbolic case because the Chow ring of orthogonal grassmannians of hyperbolic forms is very well studied and described, see e.g. [4].

We start in Section 2 with a completely elementary computation of the subring of the invariant elements under a permutation involution on a polynomial ring. It becomes clear already at this point that one can make the life much more agreeable working modulo the norms. This principle will be constantly present in the sequel.

The reason of making the above computation becomes clear in Section 3 where we are interested in the $T$-equivariant Chow ring of the point $\operatorname{Spec} F$ for certain quasi-split torus $T$: this Chow ring is identified with the subring of Section 2. Our base field $F$ appearing for the first time (after Introduction) at this point, is (and remains until the end of the paper) of arbitrary characteristic.

[1]We refer to [12] for the definition and main properties of *incompressibility* and, more generally, of *canonical dimension* of (projective homogeneous) varieties.

In Section 4, the main objects of our study begin to appear. The torus $T$ (or $\mathbb{G}_{\mathrm{m}} \times T$ – depending on parity of $n$) considered in Section 3, turns out to be a maximal torus in a Borel subgroup $B$ of a non-split quasi-split adjoint absolutely simple affine algebraic group $G$ of type $\mathcal{A}_n$. Consequently, as already noticed in [18], the $T$-equivariant Chow group $\mathrm{CH}_T(\operatorname{Spec} F)$ appears in a computation of the reduced Chow group of the complete flag variety $E/B$ given by a *generic* (in the sense of [10]) $G$-torsor $E$ (where $B \subset G$ is a Borel subgroup). Having shown that the ring $\mathrm{CH}_T(\operatorname{Spec} F)$ is generated modulo the norms by elements of codimension $\leq 2$, we get the same statement for the reduced Chow group $\overline{\mathrm{CH}}(E/B)$ (Proposition 4.5). Killing in a generic way a (certain) Tits algebra, we come to the complete flag variety of a (generic) hermitian form while keeping the above property (Corollary 4.6).

Next we would like to get from the complete flag variety to the maximal grassmannian. Clearly, this can be achieved by projecting complete flags to their maximal component. But unlikely the case of the orthogonal group, such a projection here is not a chain of projective bundles. It is a chain of quadratic Weil transfers of projective bundles. So, we make some general study of quadratic Weil transfers of projective bundles in Section 5.[2] Once again, working modulo norms makes it possible to get a very simple (and very similar to the classical case of a projective bundle) description of the Chow ring.

We apply this description in Section 6 getting a description of the reduced Chow group of a generic maximal unitary grassmannian. We still get the generation by codimension $\leq 2$, but in the same time the components of codimension 1 and 2 are 0, showing that the reduced Chow group modulo norms is trivial in positive codimensions (Proposition 6.1). This description actually shows that such a grassmannian is 2-incompressible. More than this, an essential part of its Chow motive (with coefficient in $\mathbb{F}_2$) is indecomposable. But the last two statements belong already to Section 8 and are valid for all (generic) grassmannians, not only for the maximal one. Before we come to them, we need a definition of the essential part and some general observations on isotropic grassmannians, made in Section 7.

The notion of the essential motive of a unitary grassmannian, introduced in Definition 7.5, gives one more justification for working "modulo norms": the reduced Chow group of the essential motive is identified with the reduced Chow group of the variety modulo the norms.

The proof of Theorem 8.1, the main result of the paper, closely follows [14], where a similar result is proved for the orthogonal grassmannians. The difference is in the case of the maximal grassmannian (the induction base of the proof), which constitutes the main part of the present paper while in [14] it was served by a reference.

The final Section 9 establishes and exploits the connection with the quadratic forms. It starts by (I suppose) well-known Lemma 9.1 and classically known Corollary 9.2. Corollary 9.3 is just a translation of Lemma 9.1 to the language of upper motives. The first interesting result is Corollary 9.4 (discussed above already) computing the maximal possible value of the so-called $J$-invariant of a quadratic form split by a given separable quadratic field extension. This result was not accessible by methods of quadratic forms

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[2]This is already needed in the previous section in the moment we are generically killing the Tits algebra, because this is a central simple algebra over an extension of the base field.

themselves, but transferring the problem to the world of hermitian forms, we come to a problem about a generic hermitian form (without any additional condition) which we can solve because such a hermitian form can be obtained from a generic torsor with a help of generic killing of a Tits algebra.[3]

The motivic indecomposability which we have in the generic case by Theorem 8.1 gives an isomorphism of motives of Proposition 9.5 (in generic and therefore in general case) which implies an isomorphism of Chow groups of Corollary 9.10. This isomorphism of Chow groups is interesting even in the quasi-split case: for instance, Examples 9.11 and 9.12 provide a description by generators and relations for the Chow group of a quasi-split maximal unitary grassmannian (in terms of such description known in the orthogonal case, [8, §86]).

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## 2. Invariants under a permutation involution

Let $R$ be a commutative associative unital ring. For any integer $r \geq 0$, let us consider the polynomial $R$-algebra in $2r$ variables $S := R[a_1, b_1, \dots, a_r, b_r]$. Let $\sigma$ be the involution (i.e., a self-inverse automorphism) of the $R$-algebra $S$ interchanging for every $i = 1, \dots, r$ the variables $a_i$ and $b_i$. We write $S^\sigma \subset S$ for the subalgebra of $\sigma$-invariant elements.

The subset $(1+\sigma)S := \{s + \sigma(s) \mid s \in S\} \subset S^\sigma$, the image of the norm homomorphism $N : S \to S$, $s \mapsto s + \sigma(s)$ (this is a homomorphism of $R$-modules), is contained in $S^\sigma$ and is an ideal of $S^\sigma$.

**Lemma 2.1.** *The quotient $S^\sigma/(1+\sigma)S$ is generated as $R$-algebra by the classes of the products $a_ib_i$, $i = 1, \dots, r$.*

*Proof.* Induction on $r$. The statement is trivial for $r = 0$.

To pass from $r-1$ to $r \geq 1$, let $S'$ be the $R$-subalgebra $R[a_1, b_1, \dots, a_{r-1}, b_{r-1}]$ of $S$. The $S'$-module $S$ is free with the basis $\{a_r^i b_r^j\}_{i,j\geq 0}$ so that an element $\sum_{i,j\geq 0} \alpha_{ij} a_r^i b_r^j \in S$ with $\alpha_{ij} \in S'$ is $\sigma$-invariant if and only if $\sigma(\alpha_{ij}) = \alpha_{ji}$ for any $i, j$. It follows that the $S'^\sigma/(1+\sigma)S'$-algebra $S^\sigma/(1+\sigma)S$ is generated by the class of $a_rb_r$. □

**Remark 2.2.** In general, the $R$-algebra $S^\sigma$ itself is not generated by polynomials of degree $\leq 2$. Taking for instance $R = \mathbb{Z}$ and $r = 3$, we have a $\sigma$-invariant integral polynomial $P := a_1a_2a_3 + b_1b_2b_3$ which cannot be written as a polynomial in the $\sigma$-invariant polynomials of degree $\leq 2$ in $a_1, b_1, a_2, b_2, a_3, b_3$. Indeed, let us refer as *elementary polynomial* to any $\sigma$-invariant monomial (with coefficient 1) and to any polynomial of the form $M + \sigma(M)$, where $M$ is a non-$\sigma$-invariant monomial (with coefficient 1). Any degree 3 product of elementary polynomials of degree 1 or 2 either has no monomial with repeated

[3]Actually, we use the name *generic hermitian form* in a more concrete and narrow sense (defined before Corollary 4.6) in the main body of the article. At this (non-formal) point, speaking about "generic" we mean a versality property in the spirit of [25, Definition 5.1(2)].

indices of variables (only has monomials like $a_1a_2b_3$, but no monomials like $a_1b_1a_3$ or $a_1^2a_3$) or consists only of monomials with repeated indices. If $P$ is a linear combination of such products with coefficients $\pm1$ (which is the case if $P$ is a polynomial in the $\sigma$-invariant polynomials of degree $\leq 2$), then the part of the combination given by the products of the second type is 0. Each product in the remaining part of the decomposition, as any product of the first type, has the following property: it is a sum of *even* number of degree 3 elementary polynomials. In particular, $P$ is a linear combination with coefficients $\pm1$ of an even number of elementary polynomials. Any cancelation keeps the parity of this number. However $P$ is just one elementary polynomial.

On the other hand, $S^\sigma$ *is* generated by its degree 1 and 2 components as far as 2 is invertible in $R$: this can be shown by induction using the identity

$$2(fgh + f'g'h') = (fg + f'g')(h + h') - (fh' + f'h)(g + g') + (gh + g'h')(f + f').$$

## 3. Some equivariant Chow groups of the point

Let $F$ be a field. We recall the computation of the $\mathbb{G}_{\mathrm{m}}$-equivariant Chow ring

$$\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F),$$

c.f. [6, Lemma 4]. For generalities on equivariant Chow groups and rings, we refer to [7] and [3].

The $F$-torus $\mathbb{G}_{\mathrm{m}}$ acts by multiplication on the affine line $\mathbb{A}^1 = \mathbb{A}^1_F$ over $F$. For any integer $l \geq 1$, we consider the diagonal action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^l$. By the homotopy invariance of the equivariant Chow groups, the pull-back ring homomorphism $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F) \to \mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^l)$ is an isomorphism. By the localization sequence, the pull-back ring homomorphism $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\mathbb{A}^l) \to \mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(U_l)$ to the $\mathbb{G}_{\mathrm{m}}$-invariant open subset $U_l := \mathbb{A}^l \setminus \{0\}$ is bijective in codimensions $< l$ (because the complement of $U_l$ in $\mathbb{A}^l$ is of codimension $l$). It follows that the ring homomorphism $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F) \to \mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(U_l)$ is bijective in codimensions $< l$. Since $\mathbb{G}_{\mathrm{m}}$ acts freely on $U_l$ and $U_l/\mathbb{G}_{\mathrm{m}} = \mathbb{P}^{l-1}$, the ring $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(U_l)$ is identified with the usual Chow ring $\mathrm{CH}(\mathbb{P}^{l-1})$. It follows that the ring $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F)$ is freely generated by the element $h \in \mathrm{CH}^1_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F)$ corresponding to the class of a rational point in $\mathrm{CH}^1(\mathbb{P}^1)$ under the isomorphism $\mathrm{CH}^1_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F) \to \mathrm{CH}^1(\mathbb{P}^1)$.

Now we fix an étale $F$-algebra $L$ (see [19, §18A] for generalities on étale algebras) and consider the Weil transfer $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$ (see [19, 20.5] where it is called *corestriction*) with respect to $L/F$ of the $L$-torus $\mathbb{G}_{\mathrm{m}}$. The $F$-torus $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$ is quasi-split, and any quasi-split $F$-torus is isomorphic to $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$ for an appropriate choice of $L$. We would like to describe the graded (by codimension) ring $\mathrm{CH}_{\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F)$.

We fix a separable closure $F_{\mathrm{sep}}$ of $F$, write $\Gamma$ for the Galois group of $F_{\mathrm{sep}}/F$, and $X$ for the $\Gamma$-set of the $F$-algebra homomorphisms $L \to F_{\mathrm{sep}}$. We consider the following graded rings: the polynomial ring $\mathbb{Z}[X]$ in variables indexed by the elements of $X$ and its subring $\mathbb{Z}[X]^\Gamma$ of $\Gamma$-invariant elements.

Let us do a computation for $\mathrm{CH}_{\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F)$ similar to the above computation of $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F)$. The torus $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$ acts by multiplication on the affine $F$-space $\mathcal{R}_{L/F}\mathbb{A}^1$ (of dimension $\#X = \dim_F L$). For any integer $l \geq 1$, we consider the diagonal action of $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$ on the affine $F$-space $\mathcal{R}_{L/F}(\mathbb{A}^l) = (\mathcal{R}_{L/F}\mathbb{A}^1)^l$. The action on the open $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$-invariant subset $U_l := \mathcal{R}_{L/F}(\mathbb{A}^l \setminus \{0\})$ is free, and $U_l/\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}} = \mathcal{R}_{L/F}\mathbb{P}^{l-1}$. We get a ring homomorphism $\mathrm{CH}_{\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F) \to \mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1})$ which is bijective in codimensions $< l$ (because the complement of $U_l$ in $\mathcal{R}_{L/F}\mathbb{A}^l$ is of codimension $l$).

Since the integral Chow motive (for generalities on motives wee [8, Chapter XII]) of the variety $\mathcal{R}_{L/F}\mathbb{P}^{l-1}$ is a sum of shifts of the motives of $\operatorname{Spec} K$ for certain subfields $K \subset F_{\mathrm{sep}}$ finite over $F$ (see [16] or [17]), the ring homomorphism $\mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1}) \to \mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1})_{F_{\mathrm{sep}}}$ identifies the ring $\mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1})$ with the subring of $\Gamma$-invariant elements in $\mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1})_{F_{\mathrm{sep}}}$. It remains to identify the $F_{\mathrm{sep}}$-variety $(\mathcal{R}_{L/F}\mathbb{P}^{l-1})_{F_{\mathrm{sep}}}$ with the product of copies of $\mathbb{P}^{l-1}$ indexed by the elements of the set $X$ and therefore to identify $\mathrm{CH}(\mathcal{R}_{L/F}\mathbb{P}^{l-1})_{F_{\mathrm{sep}}}$ with the tensor product of the indexed by the elements of $X$ copies of $\mathrm{CH}(\mathbb{P}^{l-1})$. Therefore we obtain an isomorphism of graded rings

$$\mathrm{CH}_{\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}}(\operatorname{Spec} F) = \mathbb{Z}[X]^{\Gamma}. \tag{3.1}$$

**Remark 3.2.** Any element $L \to F_{\mathrm{sep}}$ of $X$ gives a homomorphism of $F_{\mathrm{sep}}$-algebras $L_{F_{\mathrm{sep}}} \to F_{\mathrm{sep}}$ and therefore a character of the $F_{\mathrm{sep}}$-torus $(\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}})_{F_{\mathrm{sep}}}$. The obtained map $X \to (\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}})_{F_{\mathrm{sep}}}{}^*$ (to the group of characters) induces an isomorphism of graded $\Gamma$-rings $\mathbb{Z}[X] \to \mathcal{S}\big((\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}})_{F_{\mathrm{sep}}}{}^*\big)$ to the symmetric ring of the abelian group $(\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}})_{F_{\mathrm{sep}}}{}^*$. Therefore the isomorphism 3.1 gives rise to an (actually independent of the choice of $L$) isomorphism

$$\mathrm{CH}_T(\operatorname{Spec} F) = \mathcal{S}(T_{F_{\mathrm{sep}}}{}^*)^{\Gamma}$$

(for any quasi-split $F$-torus $T$).

Let now $K$ be a separable quadratic field extension of $F$. Applying the isomorphism 3.1 to the étale $F$-algebra $L = K^{\times r}$, since $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}} = (\mathcal{R}_{K/F}\mathbb{G}_{\mathrm{m}})^{\times r}$, we get

**Lemma 3.3.** *Let $T := (\mathcal{R}_{K/F}\mathbb{G}_{\mathrm{m}})^{\times r}$ for some $r \geq 1$. There is an isomorphism of the graded ring $\mathrm{CH}_T(\operatorname{Spec} F)$ with the subring of $\sigma$-invariant elements of the ring*

$$\mathbb{Z}[a_1, b_1, \dots, a_r, b_r]$$

*(where $\sigma$ is the ring involution exchanging $a_i$ and $b_i$ for each $i = 1, \dots, r$) such that the image of the norm map $\mathrm{CH}_T(\operatorname{Spec} K) \to \mathrm{CH}_T(\operatorname{Spec} F)$ corresponds to the image of the map $1 + \sigma : \mathbb{Z}[a_1, b_1, \dots, a_r, b_r] \to \mathbb{Z}[a_1, b_1, \dots, a_r, b_r]^{\sigma}$.* □

Note that the image of the norm map $\mathrm{CH}_T(\operatorname{Spec} K) \to \mathrm{CH}_T(\operatorname{Spec} F)$ is an ideal of the destination ring. Applying Lemma 2.1, we get

**Corollary 3.4.** *The ring $\mathrm{CH}_T(\operatorname{Spec} F)$ modulo the ideal of norms from $\operatorname{Spec} K$ is generated by elements of codimension 2.* □

Similarly, applying first Lemma 3.1 to the étale $F$-algebra $L = F \times K^{\times r}$ and then Lemma 2.1 with $R$ being a ring integral polynomials in one variable, we get

**Corollary 3.5.** *The ring $\mathrm{CH}_{\mathbb{G}_{\mathrm{m}} \times T}(\operatorname{Spec} F)$ (for $T$ as in Lemma 3.3) modulo the ideal of norms from $\operatorname{Spec} K$ is generated by elements of codimension 1 and 2.* □

## 4. Generic varieties of complete flags

Let $G$ be a non-split quasi-split adjoint absolutely simple affine algebraic group over a field $F$ of type $\mathcal{A}_{n-1}$, $n \geq 2$, becoming split over a separable quadratic field extension $K/F$. We set $r := \lfloor n/2 \rfloor$ (the floor of $n/2$).

**Lemma 4.1.** *The group $G$ contains a Borel subgroup $B$ and a maximal torus $T \subset B$ such that*

$$T \simeq \begin{cases} \mathbb{G}_{\mathrm{m}} \times (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times(r-1)}, & \textit{if } n \textit{ is even (i.e., } n = 2r\textit{);} \\ (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}, & \textit{if } n \textit{ is odd (i.e., } n = 2r+1\textit{),} \end{cases}$$

*where $\mathcal{R} = \mathcal{R}_{K/F}$ is the Weil transfer with respect to the field extension $K/F$.*

*Proof.* In the case of $n = 2r$, let $h$ be the orthogonal sum of $r$ copies of the $K/F$-hermitian *hyperbolic plane* (the $K/F$-hermitian form on the vector space $K^2$ of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). In the case of $n = 2r+1$, we let $h$ be the orthogonal sum of $r$ copies of the hyperbolic plane and of the $K/F$-hermitian space $\langle 1 \rangle$. In both cases, $h$ is a hermitian form on the $K$-vector space $K^n$. Up to an isomorphism, $G$ is the $F$-group of automorphisms of the $K$-algebra $\operatorname{End}_K(K^n)$ with the (unitary) adjoint involution. We may assume that $G$ is this group of automorphisms.

For $n = 2r$, we have a homomorphism $\alpha : (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r} \to G$, mapping an $F$-point $(\lambda_1, \dots, \lambda_r) \in (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}(F) = K^\times \times \dots \times K^\times$ to the automorphism of $\operatorname{End}_K K^n$ which is the conjugation by the diagonal automorphism $(\lambda_1, \sigma(\lambda_1)^{-1}, \dots, \lambda_r, \sigma(\lambda_r)^{-1})$ of $K^n$ preserving $h$, where $\sigma$ is the non-trivial automorphism of $K/F$. An $F$-point $(\lambda_1, \dots, \lambda_r)$ is in the kernel of $\alpha$ if and only if $\lambda_1 = \sigma(\lambda_1)^{-1} = \dots = \lambda_r = \sigma(\lambda_r)^{-1}$. It follows that $\operatorname{Ker}\alpha$ is the kernel $\mathcal{R}^{(1)}\mathbb{G}_{\mathrm{m}}$ of the norm map $\mathcal{R}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ sitting in $(\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}$ diagonally. The quotient $(\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}/\operatorname{Ker}\alpha$ is therefore a torus isomorphic to $\mathbb{G}_{\mathrm{m}} \times (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times(r-1)}$ and to a closed subgroup $T \subset G$. Since the dimension of the torus $T$ is $2r - 1 = n - 1$ and $n - 1$ is the rank of $G$, the torus $T$ is maximal in $G$.

For $n = 2r+1$, any $F$-point $(\lambda, \lambda_1, \dots, \lambda_r) \in \mathcal{R}^{(1)}\mathbb{G}_{\mathrm{m}}(F) \times (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}(F)$ produces a preserving $h$ diagonal automorphism $(\lambda, \lambda_1, \sigma(\lambda_1)^{-1}, \dots, \lambda_r, \sigma(\lambda_r)^{-1})$ of $K^n$. The kernel of the resulting homomorphism $\mathcal{R}^{(1)}\mathbb{G}_{\mathrm{m}} \times (\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r} \to G$ is $\mathcal{R}^{(1)}\mathbb{G}_{\mathrm{m}}$, and the quotient by the kernel is the required torus $(\mathcal{R}\mathbb{G}_{\mathrm{m}})^{\times r}$ (of dimension $2r = n - 1$).

The torus $T$ constructed in both cases is contained in the Borel subgroup $B$ of $G$ which is the stabilizer of the flag of length $r$ of subspaces in $K^n$ where the $i$th subspace is the sum of the first $i$ odd summands of $K^n$. □

**Remark 4.2.** Actually, for *any* quasi-split semisimple adjoint affine algebraic group $G$ and for *any* Borel subgroup $B$ of $G$ (which exists because $G$ is quasi-split), *any* maximal torus $T$ of $B$ (is maximal in $G$ and) is isomorphic to $\mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$, where $L$ is an étale $F$-algebra with the $\Gamma$-set $\operatorname{Hom}_{F\text{-alg}}(L, F_{\mathrm{sep}})$ isomorphic to the set of vertices of the Dynkin diagram of $G$. Indeed, this set of vertices is the set $X \subset (T_{F_{\mathrm{sep}}})^*$ of simple roots given by $B$, and we get that $T \simeq \mathcal{R}_{L/F}\mathbb{G}_{\mathrm{m}}$, because $X$ is a permutation basis of $(T_{F_{\mathrm{sep}}})^*$ ($X$ generates $(T_{F_{\mathrm{sep}}})^*$ because $G$ is adjoint).

In particular, for $G$ as in Lemma 4.1, $T$ has the required isomorphism type because in this case the action of $\Gamma$ on $X$ factors through the Galois group of the quadratic separable field extension $K/F$ whose the non-trivial element exchanges the vertices which are symmetric with respect to the middle of the diagram. It follows that $L \simeq K^{\times r}$ for $n = 2r+1$ and $L \simeq F \times K^{\times(r-1)}$ for $n = 2r$ (note that $\#X = n - 1$).

For the rest of this section, we fix a torus $T \subset G$ and a Borel subgroup $B \subset G$ containing $T$ as in Lemma 4.1. We are going to describe the image of $\operatorname{CH}(E/B)$ in

$\mathrm{CH}(G/B)$, where $E$ is a *generic* (in the sense of [20, §3]) principle homogenous space of $G$ (the homomorphism $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$ is defined below).

We need some preliminary observations which are valid for an arbitrary parabolic subgroup $P \subset G$ in place of $B$. Note that by [5], the Chow motive (with integer coefficients) of the $F$-variety $G/P$ decomposes in a finite direct sum of shifts of the motive of the $F$-varieties $\operatorname{Spec} F$ and $\operatorname{Spec} K$. Consequently, $\mathrm{CH}(G/P)$ injects into $\mathrm{CH}(\bar{G}/\bar{P}) = \mathrm{CH}(G_K/P_K)$ (where $\bar{G}$ and $\bar{P}$ are $G$ and $P$ over an algebraic closure of $F$) and is identified with the subring of $\sigma$-invariant elements of $\mathrm{CH}(G_K/P_K)$, where $\sigma$ is the involution of the Chow ring $\mathrm{CH}(G_K/P_K)$ induced by the non-trivial automorphism of $K/F$. More precisely, we are going to get some information on the image of $\mathrm{CH}(E/P)$ in the quotient ring $\mathrm{CH}(G/P)/\operatorname{Im}\big(\mathrm{CH}(G_K/P_K) \to \mathrm{CH}(G/P)\big)$.

For any $G$-torsor $E$ over $F$, we construct a homomorphism

$$\mathrm{CH}(E/P) \to \mathrm{CH}(G/P)$$

as the composition of three ones: the change of field homomorphism

$$\mathrm{CH}(E/P) \to \mathrm{CH}(E_{F(E)}/P_{F(E)}),$$

the pull-back

$$\mathrm{CH}(E_{F(E)}/P_{F(E)}) \to \mathrm{CH}(G_{F(E)}/P_{F(E)})$$

with respect to the morphism $G_{F(E)}/P_{F(E)} \to E_{F(E)}/P_{F(E)}$ induced by the $G$-equivariant morphism $G_{F(E)} \to E_{F(E)}$ taking the identity of $G$ to the generic point of $E$, and the inverse

$$\mathrm{CH}(G_{F(E)}/P_{F(E)}) \to \mathrm{CH}(G/P)$$

of the change of field homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G_{F(E)}/P_{F(E)})$ (which is an isomorphism because $F$ is algebraically closed in $F(E)$).

More generally, the homomorphism $\mathrm{CH}(E/P) \to \mathrm{CH}(G/P)$ is defined for any $G$-torsor $E$ over a *regular* field extension $\tilde{F}/F$: since $K \otimes_F \tilde{F}$ is a field, the change of fields homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G_{\tilde{F}}/P_{\tilde{F}})$ is an isomorphism.

**Lemma 4.3.** *Let $S$ be a smooth irreducible $F$-variety and let $E$ be a $G$-torsor over $S$. For a point $s \in S(L)$ of $S$ in a regular field extension $L/F$, let $E_s$ be the $G$-torsor over $L$ given by the fiber of $E \to S$ over $s$. Then the image of $\mathrm{CH}(E_s/P) \to \mathrm{CH}(G/P)$ contains the image of $\mathrm{CH}(E_\theta/P) \to \mathrm{CH}(G/P)$, where $\theta \in S(F(S))$ is the generic point of $S$.*

*Proof.* Let $x \in S$ be the image of the point $\operatorname{Spec} L$ under the morphism $s : \operatorname{Spec} L \to S$. Since $x$ is regular, there exists a system of local parameters on $S$ around $x$. Therefore the fields $F(S)$ and $F(x)$ are connected by a finite chain of discrete valuation fields, where each next field is the residue field of the previous one. Using the specialization homomorphisms on Chow groups as in [9, Example 20.3.2], we get a homomorphism $\mathrm{CH}(E_\theta/P) \to \mathrm{CH}(E_x/P)$. Composing it with the change of fields homomorphism $\mathrm{CH}(E_x/P) \to \mathrm{CH}(E_s/P)$, we get a homomorphism $\mathrm{CH}(E_\theta/P) \to \mathrm{CH}(E_s/P)$ which forms a commutative triangle with the homomorphisms to $\mathrm{CH}(G/P)$. □

We are turning back to the case of $P = B$. The following statement is well-known in the case of split $G$. In our quasi-split case, the proof is almost the same:

**Lemma 4.4.** *For any $G$-torsor $E$, the pull-back* $\mathrm{CH}(E/B) \to \mathrm{CH}(E/T)$ *with respect to the projection $E/T \to E/B$ is an isomorphism. In particular, the image of* $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$ *is identified with the image of* $\mathrm{CH}(E/T) \to \mathrm{CH}(G/T)$.

*Proof.* Let $U$ be the unipotent part of $B$. By [1, Theorem 10.6 of Chapter III], $U$ is a normal subgroup of $G$ possessing a finite increasing chain of normal subgroups $U_i$ with each successive quotients isomorphic to $\mathbb{G}_\mathrm{a}$, and $B$ is a semidirect product of $U$ by $T$. Since $H^1(F,T)$ is trivial, it follows that $H^1(F, U_i \rtimes T)$ is trivial for any $i$ (in particular, $H^1(F,B)$ is trivial). Therefore the fiber over any point of the projection

$$E/(U_{i-1} \rtimes T) \to E/(U_i \rtimes T)$$

is isomorphic to the affine line $U_{i-1}/U_i$ so that the pull-back of Chow groups is an isomorphism by the homotopy invariance of Chow groups, [8, Theorem 57.13]. □

We take as $E$ a generic principle homogeneous space of $G$ as defined in [20, §3]. Thus $E$ is the generic fiber of certain $G$-torsor over certain smooth absolutely irreducible $F$-variety having the versal property of [25, Definition 5.1(2)] (we only need the weak version [25, Remark 5.8] of the versality).

In particular, $E$ is a $G$-torsor over a regular field extension of $F$, not over $F$ itself. For the sake of simplicity of notation, we let now $F$ be the field of definition of $E$ (and $K$ be the quadratic extension of the new $F$ given by the tensor product of the old $K$ and the new $F$ over the old $F$).

**Proposition 4.5.** *For $E$ as right above, the image of* $\mathrm{CH}(E/B)$ *in the quotient*

$$\mathrm{CH}(G/B)/\operatorname{Im}\big(\mathrm{CH}(G_K/B_K) \to \mathrm{CH}(G/B)\big)$$

*is generated (as a ring) by elements of codimension* 1 *and* 2.

*Proof.* By Lemma 4.4, the image of

$$\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)/\operatorname{Im}\big(\mathrm{CH}(G_K/B_K) \to \mathrm{CH}(G/B)\big)$$

is identified with the image of

$$\mathrm{CH}(E/T) \to \mathrm{CH}(G/T)/\operatorname{Im}\big(\mathrm{CH}(G_K/T_K) \to \mathrm{CH}(G/T)\big).$$

By [18, Theorem 6.4], the image of $\mathrm{CH}(E/T)$ in $\mathrm{CH}(G/T)$ is identified with the image of $\mathrm{CH}_T(\operatorname{Spec} F)$ in $\mathrm{CH}_T(G) = \mathrm{CH}(G/T)$. The image of

$$\mathrm{CH}_T(\operatorname{Spec} F) \to \mathrm{CH}(G/T)/\operatorname{Im}\big(\mathrm{CH}(G_K/T_K) \to \mathrm{CH}(G/T)\big)$$

coincides with the image of the quotient

$$\mathrm{CH}_T(\operatorname{Spec} F)/\operatorname{Im}\big(\mathrm{CH}_T(\operatorname{Spec} K) \to \mathrm{CH}_T(\operatorname{Spec} F)\big).$$

This quotient is generated by its elements of codimension 1 and 2 by Corollaries 3.4 and 3.5. □

Note that for any $G$-torsor $E$ over $F$, the kernel of the homomorphism $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$ is the torsion subgroup so that its image is identified with the Chow group $\mathrm{CH}(E/B)$ modulo torsion which we denote by $\overline{\mathrm{CH}}(E/B)$ (and call the *reduced* Chow group). Furthermore, if $E$ is given by a hermitian form $h$, then it splits over $K$; therefore the image of the homomorphism of $\mathrm{CH}(E/B)$ to $\mathrm{CH}(G/B)$ modulo the norms is identified with the ring $\overline{\mathrm{CH}}(E/B)/N$, where $N := \mathrm{Im}\big(\mathrm{CH}(E_K/B_K) \to \overline{\mathrm{CH}}(E/B)\big)$ is the norm ideal.

We are going to consider a *generic hermitian form h* of dimension $n$ (for a fixed separable quadratic field extension $K/F$) by which we mean the diagonal $K(t_1,\dots,t_n)/F(t_1,\dots,t_n)$-hermitian form $\langle t_1,\dots,t_n\rangle$, where $t_1,\dots,t_n$ are variables. (For a justification of this definition, note that any hermitian form can be diagonalized, [22, Theorem 6.3 of Chapter 7].) Changing notation, we write now $F$ for $F(t_1,\dots,t_n)$ and $K$ for $K(t_1,\dots,t_n)$.

**Corollary 4.6.** *Let $Y$ be the variety of complete flags of totally isotropic subspaces of the defined above generic hermitian form $h$. Then the image of*

$$\mathrm{CH}(Y) \to \mathrm{CH}(G/B)/\,\mathrm{Im}\big(\mathrm{CH}(G_K/B_K) \to \mathrm{CH}(G/B)\big)$$

*is generated (as a ring) by the elements of codimension* 1 *and* 2*. Equivalently, the ring $\overline{\mathrm{CH}}(X)/N$ is generated by codimension* 1 *and* 2*.*

*Proof.* By the specialization argument as in Lemma 4.3, for any regular field extension $F'/F$ and any $n$-dimensional $K'/F'$-hermitian form $h'$, where $K' := K \otimes_F F'$, the image of $\mathrm{CH}(Y) \to \mathrm{CH}(G/B)$ is contained in the image of $\mathrm{CH}(Y') \to \mathrm{CH}(G/B)$, where $Y'$ is the variety of complete flags of $h'$. Similarly, the image of $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$ is contained in the image of $\mathrm{CH}(Y') \to \mathrm{CH}(G/B)$ for the $G$-torsor $E$ constructed below. It follows that $\overline{\mathrm{CH}}(Y)/N$ coincides with $\overline{\mathrm{CH}}(E/B)/N$ so that it suffices to show that the latter ring is generated by elements of codimension $\le 2$.

To construct the $G$-torsor $E$, let us take a generic principle $G$-homogeneous space $E'$ of [20, §3] and climb over the function field of the Weil transfer of the Severi-Brauer variety of the corresponding central simple algebra. In more details, the torsor $E'$ (changing notation, we may assume that it is defined over $F$) corresponds to a degree $n$ central simple $K$-algebra $A$ endowed with an $F$-linear unitary involution $\tau$, [19, §29D]. Over the function field $F(X)$ of the Weil transfer $X$ with respect to $K/F$ of the Severi-Brauer $K$-variety of $A$, the algebra $A$ splits and the involution $\tau$ becomes adjoint with respect to some $K(X)/F(X)$-hermitian form. We set $E := E'_{F(X)}$.

Writing now $Y$ for the $F$-variety of complete flags of right $\tau$-isotropic ideals in $A$, our aim is to show that $\mathrm{CH}(Y_{F(X)})$ modulo torsion and norms is generated by codimensions $\le 2$. Since the variety $Y$ is isomorphic to $E'/B$, we know by Proposition 4.5 that $\mathrm{CH}(Y)$ is so. The projection $Y \times X \to Y$ is a Weil transfer of a projective bundle like the one considered in §5. The $\mathrm{CH}(Y)$-algebra $\mathrm{CH}(Y \times X)$ is generated by codimension 2 by Proposition 5.2. Since the pull-back $\mathrm{CH}(Y \times X) \to \mathrm{CH}(Y_{F(X)})$ is surjective, the $\mathrm{CH}(Y)$-algebra $\mathrm{CH}(Y_{F(X)})$ is generated by codimension 2. □

## 5. Weil transfer of projective bundles

Let $K/F$ be a separable quadratic field extension. Let $X$ be a smooth geometrically irreducible $F$-variety. Let $\mathcal{E}$ be a vector bundle over $X_K$. We are going to consider the Weil transfer $Y \to X$ of the projective bundle $P \to X_K$ of $\mathcal{E}$ with respect to the base change morphism $X_K \to X$. We refer to [2, §7.6] and [23, §4] for generalities on Weil transfers of schemes (also called *Weil restriction* and *corestriction* in the literature).

We will need the following statement, where $P$ can be any variety with a morphism to $X_K$:

**Lemma 5.1.** *The Weil transfer of $P$ with respect to $X_K \to X$ can be obtained as the Weil transfer with respect to $K/F$ (which produces an $\mathcal{R}_{K/F}(X_K)$-scheme) followed by the base change with respect to the "diagonal" morphism $X \to \mathcal{R}_{K/F}(X_K)$ corresponding to the identity under the identification $\mathrm{Mor}_F(X, \mathcal{R}_{K/F}(X_K)) = \mathrm{Mor}_K(X_K, X_K)$.*

*Proof.* We write $\mathcal{R}$ for $\mathcal{R}_{K/F}$. Let $Y$ be the fibred product $\mathcal{R}P \times_{\mathcal{R}X_K} X$. To show that $Y$ is the Weil transfer of $P$ with respect to $X_K \to X$, it suffices to check that for any $X$-scheme $S$, the set $\mathrm{Mor}_X(S, Y)$ is naturally identified with the set $\mathrm{Mor}_{X_K}(S_K, P)$ (note that $S_K := S \times_F \operatorname{Spec} K = S \times_X X_K$). By properties of the fibred product, we have a natural identification $\mathrm{Mor}_X(S, Y) = \mathrm{Mor}_{\mathcal{R}X_K}(S, \mathcal{R}P)$. The set on the right hand side of this equality is a subset of the set $\mathrm{Mor}_F(S, \mathcal{R}P)$ which is naturally identified with the set $\mathrm{Mor}_K(S_K, P)$. The subset $\mathrm{Mor}_{\mathcal{R}X_K}(S, \mathcal{R}P) \subset \mathrm{Mor}_F(S, \mathcal{R}P)$ corresponds under this identification to the subset $\mathrm{Mor}_{X_K}(S_K, P) \subset \mathrm{Mor}_K(S_K, P)$. Composing two identifications obtained above, we get the identification required. □

Note that applying the base change $X_K \to X$ to the morphism $Y \to X$, we get the product of two projective bundles over $X_K$: one associated to $\mathcal{E}$, the other to $\mathcal{E}'$, where $\mathcal{E}' \to X_K$ is the vector bundle obtained from $\mathcal{E} \to X_K$ by the base change $X_K \to X_K$ given by the non-trivial automorphism of $K/F$.

For any integer $i$, we have a map $\mathrm{CH}^i(P) \to \mathrm{CH}^{2i}(Y)$ (just a map, not a homomorphism) defined as the composition of the map $\mathrm{CH}^i(P) \to \mathrm{CH}^{2i}(\mathcal{R}_{K/F}P)$ of [16, §3] followed by the pull-back homomorphism $\mathrm{CH}^{2i}(\mathcal{R}_{K/F}P) \to \mathrm{CH}^{2i}(Y)$ (we have in mind the construction of $Y$ given in Lemma 5.1 here). The first Chern class of the tautological line vector bundle on $P$ gives us therefore an element $c \in \mathrm{CH}^2(Y)$.

**Proposition 5.2.** *The $\mathrm{CH}(X)$-algebra $\mathrm{CH}(Y)/\operatorname{Im}\big(\mathrm{CH}(Y_K) \to \mathrm{CH}(Y)\big)$ is generated by the class of the element $c$.*

*Proof.* We start by the case where $X = \operatorname{Spec} F$. In this case, the ring $\mathrm{CH}(Y)$ is identified with the subring $R^\sigma$ of the $\sigma$-invariant elements in the ring $R := \mathrm{CH}(\mathbb{P}^{r-1} \times \mathbb{P}^{r-1})$, where $r$ is the rank of the vector bundle $\mathcal{E}$ and $\sigma$ is the factor exchange involution on $R$. The image of the norm map is the image of the homomorphism $(1+\sigma) : R \to R^\sigma$. The ring $R^\sigma$ is generated by two elements: $h \times h$ and $1 \times h + h \times 1$. Therefore the quotient $R^\sigma/(1+\sigma)R$ is generated the class of $h \times h$ which is the class corresponding to the element $c$ of the statement.

In order to do the general case, we apply a variant of [26, Statement 2.13]. Let for a moment $X$ and $Y$ be arbitrary smooth $F$-varieties with irreducible $X$. For a morphism $\pi : Y \to X$, consider the (finite) filtration

$$\mathrm{CH}(Y) = \mathcal{F}^0\,\mathrm{CH}(Y) \supset \mathcal{F}^1\,\mathrm{CH}(Y) \supset \dots$$

on $\mathrm{CH}(Y)$ with $\mathcal{F}^i\,\mathrm{CH}(Y)$ being the subgroup generated by the classes of cycles on $Y$ whose image in $X$ has codimension $\geq i$. For any point $x \in X$, let $Y_x$ be the fiber of $\pi$ over $x$. For any $i$, let $X^{(i)}$ be the set of points of $X$ of codimension $i$. There is a surjection

$$\bigoplus_{x \in X^{(i)}} \mathrm{CH}(Y_x) \to \mathcal{F}^i\,\mathrm{CH}(Y)/\mathcal{F}^{i+1}\,\mathrm{CH}(Y), \tag{5.3}$$

mapping an element $\alpha \in \mathrm{CH}(Y_x)$ to the class modulo $\mathcal{F}^{i+1}\,\mathrm{CH}(Y)$ of the image under the push-forward $\mathrm{CH}(Y_T) \to \mathcal{F}^i\,\mathrm{CH}(Y)$ of an arbitrary preimage of $\alpha$ under the pull-back epimorphism $\mathrm{CH}(Y_T) \twoheadrightarrow \mathrm{CH}(Y_x)$, where $T \subset X$ is the closure of $x$ and $Y_T := Y \times_X T \hookrightarrow Y$ is the preimage of $T$ under $Y \to X$.

**Lemma 5.4** (cf. [26, Statement 2.13])**.** *Let $\pi : Y \to X$ be as above and let $\zeta$ be the generic point of $X$. Let $B \subset \mathrm{CH}(Y)$ be a $\mathrm{CH}(X)$-submodule such that*

(a) *the composition $B \hookrightarrow \mathrm{CH}(Y) \to \mathrm{CH}(Y_\zeta)$ is surjective and*
(b) *for any $x \in X$ either the specialization homomorphism $s_x : \mathrm{CH}(Y_\zeta) \to \mathrm{CH}(Y_x)$ is surjective or the image of $\mathrm{CH}(Y_x) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$ is contained in the image of $B \cap \mathcal{F}^i \mathrm{CH}(Y) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$.*

*Then $B = \mathrm{CH}(Y)$.*

*Proof.* We are repeating the proof of [26, Statement 2.13] making necessary modifications. If the specialization homomorphism $s_x$ is surjective for a point $x \in X^{(i)}$, then the image of $\mathrm{CH}(Y_x)$ under (5.3) is contained in the image of $[T] \cdot B \subset \mathcal{F}^i \mathrm{CH}(Y) \cap B$ in the quotient $\mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$. Otherwise, we know already by the hypothesis that it is contained in the image of $B \cap \mathcal{F}^i \mathrm{CH}(Y)$. □

We are turning back to the proof of Proposition 5.2 and our particular $Y \to X$. We apply Lemma 5.4, taking for $B$ the $\mathrm{CH}(X)$-submodule of $\mathrm{CH}(Y)$ generated by the powers of $c$ and the image of $\mathrm{CH}(Y_K) \to \mathrm{CH}(Y)$. For any $x \in X$, the fiber $Y_x$ is isomorphic to the Weil transfer of $\mathbb{P}^{r-1}$ with respect to the quadratic étale $F(x)$-algebra $F(x) \otimes_F K$. If $\zeta$ is the generic point, $F(\zeta) \otimes_F K$ is a field. Therefore condition (a) of Lemma 5.4, requiring that the homomorphism $B \to \mathrm{CH}(Y_\zeta)$ is surjective, is satisfied.

Condition (b) of Lemma 5.4 is satisfied as well. Indeed, the specialization homomorphism $\mathrm{CH}(Y_\zeta) \to \mathrm{CH}(Y_x)$, $x \in X$, is surjective if the residue field of the point $x$ does not contain a subfield isomorphic to $K$. We finish the proof by showing that in the opposite case the image of $\mathrm{CH}(Y_x) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$ (for $i$ such that $x \in X^{(i)}$) is in the image of $\mathcal{F}^i \mathrm{CH}(Y_K) \to \mathcal{F}^i \mathrm{CH}(Y)\mathcal{F}^{i+1} \mathrm{CH}(Y)$.

Let $T$ be the closure of $x$ in $X$. Let $Y_T = Y \times_X T \hookrightarrow Y$ be the preimage of $T$ under $Y \to X$. The image of the homomorphism $\mathrm{CH}(Y_x) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$ coincides with the image of the homomorphism $\mathrm{CH}(Y_T) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$ induced by the push-forward. Since $x$ is the generic point of $T$ and $F(x) = F(T) \supset K$, a non-empty open subset $U \subset T$ possesses a morphism to $\operatorname{Spec} K$. Its preimage $Y_U \subset Y_T$ is open and also possesses a morphism to $\operatorname{Spec} K$. Therefore $(Y_U)_K \simeq Y_U \coprod Y_U$ (as $F$-varieties) and, in particular, the push-forward $\mathrm{CH}(Y_U)_K \to \mathrm{CH}(Y_U)$ is surjective.

The varieties and morphisms in play fit in the following commutative diagram:

$$\begin{array}{ccccc}
Y_K & \longrightarrow & Y & \longrightarrow & X \\
\uparrow & & \uparrow & & \uparrow \\
(Y_T)_K & \longrightarrow & Y_T & \longrightarrow & T \\
\uparrow & & \uparrow & & \uparrow \\
(Y_U)_K & \longrightarrow & Y_U & \longrightarrow & U
\end{array}$$

It follows that the image of the push-forward $\mathrm{CH}(Y_T)_K \to \mathrm{CH}(Y_T)$ generates $\mathrm{CH}(Y_T)$ modulo the image of $\mathrm{CH}(Y_T \setminus Y_U)$. Since the image of $\mathrm{CH}(Y_T \setminus Y_U) \to \mathrm{CH}(Y)$ is in $\mathcal{F}^{i+1} \mathrm{CH}(Y)$, it follows that the image of $\mathrm{CH}(Y_T) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1} \mathrm{CH}(Y)$ is contained in the image of $\mathcal{F}^i \mathrm{CH}(Y_K) \to \mathcal{F}^i \mathrm{CH}(Y)/\mathcal{F}^{i+1}/\mathrm{CH}(Y)$. □

We write $\overline{\mathrm{CH}}(Y)/N$ for the reduced Chow ring $\overline{\mathrm{CH}}(Y)$ modulo the norm ideal $N := \mathrm{Im}\big(\overline{\mathrm{CH}}(Y_K) \to \overline{\mathrm{CH}}(Y)\big)$. In particular, $\overline{\mathrm{CH}}(X)/N$ is defined as $\overline{\mathrm{CH}}(X)/\mathrm{Im}\big(\overline{\mathrm{CH}}(X_K) \to \overline{\mathrm{CH}}(X)\big)$.

As a particular case of the map $\mathrm{CH}^i(P) \to \mathrm{CH}^{2i}(Y)$ considered right before Proposition 5.2, we have a map $\mathrm{CH}^i(X_K) \to \mathrm{CH}^{2i}(X)$ and we write $c_i \in \overline{\mathrm{CH}}^{2i}(X)/N$ for the class of the image of the Chern class $c_i(\mathcal{E}) \in \mathrm{CH}^i(X_K)$. The element $c_i$ can be also defined as the class of the Chern class $c_{2i}(\mathcal{E} \to X)$, where $\mathcal{E} \to X$ is the vector bundle obtained by composition of the vector bundle $\mathcal{E} \to X_K$ and the trivial vector bundle $X_K \to X$ (although this $c_i$ is different from the previous one, their classes in $\overline{\mathrm{CH}}(X)/N$ coincide).

**Corollary 5.5.** *The $\overline{\mathrm{CH}}(X)/N$-algebra $\overline{\mathrm{CH}}(Y)/N$ is generated by the class of $c$ subject to the only one relation $\sum_{i=0}^{r} c_i c^{r-i} = 0$, where $r$ is the rank of the vector bundle $\mathcal{E}$.*

*Proof.* The relation between the powers of $c$ holds in $\overline{\mathrm{CH}}(Y)/N$ because it holds in $\overline{\mathrm{CH}}(Y_K)/N \supset \overline{\mathrm{CH}}(Y)/N$. Here, abusing notation, we write $\mathrm{CH}(Y_K)/N$ for the quotient of $\overline{\mathrm{CH}}(Y_K)$ by its subgroup $N := \mathrm{Im}\big(\overline{\mathrm{CH}}(Y_K) \to \overline{\mathrm{CH}}(Y)\big)$. This subgroup is actually a $\overline{\mathrm{CH}}(X)/N$-submodule so that $\overline{\mathrm{CH}}(Y_K)/N$ is a $\overline{\mathrm{CH}}(X)/N$-module (but not a $\overline{\mathrm{CH}}(X)/N$-algebra).

In particular, the $\overline{\mathrm{CH}}(X)/N$-module $\overline{\mathrm{CH}}(Y)/N$ is generated by $1, c, \dots, c^{r-1}$. The module generators are free, because they are free in $\overline{\mathrm{CH}}(Y_K)/N$. It follows that the relation is the only one. □

## 6. Generic maximal unitary grassmannians

Let now $h$ be a *generic* $K/F$-hermitian form of dimension $n \geq 2$ (defined right before Corollary 4.6). Let $r := \lfloor n/2 \rfloor$. We consider the variety $X$ of $r$-dimensional totally isotropic subspaces in $h$. We keep the notation $G$ of Section 4 for the quasi-split algebraic group and let $P$ be a parabolic subgroup in $G$ representing the conjugacy class corresponding to $X$.

The aim of this section is to describe the image of the homomorphism

$$\mathrm{CH}(X) \to \mathrm{CH}(G/P)/\mathrm{Im}\big(\mathrm{CH}(G_K/P_K) \to \mathrm{CH}(G/P)\big)$$

(constructed (in a more general situation) right before Lemma 4.3). We recall that the $G$-torsor given by $h$ splits over $K$; therefore the image of the above homomorphism is identified with the ring $\overline{\mathrm{CH}}(X)/N$, where $N := \mathrm{Im}\big(\overline{\mathrm{CH}}(X_K) \to \overline{\mathrm{CH}}(X)\big)$. (So, $\overline{\mathrm{CH}}(X)/N$ is the ring $\mathrm{CH}(X)$ modulo the sum of the ideal of the torsion elements and the ideal of the norms.)

**Proposition 6.1.** *The components of positive codimension of the ring $\overline{\mathrm{CH}}(X)/N$ are trivial.*

*Proof.* Let $I = [1,\ r] = \{1, 2, \dots, r\}$. For every $J \subset I$ we consider the variety $X_J$ of flags of totally isotropic subspaces in $h$ of dimensions given by $J$.

By induction on $l \in I$, we prove the following statement: the ring $\overline{\mathrm{CH}}(X_{[l,\,r]})/N$ (where $N$ here is the image of norm homomorphism for $X_{[l,\,r]}$) is generated by its elements of codimensions 1, its elements of codimension 2, and the Chern classes of the tautological rank $2l$ vector bundle on $X_{[l,\,r]}$. Note that this statement for $l = r$ gives the statement of

Proposition 6.1 because $X_{\{r\}} = X$ and we have the following triviality statements for the generators: according to [11, Proposition 3.9] the elements of codimension 2 as well as the Chern classes of positive codimension of the tautological vector bundle on $G/P$ are trivial in $\mathrm{CH}(G/P)/N$; for odd $n$, again by [11, Proposition 3.9], the elements of codimension 1 are also trivial in $\mathrm{CH}(G/P)/N$; finally, if $n$ is even, the elements of codimension 1 in $\mathrm{CH}(X)$ become trivial in $\mathrm{CH}(G/P)/N$ by Lemma 6.2 below (because the discriminant of our hermitian form $h$ is non-trivial).

The induction base $l = 1$ is given by Corollary 4.6. Now, assuming that $l \geq 2$, let us do the passage from $l-1$ to $l$.

The projection $X_{[l-1,\, r]} \to X_{[l,\, r]}$ is the Weil transfer of a projective bundle considered in section 5 (namely, of the projective bundle over $(X_{[l,\, r]})_K$ given by the dual of (any)one of two rank $l$ tautological vector bundles on $(X_{[l,\, r]})_K$). Therefore, by Corollary 5.5, the $\overline{\mathrm{CH}}(X_{[l,\, r]})/N$-algebra $\overline{\mathrm{CH}}(X_{[l-1,\, r]})/N$ is generated by certain codimension 2 element $c$ subject to one relation $\sum_{i=0}^{l} c_i c^{l-i} = 0$, where the coefficients $c_i$ are the even Chern classes of the rank $2l$ tautological vector bundle on $X_{[l,\, r]}$. In particular, the $\overline{\mathrm{CH}}(X_{[l,\, r]})/N$-module $\overline{\mathrm{CH}}(X_{[l-1,\, r]})/N$ is free of rank $l$.

Now let $C \subset \overline{\mathrm{CH}}(X_{[l,\, r]})/N$ be the subring generated by all $c_i$ together with the elements of codimensions $1, 2$. The coefficients of the above relation are then in $C$. Therefore the subring of $\overline{\mathrm{CH}}(X_{[l-1,\, r]})/N$ generated by $C$ and $c$ is also free (now as a $C$-module) of rank $l$. On the other hand, this subring coincides with the total ring by the induction hypothesis. Indeed, it contains all the elements of codimension $1, 2$ in $\overline{\mathrm{CH}}(X_{[l-1,\, r]})/N$ because any such element is a polynomial in $c$ with coefficients of codimension $\leq 2$ in $\overline{\mathrm{CH}}(X_{[l,\, r]})/N$. It also contains the Chern classes of the tautological rank $2(l-1)$ vector bundle on $X_{[l-1,\, r]}$ because they are expressible in terms of $c_i$ and $c$.

It follows that $C = \overline{\mathrm{CH}}(X_{[l,\, r]})/N$. □

We terminate this section by a study of the case of $n = 2r$. We do not assume anymore that our $n$-dimensional hermitian form $h$ is generic, it is arbitrary. We still write $X$ for the variety of $r$-dimensional totally isotropic subspaces in $h$. The variety $X_K$ is identified with the $K$-grassmannian of $r$-planes in $V$ (the vector space of definition of $h$). Let $\mathcal{E}$ be the tautological bundle on the grassmannian. For any $i \geq 0$, we set $c_i := c_i(\mathcal{E}) \in \mathrm{CH}^i(X_K)$. The ring $\mathrm{CH}(X_K)$ is generated by the elements $c_i$. In particular, $\mathrm{CH}^1(X_K)$ is (an infinite cyclic group) generated by $c_1$. We recall that $\overline{\mathrm{CH}}(X)$ stands for the reduced Chow group and coincides with the image of $\mathrm{CH}(X) \to \mathrm{CH}(X_K)$.

The *discriminant* $\operatorname{disc} h$ of the hermitian form $h$ is the class in $F^\times/N(K^\times)$ of the signed determinant

$$(-1)^{n(n-1)/2} \cdot \det(h(e_i, e_j))_{i,j \in [1,\, n]}$$

of the Gram matrix of $h$ in a basis $e_1, \dots, e_n$ of $V$.

**Lemma 6.2.** *The group* $\overline{\mathrm{CH}}^1(X)$ *is generated by*

$c_1$, *if* $\operatorname{disc} h$ *is trivial;*
$2c_1$, *if* $\operatorname{disc} h$ *is non-trivial.*

*Proof.* The cokernel of $\mathrm{CH}^1(X) \to \mathrm{CH}^1(X_K)$ is isomorphic to the kernel of $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$. This kernel is generated by the class of the *discriminant algebra* of $h$, [21].

The discriminant algebra of $h$ is the quaternion algebra given by the quadratic extension $K/F$, the non-trivial automorphism of $K/F$ and $\operatorname{disc} h$. Its Brauer class is always killed by 2 and is non-trivial if and only if $\operatorname{disc} h$ is non-trivial. □

## 7. Essential motives of unitary grassmannians

We fix the following notation. Let $K/F$ be a separable quadratic field extension. Let $n$ be an integer $\geq 0$. Let $V$ be a vector space over $K$ of dimension $n$. Let $h$ be a $K/F$-hermitian form on $V$. For any integer $r$, let $X_r$ be the $F$-variety of $r$-dimensional totally isotropic subspaces in $V$ ($X_r$ is a closed subvariety of the Weil transfer with respect to $K/F$ of the $K$-grassmannian of $r$-planes in $V$; $X_0 = \operatorname{Spec} F$; $X_r = \emptyset$ for $r$ outside of the interval $[0,\ n/2]$).

We are working with the Grothendieck Chow motives with coefficients in $\mathbb{F}_2$, [8, Chapter XII]. We write $M(X)$ for the motive of a smooth projective $F$-variety $X$. We are constantly (and without any further reference) using the Krull-Schmidt principle for the motives of quasi-homogeneous varieties, [17, Corollary 2.2].

**Lemma 7.1** ([15, Theorem 15.8]). *Assume that the hermitian form $h$ is isotropic: $n$ is $\geq 2$ and $h \simeq \mathbb{H} \perp h'$, where $\mathbb{H}$ is the hyperbolic plane, $h'$ a hermitian form of dimension $n-2$. For any integer $r$ one has*

$$M(X_r) \simeq M(X'_{r-1}) \oplus M(X'_r)(i) \oplus M(X'_{r-1})(j) \oplus M,$$

*where $X'_{r-1}$ and $X'_r$ are the varieties of $h'$, $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$, and $M$ is a sum of shifts of the $F$-motive of* $\operatorname{Spec} K$.

The following Corollary is also a consequence of a general result of [17] or of [5]:

**Corollary 7.2.** *If $h$ is* split *(meaning* hyperbolic *for even $n$ or* "almost hyperbolic" *for odd $n$), then $M(X_r)$ is a sum of shifts of the motives of* $\operatorname{Spec} F$ *and of* $\operatorname{Spec} K$. □

**Corollary 7.3.** *There is a decomposition $M(X_r) \simeq M_r \oplus M$ such that the motive $M$ is a sum of shifts of $M(\operatorname{Spec} K)$ and for any field extension $L/F$ with split $h_L$ the motive $M_r$ is* split *(meaning* is a sum of Tate motives*).*

*Proof.* Apply [13, Proposition 4.1] inductively to $E := F(X_{\lfloor n/2 \rfloor})$ and $S := \operatorname{Spec} K$. Note that the variety $S_E$ is still irreducible and has indecomposable motive because $K \otimes_F E$ is a field. The hermitian form $h_E$ is split. Since $X_{\lfloor n/2 \rfloor}(K) \neq \emptyset$, the sum $M$ of all copies of shifts of $M(\operatorname{Spec} K)$ present in the complete decomposition of $M(X_r)$ over $E$, can be extracted from $M(X_r)$ over $F$. The remaining part $M_r$ of the motive of $X_r$ has the desired property. □

**Remark 7.4.** The reduced Chow group (homological or cohomological one) of the motive $M$ (as a subgroup of $\overline{\operatorname{Ch}}(X_r)$) is the image $N$ of the norm map $\overline{\operatorname{Ch}}((X_r)_K) \to \overline{\operatorname{Ch}}(X_r)$ (it is evidently contained in $N$ and coincides in fact with $N$ because $N$ intersects the reduced Chow group of $M_r$ trivially). Therefore the reduced Chow group of $M_r$ is identified with the quotient $\overline{\operatorname{Ch}}(X_r)/N$. Here $\operatorname{Ch}(X_r)$ stands for the Chow group with coefficients in $\mathbb{F}_2$ of the variety $X_r$ and the reduced Chow group $\overline{\operatorname{Ch}}(X_r)$ is $\overline{\operatorname{CH}}(X_r)$ modulo 2 or, equivalently, the image of $\operatorname{Ch}(X_r) \to \operatorname{Ch}(\bar{X}_r)$. In the quasi-split case (the case where $h$ is hyperbolic or almost hyperbolic), the reduced Chow group in the above statements can be replaced by the usual Chow group (still modulo 2). We refer to [8, §64] for the definition of homological and cohomological Chow group of a motive. The coincidence of descriptions of homological and cohomological Chow groups for the motives $M$ and $M_r$ is explained by their symmetry: $M \simeq M^*(\dim X_r)$ (and the same for $M_r$), where $M^*$ is the *dual* motive, [8, §65].

**Definition 7.5.** The motive $M_r$ (defined by $X_r$ uniquely up to an isomorphism) will be called the *essential motive* of $X_r$ (or the *essential part* of the motive of $X_r$).

It follows that the decomposition of the essential motive in the isotropic case has precisely the same shape as the decomposition of the motive of an isotropic orthogonal grassmannian [14]:

**Corollary 7.6.** *Under the hypotheses of Lemma 7.1, one has*

$$M_r \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j),$$

*where* $M'_{r-1}$ *and* $M'_r$ *are the essential motives of* $X'_{r-1}$ *and* $X'_r$, $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$. □

According to the general result of [17], any summand of the complete motivic decomposition of the variety $X_r$ is a shift of the upper motive $U(X_s)$ for some $s \geq r$ or a shift of the motive of the $F$-variety $\operatorname{Spec} K$. Therefore we get

**Corollary 7.7.** *Any summand of the complete decomposition of the essential motive* $M_r$ *is a shift of the upper motive* $U(X_s)$ *for some* $s \geq r$. □

## 8. Generic unitary grassmannians

In the statement below we use the notion of the *essential motive* $M_r$ of the variety $X_r$, introduced in the previous section. It turns out that in the generic case, this motive is indecomposable:

**Theorem 8.1.** *Let $h$ be a* generic *$K/F$-hermitian form of an arbitrary dimension $n \geq 0$. For $r = 0, 1, \ldots, \lfloor n/2 \rfloor$, the essential motive $M_r$ of the variety $X_r$ is indecomposable, the variety $X_r$ is 2-incompressible.*

*Proof.* We induct on $n$ in the proof of the first statement. The induction base is the trivial case of $n < 2$. Now we assume that $n \geq 2$.

We do a descending induction on $r$. The case of the maximal $r = \lfloor n/2 \lfloor$ is an immediate consequence of Proposition 6.1. Indeed, one summand of the complete decomposition of the motive $M_r$ for such $r$ is the upper motive $U(X_r)$ of $X_r$. The remaining summands (if any) are positive shifts $U(X_r)(i)$ ($i > 0$) of the upper motive, see Corollary 7.7. But if we have a summand $U(X_r)(i)$, then the reduced Chow group $\overline{\operatorname{Ch}}^i(M_r)$ is non-zero. However by Remark 7.4, $\overline{\operatorname{Ch}}(M_r)$ is isomorphic to $\overline{\operatorname{CH}}(X_r)/N$ which is 0 in positive codimensions by Proposition 6.1.

Now we assume that $r < \lfloor n/2 \rfloor$. Since the case of $r = 0$ is trivial, we may assume that $r \geq 1$ and $n \geq 4$.

Let $L := F(X_1)$. We have $h_L \simeq \mathbb{H} \perp h'$, where $h'$ is a $K(X_1)/F(X_1)$-hermitian form of dimension $n - 2$ and $\mathbb{H}$ is the hyperbolic plane. Note that the hermitian form $h'$ is still generic.

For any integer $s$, we write $X'_s$ for the variety $X_s$ of the hermitian form $h'$, and we write $M'_s$ for the essential motive of the variety $X'_s$. By Corollary 7.6, the motive $(M_r)_L$ decomposes in a sum of three summands:

$$(M_r)_L \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j),$$

where $i := (\dim X_r - \dim X'_r)/2$ and $j := \dim X_r - \dim X'_{r-1}$. By the induction hypothesis, each of these three summands is indecomposable. It follows (taking into account the duality like in [14, Remark 1.3]) that if the motive $M_r$ is decomposable (over $F$), then it has a summand $M$ with $M_L \simeq M'_r(i) = U(X'_r)(i)$. Note that $U(X'_r) \simeq U((X_{r+1})_L)$. By an analogue of [14, Lemma 1.2], $M \simeq U(X_{r+1})(i)$, showing that $U(X_{r+1})_L \simeq M'_r$. By the induction hypothesis, the motive $M_{r+1}$ is indecomposable, that is, $U(X_{r+1}) = M_{r+1}$. Therefore we have an isomorphism $(M_{r+1})_L \simeq M'_r$ and, in particular, $\dim X_{r+1} = \dim X'_r$. However $\dim X_{r+1} = (r+1)\big(2n-3(r+1)\big)$, $\dim X'_r = r\big(2(n-2)-3r\big)$, and the difference is $2n-2r-3 \geq n-3 > 0$ (recall that $n \geq 4$ now).

To show that $X_r$ is 2-incompressible, we show that its canonical 2-dimension $\operatorname{cdim}_2 X_r$ equals $\dim X_r$. By [12, Theorem 5.1], $\operatorname{cdim}_2 X_r = \dim U(X_r)$. By the first part of Theorem 8.1, $U(X_r) = M_r$. Finally, $\dim M_r = \dim X_r$ for an arbitrary (not only for a generic) $h$. □

**Remark 8.2.** For arbitrary $h$, the motive $M_{\lfloor n/2\rfloor}$ is indecomposable if and only if the variety $X_{\lfloor n/2\rfloor}$ is 2-incompressible. If this is the case, then, as actually showed in the proof of Theorem 8.1, the motives $M_r$ are indecomposable and the varieties $X_r$ are 2-incompressible for all $r = 0, 1, \dots, \lfloor n/2\rfloor$.

## 9. Connection with quadratic forms

Let $K/F$ be a separable quadratic field extension extension, $V$ a finite-dimensional vector space over $K$, $h$ a non-degenerate $K/F$-hermitian form on $V$. For any $v \in V$ the value $h(v,v)$ is in $F$ and the map $q : V \to F$, $v \mapsto h(v,v)$ is a non-degenerate quadratic form on $V$ considered as a vector over $F$. Note that the dimension of $q$ is even. The Witt indices of $h$ and $q$ are related as follows:

**Lemma 9.1.** *$i(q) = 2i(h)$.*

*Proof.* For any integer $r \geq 0$, the inequality $i(h) \geq r$ implies $i(q) \geq 2r$. Indeed, if $i(h) \geq r$, $V$ contains a totally $h$-isotropic $L$-subspace $W$ of dimension $r$. This $W$ is also totally $q$-isotropic and has dimension $2r$ over $F$. Therefore $i(q) \geq 2r$.

To finish we prove by induction on $r \geq 0$ that $i(q) \geq 2r-1$ implies $i(h) \geq r$. This is trivial for $r = 0$. If $r > 0$ and $i(q) \geq 2r-1$, then $q$ is isotropic. But any $q$-isotropic vector is also $h$ isotropic, therefore the $K$-vector space $V$ decomposes in a direct sum of $h$-orthogonal subspaces $V = U \oplus V'$ such that $h|_U$ is a hyperbolic plane. The subspaces $U$ and $V'$ are also $q$-orthogonal and $q|_U$ is hyperbolic (of dimension 4). For $h' := h|_{V'}$ and $q' := q|_{V'}$ it follows that $i(h') = i(h) - 1$ and $i(q') = i(q) - 2$, and we are done by the induction hypothesis applied to $h'$ (of course, $q'$ is the quadratic form given by $h'$). □

**Corollary 9.2** (N. Jacobson, 1940)**.** *If the quadratic forms $q_1$ and $q_2$ corresponding to $K/F$-hermitian forms $h_1$ and $h_2$ are isomorphic, then $h_1$ and $h_2$ are also isomorphic.*

*Proof.* The orthogonal sum $q_1\perp -q_2$ is the quadratic form corresponding to $h_1\perp -h_2$. If $q_1$ and $q_2$ are isomorphic, $q_1\perp -q_2$ is hyperbolic, therefore $h_1\perp -h_2$ is also hyperbolic implying that $h_1$ and $h_2$ are isomorphic. □

For any integer $r$, let $X_r$ be the variety of totally $h$-isotropic $r$-dimensional $K$-subspaces in $V$ and let $Y_r$ be the variety of totally $q$-isotropic $r$-dimensional $F$-subspaces in $V$. The variety $Y_n$, where $n := \dim V$, is not connected and has two isomorphic connected components; changing notation, we let $Y_n$ be one of its connected component in this case.

**Corollary 9.3.** *For any $r$, the upper motives of the varieties $X_r$, $Y_{2r}$, and $Y_{2r-1}$ are isomorphic. In particular, these varieties have the same canonical dimension.* □

Theorem 8.1 with Corollary 9.3 makes it possible to compute for any $n$ the greatest (in the sense of inclusion) value of the *$J$-invariant* (see [8, §88]) of a non-degenerate quadratic form over a field extension of a fixed base field $F$ given by tensor product an $n$-dimensional bilinear form by a fixed binary quadratic form (the fixed binary quadratic form should be defined over $F$, the bilinear form needs not to be defined over $F$). Only the case of even $n$ is of interest, because the $J$-invariant is $\{0\}$ for odd $n$.

**Corollary 9.4.** *For any even $n \geq 2$, the greatest value of $J$-invariant discussed right above is $\{1, 3, 5, \dots, n-1\}$ (the set of odd integers from 1 till $n-1$).*

*Proof.* By [8, Proposition 88.8], the $J$-invariant of any non-degenerate quadratic form given by tensor product of an $n$-dimensional bilinear form by a binary quadratic form, is a subset of the given set. Let $K/F$ be the separable quadratic field extension given by the discriminant of the fixed binary quadratic form. We consider the generic $n$-dimensional $K/F$-hermitian form $h$ and calculate the $J$-invariant of the associated quadratic form $q$. We are using the above notation for the varieties associated to $h$ and to $q$.

By [8, Theorem 90.3], the canonical dimension of $Y_n$ is the sum of the elements of the $J$-invariant. On the other hand, by Corollary 9.3 and Theorem 8.1, the canonical dimension of $Y_n$ is equal to the dimension of $X_{n/2}$ which is

$$\dim X_{n/2} = n^2/4 = 1 + 3 + 5 + \cdots + (n-1).$$

Therefore the $J$-invariant is equal to the given set. □

Turning back to an arbitrary (not necessarily generic) hermitian form, we have

**Proposition 9.5.** *For any $r > 0$, the motive $M_r \oplus M_r(\dim Y_{2r-1} - \dim X_r)$ is isomorphic to a summand of the motive of $Y_{2r-1}$. If $n > 2$, then the motive $M_r \oplus M_r(\dim Y_{2r} - \dim X_r)$ is isomorphic to a summand of the motive of $Y_{2r}$. (For $n = 2$, the motive $M_r$ is isomorphic to a summand of the motive of $Y_{2r}$.)*

*Proof.* By specialization argument (together with nilpotence principle for projective homogeneous varieties, see [5, Theorem 8.2 and Corollary 8.4]), it suffices to show that the statement holds in the *generic* case. In this case, by Theorem 8.1, the motive $M_r$ is equal to the upper motive $U(X_r)$ which, by Corollary 9.3, is isomorphic to $U(Y_{2r-1})$ as well as to $U(Y_{2r})$, indecomposable summands of $M(Y_{2r-1})$ and of $M(Y_{2r})$. So, $M_r$ is an indecomposable summand of $M(Y_{2r-1})$ and $M(Y_{2r})$. By duality [8, §65], $M_r(\dim Y_{2r-1} - \dim X_r)$

is also an indecomposable summand of $M(Y_{2r-1})$ and $M_r(\dim Y_{2r} - \dim X_r)$ is also an indecomposable summand of $M(Y_{2r})$. It remains to notice that

$$\dim Y_{2r-1} - \dim X_r = (2r-1)(2n-3r+1) - r(2n-3r) > 0$$

for any $n$ and any $r \le n/2$ and that

$$\dim Y_{2r} - \dim X_r = r(4n-6r-1) - r(2n-3r) > 0$$

for any $n > 2$ and any $r \le n/2$. □

The following statement has been proved in [24]:

**Corollary 9.6.** $M(Y_1) = M_1 \oplus M_1(1)$.

*Proof.* Since $\dim Y_1 - \dim X_1 = 1$, we know by Proposition 9.5 that $M_1 \oplus M_1(1)$ is a summand of $M(Y_1)$. Comparing the *ranks* (the number of summands in the complete decomposition over an algebraic closure) of the motives , we see that $M(Y_1) = M_1 \oplus M_1(1)$. □

**Remark 9.7.** By Theorem 8.1, the decomposition of Corollary 9.6 is complete in the generic case (that is to say, the summands of the decomposition are indecomposable).

Let us consider the natural closed imbedding $in : X_r \hookrightarrow Y_{2r}$ which we have because a totally $h$-isotropic $K$-subspace is also a $q$-isotropic $F$-subspace. We will assume that the dimension $n$ of the hermitian form $h$ is even. In this case the image $N$ of the norm homomorphism $\overline{\mathrm{Ch}}(Y_{2r})_K \to \overline{\mathrm{Ch}}(Y_{2r})$ is 0 (because the discriminant of $q$ is trivial and therefore $Y_{2r}$ is a projective homogeneous variety of *inner* type), so that $\overline{\mathrm{Ch}}(Y_{2r})/N = \overline{\mathrm{Ch}}(Y_{2r})$.

The following observation is due to Maksim Zhykhovich:

**Lemma 9.8.** *The pull-back* $in^* : \overline{\mathrm{Ch}}(Y_{2r}) \to \overline{\mathrm{Ch}}(X_r)/N$ *is surjective, the push-forward* $in_* : \overline{\mathrm{Ch}}(X_r)/N \to \overline{\mathrm{Ch}}(Y_{2r})$ *is injective.*

**Remark 9.9.** In the quasi-split case, another proof of surjectivity of $in^*$ is given in [11, Lemma 4.1].

*Proof of Lemma 9.8.* We are working with motives in this proof. One may use the Chow motives with coefficients in $\mathbb{F}_2$, but since we are interested in the reduced Chow groups in the end, it is more appropriate to use the $\overline{\mathrm{Ch}}$-motives: they are obtained by replacing Ch by $\overline{\mathrm{Ch}}$ in the construction of the Chow motives with coefficients in $\mathbb{F}_2$.

Let us also write $in$ for the graph of the imbedding $in$. Let $\pi$ be a projector on $X_r$ giving the motive $M_r$. There exists a correspondence $\alpha$ such that $\pi \circ in^t \circ \alpha = \pi$. Indeed, such a correspondence exists in the generic case and can be obtained in the general case by specialization. Since $\overline{\mathrm{Ch}}(M_r)$ is identified with $\overline{\mathrm{Ch}}(X_r)/N$, we have $\pi_*(x) \equiv x \pmod N$ for any $x \in \overline{\mathrm{Ch}}(X_r)$. It follows that $x \equiv \pi_*(x) = \pi_*\, in^*\, \alpha_*(x) \equiv in^*\, \alpha_*(x)$ showing that $in^* : \overline{\mathrm{Ch}}(Y_{2r}) \to \overline{\mathrm{Ch}}(X_r)/N$ is surjective.

Similarly, there exists a correspondence $\beta$ such that $\beta \circ in \circ \pi = \pi$. It follows that $x \equiv \beta_*\, in_*(x)$ for any $x \in \overline{\mathrm{Ch}}(X_r)$ showing that $in_* : \overline{\mathrm{Ch}}(X_r)/N \to \overline{\mathrm{Ch}}(Y_{2r})$ is injective. □

Since the composition $in_* \circ in^*$ is the multiplication by the class $[X_r] \in \mathrm{Ch}(Y_{2r})$ and $in^*$ is a ring homomorphism, we get

**Corollary 9.10.** *The quotient ring* $\overline{\mathrm{Ch}}(X_r)/N$ *is isomorphic to* $\overline{\mathrm{Ch}}(Y_{2r})/I$*, where the ideal* $I \subset \overline{\mathrm{Ch}}(Y_{2r})$ *is the annihilator of the element* $[X_r] \in \overline{\mathrm{Ch}}(Y_{2r})$. □

**Example 9.11.** Let us consider the case of $n = 2r$. The ring $\mathrm{Ch}(\bar{Y}_{2r})$ is generated by certain elements $e_1, \dots, e_{2r-1}$ of codimensions $1, \dots, 2r-1$ subject to the relations $e_i^2 = e_{2i}$, where $e_j := 0$ for $j > 2r-1$, [8, §86]. If $h$ is generic, then, according to [8, Theorem 87.7] and Corollary 9.4, the subring $\overline{\mathrm{Ch}}(Y_{2r}) \subset \mathrm{Ch}(\bar{Y}_{2r})$ is generated by the even-codimensional elements $e_2, e_4, \dots, e_{2r-2}$. The class $[X_r] \in \mathrm{Ch}(\bar{Y}_{2r})$ is *rational* (that means *belongs to* $\overline{\mathrm{Ch}}(Y_{2r})$) and non-zero (by injectivity of $in_*$) so that $[X_r] = e_2 e_4 \dots e_{2r-2}$ as this product is the only non-zero rational element in codimension

$$\mathrm{codim}_{Y_{2r}} X_r = r(2r-1) - r^2 = r(r-1) = 2 + 4 + \dots + (2r-2).$$

It follows that for $n = 2r$ and hyperbolic $h$, the ring $\mathrm{Ch}(X_r)/N$ (we put the usual Chow group instead of the reduced one because they both coincide by the reason that $h$ is hyperbolic) is generated by elements $e_1, e_3, e_5, \dots, e_{2r-1}$ of codimensions $1, 3, 5, \dots, 2r-1$ subject to relations $e_i^2 = 0$ for any $i$.

**Example 9.12.** We briefly describe the situation with an *odd* $n$. In this situation, the norm homomorphism $\overline{\mathrm{Ch}}(Y_{2r})_K \to \overline{\mathrm{Ch}}(Y_{2r})$ can be non-zero. By this reason, $\overline{\mathrm{Ch}}(Y_{2r})$ in the statements of Lemma 9.8 and Corollary 9.10 has to be replaced by the quotient $\overline{\mathrm{Ch}}(Y_{2r})/N$. In particular, the ring $\overline{\mathrm{Ch}}(X_r)/N$ is naturally isomorphic to the quotient of the ring $\overline{\mathrm{Ch}}(Y_{2r})/N$ by the annihilator of $[X_r] \in \overline{\mathrm{Ch}}(Y_{2r})/N$.

Now we assume that $n = 2r+1$. The variety $\bar{Y}_{2r}$ is a rank $2r$ projective bundle over $\bar{Y}_{2r+1}$. The ring $\mathrm{Ch}(\bar{Y}_{2r})$ is generated by elements $e_i \in \mathrm{Ch}^i(\bar{Y}_{2r})$, $i = 1, 2, \dots, 2r$ and $e \in \mathrm{Ch}^1(\bar{Y}_{2r})$ subject to the relations $e_i^2 = e_{2i}$ and $e^{2r+1} = 0$. Let now $Y'_{2r}$ be the variety $Y_{2r}$ over a field extension of $F$ such that $h$ is almost hyperbolic but $K$ is still a field. The subring $\mathrm{Ch}(Y'_{2r}) \subset \mathrm{Ch}(\bar{Y}_{2r})$ is generated by $e, e_2, \dots, e_{2r}$ (all the above generators without $e_1$). The image of the norm map is the ideal generated by $e$. The ring $\mathrm{Ch}(Y'_{2r})/N$ is generated by all $e_i$ with $i \geq 2$ subject to the relations $e_i^2 = e_{2i}$. The subring $\overline{\mathrm{Ch}}(Y_{2r})/N \subset \mathrm{Ch}(Y'_{2r})/N$ contains the elements $e_2, e_4, \dots, e_{2r}$. In the case of *generic* $h$, the subring $\overline{\mathrm{Ch}}(Y_{2r})/N$ does not contain any $e_i$ with odd $i$: otherwise the canonical dimension of $Y_{2r}$ (and therefore of $X_r$) would be smaller than

$$\dim Y_{2r} - (2 + 4 + \dots + 2r) = r(r+2) = \dim X_r.$$

It follows that the subring $\overline{\mathrm{Ch}}(Y_{2r})/N$ is generated by the elements $e_2, e_4, \dots, e_{2r}$. In particular, the only non-zero homogeneous element of dimension $\dim X_r$ in $\overline{\mathrm{Ch}}(Y_{2r})/N$ is the product $e_2 e_4 \dots e_{2r}$ and therefore

$$[X_r] = e_2 e_4 \dots e_{2r} \in \mathrm{Ch}(Y'_{2r})/N.$$

The annihilator of $[X_r]$ in $\mathrm{Ch}(Y'_{2r})/N$ is therefore the ideal generated by $e_2, \dots, e_{2r}$ and it follows that for $n = 2r+1$ and *almost hyperbolic* $h$ the ring $\mathrm{Ch}(X_r)/N$ is generated by elements $e_3, e_5, \dots, e_{2r-1}$ subject to relations $e_i^2 = 0$.

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UPMC Sorbonne Universités, Institut de Mathématiques de Jussieu, Paris, FRANCE

*Web page:* `www.math.jussieu.fr/~karpenko`
*E-mail address*: `karpenko` *at* `math.jussieu.fr`