Cohomology of buildings and finiteness properties of S-arithmetic groups over function fields.

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S-arithmetic subgroups Γ of reductive algebraic groups G over *number fields* are finitely presented and contain a torsion-free subgroup of finite index, which is of type FL (Ragunathan 1968, Borel-Serre 1976), therefore they are of type FP_∞, i.e. there exists a projective resolution

$$P_m \to P_{m-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of finitely generated $\mathbb{Z}\Gamma$-modules P_i for all m, and also of type F_∞, i.e. there exists an Eilenberg-MacLane complex $K(\Gamma, 1)$ with finite m-skeleton for all m (cf. [Br2], VIII).

For *function fields* F ($[F : \mathbb{F}_q(t)] < \infty,\ q = p^k,\ p = char F$) however, many counter-examples are known: $SL_2(\mathbb{F}_q[t])$ is not even finitely generated, i.e. not of type F_1 (Nagao 1959, Serre 1968), $SL_2(\mathbb{F}_q[t, t^{-1}])$ and $SL_3(\mathbb{F}_q[t])$ are finitely generated, but not finitely presented, i.e. of type F_1, not F_2 (Stuhler 1976, Behr 1977); for the S-arithmetic ring O_S (S a finite, non-empty set of primes of F), $SL_2(O_S)$ is of type $F_{|S|-1}$, but not $F_{|S|}$ (Stuhler 1980), $SL_n(\mathbb{F}_q[t])$ is of type F_{n-2} but not F_{n-1} as long as $q \geq 2^{n-2}$ (Abels 1989) or $q \geq \binom{n-2}{[\frac{n-2}{2}]}$ (Abramenko 1987) and similar results hold for absolutely almost simple $\mathbb{F}_q$-groups G and $\Gamma = G(\mathbb{F}_q[t])$ (Abramenko 1994), the positive part was proved without restriction on q for Chevalley groups G, $\Gamma = G(O_S), |S| = 1$ (Behr 2004).

All these results provided evidence for the following *conjecture*: If G is an absolutely almost simple algebraic group, defined over F with F-rank $r > 0$ ("isotropic group"), $r_v = rank_{F_v}\ G$ for the completion F_v of F ($v \in S, S$ finite, $S \neq \emptyset$), Γ S-arithmetic subgroup (discrete in $G_S = \Pi_{v \in S} G(F_v)$), then Γ is of type F_{d-1}, but not of type F_d for $d = \sum_{v \in S} r_v$.

This conjecture was proved for the classical properties finite generation (iff $d \geq 2$, Behr 1969, Keller 1980) and finite presentation (iff $d \geq 3$: Behr 1998) and the negative part for arbitrary d (Bux-Wortman 2007). Moreover for anisotropic G (i.e. $rk_F\ G = 0$) it was known (Serre 1971), that Γ is of type F_∞.

The positive part is now proved in a preprint by Bux – Gramlich – Witzel ([BGW], 2011). This paper gives a completely different and relatively short proof, using two old results on cohomology, which has the advantage not to need such precise local informations, which are necessary in (almost) all proofs of the results mentioned above, and, also in [BGW], which use filtrations of buildings, defined in very clever ways.

In 1976 Borel-Serre also computed the cohomology of spherical and affine buildings over non-archimedean local fields. If X_v is the Bruhat-Tits-building of $G(F_v)$ (G a semi-simple F-group, $v \in S$), then the product $X = \Pi_{v \in S} X_v$ is a contractible polysimplicial complex. X gives rise to a chain-complex $C = (C_n)_{n\in\mathbb{N}}$ with $\mathbb{Z}\Gamma$-modules C_n, generated by polysimplices, but these are (in general) not projective nor finitely generated. Borel-Serre proved that the reduced cohomology with compact support $\tilde{H}^i_c(X; M)$ for a $\mathbb{Z}\Gamma-$ module M vanishes in all dimensions, except for the top-dimension $d = \sum_{v\in S} d_v,\ d_v = dim X_v = r_v = rank_{F_v}\ G$. For $H^d_c(X; M)$ they gave an explicit description by locally-constant functions on unipotent groups. In section 1 we present a short version of their results.

On the other hand, K. Brown found in 1975 a very interesting cohomological criterion for finiteness properties. In his proof he constructed for a Γ-complex C_n of projective modules another complex C'_n with the same homology, but finitely generated Γ-modules. His assumptions are not all valid in our case and so we cannot use his general arguments. In section 2 we use his construction, but we must be more explicit, on the other side our situation is more special: The crucial point is Borel-Serre's vanishing result for cohomology with compact supports; in some sense, this substitutes the notion of being essentially trivial for filtrations (see [Br3],2). We obtain a partial resolution of $\mathbb{Z}$ by finitely generated free Γ-modules up to dimension $d-1$, so Γ is of type FP_{d-1}. By this method we cannot prove type F_{d-1}, but this is implied together with finite presentation for $d \geq 3$ ([B3]).

In section 3 we give a proof for the negative part, which is based on the same ideas as the proof in [BW], but the computations of Borel-Serre provide a more natural construction of cycles, that also illustrates the geometry of Bruhat-Tits-buildings: we find arbitrary big spherical holes in a mod Γ compact subcomplex X_0 of X, which only become boundaries of cones, arbitrary far from X_0.

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1 Borel-Serre: Cohomology of buildings and S-arithmetic groups (see [BS])

1.1 Spherical or Tits buildings

Let k be a non-archimedean local field, G a connected semi-simple k-group of rank $l \geq 1$ and Y the Tits-building of $G(k)$. It is well-known by the Solomon-Tits-theorem that Y has the homotopy-type of a bouquet of $(l-1)$-spheres with respect to its simplicial topology. Borel-Serre provide Y with the analytic topology, induced by the valuation on k and prove an analogue for the Alexander-Spanier-cohomology (cf. [Sp1] or [Sp2]).

Denote by P a minimal k-parabolic subgroup of G and by C the closed chamber of Y fixed by $P(k)$. Then Y can be described as $G(k)\big/P(k) \times C$ with identifications. The homogeneous space $G(k)\big/P(k) = (G\big/P)(k)$ is compact for the k-analytic topology, C carries the simplicial topology and Y gets the quotient-topology: Y_t.

We have $P(k) = Z_G(T)(k) \cdot U(k)$, where T is a maximal k-split torus of G, Z_G its centralizer and U the unipotent radical of P, W the corresponding Weyl group. The Bruhat-decomposition $G(k) = P(k)WP(k) = U(k)WP(k)$ gives a more concrete description of Y. Especially for the longest element w_0 of W this product-decomposition is unique; therefore we have a 1-1-correspondence between $U(k)$ and the set $\mathcal{C}_P$ of all chambers opposite to C, defined by $u \mapsto u \cdot w_0 C$. Since two opposite chambers determine an apartment (cf. [BT], 4), the set $\mathcal{A}_P$ of all apartments containing C is also in 1-1-correspondence with $U(k)$ and thus inherits a k-analytic structure from that of $U(k)$. $\mathcal{A}_P = \{uA_o | u \in U(k)\}$, where A_o is the apartment, defined by the opposite pair $(C, w_0 C)$ and fixed by $Z_G(T)(k) = P(k) \cap w_0 P(k) w_0^{-1}$. In this setting Borel-Serre can compute the (reduced) Alexander-Spanier cohomology $H^*(Y_t; M)$ for a $\mathbb{Z}$-module M, using the group $C_c^\infty(\mathcal{A}_P; M)$ of locally constant functions with compact support on $\mathcal{A}_P$ or $U(k)$:

Proposition 1. *(=[BS], thm. 2.6)*

(i) $H^i(Y_t; M) = 0$ *for* $i \neq l-1$

(ii) $\tilde{H}^{l-1}(Y_t; M) \simeq C_c^\infty(U(k); M)$

1.2 Affine or Bruhat-Tits-buildings

The affine (or euclidean) building X of $G(k)$ (for a non-archimedean local field k) is more conveniently defined for the simply-connected covering $\tilde{G}$ of G in order to obtain X as a product of the buildings X_j for the almost simple factors G_j of $\tilde{G}$ – so it is a polysimplicial complex (see [BS], 4). An important part of this paper

([BS], 5) consists of the construction of a compactification of X by adding Y as a boundary at infinity. Thereby the direct sum $Z = X \amalg Y$ becomes a contractible compact space Z_t, inducing the natural topology on X and Y_t on Y.

Using the long exact cohomology sequence for $Z_t \mod Y_t$

$$\cdots \to \tilde{H}^i(Z_t; M) \to \tilde{H}^i(Y_t; M) \to H_c^{i+1}(X; M) \to H^{i+1}(Z_t; M) \to \ldots$$

where M is a module over a ring R and H_c^* denotes the cohomology with compact supports and moreover by the vanishing of $\tilde{H}^*(Z_t; M)$ we can transfer proposition 1 to

Proposition 2. *(=[BS], thm. 5.6)*

(i) $H_c^i(X; M) = 0$ *for* $i \neq l$

(ii) $H_c^l(X; M) \simeq \begin{cases} H^{l-1}(Y_t; M) & \text{if } l \geq 1 \\ M & \text{if } l = 0 \end{cases}$

in particular for $R = \mathbb{Z}$ *and* $l \geq 1$*:*

$$H_c^l(X; M) \simeq C_c^\infty(U(k); M)$$

Remark: A function $f \in C_c^\infty(U(k); M)$ has compact support and is locally constant, so there exists a finite union of open subsets of $U(k)$, such that f is constant on each of them. This union corresponds to a neighbourhood of Y_t in Z_t and f is determined on its compact complement on X.

1.3 S-arithmetic groups over function fields

Let F be a function field (i.e. $[F : \mathbb{F}_q(t)] < \infty,\ q = p^m,\ p = char\ F$) with a finite non-empty set S of places of F and F_v the completion of F with respect to $v \in S$.

G denotes a connected semi-simple algebraic F-group of rank r, $r_v := rank_{F_v}\ G$ $(v \in S)$, $G_S := \Pi_{v \in S} G(F_v);\ X = \Pi_{v \in S} X_v$ with Bruhat-Tits-buildings X_v of $G(F_v)$, $dim\ X_v = d_v = r_v$ and $d = dim\ X = \sum_{v \in S} d_v$. Finally Γ is a S-arithmetic subgroup, discrete in G_S.

G is called isotropic if $r > 0$ and anisotropic if $r = 0$. It is well known from reduction theory ("Godement's compactness criterion"; cf. [H] or [B1]) that X/Γ is compact iff G is anisotropic.

For this cocompact case Borel-Serre prove (thm. 6.2 in [BS])

Proposition 3. *A S-arithmetic subgroup Γ of an anisotropic connected semi-simple group G over a function field F is finitely presented and of type FP_∞ and also of type F_∞.*

Remarks:

a) More precisely they show that Γ has a torsion free subgroup Γ_0 of finite index, which is of type FL and so of type $F(P)_\infty$, thus Γ inherits this properties.

b) Moreover $H^i(\Gamma_0;\mathbb{Z}\Gamma_0) \simeq \begin{cases} 0 & \text{for } i \neq d \\ H_c^d(X;\mathbb{Z}), & \text{thus free} \end{cases}$

c) In the "number-field-case" (over a number field K instead of F) all these results are valid for arbitrary S-arithmetic groups, i.e. also for isotropic G.

In the function-field-case Bux and Wortman proved in [BW], that Γ can be of type F_∞ only for anisotropic G: They give a bound for the "finiteness length" ($\max\{n : \Gamma \text{ has type } F_n\}$) for isotropic groups. To obtain a sharp bound, one should restrict to absolutely almost simple groups: A simply connected semi-simple group G is the direct product of its almost simple factors G_i and so is $\Gamma = \prod \Gamma_i,\ \Gamma_i$ in G_i. For instance Γ can only be finitely generated if all Γ_i are so. In this situation they show, that Γ cannot be of type FP_d (so not F_d).

1.4 Isotropic groups

Now assume that G is an isotropic, absolutely almost simple F-group. So there exists a minimal F-parabolic subgroup P with unipotent radical U. For each $v \in S$ choose a minimal F_v-parabolic group Q_v, contained in P with unipotent radical U_v, such that $U_v(F_v) \supseteq U(F_v) \supset U(F)$.

Set $\overline{U}_S := \prod_{v\in S} U_v(F_v) \supseteq U_S := \prod_{v\in S} U(F_v) \supset U(F)$ (the last inclusion by diagonal embedding) and $\Gamma \subset G(F)$ is S-arithmetic. By propositions 1 and 2 we have the isomorphisms for an arbitrary $\mathbb{Z}-$module M

$$H_c^{d_v}(X_v;M) \simeq H_c^{d_v-1}(Y_v;M) \simeq C_c^\infty(U_v(F_v);M)$$

and $H_c^i(X_v;M) = 0$ for $i < d_v$, so for $X = \prod_{v\in S} X_v$ we obtain by Künneth's formula

Theorem 1. *(cf. [BS] , 6.6)*

a) $H_c^i(X;M) = 0$ *for* $i < d = \sum_{v\in S} dim\ X_v$*;*

b) $H_c^d(X;M) = \bigotimes_{v\in S} H_c^{d_v}(X_v;M) \simeq C_c^\infty(\overline{U}_S;M)$

2 Construction of a FP_{d-1}-resolution for Γ

The Bruhat-Tits-building X provides an augmented chain-complex $C = (C_n)_{-1 \leq n \leq d}$ with $\mathbb{Z}\Gamma-$ modules C_n, generated by the $n-$dimensional polysimplices of X for $n \geq 0$ and $C_{-1} = \mathbb{Z}$, $C_n \overset{\partial n}{\to} C_{n-1}$. Since X is a contractible space, C has trivial reduced homology.

Following K. Brown (see [Br1]) we shall construct inductively a projective — or even free — resolution of $\mathbb{Z}$ by defining chain-complexes $C'(k) = \big(C'(k)_n\big)_{-1 \leq n \leq d-1}$ with finitely generated $\mathbb{Z}\Gamma-$modules $C'(k)_n$, where $C'(k)_n = C'(k-1)_n$ for $n \leq k-1$ and $C'(k)_n = 0$ for $n > k$, beginning with $C'_{-1} = \mathbb{Z}$ (derivation $\partial'_n : C'_n \to C'_{n-1}$).

Moreover we define chain-maps $f_k : C'(k) \to C$, starting with $f_{-1} = id_{\mathbb{Z}}$ and f_k an extension of f_{k-1}. Thereby we obtain finally a subcomplex C' of C, whose support in X is compact modulo Γ.

We consider the mapping-cones $C''(k)$ for f_{k-1}, given by $C''(k)_n := C_n \oplus C'(k-1)_{n-1}$, $C''(k)_{-1} = \mathbb{Z}$ and $\partial''_n(c, c') = \big(\partial_n c - f_{n-1}(c'), -\partial'_{n-1}(c')\big)$, $\partial'_{-1} = 0$.

2.1 Homology and the beginning of induction

There is a short exact sequence

$$0 \to C \to C'' \to \Sigma C' \to 0,\ (\Sigma C')_n := C'_{n-1},$$

giving rise to a long exact sequence for homology

$$(1) \ldots \to H_n(C) \to H_n\big(C''(k)\big) \to H_{n-1}\big(C'(k-1)\big) \to H_{n-1}(C) \to \ldots$$

Denote by X_o the set of vertices of X, then we have

$$C''(0)_0 = C_0(X) \oplus \mathbb{Z} = \{(\Sigma z_i x_i, z') \mid x_i \in X_0; z_i, z' \in \mathbb{Z}\}$$

and with $\partial_0(\Sigma z_i x_i) = \Sigma z_i$ (augmentation-map) we obtain $\partial''_0\big((\Sigma z_i x_i, z')\big) = 0 \Leftrightarrow \Sigma z_i = z'$. Furthermore $C''(0)_1 = C_1(X) \oplus \{0\}; H_0(C) = 0$ implies

$$Z_0(C) = B_0(C) \simeq B_0\big(C''(0)\big) = \{\Sigma z_i x_i, 0) \mid \Sigma z_i = 0\}$$

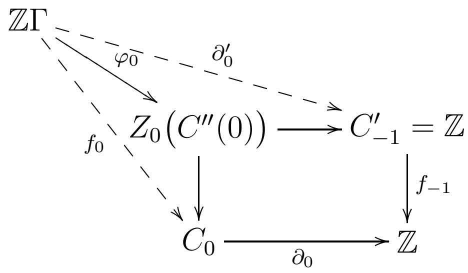
Choose now a base point $x_0 \in X_0$, then $\big(\Sigma z_i(x_0 - x_i), 0\big) \in B_0\big(C''(0)\big)$ and we see that

$$H_0\big(C''(0)\big) \simeq \{\big((\Sigma z_i) \cdot x_0, (\Sigma z_i)\big)\} = \{(z x_0, z) \mid z \in \mathbb{Z}\} \simeq \mathbb{Z}$$

We can give evidence of its $\mathbb{Z}\Gamma-$module-structure:

$$H_0\big(C''(0)\big) \simeq \{\big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big) \mid z_\gamma \in \mathbb{Z}, \gamma \in \Gamma\} \Big/ {\{\big(\Sigma z_\gamma(\gamma x_0), 0\big) \mid \Sigma z_\gamma = 0\}}$$

Using this description, we can lift the augmentation-map $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ to a $\mathbb{Z}\Gamma-$homomorphism φ_0 of $\mathbb{Z}\Gamma$ into $Z_0\big(C''(0)\big)$, defined by $\varphi_0(\Sigma z_\gamma \gamma) := \big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big)$; so φ_0 surjects onto $H_0\big(C''(0)\big)$. $Z_0\big(C''(0)\big)$ can be viewed as a fiber-product $C_0 \times_{\mathbb{Z}} C'_{-1}$, given by the maps ∂_0 and f_{-1}:



According to this diagram we define

$$C'_0 := \mathbb{Z}\Gamma, \quad f_0 : C'_0 \to C_0 \text{ by } f_0(\Sigma z_\gamma \gamma) := \Sigma z_\gamma(\gamma x_0)$$
$$\partial'_0 : C'_0 \to C'_{-1} \text{ by } \partial'_0(\Sigma z_\gamma \gamma) := \Sigma z_\gamma$$

As a consequence we have to set $C''(1)_1 = C_1 \oplus C'_0$ and $C''(1)_0 = C_0 \oplus \mathbb{Z}$, $C''(1)_n = C_n$ for $n > 1$. We confirm that

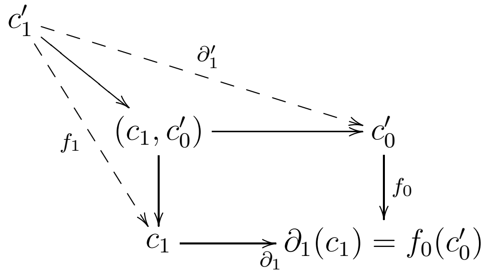
$$\begin{aligned} \partial''_0 \circ \partial''_1(c_1, c'_0) &= \partial''_0\big(\partial_1(c_1) - f_0(c'_0), -\partial'_0(c'_0)\big) \\ &= \big(\partial_0 \circ \partial_1(c_1) - \partial_0 \circ f_0(c'_0) - f_{-1} \circ (-\partial'_0(c'_0), -\partial'_1 \circ (-\partial'_0)(c'_0)\big) \\ &= 0, (\text{since } \partial_0 \circ f_0 = f_{-1} \circ \partial'_0) \end{aligned}$$

Moreover $\partial''_1(0, -c'_0) = \big(- f_0(-c'_0)\big), -\partial'_0(-c'_0)\big) = \big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big)$ for $c'_0 = \Sigma z_\gamma \gamma$, which means that $\partial''_1 : C''(1)_1 \to C''(1)_0$ is surjective on $H_0\big(C''(0)\big)$, thus $H_0\big(C''(1)\big) = 0$.

Observe that $f_0(C_0) = f_0(\mathbb{Z}\Gamma) = \mathbb{Z} \cdot (\Gamma x_0)$, whose support is the subcomplex $\Gamma \cdot x_0 =: X'_0$ of X.

For the next step consider $H_0\big(C'(0)\big) = Z_0\big(C'(0)\big) = \{\Sigma z_\gamma \cdot \gamma \mid \Sigma z_\gamma = 0\} =: I\Gamma$, the augmentation ideal of $\mathbb{Z}\Gamma$. It is well known that $I\Gamma$ is a finitely generated $\mathbb{Z}\Gamma-$module iff Γ is a finitely generated group. Now Γ is a $S-$arithmetic subgroup of $G(F)$, G an absolutely almost simple algebraic $F-$group — what we assume from now on. Therefore Γ is finitely generated iff the sum of local ranks $d \geq 2$ (see [B1] or [B3]); so we find a free module $(\mathbb{Z}\Gamma)^{r_1} =: C'(1)_1$, which surjects on $I\Gamma = H_0\big(C'(0)\big)$. By sequence (1) $H_0\big(C'(0)\big)$ is isomorphic to $H_1\big(C''(1)\big)$, because $H_1(C) = H_0(C) = 0$. We can lift the surjection of $C'(1)_1$ onto $H_1\big(C''(1)\big)$ to $Z_1\big(C''(1)\big)$, which is by definition a fiberproduct $C_1 \times_{C_0} Z_0(C'_0)$with respect to the maps ∂_1 and f_0. As above in the diagram we get the maps $f_1 : C'(1)_1 \to C_1$ and

$\partial_1' : C'(1)_1 \to C'(1)_0 = C'(0)_0$, more concretely:



with $\partial_0'(c_0') = 0$

Let us point out, that c_1 is a $1-$chain in $C_1(X)$, whose boundary is contained in $f_0(C_0')$. Since $C'(1)_1$ is finitely generated, we get finitely many elements $c_1^1, \ldots, c_1^{r_1}$, whose supports are paths $p_1, \ldots, p_{r_1}$ in X, which generate a $\Gamma-$subcomplex X_1' of X with X_1'/Γ compact.

For each $c_0' \in Z_0\big(C'(0)\big)$ there exists $c_1 \in C_1$ with $\partial_1 c_1 = f_0(c_0')$, since $H_0(C) = 0$ and also $c_1' \in C'(1)_1$ with $\partial_1'(c_1') = c_0' : H_0\big(C'(1)\big) = 0$. Alternatively we could use $C''(2)_2 := C_2 \oplus C'(1)_1$ and compute ∂_2'', proving that $H_1\big(C''(2)\big) = 0$, which implies by (1) $H_0\big(C'(1)\big) = 0$.

2.2 Some examples

1. $\Gamma = SL_2(\mathbb{F}_q[t])$ is not finitely generated, due to Nagao-Serre (see [S], II. 1.6); its Bruhat-Tits-building is a tree X, which has a half-line H as a fundamental domain mod Γ. H has vertices $x_i (i \in \mathbb{N}_0)$ with stabilizers $\Gamma_i = \mathrm{stab}_\Gamma x_i$, where $\Gamma_0 = SL_2(\mathbb{F}_q), \Gamma_i = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q[t]_i \\ 0 & \mathbb{F}_q^* \end{pmatrix}$ with $\mathbb{F}_q[t]_i = \{p \in \mathbb{F}_q[t] \mid \deg p \leq i\}$ for $i > 0$.

   We have $X_0' = \Gamma \cdot x_0$, which contains $\gamma_i x_0$ for $\gamma_i \in \Gamma_i \diagdown \Gamma_{i-1}$: the shortest path p_i in X, connecting x_0 with $\gamma_i x_0$ must contain the vertex x_i. Thus the complex X_1', generated by the p_i is not compact mod Γ. If $c_i' = \gamma_i x_0 - x_0 \in C(0)_0'$ and c_i is the chain corresponding to p_i in C_1' we have $(c_i, c_i') \in Z_1\big(C''(1)\big)$, which shows, that $H_1\big(C''(1)\big)$ cannot be a finitely generated $\mathbb{Z}\Gamma-$module — just as $H_0\big(C'(0)\big)$.

2. $\Gamma = SL_3(\mathbb{F}_q[t])$ is finitely generated: it is easy to see, that $E = SL_3(\mathbb{F}_q) \cup \{e_{12}(p), e_{23}(q) \mid p, q \in \mathbb{F}_q[t]_1\}$ is a set of generators. A standard apartment A of its Bruhat-Tits-building X is a triangulated plane and a fundamental domain for X mod Γ is given by a cone C in A with vertex x_0 and angle $\frac{\pi}{3}$. Let $\triangle_0$ be the triangle with vertex x_0 and contained in C, s.th. for each $\gamma \in E$ we have $\gamma\triangle_0 \cap \triangle_0 \neq \phi$ (γ fixes at least one vertex of $\triangle_0$). This implies, that every vertex γx_0 is connected with x_0 by a path

p, that projects into $\triangle_0$, so p is contained in $\Gamma\triangle_0 =: X_1'$, which means X_1'/Γ is compact. In the language of chains: For each $c' \in Z_0\big(C'(0)\big)$ with $f_0(c') \in C_0(X_0') \subset C_0(X), X_0' = \Gamma \cdot x_0$ we find $c \in C_1(X_1') \subset C_1(X)$ s. th. $(c, c') \in Z_1\big(C''(1)\big)$. But these pairs generate $H_1\big(C''(1)\big)$, since $(c, 0) \in Z_1\big(C''(1)\big)$ is a boundary by $H_1(X) = 0$ and for $(c_1, c'), (c_2, c') \in Z_1\big(C''(1)\big)$ we have $(c_1, c') - (c_2, c') = (c_1 - c_2, 0) \in B_1\big(C''(1)\big)$. Conclusion: $H_1\big(C''(1)\big)$ is a finitely generated $\mathbb{Z}\Gamma-$ module, because the elements $c' = \gamma x_0 - x_0$ with $\gamma \in E$ generate $Z_0\big(C'(0)\big)$.

On the other side $SL_3(\mathbb{F}_q[t])$ is not finitely presented: this is shown in [B2] by constructing an infinite series of closed paths in X_1' $\big($or $1-$cycles $c_n' \in C_1'(1)_1\big)$, which cannot be contracted in ΓC_n where $C_n (n \in \mathbb{N})$ are compact subsets of C with $\bigcup\limits_n C_n = C$. In the same way as in example 1 we obtain elements (c_n, c_n') in $Z_2\big(C''(2)\big)$, which cannot be contained in a finitely generated $\mathbb{Z}\Gamma-$module and of course are inequivalent $\mod B_2\big(C''(2)\big)$ — s.th. $H_2\big(C''(2)\big)$ is not finitely generated as a $\mathbb{Z}\Gamma-$module and the supports of all $2-$chains c_n cannot be contained in a complex X_2' with X_2'/Γ compact.

3. $\Gamma = SL_2(\mathbb{F}_q[t, t^{-1}])$ is finitely generated by the set $E = SL_2(\mathbb{F}_q) \cup \{e_{12}(p) \mid p \in \mathbb{F}_q[t]_1\} \cup \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}$. An apartment A of $X = X_1 \times X_2$ $\big($the buildings X_1 of $SL_2(\mathbb{F}_q[t])$ and X_2 of $SL_2(\mathbb{F}_q[t^{-1}])\big)$ is a plane, divided into squares. Choose $x_0 \in A$ with $\text{stab}_\Gamma x_0 = SL_2(\mathbb{F}_q)$, then there exists a finite union Q of squares, containing x_0, s.th. $\gamma Q \cap Q \neq \phi$ for all $\gamma \in E$. Then each $\gamma x_0 \in \Gamma x_0 =: X_0'$ is connected with x_0 by a path in $\Gamma Q =: X_1'$, so X_1'/Γ is compact and we conclude as in example 2, that $H_1\big(C''(1)\big)$ is generated by elements (c, c') with $c \in C_1(X_1')$, so it is a finitely generated $\mathbb{Z}\Gamma-$module.

   On the other hand, Γ is not finitely presented, which was shown in [St1] in a similar way as for example 2, s. th. $H_2\big(C''(2)\big)$ cannot be finitely generated.

4. $G = SL_2 \times SL_2$ is semi-simple, but not almost simple and $\Gamma = \Gamma_1 \times \Gamma_2 = SL_2(\mathbb{F}_q[t]) \times SL_2(\mathbb{F}_q[t])$ is not finitely generated, since the Γ_i are not and $H_0\big(C'(0)\big) = I\Gamma_1 \times I\Gamma_2$, isomorphic to $H_1\big(C''(1)\big)$, is not a finitely generated $\mathbb{Z}\Gamma-$module — although the sum d of local ranks is 2!

2.3 Cohomology

The examples show that in order to construct a resolution with free $\mathbb{Z}\Gamma-$ modules and a $\mod \Gamma$ finite subcomplex of X, one should prove that $H_k\big(C''(k)\big)$ is a finitely generated $\mathbb{Z}\Gamma-$ module for $k \leq d-1$. Brown's construction in [Br1] uses cohomology for this purpose, but he assumes, that the given modules of chains are projective, which is not true in our case - so we need more special arguments. We shall apply Borel-Serre's theorem (thm. 1) on cohomology with compact supports. They refer to Alexander-Spanier-cohomology, but for the polysimplicial complex X it can also be described by homomorphisms on the $\mathbb{Z}\Gamma-$modules $C_n(X)$ of chains (cf [Sp1], section 19 or [Sp2], 6.9). A compact support then consists of finitely many chains in $C_n(X)$, just as for the more abstract complexes C'_n and C''_n.

Recall that an element of $H^n(X;M)$ with an arbitrary coefficient-module M is given by an abelian group homomorphism φ on $C_n(X)$, which must be zero on the module $B_n(X)$ of boundaries (since $\delta_n\varphi := \varphi \circ \partial_{n+1}$) and can be modified by $\delta_{n-1}\psi = \psi \circ \partial_n$ with $\psi : C_{n-1}(X) \to M$. The action of Γ on φ is given by $(\gamma\varphi)(c) = \gamma[\varphi(\gamma^{-1}c)]$. We are interested in finitely generated $\mathbb{Z}\Gamma-$modules of φ's; sometimes it is more convenient, assuming that the supports of generators contain only one element in each Γ-orbit, to consider the Γ-orbit $\{\gamma\varphi \mid \gamma \in \Gamma\}$ as a $\Gamma-$homomorphism $\tilde{\varphi}$ with $\tilde{\varphi}(\gamma c) = \gamma\tilde{\varphi}(c)$. The finite generation of the module of φ's is equivalent to say, that the module of $\Gamma-$homomorphisms $\tilde{\varphi}$ has compact (or finite) support $\mod \Gamma$.

Now we construct the complexes C' and C'' by induction and also the subcomplex X' of X, where X'/Γ is finite and $f(C') = C(X')$. We start induction with

$$X'_0 := \Gamma \cdot x_0 (x_0 \in X), \qquad C'(0)_0 := \mathbb{Z}\Gamma, \qquad f_0\Big(\sum z_\gamma \gamma\Big) = \sum z_\gamma (\gamma x_0).$$

Suppose now $k \geqslant 1$ (I should point out, that we don't need the first step, described in 2.1, which used homology and the theorem on finite generation of Γ). Define the $\Gamma-$homomorphism φ

$$\varphi \colon C''(k)_k = C_k \times C'(k-1)_{k-1} \longrightarrow C_{k-1}, \text{ given by } \varphi(c, c') = \partial_k(c),$$

of course the support of φ is not compact $\mod \Gamma$. Since $C'(k-1)_k = 0$, we have $B_k\big(C''(k)\big) = B_k(C) = Z_k(C)$, so φ vanishes on $B_k\big(C''(k)\big)$ and defines an element of $H^k\big(C''(k); C_{k-1}\big)$. We can modify φ in its cohomology class by $(-\pi_1 \circ \partial''_k)$, π_1 the projection of $C''(k-1)$ to its first component C_{k-1}, thus we have

$$\pi_1 \circ \partial''_k(c, c') = \pi_1(\partial_k c - f_{k-1}c', \partial'_{k-1}c') = \partial_k c - f_{k-1}c'.$$

The restriction of the class of φ to homology is a well-defined element $\overline{\varphi} \in \operatorname{Hom}_\Gamma\Big(H_k\big(C''(k)\big); C_{k-1}\Big)$.

(Remark: In the more comfortable situation of [Br 1] one even has an isomorphism between $H^k\big(C''(k); M\big)$ and $\operatorname{Hom}_\Gamma\Big(H_k\big(C''(k)\big); M\Big)$ for arbitrary M.) We have

$$\overline{\varphi}(c, c') = \partial_k(c) = f_{k-1}(c').$$

The image of $\overline{\varphi}$ is $H_{k-1}\Big(f_{k-1}\big(C'(k-1)\big)\Big)$ since $\partial_{k-1}(f_{k-1}c') = \partial_{k-1} \circ \partial_k(c) = 0$ and because $H_{k-1}(C) = 0$ there exists for each cycle $f_{k-1}(c')$ an element $c \in C_k$ with $\partial_k(c) = f_{k-1}(c')$.

For isotropic groups we know, that X/Γ is not compact, so we have to consider a filtration of X, i.e. an ascending sequence of subcomplexes $X(m)$ with $X(m)/\Gamma$ compact and $\underset{m}{U} X(m) = X$. Denote by φ_m the restriction of φ to $C\big(X(m)\big)_k \times C'(k-1)_{k-1}$, then φ_m has compact support $\mod \Gamma$ (the second component $C'(k-1)_{k-1}$ is a finitely generated $\mathbb{Z}\Gamma-$ module by induction) and defines an element of $H^k_c\big(C''(k); C_{k-1}\big)$; its restriction $\overline{\varphi}_m$ may be non-trivial only for a finitely generated submodule of $H_k\big(C''(k)\big)$.

For cohomology there exists also a long exact sequence

$$(2) \ldots \to H^{n-1}_c(C;M) \to H^{n-1}_c\big(C'(k-1);M\big) \to H^n_c\big(C''(k);M\big) \to H^n_c(C;M) \to \ldots$$

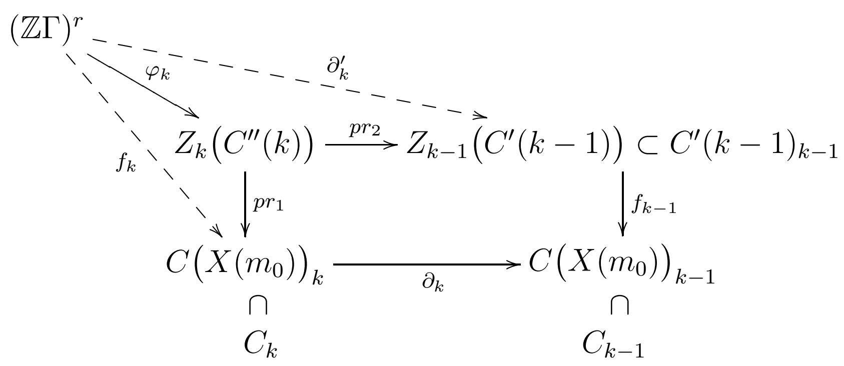
for arbitrary coefficient-modules M. For $n \leq d-1$ Borel-Serre's result says, that $H^n_c(C;M) = 0$ which implies $H^k_c\big(C''(k);M\big) \simeq H^{k-1}_c\big(C'(k-1);M\big)$ for $k \leq d-1$. By induction $C'(k-1)$ is a complex of finitely generated free $\mathbb{Z}\Gamma-$ modules and therefore its cohomology is also finitely generated, which means that the module of the corresponding $\Gamma-$homomorphisms has compact support modulo $\Gamma-$ and by the isomorphism this ist also true for $H^k_c\big(C''(k);M\big)$.

We apply this conclusion to the $\Gamma-$homomorphisms φ_m with $M = C_{k-1}$ and its cohomology classes: They vanish outside some $C\big(X(m_0)\big)_k$. In particular we have $\overline{\varphi}_m = \overline{\varphi}$ for all $m \geqslant m_0$ for the restrictions to the homology $H_k\big(C''(k)\big)$. Observe that the support of $\partial_k \colon C_k \to C_{k-1}$ is of course not compact $\mod \Gamma$, but there may be different elements c_1 and $c_2 \in C(k)$, s.th. (c_1, c') and (c_2, c') define the same homology class, which means $(c_1, c') - (c_2, c') \in B_k\big(C''(k)\big) \Rightarrow \partial_k c_1 = f_{k-1}c' = \partial_k c_2$.

We summarize: For each $c' \in Z_{k-1}\big(C(k-1)\big) = H_{k-1}\big(C'(k-1)\big), c' \neq 0$ with $\partial'_{k-1}c' = 0$ and $\partial_{k-1}(f_{k-1}c') = f_{k-2}(\partial'_{k-1}c') = 0$ we find $c \in C_k$ with $\partial_k c = f_{k-1}c'$, because $H_k(C) = 0$. Since $\overline{\varphi}(c, c') = \partial_k c = 0$ for $c \notin C\big(X(m_0)\big)$ we must have $c \in C\big(X(m_0)\big)$ for all $c' \in H_{k-1}\big(C'(k-1)\big)$. $X(m_0)/\Gamma$ is compact, so there exists a set X'_k of $k-$dimensional polysimplices, $X'_k \subset X(m_0)$ and $C(X'_k)/\Gamma$ finite, which supports the first component of $H_k\big(C''(k)\big)$.

We choose a (minimal) finite set $\{c_1, \ldots, c_r\}$ of generators for $C(X'_k)$. There may exist several elements c' with $f_{k-1}c' = \partial_k c_i$, but we may assume by induction that they are congruent $\mod \Gamma$ (even $\mod$ the finite stabilizer Γ_0 of $\partial_k c_i$). Thus we can also choose $c'_i \in C'(k-1)_{k-1}$, s.th. $\{\overline{(c_i, c'_i)} \mid i = 1, \ldots, r\}$ is a finite set of generators for the $\Gamma-$module $H_k\big(C''(k)\big)$, where $\overline{(c, c')}$ is the class of $(c, c') \in Z_k\big(C''(k)\big)$.

Now we can define $C'(k)_k := (Z\Gamma)^r$ and get the following diagram for $\varphi_k(\sum\limits_{i=1}^{r} z_{\gamma_i} \cdot \gamma_i) := \sum\limits_{1}^{r} z_{\gamma_i} \cdot \gamma_i(c_i, c'_i) \in Z_k\big(C''(k)\big)$.



thus defining $\partial'_k : C'(k)_k \to C'(k)_{k-1}$ and
$f_k : C'(k)_k \to C_k$.

Moreover we have $C''(k+1)_{k+1} := C_{k+1} \oplus C'(k)_k$ and the $k-$dimensional subcomplex $X'(k) := \bigcup_{j=1}^{k} X'_j$, adding X'_k.

We also saw, that ∂'_k surjects on $Z_{k-1}\big(C'(k-1)\big)$, which means $H_{k-1}\big(C'(k)\big) = 0$, so we obtained a resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}\Gamma-$modules up to dimension k.

Remarks

1. Under the assumptions of [Br1], one can even show, that $H^k(C;M) \simeq \mathrm{Hom}_{\mathbb{Z}\Gamma}\big(H_k(C);M\big)$ and if the cohomology of C preserves direct limits for the coefficient modules, Brown constructs the chain-complex C' with finitely generated $\mathbb{Z}\Gamma-$modules.

2. For the semi-simple group G of example 4 the construction above should not work, although for $d = 2$ the sequence (2) implies that $H^1_c\big(C''(1), C_0\big)$ is finitely generated.

   Here we have $X'_0 = \mathbb{Z}(\Gamma_1 \times \Gamma_2)\cdot(x^1_0, x^2_0)$ and the shortest path between (x^1_0, x^2_0) and $(\gamma^1_i x^1_0, x^2_0)$ must contain the vertex (x^1_i, x^2_0) (notations as in example 1 with upper index 1 or 2) s.th. the distance between x^1_0 and $\gamma^1_i x^1_0$ goes to infinity with $i-$analogous in the second component. There exists a closed path in $X = X_1 \times X_2$ with the following vertices:

   $(x^1_0, x^2_0) \to (x^1_i, x^2_0) \to (\gamma^1_i x^1_0, x^2_0) \to (\gamma^1_i x^1_0, x^2_j) \to (x^1_i, x^2_j) \to (x^1_0, x^2_j) \to (x^1_0, x^2_0)$. This path p (or chains, whose support is p) is a boundary in $X(d \geq 2)$, therefore a homomorphism from cohomology has to vanish on p, e.g. ∂_1, which should define the isomorphism on homology. Bur for fixed x^1_i we get x^i_2

arbitrary far from x_2^0, which means, that ∂_1 cannot be restricted to compact supports.

We summarize our results in the following

Proposition 4. *a) There exists a partial resolution of Z with free $\mathbb{Z}\Gamma-$modules of finite rank: $C'_{d-1} \to C'_{d-2} \to \ldots \to C'_0 \to Z$*

b) There exists a $(d-1)-$dimensional subcomplex $X' = X'(d-1)$ of X with $H_k(X') = 0$ for $k < d-1$.

Both properties imply the following finiteness theorem: Cf. for the first [Br2], VIII. 4.3 and for the second [Br3], 1.1 (observing that stabilizers in Γ of cells in X are finite).

Theorem 2. *A $S-$arithmetic subgroup Γ of an absolutely almost simple algebraic group G, defined over a function field F with $rank_F G > 0$ and $d = \sum_{r \in S} rank_{F_v} G$ is of type FP_{d-1}.*

Remarks:

1. For semi-simple groups we obtain type $FP_{d'-1}$, where d' is the minimal d for the simple factors.
2. Our (co)homological method cannot prove that Γ is also of type F_{d-1}; but this is true, since finite presentability was shown for $d = 3$ (see [B3]: unfortunately this proof is case-by-case and lengthy and part II of it was not published, but exists!): cf [Br2], VIII. 7.

3 Spherical holes in X imply, that Γ is not of type FP_d.

This theorem cannot be deduced immediately from Borel-Serre's result on the top-cohomology of X. But their computations provide a natural construction of cycles. We prove that the codim $1-$homology is not essentially trivial and use then Brown's criterion. An important tool is the existence of a section of apartments, which is compact modulo Γ — already used in [BW].

3.1 Homology and cohomology in the top-dimension

For $v \in S$ let Y_v be the Tits-building and A an apartment of Y_v in $\mathcal{A}_P$ (cf. 1.1). A is an oriented sphere and its homology $H_{l-1}(A;\mathbb{Z})$ is generated by the cycle $\sum\limits_{w \in W} (-1)^{lw} wC$ in the fundamental class [A]. An element h of the cohomology $\tilde{H}^{l-1}(A;\mathbb{Z})$ can be restricted as a function to homology: The values $h([A])$ establish the isomorphism $\tilde{H}^{l-1}(Y_{v,t};\mathbb{Z}) \simeq C_c^\infty(\mathcal{A}_p;\mathbb{Z})$, the set of locally constant functions with compact support on $\mathcal{A}_p$ (see [BS], 2.5-2.6). Roughly speaking, the cohomology may be understood as the characteristic function of supports for homology.

Recall that the standard apartment A_0 is determined by a pair (C, C^{op}) of opposite chambers and each $A \in \mathcal{A}_p$ by a pair (C, uC^{op}) with $u \in U(F_v)$, giving the $1-1-$correspondence between $\mathcal{A}_p$ and $U(F_v)$, s.th. $\tilde{H}^{l-1}(Y_{v,t};\mathbb{Z}) \simeq C_c^\infty\big(U(F_v);\mathbb{Z}\big)$. For the spherical join Y of the $Y_v (v \in S)$ we have to consider the product $U_S = \prod\limits_{v \in S} U(F_v)$ and obtain $\tilde{H}^{d-1}(Y_t;\mathbb{Z}) \simeq C_c^\infty(U_S;\mathbb{Z})$.

The transition to affine buildings with boundary is based on the sequence (cf. 1.2)

$$0 \to Y_t \to Z_t \to X \to 0$$

and its long exact sequence for cohomology shows that $H^{l-1}(Y_t;\mathbb{Z}) \simeq H_c^l(X;\mathbb{Z})$, since the cohomology of Z_t vanishes.

This isomorphism can be explained as follows: The $(l-1)-$cycles in Y_t are given by fundamental classes $[\overline{A}]$ of apartments $\overline{A} \in \mathcal{A}_P$ in Y_t and $\overline{A}$ bounds an apartment A of X. By retraction of $A \cup \overline{A}$ towards a center $z_A \in A$ we can remove a collar and can consider $[\overline{A}]$ as a $(l-1)$-cycle in X; there it is the boundary of a $l-$chain c_A. In the top-domension c_A is unique, its support is a cone with vertex z_A and edges, whose directions represent vertices in Y_t, defining chambers in $\overline{A}$. Observe that the support of c_A in X is compact. In the other direction: A given support of a $l-$chain c_A may be extended in different ways towards Y_t. But since the cohomology is locally constant, their extension is constant for a neighbourhood in Y_t.

3.2 Example: $\mathbf{\Gamma = SL_2(0_S)}$

It should be useful to see the proof at first for a simple example without too many technicalities. We choose the first example, for which the theorem was proved in 1980 by U. Stuhler, computing the $\mathbb{F}_p-$cohomology of Γ.

For $SL_2(0_S)$ the Bruhat-Tits-building X is the product of $s = \mid S \mid$ trees X_v, its apartments are $\mathbb{R}^s$ with a right-angled complex. The apartments of the boundary Y are spherical joins, its chambers $(s-1)$-simplices. A pair of opposite minimal parabolic subgroups is (P^+, P^-), the groups of upper and lower triangular matrices with Levi-decompositions $P^+ = T \ltimes U^+, P^- = T \ltimes U^-$. The standard apartments A_0 and $\overline{A}_0$ are determined by the opposite chambers C and C^-, fixed by $P_S^+ = \prod_{v \in S} P^+(F_v)$ and P_S^-. The torus T_S acts on A_0; if we choose an origin $o = (o_v)$ in A_0, we get $A_0 = \{(x_v) = (t_v o_v) \mid (t_v) \in T_S, v \in S\}$. If α is the root of T with respect to $P^+, \mid \alpha(t) \mid = \prod_v \mid \alpha(t_v) \mid_v$ and by abuse of notation $\alpha(x) = \sum_v \alpha(x_v) = \sum \log \mid \alpha(t_v) \mid_v$ for $x_v = (t_v o_v)$ is a linear weight-function on A_0 and $H = \{x \in A_0 \mid \alpha(x) = 0\}$ a hyperplane in A_0, containing o.

$SL_2(F)$ is diagonally embedded in $SL_2(F_S) = \prod_v SL_2(F_v)$ and by the product-formula for values we see that $T(0_S) = \{t \in T(F)\big| \mid \alpha(t) \mid = 1\}$, so $T(0_S)$ is by Dirichlet's unit-theorem the product of a finite group and a free abelian group of rank $(s-1)$. The latter acts on H as a lattice of translations and therefore $H/T(0_S)$ is compact. The subcomplex $X_0 := \Gamma \cdot H$ of X is then also compact mod Γ. Our aim is to project apartments into X_0.

Denote by p_v resp. p'_v the vertices of C and C' and consider them as directions in A_0, given by half-lines, starting at a center z. We choose a series of centers $z_m (m \in \mathbb{N})$ in the sector with vertex o and base C: $\alpha[(z_m)_v] = m$ for all v.

Let ρ_m be the projection of $Z_t = X \amalg Y_t$ into X_0 with center z_m. For the restriction to $A_0 \cup \overline{A}_0$ we obtain $\rho_m(p_v) = \rho_m(p'_v) =: q_{v,m}$ with the following coordinates: $\alpha[(q_{v,m})_w] = m$ for all $w \neq v$ and $\alpha[(q_{v,m})_v] = -(s-1)m$. The vertices $q_{v,m}$ span a $(s-1)-$simplex $\triangle_m$ in H and $\rho_m(\overline{A}_0) = \triangle_m$.

Consider now arbitrary apartments $uA_0 \in \mathcal{A}_p, u \in U_S$. It is well known, that $U_S = \prod_v U(F_v)$ is compact mod $U(0_S)$ (see [B1], Satz 3): $U_S = U(0_S) \cdot K, K$ compact (if the class-number of 0_S is 1, one can choose for K the product $\prod_v 0_v$ of the valuation rings). We assume that the origin o is fixed by K.

Take $u' \in \text{stab} z_m \cap U_S, u' = u \cdot k$ with $u \in U(0_S), k = (k_v) \in K$ and suppose $\log \mid u \mid_v = \alpha[(z_m)_v] = m$, so $m > \alpha[q_{v,m})_v]$, which implies that u fixes none of the vertices $q_{v,m}$ — but $u(q_{v,m}) \in \Gamma \cdot H = X_0$.

Let us now describe the fundamental classes $[\overline{A}_0]$ and $u[\overline{A}_0]$, elements of $H_{s-1}(Y_t; \mathbb{Z})$. $[\overline{A}_0]$ is given by $\sum_{w \in W} (-1)^{lw} \cdot wC$; the Weyl-group W of $SL_2(F_S)$ is $(\mathbb{Z}/2\mathbb{Z})^s = \{w = \prod_v w_v^{\epsilon_v} \mid \epsilon_v = 0, 1\}$, where w_v is the involution with $w_v[P^+(F_v)] = P^-(F_v)$, in particular $w_0 = \prod_v w_v$ and $C' = w_0 C$.

For $u([\overline{A}_0])$ we get $uw_v^{\epsilon_v} p_v = p_v$ for $\epsilon = 0 (U$ stabilizes $C)$ and $uw_v^{\epsilon_v} p_v = up'_v$ for $\epsilon = 1$, so all chambers uwC have vertices from $\{p_v\} \cup \{up'_v\}$, especially $uC = C$ and $uw_0 C = uC'$. In geometric terms: the apartments $u\overline{A}_0$ are cross-polytopes (for $s = 3$ octahedrons) and topologically $(s-1)-$ spheres.

In the last step we retract these cycles into X_0 by ρ_m (observe that ρ_m is compatible with the action of Γ): $\rho_m(p_v) = q_v = \rho_m(p'_v), \rho_m(up'_v) = uq_v$. Thus $\rho_m(u[\overline{A}_0])$ is a cycle $c_m \in Z_{s-1}(X_0;\mathbb{Z})$, whose support is a $(s-1)-$sphere S_m, consisting of 2^s $(s-1)-$simplices. We should point out, that different simplices are in different apartments of X with boundary in $\mathcal{A}_p$. Moreover these apartments contain the retraction-center z_m and therefore the cone C_m with base S_m and vertex z_m, which supports a $s-$chain $\tilde{C}_m$, whose boundary is c_m. In the top-dimension such a chain $\tilde{C}_m$ is unique and since $z_m \notin X_0$, we conclude that c_m defines a non-trivial class in $H_{s-1}(X_0,\mathbb{Z})$. In short: S_m is a spherical hole in X_0. On the other hand $\lim\limits_{m\to\infty} z_m \in C$, which means that the cones C_m grow out of each subcomplex X' compact $\mod \Gamma$: $H_{s-1}(X;\mathbb{Z})$ is not essentially trivial (using a filtration $(X_m)_{m\in\mathbb{N}_0}$ of X with X_0 and X_m/Γ compact). Then Brown's criterion ([Br3], thm. 2.2) shows, that $SL_2(0_S)$ is not of type $F_{|S|}$.

3.3 The general case

Most problems arise already for a fixed place $v \in S$. For the construction of spheres and cycles it seems convenient not to use complete apartments but only the links of two opposite vertices in Y_v. These links have good projections into a hyperplane H in an apartment A of X, on which an arithmetic torus acts cocompactly. The proof of the following proposition can be found in [BW], thm. 2.2 and the literature quoted there.

Proposition 5. *Let Q be a maximal $F-$parabolic subgroup of G, containing a $1-$dimensional $F-$split torus T_1 with Levi-decomposition $Q = Z_G(T_1) \ltimes R_u(Q)$.*

a) *There exists a maximal $F-$torus T in Q, such that:*

   (i) *The maximal $F-$split torus of T is T_1;*

   (ii) *T contains a maximal F_v- split torus T_v for all $v \in S$.*

b) *$T_S = \prod\limits_{v\in S} T(F_v)$ acts on an apartment $A = \prod\limits_{v} A_v$ of dimension $d = \sum\limits_{v} d_v$ in $X = \prod\limits_{v} X_v$.*

c) *T_S is the product of a compact group and a free abelian group of rank d, which acts on A by translations.*

d) *$T(0_S)$ is discrete in T_S and is the product of a finite group and a free abelian group of rank $\sum\limits_{v}$ rank$_{F_v}(T)-$ rank$_F T = d-1$ ("generalized Dirichlet-unit-theorem").*

e) *Fixung an origin o in A, we get a hyperplane $H \subset A$ on which $T(0_S)$ acts cocompactly.*

We concentrate now on a fixed place $v \in S$ and consider an apartment $A_v \subset X_v$ with boundary $\overline{A}_v \subset Y_v$ (for simplicity we omit the index v for the details).

$\overline{A}_v$ is determined by a pair of opposite chambers C and C', fixed by F_v- minimal parabolic subgroups P and P'. The vertices of C and C' are stabilized by F_v- maximal parabolics, containing P resp. P'. We assume that the $F-$group Q of prop. 5 is one of them and denote it by Q_0, fixing $p_0 \in C$ and its opposites by Q'_0 and p'_0.

There exist several minimal parabolic groups P'_i, contained in Q'_0 (their number depends on the local Weyl-group), fixing chambers $C'_1, \ldots, C'_k$, so the union $\bigcup\limits_1^k C'_i$ is the star of p'_0 in $\overline{A}_v$ and the faces of the C'_i, which do not contain p'_0, establish the link $L(p'_0)$. On the opposite side we have the link $L(p_0)$.

Remember that the set of all apartments in $\overline{A}_v$, containing the chamber C is given by the elements of $\overline{U}(F_v)$, where $\overline{U}$ is the unipotent radical of P. Set $U_0 := R_u(Q_0)$, then $U_0(F_v) \leq \overline{U}(F_v)$, since $Q_0 \geq P$. This is also true for the other minimal parabolics $P_i \leq Q_0$ and if some maximal $Q_i \geq P_i$ fixes a vertex $p_i \in L(p_0)$, then the unipotent radical $U_i(F_v)$ of $Q_i(F_v)$ has a non-trivial intersection with $U_0(F_v)$. An element $u \neq 1$ from this intersection is not contained in $U'_i(F_v)$, where U'_i is the unipotent radical of Q'_i, fixing the vertex p'_i opposite to p_i — but then u moves $p' \in L(p'_o)$.

Now we turn the attention to A_v, interpreting the vertices of $\overline{A}_v$ as directions in A_v. $T(F_v)$ acts by translations on A_v, in particular $T_1(F_v)$ on lines with ends p_0 and p'_0. Each of the minimal parabolic subgroups P'_i determines a so-called dual root-systems R_i^V of characters on $T(F_v)$. Its basis consists of fundamental weights $\omega'_{ij}(i = 1, \ldots, k; j = 0, \ldots, d_v - 1)$, which describe the action of $T(F_v)$ on the unipotent radicals of maximal parabolic subgroups containing P'_i. In particular $\omega'_0 := \omega'_{i0}$ for all i belongs to Q_0.

If we choose an origin $o_v \in A_v$, then every $x \in A$ is given as $x = t_x o_v$ with $t_x \in T(F_v)$ and we define $\omega'_{ij}(x) = \log \mid \omega'_{ij}(t_x) \mid_v$ as linear functions on A.

Lemma 1. *For H from Prop. 5 $H \cap A_v$ is a hyperplane in A_v.*

a) The vector ω'_0 is orthogonal to $H \cap A_v$.

b) All half-lines in A_v with direction ω'_{ij} — starting at a vertex on the same side of $H \cap A_v$ as p_0 — intersect $H \cap A_v$.

Proof: a) For a sequence $x_m (m \in \mathbb{N})$ of vertices in A_v with $\lim\limits_{m\to\infty} x_m = p'_0$ we have $\lim\limits_{m\to\infty}$ vol [stab $x_m \cap U'_0(F_v)] = \infty$. By reduction theory we know that $(U'_0)_S = \prod\limits_v U'_0(F_v) = U'_0(0_S) \cdot K$ with a compact set K (cf 3.2). Therefore the finite intersections stab $x_m \cap U'_0(O_S)$ grow also for $m \to \infty$, so the vertices

x_m cannot be congruent $\mod \Gamma$ to finitely many ones. Since H is compact $\mod \Gamma$, the direction w'_0 can not have a component in $H \cap A_v$.

b) It is well known that the angles between fundamental weights are acute, if R^V is irreducible (cf [B3], 2.2).

□

If we assume that the origin $o = (o_v)_v \in \Pi A_v = A$ lies in H and observe that the linear function ω'_0 is defined on all A_v, s.th. $\omega'_0[(x_v)_v] = \sum\limits_v \omega'_0(x_v)$, we obtain — using the product formula for values — an explicit description for H : $H = \{x \in A \mid \omega'_0(x) = 0\}$.

Construction of spheres and cycles in the subcomplex $X_0 := \Gamma \cdot H$ of X, which is compact $\mod \Gamma$ — just as H. It will be enough to work in a fixed building X_v and take spherical joins in the end. In X_v we are not interested in full apartments but only in links of opposite vertices.

In the first step we project the link $L(p'_0) \subset Y_v$ into $H \cap A_v$, using a series of projection-centers z_m, lying on the half-line with vertex o_v and direction ρ_o with $\lim\limits_{m\to\infty} z_m = p_0$. Then all half-lines $h_{m,p'}$ with vertex z_m and direction $p' \in L(p'_0)$ — or with vector ω'_{ij} — intersect $H \cap A_v$ in a vertex q_m. This provides a map of $L(p'_0)$ onto the link $L(q_{0,m})$ in $H \cap A_v$. In order to simplify the situation we shall not consider the whole link, but only a $(d_v - 1)-$simplex $\sum_m$ with vertices from $L(q_{0,m})$, that will be specified in the following lemma.

In a second step we map $\sum_m$ by an element $u \in \text{stab } z_m \cap U_0(O_S)$ into $X_0 \cap A_v$. Over F_v the element u splits up into root-factors, which are needed to define further simplices connecting $\sum_m$ and $u(\sum_m)$.

Thus the lemma will be an exercise on root-systems. The root-system R for $T(F_v)$ with respect to the F_v-minimal parabolic group P has a generating set $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots, which contains the set $\triangle = \{\alpha_1, \ldots, \alpha_{r_0}\}$ of simple roots over F. We specialize the definition of Q_0, assuming that $T_1 = \bigcap\limits_{i\in\triangle-\{\alpha_1\}} (\ker \alpha_i)^0$ (where ^{0} denotes the connected component of identity: $Q_0 = Z_G(T) \ltimes U_0$.

Lemma 2. *There exist F_v- maximal parabolic subgroups $Q_i (i = 1, \ldots, r$; $r = rank_{F_v} G)$, which fix $p_i \in L(p_0)$, and opposites Q'_i, fixing $p'_i \in L(p'_0)$ and root-factors $u_j \in U_0(0_S) \cap U_\beta(F_v)$ for some positive root $\beta \in R_+$, also for $j = 1, \ldots, r$, such that $u_j(p'_i) = p'_i$ for $i \neq j$ and $u_j(p'_j) \neq p'_j$.*

For the **proof** we describe the roots, who contribute factors to the unipotent radicals U_i of the groups $Q_i (i = 0, \ldots, r)$; we use a suitable order of the simple roots (see the remark afterwards).

Let $u \in U_0(F_v)$ be the product $u = \prod\limits_{\beta} u_\beta$ with $u_\beta \in U_\beta(F_v)$ for $\beta \in \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_r\}$ — so we admit no factors, where β has a coefficient > 1.

Now take Q_1 with $U_1 = R_u(Q_1)$, s.th. U_1 has root-factors U_j with $j \in \{\alpha_r, \alpha_r + \alpha_{r-1}, \ldots, \alpha_r + \alpha_{r-1} + \ldots + \alpha_1\}$, perhaps additional ones.

Denote the generators of the local Weyl-group W_v with $s_i (i = 1, \ldots, r)$, s.th. $s_i(\alpha_i) = -\alpha_i$ and define the groups Q_i for $i > 1$ by $Q_2 = s_r(Q_1), Q_3 = s_{r-1}(Q_2)$, $\ldots, Q_r = s_2(Q_{r-1})$. We indicate briefly the roots for the unipotent radicals U_i. $U_2 : \{-\alpha_r, \alpha_{r-1}, \alpha_{r-1} + \alpha_{r-2}, \ldots\}, U_3 : \{-\alpha_r - \alpha_{r-1}, -\alpha_{r-1}, \alpha_{r-2}, \ldots\}, \ldots,$ and finally $U_r : \{-\alpha_r - \ldots - \alpha_{r-2}, \ldots - \alpha_2, \alpha_1\}$.

Remark: The numbering of roots is not always the usual one. α_r must be a long root and for E_6, E_7, E_8 the "extra" vertice must be α_4.

The result is as follows: Each factor u_β from the product $u = \prod\limits_{\beta} u_\beta$ is contained in exactly one of the unipotent radicals U_i(for $i = 1, \ldots, v$). If we rewrite $u = \prod\limits_{j=1}^{v} u_j$, we can assume that $u_j \in U_j(F_v)$, but $u_j \notin U_i(F_v)$ for all $i \neq j$. Now $u_j \notin U_i(F_v)$ implies that $u_j \in U_i'(F_v) \subset Q_i'(F_v)$ for the opposite group, which means $u_j(p_i') = p_i'$ for $i \neq j$ and since $u_j \notin U_j'(F_v)$ also $u_j(p_j') \neq p_j'$.

It remains to guarantee, that the factors $u_j = u_\beta$ can be chosen in $U_0(0_S)$. For this purpose we have to make precise the positions of o_v and $(z_m)_v$. We know that $U_0(F_v) = U(0_S) \cdot K_v$ with K_v compact, so we can suppose that K_v fixes o_v. For vertices $x_v \in T_1(F_v) \cdot o_v$ we have $\beta(x_v) = \alpha_1(x_v)$ for all $\beta \in \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_r\}$, especially we get $\beta(o_v) = 0$ and we define $(z_m)_v$ by $\alpha_1[(z_m)_v] = m = \beta[(z_m)_v]$. An arbitrary element $u_\beta \in U_\beta(F_v)$ can be written in the form $u_\beta = u_\beta' \cdot k_\beta$ with $u_\beta' \in U_0(O_S)$ and $k_\beta \in K_v$ with $\beta(u_\beta') = \beta(u_\beta)$ and we have also $u_\beta'(p_i') = p_i'$ for $u_\beta = u_j, i \neq j$ and $u_\beta'(p_j') \neq p_j$ — but not $u_\beta'(p_j') = u_\beta(p_j)$ in general.

The result can be transmitted to the image of projections into $X_0 = \Gamma \cdot H$.

Corollary 1. *For* $u = \prod\limits_{j=1}^{r} u_j \in U_0(0_S) \cap \prod\limits_{\beta} U_\beta(F_v) \cap stab\ z_m$ *and* $q_i = h_{m,p_i'} \cap H \cap A_v$ *there is* $u(q_i) = u_i(q_i) \in X_0 \cap A_v$*, since* $u_j(q_i) = q_i$ *for* $i \neq j$*.*

Remark: A simple, but instructive example for this proof is $\Gamma = SL_4(0_S)$, $s = 1$ with $r = 3$, where q_1, q_2, q_3 are vertices of a triangle in $L(q_0)$, cutting off the three other vertices of $L(q_0)$.

Now it is easy to construct spherical complexes in $X_0 \cap A_v$, following a similar pattern as in the example 3.2. We start with the $(d_v - 1)-$simplex $\sum_m \subseteq L(q_0) \subset H \cap A_v$ with vertices $q_1, \ldots, q_{dv}(d_v = r)$, which come from the projection of $L(p_0') \subset Y_v$ towards the center $(z_m)_v \in T_1(F_v) \cdot o_v$. Now we use a product $u = \prod\limits_{j=1}^{r} u_j \in U_0(0_S) \cap \text{stab } z_m$ from the corrollary and map $\sum_m$ by partial products:

Set $J = \{1, \ldots, r\} = J' \dot{\cup} J''$ and $u_{J'} := \prod_{j \in J'} u_j$, thus $u_{J'}$ fixes the vertices q_j with $j \in J''$ and for $j \in J'$ we get $u_{J'}(q_j) = u(q_j) \in u(\sum_m) \subseteq u\big(L(q_0)\big)$. We get 2^{d_v} $(d_v - 1)-$simplices, which fit together to a topological $(d_v - 1)-$sphere $S_m^{d_v-1}$. Its faces $U_{J'}(\sum_m)$ are contained in different apartments $u_{J'}(A_v)$, determined by the pair $\big(C, u_{J'}(C')\big)$ of opposite chambers in Y_v. Moreover all these apartments contain the center $(z_m)_v$, so $S_m^{d_v-1}$ can be retracted to $(z_m)_v$. For $(z_m)_v$ we have $\alpha_1[(z_m)_v] = m > 0$, which means $(z_m)_v \notin X_0 \cap A_v$ and implies that $S_m^{d_v-1}$ cannot be a boundary in $X_0 \cap X_v$ — by uniqueness in the top-dimension for killing homology: We have a spherical hole in $X_0 \cap X_V$.

For the combination of all places $v \in S$ we simply have to define the spherical joins of all $S_m^{d_v-1}$ to obtain topological $(d-1)-$spheres S_m^{d-1} in X_0, supporting non-trivial elements in $H_{d-1}(X_0; \mathbb{Z})$. For $\lim_{m \to \infty} z_m = p_0 \in Y$ with $\alpha_1[(z_m)_v] = m \to \infty$ for all $v \in S$ we have therefore shown that $H_{d-1}(X_0; \mathbb{Z})$ is not essentially trivial.

Theorem 3. *A $S-$arithmetic subgroup of an absolutely almost simple algebraic group G, defined over a function field F with $\mathrm{rank}_F G > 0$ and $d = \sum_{v \in S} \mathrm{rank}_{F_v} G$ is not of type FP_d and so not of type F_d.*

Remark: The result can be extended to reductive groups, taking the minimum of the sums of local ranks for the simple factors.

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