

CONJUGACY CLASSES OF TRIALITARIAN AUTOMORPHISMS AND SYMMETRIC COMPOSITIONS

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Dedicated with great friendship to Eva Bayer on the occasion of her 60th birthday

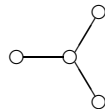
ABSTRACT. The trialitarian automorphisms considered in this paper are the outer automorphisms of order 3 of adjoint classical groups of type D_4 over arbitrary fields. A one-to-one correspondence is established between their conjugacy classes and similarity classes of symmetric compositions on 8-dimensional quadratic spaces. Using the known classification of symmetric compositions, we distinguish two conjugacy classes of trialitarian automorphisms over algebraically closed fields. For type I, the group of fixed points is of type G_2 , whereas it is of type A_2 for trialitarian automorphisms of type II.

1. INTRODUCTION

Among simple algebraic groups of classical type only the simple adjoint algebraic groups $G = \mathbf{PGO}^+(n)$ and the simple simply connected algebraic groups $G = \mathbf{Spin}(n)$, where n is the norm of an octonion algebra, admit outer automorphisms of order 3, known as trialitarian automorphisms, see [KMRT98, (42.7)] or [Jac64]. These groups are of type D_4 and there is a split exact sequence of algebraic groups

$$(1.1) \quad 1 \rightarrow \mathbf{Int}(G) \rightarrow \mathbf{Aut}(G) \rightarrow \mathfrak{S}_3 \rightarrow 1$$

where the permutation group of three elements \mathfrak{S}_3 is viewed as the group of automorphisms of the Dynkin diagram of type D_4 :



In view of the exact sequence (1.1), all the trialitarian automorphisms of G can be obtained from a fixed one ρ by composing ρ or ρ^{-1} with inner automorphisms. They are not necessarily conjugate to ρ , however. Our goal is to classify trialitarian automorphisms defined over an arbitrary field F up to conjugation in the group $\mathbf{Aut}(G)(F)$ of F -automorphisms of G . We achieve this goal by relating trialitarian automorphisms to symmetric compositions and using the known classification of composition algebras.

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We take as guiding principle the analogous description of outer automorphisms of order 2 of \mathbf{PGL}_n for $n \geq 3$, which could be termed *dualitarian automorphisms*: they have the form $f \mapsto \sigma(f)^{-1}$, where σ is the adjoint involution of a nonsingular symmetric or skew-symmetric bilinear form. Dualitarian automorphisms on \mathbf{PGL}_n are thus in one-to-one correspondence with nonsingular symmetric or skew-symmetric bilinear forms on an n -dimensional vector space up to scalar multiples, and dualitarian automorphisms are conjugate if and only if the corresponding bilinear forms are similar. There are two types of dualitarian automorphisms, distinguished by the type of their groups of fixed points, which can be either symplectic or (in characteristic different from 2) orthogonal. In characteristic 2, the non-symplectic case leads to group schemes that are not smooth.

Likewise, we set up a one-to-one correspondence between trialitarian automorphisms of $\mathbf{PGO}^+(n)$, for n a 3-fold Pfister quadratic form, and symmetric compositions up to scalar multiples on the underlying vector space of n , and use it to define a bijection between conjugacy classes of trialitarian automorphisms and similarity classes of symmetric compositions. We isolate two types of symmetric compositions. Type I is related to octonion algebras; the fixed subgroups are of type G_2 . When the characteristic is not 3, type II is related to central simple algebras of degree 3; the fixed subgroups are of type 1A_2 or 2A_2 . In characteristic 3, the fixed subgroups under trialitarian automorphisms of type II are not smooth.

Triality for simple Lie groups first appears in the paper [Car25] of É. Cartan, who already noticed that octonions can be used to explicitly define trialitarian automorphisms.¹ The observation that symmetric compositions are particularly well suited for that purpose is due to M. Rost.

As far as we know a complete classification of trialitarian automorphisms of simple groups of type D_4 had only been obtained over finite fields, in [GL83, (9.1)], and in [GLS98, Theorem (4.7.1)] for the algebraic closure of a finite field of characteristic different from 3. A summary of known results for Lie algebras can be found in [Knu09]. Like for duality, there is also a projective geometric version of triality, which is in fact older than Cartan's triality and goes back to Study [Stu13]. A classification of geometric trialities, as well as of their groups of automorphisms, was done by Tits in [Tit59].

If not explicitly mentioned F denotes throughout the paper an arbitrary field.

2. SIMILARITIES OF QUADRATIC SPACES

A quadratic space over F is a finite-dimensional vector space V over F with a quadratic form $q: V \rightarrow F$. We always assume that q is nonsingular, in the sense that the *polar bilinear form* b_q defined by

$$b_q(x, y) = q(x + y) - q(x) - q(y) \quad \text{for } x, y \in V$$

has radical $\{0\}$. We also assume throughout that $\dim V$ is *even*. Let ad_q denote the involution on $\text{End}_F V$ such that

$$b_q(f(x), y) = b_q(x, \text{ad}_q(f)(y)) \quad \text{for all } f \in \text{End}_F V \text{ and } x, y \in V.$$

This involution is said to be *adjoint* to q . Let $\mathbf{GO}(q)$ be the F -algebraic group of similarities of (V, q) , whose group of rational points $\mathbf{GO}(q)(F)$ consists of linear

¹We refer to [KMRT98, pp. 510–511] and [SV00, §3.8] for historical comments on triality.

maps $f: V \rightarrow V$ for which there exists a scalar $\mu(f) \in F^\times$, called the *multiplier* of f , such that

$$q(f(x)) = \mu(f)q(x) \quad \text{for all } x \in V.$$

The center of $\mathbf{GO}(q)$ is the multiplicative group \mathbf{G}_m , whose rational points are viewed as homotheties. Let $\mathbf{PGO}(q)$ be the F -algebraic group of automorphisms of $\text{End}_F V$ that commute with the adjoint involution ad_q . This group is identified with the quotient $\mathbf{GO}(q)/\mathbf{G}_m$, acting on $\text{End}_F V$ by inner automorphisms: for $f \in \mathbf{GO}(q)(F)$, we let $[f]$ be the image of f in $\mathbf{GO}(q)(F)/\mathbf{G}_m(F)$ and identify $[f]$ with

$$\text{Int}[f]: \text{End}_F V \rightarrow \text{End}_F V, \quad \phi \mapsto f\phi f^{-1},$$

see [KMRT98, §23]. For simplicity, we write

$$\text{GO}(q) = \mathbf{GO}(q)(F) \quad \text{and} \quad \text{PGO}(q) = \mathbf{PGO}(q)(F) = \text{GO}(q)/F^\times.$$

Let $C(V, q)$ be the Clifford algebra of the quadratic space (V, q) and let $C_0(V, q)$ be the even Clifford algebra. We let σ be the canonical involution of $C(V, q)$, such that $\sigma(x) = x$ for $x \in V$, and use the same notation for its restriction to $C_0(V, q)$. Every similarity $f \in \text{GO}(q)$ induces an automorphism $C_0(f)$ of $(C_0(V, q), \sigma)$ such that

$$(2.1) \quad C_0(f)(xy) = \mu(f)^{-1}f(x)f(y) \quad \text{for } x, y \in V,$$

see [KMRT98, (13.1)]. This automorphism depends only on the image $[f] = fF^\times$ of f in $\text{PGO}(q)$, and we shall use the notation $C_0[f]$ for $C_0(f)$. The similarity f is *proper* if $C_0[f]$ fixes the center of $C_0(V, q)$ and *improper* if it induces a nontrivial automorphism of the center of $C_0(V, q)$ (see [KMRT98, (13.2)]). Proper similarities define an algebraic subgroup $\mathbf{GO}^+(q)$ in $\mathbf{GO}(q)$, and we let $\mathbf{PGO}^+(q) = \mathbf{GO}^+(q)/\mathbf{G}_m$, a subgroup of $\mathbf{PGO}(q)$. The groups $\mathbf{GO}^+(q)$ and $\mathbf{PGO}^+(q)$ are the connected components of the identity in $\mathbf{GO}(q)$ and $\mathbf{PGO}(q)$ respectively, see [KMRT98, §23.B]. Conjugation by an improper similarity is an outer automorphism of $\mathbf{PGO}^+(q)$, since the induced automorphism on the center of $C_0(V, q)$ is nontrivial. As pointed out in the introduction, more outer automorphisms can be defined when the form q is the norm of an octonion algebra, i.e., a 3-fold Pfister form. In this case, we call the quadratic space a *3-fold Pfister quadratic space*.

3. SYMMETRIC COMPOSITIONS

Let (S, n) be a quadratic space of dimension 8 over F .

Definition 3.1. A *symmetric composition* on (S, n) is an F -bilinear map

$$\star: S \times S \rightarrow S, \quad (x, y) \mapsto x \star y \quad \text{for } x, y \in S$$

subject to the following conditions:

- (1) there exists $\lambda_\star \in F^\times$, called the *multiplier* of the symmetric composition \star , such that

$$n(x \star y) = \lambda_\star n(x)n(y) \quad \text{for all } x, y \in S;$$

- (2) for all $x, y, z \in S$,

$$b_n(x \star y, z) = b_n(x, y \star z).$$

A symmetric composition with multiplier $\lambda_\star = 1$ is called *normalized*.

This definition of symmetric compositions is not identical to the one given in [KMRT98, §34], where (S, n) can *a priori* be a nonsingular quadratic space of arbitrary finite dimension (but in fact $\dim S = 1, 2, 4$ or 8 by a theorem of Hurwitz, see [KMRT98, (33.28)]), and all the symmetric compositions are normalized.

The following lemma is an immediate consequence of the definition of a symmetric composition:

Lemma 3.2. *Let \star be a symmetric composition on (S, n) with multiplier λ_\star .*

- (1) *For any scalar $\nu \in F^\times$, \star is a symmetric composition on the quadratic space $(S, \nu n)$ with multiplier $\nu^{-1}\lambda_\star$.*
- (2) *For any scalar $\nu \in F^\times$, the bilinear map $\nu \cdot \star: (x, y) \mapsto \nu x \star y$ is a symmetric composition on the quadratic space (S, n) with multiplier $\nu^2\lambda_\star$.*

Lemma 3.3. *Under condition (1), condition (2) of the definition of a symmetric composition is equivalent to*

$$x \star (y \star x) = \lambda_\star n(x)y = (x \star y) \star x \quad \text{for all } x, y \in S.$$

Proof. The claim follows by applying [KMRT98, (34.1)] to the (normalized) composition \star on $(S, \lambda_\star n)$. \square

Since the symmetric compositions do not change when the quadratic form n is scaled, we may and will always assume without loss of generality that n represents 1. It is then a 3-fold Pfister form, by [KMRT98, (33.18), (33.29)].

Example 3.4. Let (\mathbb{O}, n) be an octonion algebra with norm n , multiplication $(x, y) \mapsto x \cdot y$, identity 1, and conjugation $x \mapsto \bar{x}$. The multiplication

$$x \star y = \bar{x} \cdot \bar{y}$$

defines a normalized symmetric composition on (\mathbb{O}, n) , called the *para-octonion composition* (see for example [KMRT98, §34.A]). Observe that $\overline{x \star y} = \bar{y} \star \bar{x}$ for all $x, y \in \mathbb{O}$.

More examples—and a complete classification of symmetric compositions—are given in Section 9.

Definitions 3.5. Let (S, n) be a 3-fold Pfister quadratic space and let \star and \diamond be symmetric compositions on (S, n) . A *similarity* $f: \star \rightarrow \diamond$ is an element $f \in \text{GO}(n)$ such that

$$f(x \star y) = f(x) \diamond f(y) \quad \text{for all } x, y \in S.$$

The multipliers of f , \star and \diamond are then related by $\lambda_\star = \lambda_\diamond \mu(f)$. Similarities with multiplier $\mu(f) = 1$ are called *isomorphisms*. In particular, similarities between symmetric compositions with the same multiplier are isomorphisms.

The *opposite* of the symmetric composition \star on (S, n) is the symmetric composition \star^{op} on (S, n) defined by

$$x \star^{\text{op}} y = y \star x \quad \text{for } x, y \in S.$$

The multiplier of \star^{op} is the same as the multiplier of \star .

Symmetric compositions \star and \diamond are said to be *similar* if there is a similarity $\star \rightarrow \diamond$; they are said to be *antisimilar* if \star^{op} and \diamond are similar.

Proposition 3.6. *Let n be a 3-fold quadratic Pfister form on a vector space S , and let \star be a symmetric composition on (S, n) . There exists up to isomorphism a unique normalized composition \diamond on (S, n) similar to \star .*

Proof. Let $u \in S$ be such that $n(u) = 1$. Consider the maps $\ell_u^*, r_u^* : S \rightarrow S$ defined by

$$\ell_u^*(x) = u \star x \quad \text{and} \quad r_u^*(x) = x \star u \quad \text{for } x \in S.$$

Define a new multiplication \diamond on S by

$$x \diamond y = \lambda_\star^{-2} \ell_u^*(r_u^*(x) \star r_u^*(y)) \quad \text{for } x, y \in S.$$

Condition (1) for a symmetric composition yields

$$n(\ell_u^*(x)) = \lambda_\star n(x) = n(r_u^*(x)) \quad \text{for all } x \in S.$$

It is then easy to check that \diamond is a normalized symmetric composition on (S, n) . Moreover, by Lemma 3.3, we have

$$\ell_u^* \circ r_u^* = r_u^* \circ \ell_u^* = \lambda_\star \text{Id}_S.$$

From the definition of \diamond , it follows that

$$\lambda_\star^{-1} r_u^*(x \diamond y) = (\lambda_\star^{-1} r_u^*(x)) \star (\lambda_\star^{-1} r_u^*(y)) \quad \text{for all } x, y \in S,$$

hence $\lambda_\star^{-1} r_u^* : \diamond \rightarrow \star$ is a similarity. If \diamond' is another normalized composition similar to \star , then \diamond and \diamond' are similar, hence isomorphic since they have the same multiplier. \square

4. FROM SYMMETRIC COMPOSITIONS TO TRIALITARIAN AUTOMORPHISMS

Throughout this section, we fix a 3-fold Pfister quadratic space (S, n) and a symmetric composition \star on (S, n) with multiplier λ_\star . We show how to associate to this composition a trialitarian automorphism ρ_\star of $\mathbf{PGO}^+(n)$ defined over F , i.e., an outer automorphism of order 3 in $\mathbf{Aut}(\mathbf{PGO}^+(n))(F)$.

For each element $x \in S$ we define linear maps $\ell_x^*, r_x^* : S \rightarrow S$ by

$$\ell_x^*(y) = x \star y \quad \text{and} \quad r_x^*(y) = y \star x \quad \text{for } y \in S.$$

Consider $\begin{pmatrix} 0 & \lambda_\star^{-1} r_x^* \\ \ell_x^* & 0 \end{pmatrix} \in M_2(\text{End } S) = \text{End}_F(S \oplus S)$. By Lemma 3.3, we have

$$\begin{pmatrix} 0 & \lambda_\star^{-1} r_x^* \\ \ell_x^* & 0 \end{pmatrix}^2 = n(x) \cdot \text{Id}_{S \oplus S}.$$

Therefore, by the universal property of Clifford algebras, the map

$$x \mapsto \begin{pmatrix} 0 & \lambda_\star^{-1} r_x^* \\ \ell_x^* & 0 \end{pmatrix}, \quad x \in S$$

induces an F -algebra homomorphism

$$\alpha^\star : C(S, n) \rightarrow \text{End}_F(S \oplus S).$$

Proposition 4.1. *The map α^\star is an isomorphism of F -algebras*

$$\alpha^\star : C(S, n) \xrightarrow{\sim} \text{End}_F(S \oplus S).$$

It restricts to an isomorphism of F -algebras with involution

$$\alpha_0^\star : (C_0(S, n), \sigma) \xrightarrow{\sim} (\text{End}_F S, \text{ad}_n) \times (\text{End}_F S, \text{ad}_n)$$

such that for $x, y \in S$

$$\alpha_0^\star(x \cdot y) = (\lambda_\star^{-1} r_x^* \ell_y^*, \lambda_\star^{-1} \ell_x^* r_y^*).$$

Proof. This is shown in [KMRT98, (35.1)]. For completeness, we reproduce the easy argument. The map α^* is injective since the Clifford algebra $C(S, n)$ is simple. It restricts to an F -algebra embedding $C_0(S, n) \hookrightarrow (\text{End } S) \times (\text{End } S)$. Since $\dim S = 8$, dimension count shows that this embedding is an isomorphism. Using Lemma 3.3, it is easy to check that the involution σ corresponds to $\text{ad}_n \times \text{ad}_n$ under α_0^* . \square

For $f, g, h \in \text{GO}(n)$, define the F -algebra automorphism $\Theta(g, h)$ of $(\text{End}_F S) \times (\text{End}_F S)$ by

$$\Theta(g, h): (\varphi, \psi) \mapsto (g\psi g^{-1}, h\varphi h^{-1})$$

and consider the following diagrams:

$$D_\star^+(f, g, h) \quad \begin{array}{ccc} C_0(S, n) & \xrightarrow{\alpha_0^*} & (\text{End}_F S) \times (\text{End}_F S) \\ C_0[f] \downarrow & & \downarrow \text{Int}[g] \times \text{Int}[h] \\ C_0(S, n) & \xrightarrow{\alpha_0^*} & (\text{End}_F S) \times (\text{End}_F S) \end{array}$$

and

$$D_\star^-(f, g, h) \quad \begin{array}{ccc} C_0(S, n) & \xrightarrow{\alpha_0^*} & (\text{End}_F S) \times (\text{End}_F S) \\ C_0[f] \downarrow & & \downarrow \Theta(g, h) \\ C_0(S, n) & \xrightarrow{\alpha_0^*} & (\text{End}_F S) \times (\text{End}_F S). \end{array}$$

Lemma 4.2. *Let \star be a symmetric composition on (S, n) . For $f, g, h \in \text{GO}(n)$, the following statements are equivalent:*

(a⁺) *there exists a scalar $\lambda \in F^\times$ such that*

$$\lambda f(x \star y) = g(x) \star h(y) \quad \text{for all } x, y \in S;$$

(b⁺) *there exists a scalar $\mu \in F^\times$ such that*

$$\mu g(x \star y) = h(x) \star f(y) \quad \text{for all } x, y \in S;$$

(c⁺) *there exists a scalar $\nu \in F^\times$ such that*

$$\nu h(x \star y) = f(x) \star g(y) \quad \text{for all } x, y \in S;$$

(d⁺) *the diagram $D_\star^+(f, g, h)$ commutes;*

(e⁺) *the diagram $D_\star^+(g, h, f)$ commutes;*

(f⁺) *the diagram $D_\star^+(h, f, g)$ commutes.*

When they hold, the scalars λ, μ, ν and the multipliers of f, g, h are related by

$$\lambda\mu = \mu(h), \quad \mu\nu = \mu(f), \quad \lambda\nu = \mu(g).$$

Moreover, the similarities $f, g,$ and h are all proper in this case.

Likewise, for $f, g, h \in \text{GO}(n)$ the following statements are equivalent:

(a⁻) *there exists a scalar $\lambda \in F^\times$ such that*

$$\lambda f(x \star y) = h(y) \star g(x) \quad \text{for all } x, y \in S;$$

(b⁻) *there exists a scalar $\mu \in F^\times$ such that*

$$\mu g(x \star y) = f(y) \star h(x) \quad \text{for all } x, y \in S;$$

(c⁻) *there exists a scalar $\nu \in F^\times$ such that*

$$\nu h(x \star y) = g(y) \star f(x) \quad \text{for all } x, y \in S;$$

- (d^-) the diagram $D_{\star}^-(f, h, g)$ commutes;
- (e^-) the diagram $D_{\star}^-(g, f, h)$ commutes;
- (f^-) the diagram $D_{\star}^-(h, g, f)$ commutes.

When they hold, the scalars λ, μ, ν and the multipliers of f, g, h are related by

$$\lambda\mu = \mu(h), \quad \mu\nu = \mu(f), \quad \lambda\nu = \mu(g).$$

Moreover, the similarities $f, g,$ and h are all improper in this case.

Proof. This is essentially proved in [KMRT98, (35.4)]. We give a proof for the reader's convenience.

($a^+ \Rightarrow b^+$) Multiplying each side of (a^+) on the left by $h(y)$ and using Lemma 3.3, we obtain

$$\lambda h(y) \star f(x \star y) = \lambda_{\star} n(h(y)) g(x) \quad \text{for } x, y \in S.$$

If y is anisotropic, the map r_y^{\star} is a bijection whose inverse is $\lambda_{\star}^{-1} n(y)^{-1} \ell_y^{\star}$. Letting $X = y$ and $Y = x \star y$, we derive from the preceding equation

$$\lambda h(X) \star f(Y) = \mu(h) g(X \star Y) \quad \text{for } X, Y \in S \text{ with } Y \text{ anisotropic.}$$

Since generic vectors in S are anisotropic, (b^+) follows with $\mu = \mu(h)\lambda^{-1}$. Similar arguments yield ($b^+ \Rightarrow c^+$) with $\nu = \mu(f)\mu^{-1}$ and ($c^+ \Rightarrow a^+$) with $\lambda = \mu(g)\nu^{-1}$.

Now, assume (a^+), (b^+), and (c^+) hold. From (b^+) and (c^+), and from $\mu\nu = \mu(f)$, we readily derive

$$g((y \star z) \star x) = \mu^{-1} h(y \star z) \star f(x) = \mu(f)^{-1} (f(y) \star g(z)) \star f(x)$$

and

$$h(x \star (z \star y)) = \nu^{-1} f(x) \star g(z \star y) = \mu(f)^{-1} f(x) \star (h(z) \star f(y))$$

for all $x, y, z \in S$. These equations mean that diagram $D_{\star}^+(f, g, h)$ commutes, hence (d^+) holds. Similar computations show that (e^+) and (f^+) hold.

Now, assume (d^+) holds. The map

$$x \mapsto \begin{pmatrix} 0 & \lambda_{\star}^{-1} r_{f(x)}^{\star} \\ \mu(f)^{-1} \ell_{f(x)}^{\star} & 0 \end{pmatrix}$$

also yields by the universal property of Clifford algebras an F -algebra isomorphism $\beta: C(S, n) \xrightarrow{\sim} \text{End}_F(S \oplus S)$. The automorphism $\beta \circ (\alpha^{\star})^{-1}$ of $\text{End}_F(S \oplus S)$ is inner by the Skolem–Noether theorem, and it preserves $(\text{End}_F S) \times (\text{End}_F S)$ diagonally embedded, hence

$$\beta \circ (\alpha^{\star})^{-1} = \text{Int} \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \quad \text{for some invertible } \varphi, \psi \in \text{End}_F S.$$

It follows that for $x \in S$

$$\varphi \circ r_x^{\star} \circ \psi^{-1} = r_{f(x)}^{\star} \quad \text{and} \quad \psi \circ \ell_x^{\star} \circ \varphi^{-1} = \mu(f)^{-1} \ell_{f(x)}^{\star},$$

which means that for $x, y \in S$

$$(4.3) \quad \varphi(y \star x) = \psi(y) \star f(x) \quad \text{and} \quad \psi(x \star y) = \mu(f)^{-1} f(x) \star \varphi(y).$$

For $x, y \in S$ we have

$$\beta(x \cdot y) = (\lambda_{\star}^{-1} \mu(f)^{-1} r_{f(x)}^{\star} \ell_{f(x)}^{\star}, \lambda_{\star}^{-1} \mu(f)^{-1} \ell_{f(x)}^{\star} r_{f(x)}^{\star}) = \alpha_0^{\star} \circ C_0[f](x \cdot y),$$

hence (d^+) yields

$$\text{Int} \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} = \text{Int} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}.$$

It follows that $[\varphi] = [g]$ and $[\psi] = [h]$, hence (4.3) yields (c^+) and (b^+) . Similarly, $(e^+) \Rightarrow (a^+)$, (c^+) , and $(f^+) \Rightarrow (a^+)$, (b^+) . Thus, all the statements $(a^+) - (f^+)$ are equivalent. When they hold, (d^+) shows that $C_0[f]$ is the identity on the center of $C_0(S, n)$, hence f is proper. Similarly, (e^+) shows that g is proper and (f^+) that h is proper.

The proof of the second part is similar. Details are left to the reader. \square

Corollary 4.4. *Let \star be a symmetric composition on (S, n) . Every automorphism $\star \rightarrow \star$ is a proper isometry.*

Proof. For any automorphism f , condition (a^+) of Lemma 4.2 holds with $\lambda = 1$ and $g = h = f$. \square

Theorem 4.5. *Let \star be a symmetric composition on (S, n) .*

- (1) *For any $f \in \text{GO}^+(n)$, there exist $g, h \in \text{GO}^+(n)$ such that the equivalent conditions $(a^+) - (f^+)$ in Lemma 4.2 hold.*
- (2) *For any improper similarity $f \in \text{GO}(n)$, there exist improper similarities $g, h \in \text{GO}(n)$ such that the equivalent conditions $(a^-) - (f^-)$ in Lemma 4.2 hold.*

In each case, the similarities g and h are uniquely determined up to multiplication by scalars, so $[g]$ and $[h] \in \text{PGO}(n)$ are uniquely determined.

Proof. (1) The automorphism $C_0[f]$ of $C_0(S, n)$ preserves the canonical involution σ and restricts to the identity on the center. Therefore, we may find $g, h \in \text{GO}(n)$ such that the diagram $D_\star^+(f, h, g)$ commutes. The elements $[g], [h] \in \text{PGO}(n)$ are uniquely determined by this condition (or by any other in the list $(a^+) - (f^+)$ from Lemma 4.2), and Lemma 4.2 shows that g and h are proper similarities.

(2) If f is improper, then $C_0[f]$ restricts to the nontrivial automorphism of the center of $C_0(S, n)$, hence it fits in a diagram $D_\star^-(f, h, g)$ for some similarities $g, h \in \text{GO}(n)$, which are improper by Lemma 4.2. \square

Note that scaling g and/or h we may change the scalars λ, μ, ν in conditions $(a^+) - (c^+)$ or $(a^-) - (c^-)$. We may for instance choose g and h so that $\lambda = 1$, or, as in [KMRT98, (35.4)], so that $\lambda = \mu(f)^{-1}$, $\mu = \mu(g)^{-1}$, and $\nu = \mu(h)^{-1}$. In this case, $\mu(f)\mu(g)\mu(h) = 1$.

In view of Theorem 4.5, we may define a map

$$\rho_\star: \text{PGO}^+(n) \rightarrow \text{PGO}^+(n)$$

by carrying any $[f] \in \text{PGO}^+(n)$ to the unique $[g] \in \text{PGO}^+(n)$ such that the diagram $D_\star^+(f, g, h)$ commutes for some $h \in \text{GO}^+(n)$. Since ρ_\star is defined in a functorial way, it actually defines a map of F -algebraic groups

$$\rho_\star: \mathbf{PGO}^+(n) \rightarrow \mathbf{PGO}^+(n).$$

Theorem 4.6. *The map ρ_\star is an outer automorphism of order 3 of $\mathbf{PGO}^+(n)$ and $\rho_\star^{-1} = \rho_{\star^{op}}$.*

Proof. Since commutation of $D_\star^+(f, g, h)$ implies that $D_\star^+(g, h, f)$ and $D_\star^+(h, f, g)$ commute, we have

$$\rho_\star[g] = [h] \quad \text{and} \quad \rho_\star[h] = [f],$$

hence $\rho_\star^3 = \text{Id}$. Now, let $f_1, f_2 \in \text{GO}^+(n)$ and assume $g_1, g_2, h_1, h_2 \in \text{GO}^+(n)$ are such that $D_\star^+(f_1, g_1, h_1)$ and $D_\star^+(f_2, g_2, h_2)$ commute. By Lemma 4.2 we may find scalars $\lambda_1, \lambda_2 \in F^\times$ such that for all $x, y \in S$

$$\lambda_1 f_1(x \star y) = g_1(x) \star h_1(y) \quad \text{and} \quad \lambda_2 f_2(x \star y) = g_2(x) \star h_2(y),$$

hence

$$\lambda_1 \lambda_2 f_1 f_2(x \star y) = \lambda_1 f_1(g_2(x) \star h_2(y)) = g_1 g_2(x) \star h_1 h_2(y).$$

Therefore, $\rho_\star[f_1 f_2] = [g_1 g_2]$, showing that ρ_\star is an automorphism. We refer to [KMRT98, (35.6)] for the fact that ρ_\star is an outer algebraic group automorphism. (That it is outer also follows from the fact that ρ_\star lifts to an automorphism of $\mathbf{Spin}(n)$ that is not the identity on the center, see (8.2)). \square

The definition of ρ_\star above depends on the isomorphism α_0^\star through the diagram $D_\star^+(f, g, h)$. We next show that, conversely, the automorphism ρ_\star determines α_0^\star .

Proposition 4.7. *Let \star and \diamond be symmetric compositions on (S, n) . We have $\rho_\star = \rho_\diamond$ if and only if $\alpha_0^\star = \alpha_0^\diamond$.*

Proof. If $\alpha_0^\star = \alpha_0^\diamond$, then the diagrams $D_\star^+(f, g, h)$ and $D_\diamond^+(f, g, h)$ coincide for all $f, g, h \in \text{GO}^+(n)$, hence $\rho_\star = \rho_\diamond$. Conversely, suppose that $\rho_\star = \rho_\diamond$. Then $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ is an automorphism of $(\text{End } S) \times (\text{End } S)$ that commutes with the involution $\text{ad}_n \times \text{ad}_n$ and that makes the following diagram commute for any $f \in \text{GO}^+(n)$

$$(4.8) \quad \begin{array}{ccc} (\text{End } S) \times (\text{End } S) & \xrightarrow{\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}} & (\text{End } S) \times (\text{End } S) \\ \text{Int}(\rho_\star[f]) \times \text{Int}(\rho_\star^2[f]) \downarrow & & \downarrow \text{Int}(\rho_\star[f]) \times \text{Int}(\rho_\star^2[f]) \\ (\text{End } S) \times (\text{End } S) & \xrightarrow{\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}} & (\text{End } S) \times (\text{End } S). \end{array}$$

Suppose first that $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ exchanges the two factors, so there exist automorphisms θ_1, θ_2 of $(\text{End } S, \text{ad}_n)$ such that

$$\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}(a, b) = (\theta_1(b), \theta_2(a)) \quad \text{for all } a, b \in \text{End } S.$$

Automorphisms of $(\text{End } S, \text{ad}_n)$ are inner automorphisms induced by similarities of (S, n) , so we may find $s_1, s_2 \in \text{GO}(n)$ such that $\theta_i = \text{Int}[s_i]$ for $i = 1, 2$. By commutativity of diagram (4.8), we have

$$\rho_\star[f] \cdot [s_1] = [s_1] \cdot \rho_\star^2[f] \quad \text{and} \quad \rho_\star^2[f] \cdot [s_2] = [s_2] \cdot \rho_\star[f] \quad \text{for all } f \in \text{GO}^+(n).$$

Thus, $\text{Int}[s_1] \circ \rho_\star^2 = \rho_\star$, hence $\text{Int}[s_1] = \rho_\star^2$ and therefore $\rho_\star = \text{Int}[s_1^2]$. Since the square of any similarity is a proper similarity, it follows that ρ_\star is an inner automorphism of $\mathbf{PGO}^+(n)$, a contradiction. Therefore, $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ preserves the two factors, and we have

$$\alpha_0^\diamond \circ (\alpha_0^\star)^{-1} = \text{Int}[s_1] \times \text{Int}[s_2]$$

for some $s_1, s_2 \in \text{GO}(n)$. Commutativity of diagram (4.8) yields

$$\rho_\star[f] \cdot [s_1] = [s_1] \cdot \rho_\star[f] \quad \text{and} \quad \rho_\star^2[f] \cdot [s_2] = [s_2] \cdot \rho_\star^2[f] \quad \text{for all } f \in \text{GO}^+(n).$$

Therefore, $[s_1]$ and $[s_2]$ centralize $\mathbf{PGO}^+(n)$. It follows that s_1 and s_2 are proper similarities, since conjugation by an improper similarity is an outer automorphism of $\mathbf{PGO}^+(n)$. Since the center of $\mathbf{PGO}^+(n)$ is trivial, it follows that $[s_1] = [s_2] = 1$, hence $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1} = \text{Id}$. \square

5. THE ONE-TO-ONE CORRESPONDENCE

As in the preceding section, we fix a 3-fold Pfister quadratic space (S, n) over F . Our goal is to show that the map $\star \mapsto \rho_\star$ defines a one-to-one correspondence between symmetric compositions on (S, n) up to a scalar factor and trialitarian automorphisms of $\mathbf{PGO}^+(n)$ defined over F .

Recall that given a symmetric composition \star on (S, n) , linear operators ℓ_x^\star and r_x^\star on S are defined for any $x \in S$ by

$$\ell_x^\star(y) = x \star y \quad \text{and} \quad r_x^\star(y) = y \star x \quad \text{for } y \in S.$$

If x is anisotropic, condition (1) in the definition of symmetric compositions shows that ℓ_x^\star and r_x^\star are similarities of (S, n) with multiplier $\lambda_\star n(x)$. We shall see in Corollary 7.4 below that ℓ_x^\star and r_x^\star are improper similarities for every anisotropic vector $x \in S$. At this stage, we can at least prove:

Lemma 5.1. *The following conditions are equivalent:*

- (a) *there exists an anisotropic vector $x \in S$ such that ℓ_x^\star is an improper similarity;*
- (b) *there exists an anisotropic vector $x \in S$ such that r_x^\star is an improper similarity;*
- (c) *for every anisotropic vector $x \in S$, the similarity ℓ_x^\star is improper;*
- (d) *for every anisotropic vector $x \in S$, the similarity r_x^\star is improper.*

Proof. By Lemma 3.3 we have for all $x \in S$

$$r_x^\star \circ \ell_x^\star = \lambda_\star n(x) \cdot \text{Id}_S,$$

and $\lambda_\star n(x) \cdot \text{Id}_S$ is a proper similarity. Thus r_x^\star and ℓ_x^\star are either both proper or improper, proving (a) \iff (b) and (c) \iff (d). Since (c) \implies (a) is clear, it only remains to show (a) \implies (c). For this, we use a homotopy argument. Let $x, y \in S$ be anisotropic vectors, and let t be an indeterminate over F . Consider the vector $u(t) = x(1-t) + yt \in S(t) = S_{F(t)}$. It gives rise to a rational morphism

$$\phi: \mathbb{A}^1 \dashrightarrow \mathbf{GO}(S, n),$$

induced by $t \mapsto \ell_{u(t)}^\star$, of the affine line over F to $\mathbf{GO}(S, n)$. Note that ϕ is defined at a if the vector $u(a) = x(1-a) + ya$ is anisotropic. Since $\mathbf{GO}(S, n)$ contains an improper similarity g of S , the variety of $\mathbf{GO}(S, n)$ is the disjoint union of two irreducible subvarieties $\mathbf{GO}^+(S, n)$ and $g\mathbf{GO}^+(S, n)$. Since \mathbb{A}^1 is irreducible the image of ϕ is contained either in $\mathbf{GO}^+(S, n)$ or in $g\mathbf{GO}^+(S, n)$. It follows immediately that the similarities $\phi(0) = x$ and $\phi(1) = y$ are both proper or both improper. \square

Lemma 5.2. *Let $f, g \in \mathbf{GO}(n)$ and let \star be a symmetric composition on (S, n) . The following conditions are equivalent:*

- (a) *the map $\diamond: S \times S \rightarrow S$ defined by*

$$x \diamond y = f(x) \star g(y) \quad \text{for } x, y \in S$$

is a symmetric composition on (S, n) ;

- (b) *for all $x, y \in S$,*

$$f(x) \star g(y) = \mu(g)g^{-1}(x \star f(y));$$

(c) f and g are proper similarities such that

$$\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1 \quad \text{and} \quad [g] = \rho_\star^2[f]^{-1} \quad \text{in } \text{PGO}^+(n).$$

When they hold, the multiplier of \diamond is $\lambda_\diamond = \mu(f)\mu(g)\lambda_\star$ and

$$\rho_\diamond = \text{Int}[f^{-1}] \circ \rho_\star = \text{Int}[g^{-1}] \circ \rho_\star \circ \text{Int}[f].$$

Moreover, assuming (a)–(c) hold, the following conditions on $[h] \in \text{PGO}^+(n)$ are equivalent:

- (i) $\rho_\diamond = \text{Int}[h^{-1}] \circ \rho_\star \circ \text{Int}[h]$,
- (ii) $[f] = \rho_\star[h]^{-1} \cdot [h] = [h] \cdot \rho_\diamond[h]^{-1}$.

Proof. We first show (a) \iff (b). Since f and g are similarities and \star is a symmetric composition, we have

$$(5.3) \quad n(f(x) \star g(y)) = \lambda_\star \mu(f)\mu(g)n(x)n(y) \quad \text{for all } x, y \in S,$$

hence the map \diamond satisfies condition (1) in the definition of symmetric compositions. Therefore, (a) is equivalent to

$$b_n(f(x) \star g(y), z) = b_n(x, f(y) \star g(z)) \quad \text{for all } x, y, z \in S.$$

Since \star is a symmetric composition and g is a similarity, we may rewrite the right side as

$$b_n(x \star f(y), g(z)) = \mu(g)b_n(g^{-1}(x \star f(y)), z).$$

Therefore, (a) is equivalent to

$$b_n(f(x) \star g(y), z) = \mu(g)b_n(g^{-1}(x \star f(y)), z) \quad \text{for all } x, y, z \in S.$$

Since n is nonsingular, this condition is also equivalent to (b).

Now, assume (b) holds. Fixing an anisotropic vector $x \in S$ and considering each side of the equation in (b) as a function of y , we have

$$\ell_{f(x)}^\star \circ g = \mu(g)g^{-1} \circ \ell_x^\star \circ f.$$

By Lemma 5.1, the similarities $\ell_{f(x)}^\star \circ g$ and $\mu(g)g^{-1} \circ \ell_x^\star$ are both proper or both improper, hence this equation shows that f is proper. Likewise, fixing an anisotropic vector y and considering each side of the equation in (b) as a function of x , we have

$$r_{g(y)}^\star \circ f = \mu(g)g^{-1} \circ r_{f(y)}^\star.$$

Since by Lemma 5.1 the similarities $r_{g(y)}^\star$ and $r_{f(y)}^\star$ are either both proper or both improper, this equation shows that g is proper because f is proper.

By Lemma 4.2, condition (b) is equivalent to the commutativity of diagram $D_\star^+(g^{-1}, f, gf^{-1})$, hence to

$$(5.4) \quad \rho_\star[g^{-1}] = [f] \quad \text{and} \quad \rho_\star^2[g^{-1}] = [gf^{-1}].$$

Since $\rho_\star^3 = \text{Id}$, it follows that $[g] = \rho_\star^2[f]^{-1}$ and $\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1$. We have thus proved (b) \implies (c). Conversely, we readily deduce (5.4) from (c), hence the diagram $D_\star^+(g^{-1}, f, gf^{-1})$ commutes. By Lemma 4.2, we may find $\lambda, \mu, \nu \in F^\star$ such that for all $x, y \in S$

$$(5.5) \quad \lambda g^{-1}(x \star y) = f(x) \star gf^{-1}(y),$$

$$(5.6) \quad \mu f(x \star y) = gf^{-1}(x) \star g^{-1}(y),$$

$$(5.7) \quad \nu gf^{-1}(x \star y) = g^{-1}(x) \star f(y),$$

and

$$\lambda\mu = \mu(gf^{-1}), \quad \mu\nu = \mu(g^{-1}), \quad \lambda\nu = \mu(f).$$

These last equations yield $\lambda^2\mu\nu = \mu(g)$. To obtain (b) from (5.5), it suffices to prove $\lambda = \mu(g)$, which amounts to $\lambda\mu\nu = 1$. For this, observe that

$$\lambda\mu\nu x \star y = \lambda\mu\nu gf^{-1} \circ f \circ g^{-1}(x \star y) \quad \text{for all } x, y \in S.$$

Compute the right side using successively (5.5), (5.6), and (5.7):

$$\begin{aligned} \lambda\mu\nu gf^{-1} \circ f \circ g^{-1}(x \star y) &= \mu\nu gf^{-1} \circ f(f(x) \star gf^{-1}(y)) \\ &= \nu gf^{-1}(g(x) \star f^{-1}(y)) \\ &= x \star y. \end{aligned}$$

Thus, $\lambda\mu\nu x \star y = x \star y$ for all $x, y \in S$, hence $\lambda\mu\nu = 1$ and it follows that (c) \Rightarrow (b).

Now, assume (a), (b), and (c) hold. The equation $\lambda_\diamond = \mu(f)\mu(g)\lambda_\star$ easily follows from (5.3). For $\varphi \in \text{GO}^+(n)$ Theorem 4.5 yields $\varphi', \varphi'' \in \text{GO}^+(n)$ such that

$$\varphi(x \star y) = \varphi'(x) \star \varphi''(y) \quad \text{for all } x, y \in S,$$

so $\rho_\star[\varphi] = [\varphi']$. Then

$$\varphi(x \diamond y) = \varphi(f(x) \star g(y)) = \varphi'f(x) \star \varphi''g(y) = f^{-1}\varphi'f(x) \diamond g^{-1}\varphi''g(y),$$

hence $\rho_\diamond[\varphi] = [f^{-1}\varphi'f]$. It follows that $\rho_\diamond = \text{Int}[f^{-1}] \circ \rho_\star$. From (c) it is easily derived that $[f]^{-1} = [g^{-1}] \cdot \rho_\star[f]$, hence we also have $\rho_\diamond = \text{Int}[g^{-1}] \circ \rho_\star \circ \text{Int}[f]$.

Finally, (i) holds if and only if $\rho_\diamond = \text{Int}([h]^{-1} \cdot \rho_\star[h]) \circ \rho_\star$. This equation is equivalent to (ii) since $\rho_\diamond = \text{Int}[f^{-1}] \circ \rho_\star$ and the center of $\mathbf{PGO}^+(n)$ is trivial. \square

Theorem 5.8. *The assignment $\star \mapsto \rho_\star$ defines a one-to-one correspondence between symmetric compositions on (S, n) up to scalars and trialitarian automorphisms of $\mathbf{PGO}^+(n)$ defined over F .*

Proof. We first show that the map is onto. Let τ be a trialitarian automorphism of $\mathbf{PGO}^+(n)$ over F and let \star be a symmetric composition on (S, n) , so ρ_\star also is a trialitarian automorphism. In view of the exact sequence (1.1), we have either $\tau \equiv \rho_\star$ or $\tau \equiv \rho_\star^{-1} \pmod{\text{Int}(\mathbf{PGO}^+(n))}$. Substituting \star^{op} for \star if necessary, we may assume $\tau \equiv \rho_\star \pmod{\text{Int}(\mathbf{PGO}^+(n))}$, hence there exists $f \in \text{GO}^+(n)$ such that

$$\tau = \text{Int}[f^{-1}] \circ \rho_\star.$$

Since $\tau^3 = \rho_\star^3 = \text{Id}$, we must have $[f]^{-1} \cdot \rho_\star[f]^{-1} \cdot \rho_\star^2[f]^{-1} = 1$. Let $g \in \text{GO}^+(n)$ be such that $[g] = \rho_\star^2[f]^{-1}$. Lemma 5.2 then shows that $\tau = \rho_\diamond$ for the symmetric composition \diamond defined by

$$x \diamond y = f(x) \star g(y) \quad \text{for } x, y \in S.$$

Now, let \star and \diamond be two symmetric compositions on (S, n) . Suppose \diamond is a scalar multiple of \star , say there exists $\mu \in F^\times$ such that $x \diamond y = \mu x \star y$ for all $x, y \in S$. Then $\lambda_\diamond = \mu^2 \lambda_\star$, and for all $x \in S$ we have $\ell_x^\diamond = \mu \ell_x^\star$ and $r_x^\diamond = \mu r_x^\star$. Therefore, $\alpha_0^\diamond = \alpha_0^\star$, hence $\rho_\diamond = \rho_\star$ by Proposition 4.7. Thus, symmetric compositions that are scalar multiples of each other define the same trialitarian automorphism, and it only remains to show the converse: if \star and \diamond are symmetric compositions such that $\rho_\star = \rho_\diamond$, then \star and \diamond are multiples of each other. By Proposition 4.7, the hypothesis $\rho_\star = \rho_\diamond$ implies $\alpha_0^\star = \alpha_0^\diamond$, hence

$$\lambda_\star^{-1} \ell_x^\star r_y^\star = \lambda_\diamond^{-1} \ell_x^\diamond r_y^\diamond \quad \text{for all } x, y \in S.$$

Fix an anisotropic vector $x \in S$ and let $\varphi = (\ell_x^\diamond)^{-1}\ell_x^\star$, so

$$\lambda_\diamond \lambda_\star^{-1} \varphi r_y^\star = r_y^\diamond \quad \text{for all } y \in S.$$

This equation means that

$$(5.9) \quad z \diamond y = \lambda_\diamond \lambda_\star^{-1} \varphi(z \star y) \quad \text{for all } y, z \in S.$$

The map φ is a similarity since ℓ_x^\diamond and ℓ_x^\star are similarities. Its multiplier is

$$\mu(\varphi) = \mu(\ell_x^\diamond)^{-1} \mu(\ell_x^\star) = \lambda_\diamond^{-1} \lambda_\star.$$

Suppose first φ is improper. Theorem 4.5 then yields improper similarities φ' , φ'' such that

$$\varphi(z \star y) = \mu(\varphi) \varphi'(y) \star \varphi''(z) \quad \text{for all } y, z \in S,$$

hence

$$z \diamond y = \varphi''(z) \star^{\text{op}} \varphi'(y) \quad \text{for all } y, z \in S.$$

Lemma 5.2 yields a contradiction since φ' and φ'' are improper. Therefore, φ is proper and Theorem 4.5 yields φ' , $\varphi'' \in \text{GO}^+(n)$ such that

$$\varphi(z \star y) = \mu(\varphi) \varphi'(z) \star \varphi''(y) \quad \text{for all } y, z \in S,$$

hence

$$z \diamond y = \varphi'(z) \star \varphi''(y) \quad \text{for all } y, z \in S.$$

By Lemma 5.2 it follows that $\rho_\diamond = \text{Int}[\varphi'^{-1}] \circ \rho_\star$. Since by hypothesis $\rho_\star = \rho_\diamond$, this equation implies $[\varphi'] = 1$. But $[\varphi'] = \rho_\star[\varphi]$, so also $[\varphi] = 1$. Equation (5.9) then shows that \diamond and \star are scalar multiples of each other. \square

6. CLASSIFICATION OF CONJUGACY CLASSES OF TRIALITARIAN AUTOMORPHISMS

We next show that under the one-to-one correspondence of Theorem 5.8 similarity of symmetric compositions corresponds to conjugacy of trialitarian automorphisms.

Proposition 6.1. *Let $h: \diamond \rightarrow \star$ be a similarity of symmetric compositions on (S, n) . Then*

$$(6.2) \quad \rho_\star = \text{Int}[h] \circ \rho_\diamond \circ \text{Int}[h]^{-1}.$$

Conversely, if (6.2) holds for some $h \in \text{GO}(n)$, then some scalar multiple of h is a similarity $\diamond \rightarrow \star$.

Proof. Suppose first $h: \diamond \rightarrow \star$ is a similarity of symmetric compositions, so

$$(6.3) \quad h(x \diamond y) = h(x) \star h(y) \quad \text{for all } x, y \in S.$$

Let $\varphi \in \text{GO}^+(n)$. Theorem 4.5 yields φ' , $\varphi'' \in \text{GO}^+(n)$ such that

$$\varphi(x \star y) = \varphi'(x) \star \varphi''(y) \quad \text{for all } x, y \in S,$$

and $\rho_\star[\varphi] = [\varphi']$ by definition. Applying φ to each side of (6.3), we find

$$\varphi h(x \diamond y) = \varphi' h(x) \star \varphi'' h(y).$$

The right side is

$$h h^{-1} \varphi' h(x) \star h h^{-1} \varphi'' h(y) = h(h^{-1} \varphi' h(x) \diamond h^{-1} \varphi'' h(y)),$$

hence

$$h^{-1} \varphi h(x \diamond y) = h^{-1} \varphi' h(x) \diamond h^{-1} \varphi'' h(y) \quad \text{for all } x, y \in S.$$

Therefore, $\rho_\diamond([h^{-1} \varphi h]) = [h^{-1}] \cdot \rho_\star[\varphi] \cdot [h]$, proving (6.2).

Conversely, assume (6.2) holds, and set

$$x \triangleright y = h^{-1}(h(x) \star h(y)) \quad \text{for } x, y \in S,$$

so \triangleright is a symmetric composition similar to \star under h . The arguments above show that

$$\rho_{\triangleright} = \text{Int}[h^{-1}] \circ \rho_{\star} \circ \text{Int}[h] = \rho_{\diamond},$$

hence Theorem 5.8 implies that \diamond is a multiple of \triangleright . If $\lambda \in F^{\times}$ is such that $x \triangleright y = \lambda x \diamond y$ for all $x, y \in S$, then $\lambda h(x \diamond y) = h(x) \star h(y)$, hence

$$\lambda^{-1} h(x \diamond y) = \lambda^{-1} h(x) \star \lambda^{-1} h(y) \quad \text{for all } x, y \in S.$$

This equation shows that $\lambda^{-1} h: \diamond \rightarrow \star$ is a similarity. \square

For the following statement, recall from Proposition 3.6 that every symmetric composition is similar to a normalized symmetric composition, which is unique up to isomorphism.

Theorem 6.4. *Let \star and \diamond be symmetric compositions on (S, n) , and let $\bar{\star}, \bar{\diamond}$ be normalized symmetric compositions similar to \star and to \diamond respectively. The following conditions are equivalent:*

- (a) \star and \diamond are similar;
- (b) $\bar{\star}$ and $\bar{\diamond}$ are isomorphic;
- (c) ρ_{\star} and ρ_{\diamond} are conjugate in $\text{Aut}(\mathbf{PGO}^+(n))$ over F ;
- (d) $\rho_{\bar{\star}}$ and $\rho_{\bar{\diamond}}$ are conjugate in $\text{Aut}(\mathbf{PGO}^+(n))$ over F .

Proof. It is clear from the definition of $\bar{\star}$ and $\bar{\diamond}$ that (a) \iff (b). Since \star and $\bar{\star}$ are similar, Proposition 6.1 shows that ρ_{\star} and $\rho_{\bar{\star}}$ are conjugate in $\text{Aut}(\mathbf{PGO}^+(n))$ over F . Similarly, ρ_{\diamond} and $\rho_{\bar{\diamond}}$ are conjugate, hence it is clear that (c) \iff (d). Proposition 6.1 also yields (a) \implies (c), so it only remains to show (c) \implies (a).

Suppose ϕ is an automorphism of $\mathbf{PGO}^+(n)$ defined over F such that

$$\rho_{\star} = \phi \circ \rho_{\diamond} \circ \phi^{-1}.$$

Let $f \in \text{GO}(n)$ be an improper similitude. The restriction of $\text{Int}[f]$ to $\mathbf{PGO}^+(n)$ is an outer automorphism whose square is inner, hence each coset of $\text{Aut}(\mathbf{PGO}^+(n))$ modulo $\text{Int}(\mathbf{PGO}^+(n))$ is represented over F by an element from the set

$$\{\text{Id}, \rho_{\diamond}, \rho_{\diamond}^2, \text{Int}[f], \text{Int}[f] \circ \rho_{\diamond}, \text{Int}[f] \circ \rho_{\diamond}^2\}.$$

Since $\phi \circ \rho_{\diamond} \circ \phi^{-1}$ does not change when ϕ is multiplied on the right by ρ_{\diamond} or ρ_{\diamond}^{-1} , we may assume the coset of ϕ is represented by Id or by $\text{Int}[f]$, hence $\phi = \text{Int}[h]$ for some $h \in \text{GO}(n)$. It follows from Proposition 6.1 that \star and \diamond are similar. \square

Theorem 6.4 shows that the correspondence $\star \mapsto \rho_{\star}$ induces a one-to-one correspondence between similarity classes of symmetric compositions, or isomorphism classes of normalized symmetric compositions, and conjugacy classes of trialitarian automorphisms over F in $\text{Aut}(\mathbf{PGO}^+(n))$.

Corollary 6.5. (1) *Two symmetric compositions \star and \diamond on (S, n) are similar or antisimilar if and only if the subgroups generated by ρ_{\star} and ρ_{\diamond} in $\text{Aut}(\mathbf{PGO}^+(n))$ are conjugate.*
 (2) *Two normalized symmetric compositions \star and \diamond on (S, n) are isomorphic or anti-isomorphic if and only if the subgroups generated by ρ_{\star} and ρ_{\diamond} in $\text{Aut}(\mathbf{PGO}^+(n))$ are conjugate.*

Proof. The subgroups generated by ρ_\star and ρ_\diamond are conjugate if and only if ρ_\star is conjugate to ρ_\diamond or to ρ_\diamond^{-1} . Since $\rho_\diamond^{-1} = \rho_{\diamond^{\text{op}}}$ by Theorem 4.6, the corollary follows from Theorem 6.4. \square

To complete this section, we discuss fixed points of trialitarian automorphisms, which will be used to classify trialitarian automorphisms in Section 9. For any symmetric composition \star on (S, n) , we let $\mathbf{Aut}(\star)$ be the F -algebraic group of automorphisms of \star , whose group of F -rational points $\text{Aut}(\star)$ consists of the isomorphisms $f: \star \rightarrow \star$. Corollary 4.4 shows that $\mathbf{Aut}(\star) \subset \mathbf{GO}^+(n)$.

Theorem 6.6. *Let \star be a symmetric composition on (S, n) . The canonical map $\phi: \mathbf{GO}^+(n) \rightarrow \mathbf{PGO}^+(n)$ induces an isomorphism from $\mathbf{Aut}(\star)$ to the subgroup $\mathbf{PGO}^+(n)^{\rho_\star}$ of $\mathbf{PGO}^+(n)$ fixed under the trialitarian automorphism ρ_\star .*

Proof. If $f \in \text{Aut}(\star)$, then $\rho_\star[f] = [f]$, hence ϕ gives rise to a canonical map

$$\psi: \mathbf{Aut}(\star) \rightarrow \mathbf{PGO}^+(n)^{\rho_\star}.$$

To prove that ψ is an isomorphism it suffices to show that $\text{Ker } \psi = 1$ and that ψ is a quotient map (see [Wat79, Corollary 15.4]). The kernel of ψ represents the functor which takes a commutative F -algebra A to $\text{Ker}[\mathbf{Aut}(\star)(A) \rightarrow \mathbf{PGO}^+(n)^{\rho_\star}(A)]$. Let a be in this kernel. Note that $\text{Ker } \psi \subset \text{Ker } \phi = \mathbf{G}_m$. Hence a is a homothety $r \text{Id}_{S_A}$ for some $r \in A^\times$. But such a map is in $\mathbf{Aut}(\star)(A)$ if and only if $r = 1$. Thus $\text{Ker } \phi$ represents the trivial functor and so $\text{Ker } \phi = 1$. To see that ψ is a quotient map we need to show that for every commutative F -algebra A and every $g \in \mathbf{PGO}^+(n)^{\rho_\star}(A)$ there exists a faithfully flat extension $A \rightarrow B$ and an element $f \in \mathbf{Aut}(\star)(B)$ such that $\psi(f) = g$ (see [Wat79, Theorem 15.5]). Since ϕ is a quotient map we can find an F -algebra B and an element $f_0 \in \mathbf{GO}^+(n)(B)$ such that $\phi(f_0) = g$. Since g is fixed by ρ_\star there are scalars $\alpha, \beta \in B^\times$ such that

$$f_0(x \star y) = \alpha f_0(x) \star \beta f_0(y) \quad \text{for all } x, y \in S_B.$$

Let $f = \alpha\beta f_0$. Then f is an automorphism of \star and $\psi(f) = \phi(f_0) = g$. \square

7. THE TWIST OF A SYMMETRIC COMPOSITION

As in the preceding sections, we fix a 3-fold Pfister quadratic space (S, n) over F . If \star and \diamond are symmetric compositions on (S, n) , then the trialitarian automorphisms ρ_\star and ρ_\diamond are related by inner automorphisms of $\mathbf{PGO}^+(n)$: in view of the exact sequence (1.1), we have either

$$\rho_\star \equiv \rho_\diamond \pmod{\text{Int}(\mathbf{PGO}^+(n))} \quad \text{or} \quad \rho_\star \equiv \rho_\diamond^{-1} \pmod{\text{Int}(\mathbf{PGO}^+(n))}.$$

We use this observation to relate the symmetric compositions \star and \diamond .

Proposition 7.1. *Let \star and \diamond be two symmetric compositions on (S, n) . If $\rho_\star \equiv \rho_\diamond \pmod{\text{Int}(\mathbf{PGO}^+(n))}$, then there exist $f, g \in \mathbf{GO}^+(n)$ such that*

$$(7.2) \quad x \diamond y = f(x) \star g(y) \quad \text{for all } x, y \in S.$$

If $\rho_\star \equiv \rho_\diamond^{-1} \pmod{\text{Int}(\mathbf{PGO}^+(n))}$, then there exist $f, g \in \mathbf{GO}^+(n)$ such that

$$x \diamond y = g(y) \star f(x) \quad \text{for all } x, y \in S.$$

Proof. Suppose first $\rho_\diamond = \text{Int}[f^{-1}] \circ \rho_\star$ for some $f \in \text{GO}^+(n)$. Since $\rho_\diamond^3 = \text{Id}$, we have $\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1$. Let $g \in \text{GO}^+(n)$ be such that $[g] = \rho_\star^2[f^{-1}]$, and define

$$x \triangleright y = f(x) \star g(y) \quad \text{for } x, y \in S.$$

By Lemma 5.2, \triangleright is a symmetric composition on (S, n) , and

$$\rho_{\triangleright} = \text{Int}[f^{-1}] \circ \rho_\star = \rho_\diamond.$$

Therefore, Theorem 5.8 shows that \triangleright is a scalar multiple of \diamond . Scaling g , we may assume (7.2) holds.

If $\rho_\star \equiv \rho_\diamond^{-1} \pmod{\text{Int}(\mathbf{PGO}^+(n))}$, then $\rho_{\star^{\text{op}}} \equiv \rho_\diamond \pmod{\text{Int}(\mathbf{PGO}^+(n))}$ since $\rho_{\star^{\text{op}}} = \rho_\star^{-1}$ by Theorem 4.6. The first part of the proof yields $f, g \in \text{GO}^+(n)$ such that

$$x \diamond y = f(x) \star^{\text{op}} g(y) = g(y) \star f(x) \quad \text{for } x, y \in S.$$

□

When the symmetric composition \diamond is given by (7.2), then we must have $[g] = \rho_\star^2[f^{-1}]$, by Lemma 5.2. The map g is therefore uniquely determined by f up to a scalar factor, and we say \diamond is a *twist* of \star through the similarity f . Thus, by Proposition 7.1, given a symmetric composition \star on (S, n) , every symmetric composition on (S, n) is a twist of \star or \star^{op} .

Twisting has its origin in Petersson [Pet69], where the following special case is considered: suppose $f: \star \rightarrow \star$ is a similarity (i.e., f is an automorphism of \star), and $f^3 = \text{Id}_S$. Since f is an automorphism, we have $f \in \text{GO}^+(n)$ and $\rho_\star[f] = [f]$: see Theorem 6.6. Since $f^3 = \text{Id}_S$, we have $\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1$, hence we may choose $g = f^{-1}$ in the discussion above. By Lemma 5.2, the product

$$(7.3) \quad x \star_f y = f(x) \star f^{-1}(y) \quad \text{for } x, y \in S$$

defines a symmetric composition.

The idea of twisting was further developed in [Eld00].

Corollary 7.4. *For every symmetric composition \diamond on (S, n) and every anisotropic vector $x \in S$, the similarities ℓ_x^\diamond and r_x^\diamond are improper.*

Proof. Let \star be a para-octonion composition on (S, n) (see Example 3.4). The similarities ℓ_1^\star and r_1^\star , where 1 is the identity of the octonion algebra, coincide with the conjugation map, which is an improper isometry. Therefore, by Lemma 5.1, the similarities ℓ_x^\star and r_x^\star are improper for every anisotropic vector $x \in S$. Proposition 7.1 shows that there exist $f, g \in \text{GO}^+(n)$ such that either

$$x \diamond y = f(x) \star g(y) \quad \text{for all } x, y \in S \quad \text{or} \quad x \diamond y = g(y) \star f(x) \quad \text{for all } x, y \in S.$$

Therefore, for every anisotropic vector $x \in S$ we have either $\ell_x^\diamond = \ell_{f(x)}^\star \circ g$ and $r_x^\diamond = r_{g(x)}^\star \circ f$, or $\ell_x^\diamond = \ell_{f(x)}^\star \circ g$ and $r_x^\diamond = \ell_{g(x)}^\star \circ f$. The similarities ℓ_x^\diamond and r_x^\diamond are therefore improper. □

8. SPIN GROUPS AND SYMMETRIC COMPOSITIONS

Let \star be a normalized symmetric composition on a 3-fold Pfister quadratic space (S, n) . We briefly point out in this section how the results of the preceding sections can be modified to apply to the group $\mathbf{Spin}(n)$ instead of $\mathbf{PGO}^+(n)$.

Recall that the F -rational points of the F -algebraic group $\mathbf{Spin}(n)$ are given by

$$\text{Spin}(n) = \{c \in C_0(S, n)^\times \mid cSc^{-1} \subset S \quad \text{and} \quad c\sigma(c) = 1\}$$

(see for example [KMRT98, 35.C.]). The isomorphism α_0^* introduced in Proposition 4.1 can be used to give a convenient description of $\mathbf{Spin}(n)$: for $c \in \mathbf{Spin}(n)$, define $f, f_1, f_2 \in \text{End}_F S$ as follows:

$$f(x) = cxc^{-1} \text{ for } x \in S, \quad \alpha_0^*(c) = (f_1, f_2).$$

The map f is a proper isometry. Similarly, since $c\sigma(c) = 1$ and α_0^* is an isomorphism of algebras with involution, f_1 and f_2 are isometries. We have $C_0[f] = \text{Int}(c)|_{C_0(S,n)}$, hence the diagram $D_\star^+(f, f_1, f_2)$ commutes. Therefore, f_1 and f_2 are proper isometries. Let $\mathbf{O}^+(n)$ be the group of proper isometries of (S, n) . The following result is shown in [KMRT98, (35.7)] (assuming $\text{char } F \neq 2$):

Proposition 8.1. *The map $c \mapsto (f, f_1, f_2)_\star$ defines an isomorphism*

$$\mathbf{Spin}(n) \xrightarrow{\sim} \{(f, f_1, f_2)_\star \mid f_i \in \mathbf{O}^+(n)(F), f(x \star y) = f_1(x) \star f_2(y), x, y \in S\}.$$

Moreover any of the three relations

$$\begin{aligned} f(x \star y) &= f_1(x) \star f_2(y) \\ f_1(x \star y) &= f_2(x) \star f(y) \\ f_2(x \star y) &= f(x) \star f_1(y) \end{aligned}$$

implies the two others.

Observe that the representation of elements of $\mathbf{Spin}(n)$ as triples of elements of $\mathbf{O}^+(n)$ depends on the choice of the composition \star , hence the notation $(f, f_1, f_2)_\star$.

We have an obvious trialitarian automorphism $\widehat{\rho}_\star$ of $\mathbf{Spin}(n)$ defined over F by

$$\widehat{\rho}_\star : (f, f_1, f_2)_\star \mapsto (f_1, f_2, f)_\star.$$

The center \mathbf{Z} of $\mathbf{Spin}(n)$ is the scheme μ_2^2 with $\mu_2(F) = \pm 1$. Viewing $\mathbf{Z}(F)$ as kernel of the multiplication map

$$\mu_2^3(F) \rightarrow \mu_2(F), \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \varepsilon_1\varepsilon_2\varepsilon_3,$$

the group scheme \mathbf{Z} admits a natural \mathfrak{S}_3 -action. The exact sequence of schemes

$$(8.2) \quad 1 \rightarrow \mathbf{Z} \rightarrow \mathbf{Spin}(n) \rightarrow \mathbf{PGO}^+(n) \rightarrow 1,$$

where the morphism $\mathbf{Spin}(n) \rightarrow \mathbf{PGO}^+(n)$ is induced by $(f, f_1, f_2)_\star \mapsto [f]$, is \mathfrak{S}_3 -equivariant (see [KMRT98, (35.13)]). Thus the trialitarian action on $\mathbf{Spin}(n)$ is a lift of the trialitarian action on $\mathbf{PGO}^+(n)$.

Most of the results on trialitarian actions on $\mathbf{PGO}^+(n)$ hold for $\mathbf{Spin}(n)$. Moreover, in a similar way we have a natural action of \mathfrak{S}_3 on $\mathbf{Spin}(n)$. Let \mathbf{H} be the semidirect product of $\mathbf{Spin}(n)$ and \mathfrak{S}_3 .

Theorem 8.3. *Two normalized symmetric compositions \star and \diamond on (S, n) are isomorphic if and only if the trialitarian automorphisms $\widehat{\rho}_\star$ and $\widehat{\rho}_\diamond$ are conjugate over F in \mathbf{H} .*

One of the main steps in the proof for $\mathbf{PGO}^+(n)$ was to show that any trialitarian automorphism is induced by a symmetric composition. We describe this step for $\mathbf{Spin}(n)$.

Proposition 8.4. *Every trialitarian automorphism of $\mathbf{Spin}(n)$ over F in \mathbf{H} has the form $\widehat{\rho}_\diamond$ for some normalized symmetric composition \diamond .*

Proof. We show how to modify the proof of the corresponding statement in Theorem 5.8 for $\mathbf{PGO}^+(n)$. Let τ be a trialitarian automorphism of $\mathbf{Spin}(n)$ over F , let \star be a symmetric composition on (S, n) and let $\widehat{\rho}_\star$ be the associated trialitarian automorphism. In view of the exact sequence (1.1), we have either $\tau \equiv \widehat{\rho}_\star$ or $\tau \equiv \widehat{\rho}_\star^{-1} \pmod{\mathbf{Spin}(n)}$. Substituting \star^{op} for \star if necessary, we may assume $\tau \equiv \widehat{\rho}_\star \pmod{\mathbf{Spin}(n)}$, hence there exists $(h, h_1, h_2)_\star \in \mathbf{Spin}(n)$ such that

$$\tau = \text{Int}((h, h_1, h_2)_\star^{-1}) \circ \widehat{\rho}_\star,$$

which means that for all $(f, f_1, f_2)_\star \in \mathbf{Spin}(n)$,

$$\tau((f, f_1, f_2)_\star) = (h^{-1}f_1h, h_1^{-1}f_2h_1, h_2^{-1}fh_2)_\star.$$

It follows from $\tau^3 = \widehat{\rho}_\star^3 = \text{Id}$ that $h_2h_1h = \text{Id}_S$. Let

$$x \diamond y = h(x) \star h_2^{-1}(y) \quad \text{for } x, y \in S.$$

One deduces easily from $h_2h_1h = \text{Id}_S$ that $(x \diamond y) \diamond x = x \diamond (y \diamond x) = n(x)y$ for $x, y \in S$. Thus \diamond is a normalized symmetric composition on (S, n) . For $(f, f_1, f_2)_\star \in \mathbf{Spin}(n)$ and $x, y \in S$ we have

$$\begin{aligned} (8.5) \quad f(x \diamond y) &= f(h(x) \star h_2^{-1}(y)) \\ &= f_1h(x) \star f_2h_2^{-1}(y) \\ &= h^{-1}f_1h(x) \diamond h_2f_2h_2^{-1}(y) \end{aligned}$$

Let

$$\mathbf{Spin}(n) = \{(f, f'_1, f'_2)_\diamond \mid f, f'_1, f'_2 \in \mathbf{O}^+(n), f(x \diamond y) = f'_1(x) \star f'_2(y), x, y \in S\}$$

be the presentation of $\mathbf{Spin}(n)$ using the symmetric composition \diamond . It follows from (8.5) that

$$f'_1 = h^{-1}f_1h \quad \text{and} \quad f'_2 = h_2^{-1}f_2h_2,$$

so that the passage from the presentation of $\mathbf{Spin}(n)$ using \star to the presentation of $\mathbf{Spin}(n)$ using \diamond is described as

$$(8.6) \quad (f, f_1, f_2)_\star \mapsto (f, h^{-1}f_1h, h_2f_2h_2^{-1})_\diamond.$$

Since $\widehat{\rho}_\diamond(f, f'_1, f'_2)_\diamond = (f'_1, f'_2, f)_\diamond$, we get, using (8.6),

$$\widehat{\rho}_\diamond((f, f_1, f_2)_\star) = (h^{-1}f_1h, hh_2f_2h_2^{-1}h^{-1}, h_2^{-1}fh_2)_\star$$

so that the relation $h_2h_1h = \text{Id}_S$ implies $\tau = \widehat{\rho}_\diamond$. \square

9. CLASSIFICATION OF SYMMETRIC COMPOSITIONS

It follows from Theorem 6.4 that the classification of symmetric compositions up to similarity, the classification of *normalized* symmetric compositions up to isomorphism, and the classification of trialitarian automorphisms up to conjugation are essentially equivalent. In this section we recall the classification of normalized symmetric compositions of dimension 8 over arbitrary fields.

We first consider symmetric compositions over algebraically closed fields. The norm form n is then hyperbolic. Octonion algebras over such fields are split and it is convenient to choose as a model the Zorn algebra, which is defined in arbitrary characteristic.

The Zorn algebra. We denote by \bullet the usual scalar product on $F^3 = F \times F \times F$, and by \times the vector product: for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in F^3$,

$$a \bullet b = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{and} \quad a \times b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The Zorn algebra is the set of matrices

$$\mathfrak{Z} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in F, a, b \in F^3 \right\}$$

with the product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a \bullet d & \alpha c + \delta a - b \times d \\ \gamma b + \beta d + a \times c & \beta\delta + b \bullet c \end{pmatrix},$$

the norm

$$n \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = \alpha\beta - a \bullet b,$$

and the conjugation

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix},$$

which is such that $\xi \cdot \bar{\xi} = \bar{\xi} \cdot \xi = n(\xi)$ for all $\xi \in \mathfrak{Z}$ (see [Zor30, p. 144]).

In view of Example 3.4 the product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \star \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} \cdot \overline{\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}} = \begin{pmatrix} \beta\delta + a \bullet d & -\beta c - \gamma a - b \times d \\ -\delta b - \alpha d + a \times c & \alpha\gamma + b \bullet c \end{pmatrix},$$

defines a symmetric composition on the quadratic space (\mathfrak{Z}, n) . We call \star the *para-Zorn composition*.

We use the technique of Section 7 to twist the para-Zorn composition. Let $p: F^3 \rightarrow F^3$ be the map $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$, and let

$$\pi: \mathfrak{Z} \rightarrow \mathfrak{Z}, \quad \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = \begin{pmatrix} \alpha & p(a) \\ p(b) & \beta \end{pmatrix}.$$

The map π is an automorphism of \star of order 3, so we may consider the twisted composition \star_π as in (7.3), which we call the *split Petersson symmetric composition*. (The algebra $(\mathfrak{Z}, \star_\pi)$ is also known as the *split pseudo-octonion algebra*.)

Theorem 9.1. *Over an algebraically closed field, there are exactly two symmetric compositions up to isomorphism: the para-Zorn composition and the split Petersson composition.*

Proof. The claim is a result of Petersson [Pet69, Satz 2.7] if the field has characteristic different from 2 and 3, and is due to Elduque-Pérez [EP96] in arbitrary characteristic. \square

Over arbitrary fields we consider two kinds of symmetric compositions, which we call type I and type II. Symmetric compositions of *type I* are forms of the para-Zorn composition, i.e., \star is of *type I* if and only if \star is isomorphic to the para-Zorn composition after scalar extension to an algebraic closure. Similarly, we say that \star is of *type II* if it is a form of the split Petersson composition. Thus, Theorem 9.1 shows that every symmetric composition is either of type I or of type II.

Type I. The classification of symmetric compositions of type I is particularly simple.

Theorem 9.2. *Let (S, n) be a 3-fold Pfister quadratic space over an arbitrary field F . Up to isomorphism, there is a unique normalized symmetric composition of type I on (S, n) , given by the para-octonion composition.*

Proof. Forms of the split para-octonion algebra are para-octonion algebras (see for example [KMRT98, §34 A]), hence symmetric compositions of type I are para-octonion compositions as defined in Example 3.4. Isomorphisms of octonion algebras are isomorphisms of the corresponding para-octonion algebras, and octonion algebras with isometric norms are isomorphic (see for example [KMRT98, (33.19)]). \square

The automorphism group of a symmetric composition of type I is the automorphism group of the corresponding octonion algebra; it is a simple algebraic group of type G_2 . Accordingly, symmetric compositions of type I are also called *symmetric compositions of type G_2* .

Type II. The classification of symmetric compositions of type II has a completely different flavor in characteristic 3. We first discuss the case where the characteristic is different from 3, and distinguish two subcases, depending on whether the base field contains a primitive cube root of unity or not.

Suppose first F is a field of characteristic different from 3 containing a primitive cube root of unity ω . Let A be a central simple F -algebra of degree 3. For the reduced characteristic polynomial of $a \in A$, we use the notation

$$X^3 - \text{Trd}(a)X^2 + \text{Srd}(a)X - \text{Nrd}(a)1,$$

so Trd is the reduced trace map on A , Srd is the reduced quadratic trace map, and Nrd is the reduced norm. Let $A^0 \subset A$ be the kernel of Trd . We define a multiplication \star on A^0 by

$$(9.3) \quad x \star y = \frac{yx - \omega xy}{1 - \omega} - \frac{1}{3} \text{Trd}(xy)1$$

and a quadratic form n by

$$(9.4) \quad n(x) = -\frac{1}{3} \text{Srd}(x).$$

The following result can be found for instance in [KMRT98, (34.19), (34.25)]:

Proposition 9.5. *The quadratic space (A^0, n) is hyperbolic, and \star is a normalized symmetric composition on (A^0, n) .*

When the algebra A is split, the composition \star is isomorphic to the split Petersson composition. Symmetric compositions as in Proposition 9.5 are called *Okubo compositions*. The automorphism group of an Okubo composition associated to a central simple algebra A is $\mathbf{PGL}_1(A)$. Therefore, these compositions are also called *symmetric compositions of type 1A_2* .

Suppose next F is a field of characteristic different from 3 that does not contain a primitive cube root of unity. Let ω be a primitive cube root of unity in some separable closure of F , and let $K = F(\omega)$, a separable quadratic extension of F . Let B be a central simple K -algebra of degree 3 with a unitary involution τ leaving F fixed. We let $\text{Sym}(B, \tau)^0$ denote the F -vector space of τ -symmetric elements of

reduced trace zero. Formula (9.3) defines a multiplication on $\text{Sym}(B, \tau)^0$, and the form n of (9.4) is a quadratic form on $\text{Sym}(B, \tau)^0$.

The following result is proved in [KMRT98, (34.35)]:

Proposition 9.6. *The quadratic space $(\text{Sym}(B, \tau)^0, n)$ is a 3-fold Pfister quadratic space that becomes hyperbolic over K , and \star is a normalized symmetric composition on this space.*

Normalized symmetric compositions as in Proposition 9.6 are also called Okubo compositions. The automorphism group of an Okubo composition associated to a unitary involution τ on a central simple K -algebra B is $\mathbf{PGU}(B, \tau)$. Therefore, these compositions are also called *symmetric compositions of type 2A_2* .

Theorem 9.7. *Let (S, n) be a 3-fold Pfister quadratic space over a field F of characteristic different from 3. Let ω be a primitive cube root of unity in a separable closure of F .*

- (1) *If $\omega \in F$, then every normalized symmetric composition of type II on (S, n) is isomorphic to the Okubo composition associated to some central simple F -algebra of degree 3, uniquely determined up to isomorphism. Such compositions exist if and only if n is hyperbolic.*
- (2) *If $\omega \notin F$, let $K = F(\omega)$. Every normalized symmetric composition of type II on (S, n) is isomorphic to the Okubo composition associated to some central simple K -algebra of degree 3 with unitary involution, uniquely determined up to isomorphism. Such compositions exist if and only if n is split by K .*

Proof. See Elduque-Myung [EM93, p. 2487] or [KMRT98, (34.37)]. \square

Now, suppose the characteristic of F is 3. Normalized symmetric compositions of type II over F are extensively discussed in [EP96], [Eld97], [Eld99] and [Eld00]. They can be viewed as a specialization of symmetric compositions of type II over fields containing a primitive cube root of unity. To introduce them we first consider symbol algebras of degree 3. Any central simple algebra A of degree 3 over a field containing a cubic primitive root of unity ω is a symbol algebra, i.e., A has generators x, y such that $x^3 = a, y^3 = b$ and $yx = \omega xy$ for some $a, b \in F^\times$. We consider the Okubo composition \star associated with A . The following basis of A^0 :

$$(9.8) \quad \begin{array}{cccc} e_1 = x & e_2 = y & e_3 = \omega^2 xy & e_4 = \omega xy^{-1} \\ f_1 = x^{-1} & f_2 = y^{-1} & f_3 = \omega^2 x^{-1}y^{-1} & f_4 = \omega x^{-1}y \end{array}$$

is hyperbolic for the form n and the multiplication table with respect to this basis of the product \star is given by

	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	af_1	0	0	$-e_4$	0	$-f_2$	$-af_3$	0
f_1	0	$a^{-1}e_1$	$-f_4$	0	$-e_2$	0	0	$-a^{-1}e_3$
e_2	$-e_3$	0	bf_2	0	$-be_4$	0	$-e_1$	0
f_2	0	$-f_3$	0	$b^{-1}e_2$	0	$-b^{-1}f_4$	0	$-f_1$
e_3	$-af_4$	0	0	$-e_1$	abf_3	0	0	$-bf_2$
f_3	0	$-a^{-1}e_4$	$-f_1$	0	0	$a^{-1}b^{-1}e_3$	$-b^{-1}e_2$	0
e_4	0	$-f_2$	0	$-b^{-1}e_3$	$-af_1$	0	$a^{-1}bf_4$	0
f_4	$-e_2$	0	$-bf_3$	0	0	$-a^{-1}e_1$	0	$ab^{-1}e_4$

This multiplication table does not involve ω , hence specialising gives a symmetric composition $\diamond_{a,b}$ over arbitrary fields, also of characteristic 3.

Proposition 9.9. 1) *The symmetric composition $\diamond_{1,1}$ is isomorphic to the split Petersson composition.*

2) *Let \overline{F} be an algebraic closure of F and let $c, d \in \overline{F}$ be such that $c^3 = a$ and $d^3 = b$. The symmetric compositions $\diamond_{a,b}$ and $\diamond_{1,1}$ are isomorphic over $F(c,d)$. Thus the symmetric composition $\diamond_{a,b}$ splits over $F(c,d)$.*

Proof. 1) and 2) follow by comparing multiplication tables. \square

Theorem 9.10. *Let (S, n) be a 3-fold Pfister quadratic space over a field F of characteristic 3. Every normalized symmetric composition of type II on (S, n) is isomorphic to a symmetric composition of the form $\diamond_{a,b}$ for some $a, b \in F^\times$.*

Proof. The claim follows from [Eld99, p. 291] and a comparison of multiplication tables. \square

Conditions for isomorphism $\diamond_{a,b} \xrightarrow{\sim} \diamond_{a',b'}$ are discussed in [Eld99].

The automorphism groups (as group schemes) of the compositions $\diamond_{a,b}$ are not smooth in characteristic 3 and are, as far as we know, not studied in the literature. Their groups of rational points are described in [Eld99].

Observe that symmetric compositions of type II in characteristic 3 are not necessarily split over separably closed fields, in contrast to symmetric compositions of type I and to symmetric compositions of type II in characteristic different from 3.

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