Sums of values represented by a quadratic form

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Abstract

Let q be a quadratic form over a field K of characteristic different from 2. We investigate the properties of the smallest positive integer n such that -1 is a sums of n values represented by q in several situations. We relate this invariant (which is called the q-level of K) to other invariants of K such as the level, the u-invariant and the Pythagoras number of K. The problem of determining the numbers which can be realized as a q-level for particular q or K is studied. We also observe that the q-level naturally emerges when one tries to obtain a lower bound for the index of the subgroup of non-zero values represented by a Pfister form q. We highlight necessary and/or sufficient conditions for the q-level to be finite. Throughout the paper, special emphasis is given to the case where q is a Pfister form.

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Key words: Level of a field, quadratic form, Pfister form, round form, Pythagoras number, sums of squares, formally real field, u-invariant, hermitian level, sublevel, weakly isotropic form, signature of a quadratic form, strong approximation property, ordering of a field.

1 Introduction

A celebrated theorem of E. Artin and O. Schreier states that a field K has an ordering if and only if -1 cannot be written as a sum of squares in K. In this case, the field K is called *formally real*. In the situation where K is not formally real, one may wonder how many squares are actually needed to write -1 as a sum of squares in K. This leads to the following definition:

Definition 1.1. The *level* $\mathrm{s}(K)$ of a field K is defined by

$$\mathrm{s}(K) = \min\{n \mid \exists x_1, \cdots, x_n \in K,\ -1 = x_1^2 + \cdots + x_n^2\}$$

if -1 is a sum of squares in K, otherwise one defines $\mathrm{s}(K) = +\infty$.

A question raised by B. L. van der Waerden in the 1930s asks for the integers that can occur as the level of a field K (cf. [20]). H. Kneser obtained a partial answer to this question in 1934 by showing that the possible values for the level are 1, 2, 4, 8 or the multiples of 16 (cf. [12]). In 1965, A. Pfister developed the theory of multiplicative forms which furnished a complete answer to this question: if finite, the level of a field is always a power of 2 and every prescribed 2-power can be realized as the level of a field, see [23].

The level of a ring R with unity which can be defined in the same manner as for commutative fields, has been studied at least since the early 20th century. See the survey paper [20] which provides a historical overview of different notions of the level as well as an extensive bibliography. The *sublevel* $\underline{\mathrm{s}}(R)$ of a ring R has also been defined in the following way:

$$\underline{\mathrm{s}}(R) = \min\{n \mid \exists x_1, \cdots, x_{n+1} \in R \setminus \{0\},\ 0 = x_1^2 + \cdots + x_{n+1}^2\}$$

if 0 is a sum of nonzero squares in R and $\underline{\mathrm{s}}(R) = +\infty$ otherwise. The question of how the level and the sublevel are related to each other is a natural matter of interest. It is clear from the definition that $\underline{\mathrm{s}}(R) \leqslant \mathrm{s}(R)$ and that $\underline{\mathrm{s}}(R) = \mathrm{s}(R)$ in the case of commutative fields. In [17] and [19], D. W. Lewis constructed quaternion division algebras with $\mathrm{s} = \underline{\mathrm{s}} = 2^k$ and with $\mathrm{s} = \underline{\mathrm{s}}+1 = 2^k + 1$ for all $k \in \mathbb{N}$. In [10], D. W. Hoffmann showed that $\underline{\mathrm{s}}(R) \leqslant \mathrm{s}(R) \leqslant \underline{\mathrm{s}}(R) + 1$ if R is a quaternion or an octonion division algebra. However, the general problem of determining the numbers attainable as the levels and sublevels of quaternion and division algebras - and thus of other specific rings - remains widely open although several results have been obtained: see for example [13], [9] or [22].

In this paper, we intend to study a natural generalization of the level of a field together with its associate notion of sublevel:

Definition 1.2. Let (V, q) be a quadratic form over a field K.
(1) The *level of K with respect to q* (or the q-level for short) denoted by $\mathrm{s}_q(K)$ is defined by

$$\mathrm{s}_q(K) = \min\{n \mid \exists v_1, \cdots, v_n \in V,\ -1 = q(v_1) + \cdots + q(v_n)\},$$

if such an n exists and by $\mathrm{s}_q(K) = +\infty$ otherwise.
(2) The *sublevel of K with respect to q* is denoted by $\underline{\mathrm{s}}_q(K)$ and defined by

$$\underline{\mathrm{s}}_q(K) = \min\{n \mid \exists v_1, \cdots, v_{n+1} \in V \setminus \{0\},\ 0 = q(v_1) + \cdots + q(v_{n+1})\},$$

if such an n exists and by $\underline{\mathrm{s}}_q(K) = +\infty$ otherwise.

Under this setting, the (usual) level of K corresponds to the level of K with respect to the quadratic form X^2 over K. More generally, the length (see §2) of a non-zero element $a \in K$ coincides with the q-level of K where q is the quadratic form $-aX^2$ over K. Studying the level of K with respect to one-dimensional quadratic forms over K is thus nothing but investigating the length of any element of K.

To our best knowledge, the notion of q-level has not been explicitly defined - at least in the general case of a quadratic form - but it appears implicitly in many places and it is closely related to some other invariants appearing in the literature.

It is already relevant to point out that q-levels are related to some hermitian levels studied by D. W. Lewis (see [18] and [20]). For a ring R with an identity and a non trivial involution σ, recall that the *hermitian level* of R is defined as the least integer n such that -1 is a sum of n hermitian squares in R, i.e., elements of the form $\sigma(x_1)x_1 + \cdots + \sigma(x_n)x_n$ where $x_1, \cdots, x_n \in R$. The hermitian level of (R, σ) is denoted by $\mathrm{s}(R, \sigma)$. If L/K (resp. Q) is a quadratic extension with $L = K(\sqrt{a})$ (resp. a K-quaternion algebra $(a,b)_K$) and if $-$ is the canonical involution of L (resp. of Q), it is easy to see that $\mathrm{s}(L,-) = \mathrm{s}_q(K)$ (resp. $\mathrm{s}(Q,-) = \mathrm{s}_q(K)$) where $q = \langle 1,-a\rangle$ (resp. $q = \langle 1,-a,-b,ab\rangle$) is the norm form of L/K (resp. of Q). Note that in both cases the hermitian level is a power of two (see [18, Prop. 1.5]) and the quadratic form q is a Pfister form over K. More generally, we observe that $\mathrm{s}_q(K)$ is always a 2-power or infinite whenever q is a Pfister form (see Proposition 4.3).

To give an example where we can relate the level of a ring with a q-level, consider the Clifford algebra D of a nondegenerate quadratic form q over a field K. We obviously have $\mathrm{s}(D) \leqslant \mathrm{s}_q(K)$; in particular if D is the quaternion algebra $(a,b)_K$, generated by the elements i and j subject to the relations $i^2 = a \in K^\times$, $j^2 = b \in K^\times$ and $ij = -ji$, then we have $s(D) \leqslant \mathrm{s}_q(K)$ where $q = \langle a,b\rangle$.

The essential trait of the concept of the q-sublevel is already present in the literature. In [2], K. J. Becher studies an invariant $\mathrm{wi}(q)$ which is called the *weak isotropy index* of q. One has $\underline{\mathrm{s}}_q(K) = \mathrm{wi}(q) - 1$. In this work, the q-sublevel is used as an auxiliary tool in some places.

A substantial part of this paper is devoted to investigating

– the properties of $\mathrm{s}_q(K)$, i.e., the q-level of a field K,
– the relations of $\mathrm{s}_q(K)$ with other invariants of K such as the (usual) level, the u-invariant, the Pythagoras number,
–the relations between $\mathrm{s}_q(K)$ and $\underline{\mathrm{s}}_q(K)$,
– the calculation of $\mathrm{s}_q(K)$ for particular q or K,
– the behavior of $\mathrm{s}_q(\cdot)$ under field extensions,
– for a given n, the possible values of $\mathrm{s}_q(K)$ that are attained when q runs over all quadratic forms of dimension n over K,
– for a given q, the possible values of $\mathrm{s}_{q_{K'}}(K')$ when K'/K runs over all field extensions of K.
– criteria for the finiteness or infiniteness of $\mathrm{s}_q(K)$ for particular q or K.

When q is a Pfister form there are several strong analogies between the properties of $\mathrm{s}_q(K)$ and that of $\mathrm{s}(K)$. However this does not mean that when q is a Pfister form, every result on $\mathrm{s}(K)$ can be generalized to $\mathrm{s}_q(K)$, see §4.3, Remark 4.11.

The paper is structured as follows. In the next section, we collect some definitions and preliminary observations about the q-level of a field and define the q-length of $a \in K$ and the Pythagoras q-number of K which are respective generalizations of the length of a and of the Pythagoras number of K.

Section 3 is devoted to the study of the q-level for an arbitrary quadratic form q. We give upper bounds for the q-level in terms of some familiar invariants (e.g., the usual level, the Pythagoras number and the u-invariant). We also determine the relations between the q-level and the q-sublevel of a field. Then we study the behavior of the q-level and the q-length of an element with respect to purely transcendental field extensions; this leads to a generalization of a theorem due to Cassels, see Theorem 3.7. One of the applications of this generalization is the construction of elements with prescribed q-length (see Corollaries 3.8 and 3.9).

Next we prove some results about the integers which belong to the set $\{\mathrm{s}_q(K) | \dim q = n\}$, see §3.3.1. After that, we show the following result:

Let q be a quadratic form with $\dim q \leqslant 3$ such that $\mathrm{s}_q(K) = +\infty$, then
(1) If $\dim q = 1$ or 2 then for any $k \in \mathbb{N}$ there is a field extension K'/K

such that $\mathrm{s}_q(K') = 2^k$.

(2) If $\dim q = 3$ then for any $k \in \mathbb{N}$ there exist field extensions K'/K and K''/K such that $\mathrm{s}_q(K') = \frac{2^{2k}+2}{3}$ and $\mathrm{s}_q(K'') = \frac{2^{2k+1}+1}{3}$.

These results are proved in Corollary 3.14: for this, the key ingredient is Hoffmann's separation Theorem (see Theorem 3.12). We make explicit calculations of q-levels for familiar fields, whenever possible, see §3.4

Section 4 concentrates on the particular case of Pfister forms. We prove that the q-level and the q-sublevel coincide whenever q is an anisotropic group form, see Proposition 4.1. We next prove that (see Proposition 4.3 and Theorem 4.4):

> If φ is a Pfister form over K, then $\mathrm{s}_\varphi(K)$ is a 2-power or infinite. Moreover $\{\mathrm{s}_\varphi(K') : K'/K \text{ field extension}\} = \{1, 2, \cdots, 2^i, \cdots, \mathrm{s}_\varphi(K)\}$.

Then, we investigate the behavior of the q-level with respect to quadratic field extensions: it is described in Proposition 4.10. We next show that for the case of Pfister forms the q-level, q-length and Pythagoras q-number share many similar properties with their usual counterparts (see Proposition 4.12 and Proposition 4.19).

A sharp lower bound for the cardinality of the group $K^\times/K^{\times^2}$ in terms of the level of K was found by A. Pfister who proved that if K is a field whose level is 2^n then $|\frac{K^\times}{K^{\times^2}}| \geqslant 2^{n(n+1)/2}$ (see [24, Satz 18 (d)]). The examples $K = \mathbb{F}_q$, $q \equiv 3 \mod 4$ or $K = \mathbb{Q}_2$ show that this inequality is best possible for $n \leqslant 2$. Recall that an element a of $K^\times$ is said to be *represented by q* if there exists $v \in V$ such that $q(v) = a$. Denote by $D_K(q)$ the set of values represented by q. In the case where q is a Pfister form, $D_K(q)$ is a subgroup of $K^\times$ ([14, Proposition X.2.5]) which contains $K^{\times^2}$, hence it is a natural thing to wonder if one may obtain a lower bound for the cardinality of the group $K^\times/D_K(q)$ in terms of the q-level of K. In fact we obtain the following result (see Theorem 4.14):

> If q is a Pfister form over a field K whose q-level is 2^n then $|K^\times/D_K(q)| \geqslant 2^{n(n+1)/2}$ and this lower bound is sharp.

We also draw some direct consequences of this lower bound. We then study the behavior of the Pythagoras q-number with respect to field extensions L/K of finite degree in Proposition 4.22: in fact $p_q(L)$ is smaller than $[L : K]p_q(K)$ thus generalizing a classical result due to Pfister.

In Section 5, we investigate possible characterizations of the finiteness of the q-level. In the case of Pfister forms, a complete characterization can be given (see Proposition 5.1): this also leads to an analogue of Artin-Schreier's characterization of the existence of an ordering in this framework. In the general case, we easily observe that if $\mathrm{s}_q(K)$ is finite then q is not totally positive. It turns out that when $\dim q = 1$ or 2 the converse is always true (see Proposition 5.6) but this is not the case anymore if $\dim q = 3$. We also characterize all fields K for which for every q, the finiteness of $\mathrm{s}_q(K)$ implies that q is not totally positive (see Theorem 5.9).

Finally, in Section 6, we assemble a few open questions in relation with some of the topics covered in this paper.

2 Preliminaries

In this paper, the characteristic of the base field K will always supposed to be different from 2 and all quadratic forms are implicitly supposed to be nondegenerate. The notation $\langle a_1, \cdots, a_n\rangle$ will refer to the diagonal quadratic form $a_1{X_1}^2 + \cdots + a_n{X_n}^2$. Every quadratic form q over K can be diagonalized, that is q is isometric to a diagonal quadratic form $\langle a_1, \cdots, a_n\rangle$ which we denote by $q \simeq \langle a_1, \cdots, a_n\rangle$. For a quadratic form q and an scalar $a \in K^\times$, $a \cdot q$ denotes the form q scaled by a. We will denote by $W(K)$ the Witt ring of K and by $I(K)$ its fundamental ideal. Its nth power is denoted by $I^n(K)$ and is additively generated by the *(n-fold) Pfister forms* $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. A quadratic form π over K is a *Pfister neighbor* if there exists a Pfister form φ and $a \in K^\times$ such that $2\dim(\pi) > \dim(\varphi)$ and $a \cdot \pi$ is a subform of φ. In this case, it is known that π is isotropic if and only if φ is isotropic if and only if φ is hyperbolic. For a positive integer n and a quadratic form q, we use the notations

$$\sigma_n = n \times \langle 1\rangle \text{ and } \sigma_{n,q} = n \times q.$$

We also denote $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

The *length* of an element $a \in K^\times$ denoted by $\ell(a)$ is the smallest integer n such that a is a sum of n squares; if such an n does not exist, we put $\ell(a) = +\infty$. Note that $\mathrm{s}(K) = \ell(-1)$. Following [26, Ch. 6, p.75], we denote by $\Sigma K^\bullet$ the set of all elements in $K^\times$ which can be written as a sum of squares in K.

The *Pythagoras number* of K is defined to be

$$p(K) = \sup\{\ell(a) \mid a \in \Sigma K^\bullet\} \in \mathbb{N}_0 \cup \{+\infty\}.$$

Recall that K is non formally real if and only if $K^\times = \Sigma K^\bullet$; in that case, $p(K)$ is always finite. To see this, if we put $s = \mathrm{s}(K)$ then the form σ_{s+1} is isotropic, hence universal and $p(K) \leqslant s + 1$. As -1 is not a sum of $s - 1$ squares, we obtain that $p(K) \in \{\mathrm{s}(K), \mathrm{s}(K) + 1\}$. If K is formally real, $p(K)$ can either be finite or infinite. D. W. Hoffmann has shown that each integer can in fact be realized as the Pythagoras number of a certain (formally real) field: see [7] or [8, Theorem 5.5]. For further details about the level and the Pythagoras number, the reader may also consult [26, Ch. 3, Ch. 7] or [15].

The *u-invariant of* K, which is denoted by $u(K)$, is defined to be $\max(\dim q)$ where q ranges over all anisotropic quadratic forms over K if such a maximum exists, and we define $u(K) = +\infty$ otherwise. Note that $u(K)$ is also the minimal integer n for which all quadratic forms of dimension strictly greater than n (resp. greater or equal than n) are isotropic (resp. universal) over K.

Let (V, q) be a quadratic form over K. We define the q-length of an element of K and the Pythagoras q-number of K as follows.

Definition 2.1. (1) The *q-length* of an element $a \in K^\times$ denoted by $\ell_q(a)$ is defined by

$$\ell_q(a) = \min\{n \mid \exists (v_1, \cdots, v_n) \in V^n,\ a = q(v_1) + \cdots + q(v_n)\}$$

if such an n exists and by $\ell_q(a) = +\infty$ otherwise.
(2) Let $\Sigma_q K^\bullet$ be the set of all elements x in $K^\times$ for which there exists an integer n such that the form $\sigma_{n,q}$ represents x. The Pythagoras q-number is defined by

$$p_q(K) = \sup\{\ell_q(a) \mid a \in \Sigma_q K^\bullet\},$$

if such an n exists and by $p_q(K) = +\infty$ otherwise.

As $\mathrm{s}_{\langle 1\rangle}(K) = \mathrm{s}(K)$, $\ell_{\langle 1\rangle}(a) = \ell(a)$ and $p_{\langle 1\rangle}(K) = p(K)$, the q-level, the q-length of a and the Pythagoras q-number can be regarded as respective generalizations of the level, the length of a and the Pythagoras number. Note also that $\mathrm{s}_q(K) = 1$ if and only if q represents -1, the number $\mathrm{s}_q(K)$ only depends on the isometry class of q, $\mathrm{s}_{\langle a\rangle}(K) = \ell(-a)$ and $\mathrm{s}_q(K) = \ell_q(-1)$.

Remarks 2.2. (1) For all $a \in K^\times$, we have $p_{a\cdot q}(K) = p_q(K)$, since we obviously have $\ell_{a\cdot q}(b) = \ell_q(a^{-1}b)$ for all $b \in K$.
(2) For every positive integer n, we can easily check that $\ell_{n\times q}(a) = \lceil \frac{1}{n}\ell_q(a)\rceil$ and $p_{n\times q}(a) = \lceil \frac{1}{n} p_q(a)\rceil$.
(3) If $a \in K^\times$ and L/K is a field extension of odd degree then $\ell_q(a) = \ell_{q_L}(a)$.

In the sequel it is convenient to introduce the following notations. If K is a field and n is a positive integer greater or equal than 1, we put

$$L(n, K) = \{\mathrm{s}_q(K) \mid q \text{ is a quadratic form of dimension } n \text{ over } K\}$$

and we set $L(K) = \bigcup_{n\in\mathbb{N}\setminus\{0\}} L(n, K)$. If q is a quadratic form over K, let

$$L_q(K) = \{\mathrm{s}_{q_{K'}}(K') \mid K'/K \text{ field extension}\}.$$

3 General results

3.1 Comparison of the q-level with some other invariants

In the following lemma, we list some properties concerning the q-level of a field.

Lemma 3.1. *Let K be a field and q be a quadratic form over K.*
(1) *We have $1 \leqslant \mathrm{s}_q(K) \leqslant \mathrm{s}(K) + 1$.*
(2) *We have $1 \leqslant \mathrm{s}_q(K) \leqslant \inf\{\mathrm{s}_{\langle a\rangle}(K) :\ a \in D_K(q)\} = \inf\{\ell(-a) :\ a \in D_K(q)\}$. More generally, if q' is a subform of q, then $\mathrm{s}_q(K) \leqslant \mathrm{s}_{q'}(K)$.*
(3) *If L/K is a field extension then $\mathrm{s}_q(K) \geqslant \mathrm{s}_{q_L}(L)$.*
(4) *If L/K is a field extension whose degree is odd then $\mathrm{s}_q(K) = \mathrm{s}_{q_L}(L)$.*
(5) *For every positive integer n we have $\mathrm{s}_{n\times q}(K) = \left\lceil \frac{\mathrm{s}_q(K)}{n} \right\rceil$.*
(6) *If q is isotropic then $\underline{\mathrm{s}}_q(K) = 0$ and $\mathrm{s}_q(K) = 1$.*
(7) *If q is anisotropic then $\mathrm{s}_q(K) \leqslant \underline{\mathrm{s}}_q(K) + 1$.*
(8) *If q is anisotropic and if $1 \in D_K(q)$, then $\underline{\mathrm{s}}_q(K) \leqslant \mathrm{s}_q(K) \leqslant \underline{\mathrm{s}}_q(K) + 1$.*

Proof. To prove (1), we may assume that $\mathrm{s}(K) = n < +\infty$. In this case, the quadratic form σ_{n+1} is isotropic so the quadratic form $\sigma_{n+1,q}$ is isotropic, hence universal. In particular, $\sigma_{n+1,q}$ represents -1. This proves (1).

For statement (2), it suffices to prove the second property. If $\sigma_{n,q'}$ represents -1 for a certain n, then it is also the case for $\sigma_{n,q}$, hence (2).

The statement (3) is trivial, (4) follows from a theorem of T. A. Springer (see [26, Ch. 6, 1.12]) and (5) follows from 2.2 (2).

The assertion (6) is obvious. For (7), if $\mathrm{s}_q(K) = +\infty$ then we obviously have $\underline{\mathrm{s}}_q(K) = +\infty$ so we may assume that $s = \mathrm{s}_q(K) < +\infty$. Then $\sigma_{s-1,q}$ is not isotropic as it would represent -1 otherwise, thus contradicting the minimality of s. This means that $s - 1 \leqslant \underline{\mathrm{s}}_q(K)$ and (7) follows.

To prove (8), we only have to show that $\underline{\mathrm{s}}_q(K) \leqslant \mathrm{s}_q(K)$ with $s = \mathrm{s}_q(K) < +\infty$. For this, it suffices to remark that $\langle 1\rangle \perp \sigma_{s,q}$ is a subform of $\sigma_{s+1,q}$. □

Remarks 3.2. (1) The upper bound $\mathrm{s}_q(K) \leqslant \mathrm{s}(K)+1$ given in 3.1 (1) is sharp. If $K = \mathbb{Q}(i)$ and $q = \langle 2 \rangle$, we easily see that $\mathrm{s}(K) = 1$ and $\mathrm{s}_q(K) = \ell(-2) = 2$.
(2) In the second statement of 3.1 (2), it is possible to have $\mathrm{s}_{q'}(K) = \mathrm{s}_q(K)$ for a proper subform q' of q. For instance consider the forms $q' = \langle -3 \rangle$ and $q = \langle -3, X \rangle$ over $\mathbb{Q}(X)$, we have $\mathrm{s}_q(\mathbb{Q}(X)) = \mathrm{s}_{q'}(\mathbb{Q}(X)) = 3$.
(3) When $\underline{\mathrm{s}}_q(K) < +\infty$ the above result shows that $\mathrm{s}_q(K) < +\infty$ but the converse is false as the following simple example shows: if $K = \mathbb{R}$, $q = \langle -1 \rangle$ then $\underline{\mathrm{s}}_q(K) = +\infty$ and $\mathrm{s}_q(K) = 1$.
(4) Note that the assertion (8) of the previous lemma is analogous to Hoffmann's result mentioned in the Introduction relating the level and the sublevel of a quaternion or of an octonion division algebra.
(5) For an example which shows that both values in the assertion (8) of the previous lemma occur see Remark 4.2.

The purpose of the following proposition is to give upper bounds for the q-level of a field K in terms of some classical invariants of K.

Proposition 3.3. *Let K be a field and let q be a quadratic form over K.*
(1) If $\mathrm{s}_q(K) < +\infty$ then $\mathrm{s}_q(K) \leqslant p(K)$ (see also Proposition 4.12 (2)).
(2) If K is not formally real, we have $\mathrm{s}_q(K) \leqslant \left\lceil \dfrac{u(K)}{\dim(q)} \right\rceil \leqslant u(K)$.

Proof. To prove (1), one may assume that $p = p(K) < +\infty$. Let $q = \langle a_1, \cdots, a_n \rangle$ be a diagonalization of q. By assumption, there exists an integer m and vectors $v_1, \cdots, v_m$ such that $-1 = q(v_1) + \cdots + q(v_m)$. It follows that $-1 = a_1\Sigma_1 + \cdots + a_n\Sigma_n$ where $\Sigma_1, \cdots, \Sigma_n \in \Sigma K^\bullet$ are sums of at most m squares. By the definition of the Pythagoras number, each Σ_i can be written as a sum of at most p squares. This fact readily implies that $\mathrm{s}_q(K) \leqslant p(K)$. An alternative proof can also be obtained Corollary 2.5 of [2] and Lemme 3.1 (7).

(2) One may assume that $u(K) < +\infty$. Then every quadratic form q of dimension greater or equal than $u(K)$ is universal. It follows that if $n \times \dim(q) \geqslant u(K)$ then $\mathrm{s}_q(K) \leqslant n$, hence the result. □

Remark 3.4. In the previous proposition, the bound given in (1) is sharp for any non formally real field K as Proposition 3.10 shows. We now show that the inequality $\mathrm{s}_q(K) \leqslant \left\lceil \frac{u(K)}{\dim(q)} \right\rceil$ is sharp for any prescribed dimension. Let n be a positive integer and choose m such that $n < 2^m$. Let F be a field such that $\mathrm{s}(F) = u(F) = 2^m$ (it is even possible to construct a field F such that $\mathrm{s}(F) = u(F) = p(F) = 2^m$, see [8, §5.2]). If we put $r = \left\lceil \frac{u(F)}{n} \right\rceil$ then

$$\frac{u(F)}{n} \leqslant r < \frac{u(F)}{n} + 1$$

so $rn \geqslant u(F)$ which implies that $\mathrm{s}_{\sigma_n}(F) \leqslant r$. Moreover $(r-1)n < u(F) = s(F)$ so $\sigma_{(r-1)n+1}$ is anisotropic which shows that $\mathrm{s}_{\sigma_n}(F) = r$ as claimed.

Corollary 3.5. *Let K be a field.*
(1) If q is a quadratic form over K, we have $L_q(K) \subseteq \{1, \cdots, \mathrm{s}_q(K)\}$.
(2) If $n \geqslant 1$ is an integer then $L(n, K) \subseteq \{1, \cdots, h_n\} \cup \{+\infty\}$ where $h_n = \min(p(K), \left\lceil \frac{u(K)}{n} \right\rceil)$. In particular, if K is non formally real and $n \geqslant u(K)$ then $L(n, K) = \{1\}$.

Proof. This follows from Lemma 3.1 (3) and Proposition 3.3. □

3.2 On a theorem of Cassels

The following two results generalize two classical theorems concerning quadratic forms under transcendental field extensions in the framework of q-levels.

Proposition 3.6. *Let q be a quadratic form over K and let K'/K be a purely transcendental field extension of K. Then $\mathrm{s}_q(K) = \mathrm{s}_q(K') = \mathrm{s}_q(K((t)))$.*

Proof. By Lemma 3.1 (3), we have $\mathrm{s}_q(K) \geqslant \mathrm{s}_q(K')$, hence we may assume that $\mathrm{s}_q(K') = n < +\infty$. The quadratic form $\sigma_{n,q}$ represents -1 over K' and by Cassels-Pfister's Theorem (see [14, Theorem IX.1.3]), $\sigma_{n,q}$ already represents -1 over K, hence the first equality. The second equality is proved similarly. □

A theorem due to J. W. S. Cassels asserts that if K is formally real, the polynomial $P = 1+X_1^2+\cdots+X_n^2$ is not a sum of n squares in $L = K(X_1, \cdots, X_n)$ (see [14, Corollary IX.2.4]). Note that this is equivalent to $\ell(P) = n+1$ over L. In the same vein we obtain the following result:

Theorem 3.7. *For $m \geqslant 1$ and $n \geqslant 0$, let $q = \langle a_1, \cdots, a_m\rangle$ be a quadratic form over a field K and set $L = K(X_j^{(i)}, i = 1, \cdots, n, j = 1, \cdots, m)$. If $\sigma_{n,q}$ is anisotropic, then*

$$\ell_{q_L}(a_1 + \sum_{i,j} a_j (X_j^{(i)})^2) = n + 1.$$

In particular, this is the case if $\mathrm{s}_q(K) = +\infty$.

Proof. If $n = 0$, we have to prove that $\ell_q(a_1) = 1$, which is obvious since a_1 is represented by q. Assume now that $n \geqslant 1$, and set $a = a_1 + \sum_{i,j} a_j (X_j^{(i)})^2$.

Since a_1 is represented by q, and thus by q_L, we have $\ell_{q_L}(a) \leqslant n+1$. Assume that $\ell_{q_L}(a) < n + 1$, so that a is represented by σ_{n,q_L}. Since $\sigma_{n,q}$ is anisotropic over K by assumption, any subform q' of $\sigma_{n,q}$ is also anisotropic over K. This implies that q'_L is anisotropic over L, since L/K is purely transcendental. Consequently, q'_M is anisotropic over M for any subfield M of L, and any subform q' of $\sigma_{n,q}$. A repeated application of a theorem due to Cassels (cf. [29, Theorem 3.4, p.150]) then shows that $a_1 + a_1(X_1^{(1)})^2$ is represented by $\langle a_1\rangle$ over $K(X_1^{(1)})$. This implies that $1 + (X_1^{(1)})^2$ is a square in $K(X_1^{(1)})$, hence a contradiction.

Let us prove the last part of the proposition. If $\mathrm{s}_q(K) = +\infty$ but $\sigma_{n,q}$ is isotropic for $n \geqslant 1$ then $\sigma_{n,q}$ is universal, hence represents -1, so $\mathrm{s}_q(K) \leqslant n$, and we have a contradiction. Now apply the first part to conclude. □

Corollary 3.8. *With the same hypotheses as in 3.7, we have*

$$\ell_{q_L}(\sum_{i,j} a_j (X_j^{(i)})^2) = n.$$

Proof. As $p(X) = \sum_{i,j} a_j (X_j^{(i)})^2$ is represented by $\sigma_{n,q}$ we have $\ell_{q_L}(p(X)) \leqslant n$. If $\ell_{q_L}(p(X)) < n$ then $p(X)$ is represented by $\sigma_{n-1,q}$. Since a_1 is represented by q, it follows that $\ell_{q_L}(a_1 + p(X)) < n + 1$ which contradicts the conclusion of Theorem 3.7. □

Corollary 3.9. *Let q be a quadratic form over a field K such that for every n the form $\sigma_{n,q}$ is anisotropic (in other words q is supposed to be strongly anisotropic). Then one can find elements with prescribed q-length in a suitable purely transcendental extension of K.*

3.3 Values of the q-level

3.3.1 Values of the q-level when dimension of q is given

Proposition 3.10. *If K is non formally real field then $L(1, K) = L(K) = \{1, \cdots, p(K)\}$. If K is formally real then $L(1, K) = L(K) = \{1, \cdots, p(K)\} \cup \{+\infty\}$*

Proof. As $L(1, K) \subseteq L(K)$, the direct inclusions come from Proposition 3.3 (1) in both cases. It remains to show that the sets on the right-hand sides are included in $L(1, K)$.

For this, we distinguish between the cases $p(K) < +\infty$ and $p(K) = +\infty$. Note that $+\infty = \mathrm{s}_{\langle 1\rangle}(K) \in L(1, K)$ in the formally real case.

If $p = p(K) < +\infty$, there exists $a \in \Sigma K^{\bullet}$ such that $\ell(a) = p$. Write $a = x_1^2 + \cdots + x_p^2$ where $x_i \in K$. For $1 \leqslant i \leqslant \ell(a)$ put $\beta_i = x_1^2 + \cdots + x_i^2$. Then $\ell(\beta_i) = i = \mathrm{s}_{\langle -\beta_i\rangle}(K)$, hence the result.

Suppose now that $p(K) = +\infty$ and let n be a fixed integer. By definition of the Pythagoras number, there exists $a \in \Sigma K^{\bullet}$ such that $q = \ell(a) > n$. Put $a = x_1^2 + \cdots + x_q^2$ where $x_i \in K$. If we choose $b = x_1^2 + \cdots + x_n^2$ then $\ell(b) = n = \mathrm{s}_{\langle -b\rangle}(K)$ and this concludes the proof. □

Whereas the knowledge of the possible q-levels over a field K is equivalent to the knowledge of the Pythagoras number of K, the previous result does not give an explicit way to find a quadratic form q with prescribed q-level in general.

We obtain the following immediate consequences:

Corollary 3.11. (1) *There exists a field K such that for every integer n, there exists a quadratic form q over K with $\mathrm{s}_q(K) = n$.*
(2) *If $n \geqslant 1$ is an integer then $\{1, \cdots, l_n\} \subseteq L(n, K)$ where $l_n = \lfloor p(K)/n \rfloor$. In particular, if $p(K) = +\infty$ then $L(n, K) = \mathbb{N}_0 \cup \{+\infty\}$ for any $n \geqslant 1$.*

Proof. (1) By Proposition 3.10, it suffices to find K with $p(K) = +\infty$. Following the proof of [8, Theorem 5.5], put $K = F(X_1, X_2, \cdots)$ with an infinite number of variables X_i over a formally real field F. If $n \in \mathbb{N}_0$, put $P_n = 1 + X_1^2 + \cdots + X_n^2$. Then $\ell(P_n) = n+1$ over $F(X_1, \cdots, X_n)$ (by Cassels' theorem mentioned above) and K (as K/F is purely transcendental). Thus $p(K) = +\infty$. An alternative proof can be given using Theorem 3.7.

(2) It suffices to prove the first assertion. If we fix $1 \leqslant i \leqslant l_n$ then $n \times i \leqslant p(K)$. By Proposition 3.10, there exists a form $q_i = \langle a_i\rangle$ with $\mathrm{s}_{q_i}(K) = n \times i$. By Lemma 3.1 (5), the form $\sigma_{n,q_i} \in L(n, K)$ has level i, hence the result. □

3.3.2 Values of the q-level when q is fixed and the base field changes

We now focus on the proof of Corollary 3.14, mentioned in the Introduction, for which the crucial ingredient is the following result due to D. W. Hoffmann in [6, Theorem 1].

Theorem 3.12 (Hoffmann's Separation Theorem)**.** *Let q and q' be anisotropic quadratic forms over K such that $\dim(q) \leqslant 2^n < \dim(q')$. Then q is anisotropic over $K(q')$, the function field of the projective quadric defined by $q' = 0$.*

The key fact for us is the second assertion of the following proposition. The first assertion is a direct consequence of Corollary 3.11 (1) but can also be proved independently using Hoffmann's result.

Proposition 3.13. *Let n be a positive integer and K be a formally real field.*
(1) *There exist a field extension K'/K and a quadratic form q over K' such that $\mathrm{s}_q(K') = n$.*
(2) *Let q be a quadratic form over K with $\mathrm{s}_q(K) = +\infty$. If there exists a positive integer k such that $1 + (n-1)\dim q \leqslant 2^k < 1 + n \dim q$ then there exists a field extension K'/K such that $\mathrm{s}_q(K') = n$. In fact one can choose $K' = K(\langle 1\rangle \perp \sigma_{n,q})$.*

Proof. For (1), we can suppose that $n > 1$. Let m and r be two integers such that

$$\frac{2^r - 1}{n} < m \leqslant \frac{2^r - 1}{n-1}.$$

These two integers do exist: in fact, it suffices to choose m and r satisfying $r > \dfrac{\ln(n^2 - n + 1)}{\ln(2)}$ which implies that $\dfrac{2^r - 1}{n-1} - \dfrac{2^r - 1}{n} = \dfrac{2^r - 1}{n(n-1)} > 1$.

For $j > 1$, let $\varphi_j = \langle 1\rangle \perp \sigma_{j,q}$ where q is the anisotropic quadratic form $q = \sigma_m$. The quadratic forms φ_n and φ_{n-1} are anisotropic over K. If $K' = K(\varphi_n)$ then, by Theorem 3.12 and by the choice of m and r, $(\varphi_{n-1})_{K'}$ is anisotropic, hence $\mathrm{s}_q(K') \geqslant n$. As $(\varphi_n)_{K'}$ is isotropic, we obtain $\mathrm{s}_q(K') = n$, hence (1). For (2), the fact that $\mathrm{s}_q(K) = +\infty$ implies similarly that $\mathrm{s}_q(K') = n$ for $K' = K(\psi_n)$ where $\psi_n = \langle 1\rangle \perp \sigma_{n,q}$. ☐

Corollary 3.14. *Let q be a quadratic form of dimension at most 3 such that $\mathrm{s}_q(K) = +\infty$.*
(1) *If q has dimension 1 or 2 then $2^k \in L_q(K)$ for any $k \in \mathbb{N}$.*
(2) *If q has dimension 3 then $\frac{2^{2k}+2}{3}$, $\frac{2^{2k+1}+1}{3} \in L_q(K)$ for any $k \in \mathbb{N}$.*

Proof. (1) Consider the form $\varphi_{2^k} = \langle 1\rangle \perp \sigma_{2^k,q}$. As $\dim q = 1$ or 2, we have $1 + (2^k - 1)\dim q \leqslant 2^{k+\dim q - 1} < 1 + 2^k \dim q$. By Proposition 3.13 (2), we obtain $\mathrm{s}_q(K') = 2^k$ where $K' = K(\varphi_{2^k})$.

(2) If $n = \frac{2^{2k}+2}{3}$, we have $1+(n-1)\dim q \leqslant 2^{2k} < 1+(n-1)\dim q$, therefore the existence of K' follows from Proposition 3.13 (2). The proof of the second assertion is similar and is left to the reader. ☐

Corollary 3.15. *Let K be a field and let n be a nonnegative integer.*
(1) *If $\mathrm{s}(K) = +\infty$ then there exists a field extension K'/K with $\mathrm{s}(K') = 2^n$.*
(2) *If $a \in K$ and $\ell(a) = +\infty$ over K then there exists a field extension K'/K such that $\ell(a) = 2^n$ over K'.*

One may wonder if $\ell_K(a) = +\infty$ implies that for every field extension L/K, $\ell_L(a)$ is always infinite or a power of two.

3.4 Values of the q-level for some specific fields

We now make explicit calculations of q-levels in many familiar fields. Note that $\mathrm{s}_{\langle -1\rangle}(K) = 1$ for any field K and that $\mathrm{s}_{\langle 1\rangle}(K) = +\infty$ when K is formally real. The results 3.10, 3.5 (2) and 3.11 (2) are used without further mention.

3.4.1 Non formally real fields

Algebraically closed fields: in such a field K, $L(n,K) = L_q(K) = \{1\}$ for any n and q.

Finite fields: if K is a finite field, $L(K) = \{1,2\}$. As $u(K) = 2$, we have $L(n,K) = L_q(K) = \{1\}$ for $n \geqslant 2$, $\dim(q) \geqslant 2$. Moreover, the quadratic form $q = \langle a \rangle$ has q-level 1 if and only if $-a$ is a square, otherwise it has q-level 2.

Non dyadic local fields: in such a field K with residue field $\overline{K}$, denote by U the group of units and choose a uniformizer π. Any quadratic form q can be written $\partial_1(q) \perp \pi\partial_2(q)$ where $\partial_1(q) = \langle u_1, \cdots, u_r \rangle$, $\partial_2(q) = \langle u_{r+1}, \cdots u_n \rangle$ for $u_i \in U$, $i = 1, \cdots, n$. The forms $\partial_1(q)$ and $\partial_2(q)$ are respectively called the first and second residue forms of q. By a Theorem of T. A. Springer, q is anisotropic over K if and only if $\overline{\partial_1(q)}$ and $\overline{\partial_2(q)}$ are anisotropic over $\overline{K}$ (see [14, Proposition VI.1.9]). This reduces the calculation of q-levels over K to calculations of some levels over $\overline{K}$.

Now, $p(K) = \min(\mathrm{s}(K)+1, 4)$ (see [26, Ch. 7, Examples 1.4 (1)]), hence $L(K) = \{1,2\}$ (resp. $\{1,2,3\}$) if $|\overline{K}| \equiv 1 \mod 4$ (resp. $|\overline{K}| \equiv 3 \mod 4$).

Take $u \in U$ with $\overline{u} \notin \overline{K}^2$ and put $q = \langle -u \rangle$. Then $\langle 1, -u \rangle$ is anisotropic but $\langle 1, -u, -u \rangle$ is isotropic by Springer's Theorem, hence $\mathrm{s}_q(K) = 2$. If $|\overline{K}| \equiv 3 \mod 4$, take $q' = \langle \pi \rangle$. As -1 is not a square in $\overline{K}$, Springer's Theorem shows that $\mathrm{s}_{q'}(K) > 2$, hence $\mathrm{s}_{q'}(K) = 3$.

Dyadic local fields: we have $p(K) = \min(\mathrm{s}(K)+1, 4)$ and $\mathrm{s}(K) = 1, 2$ or 4. If $K = \mathbb{Q}_2$, we have $\mathrm{s}(K) = 4$ (see [14, Examples XI.2.4]), hence $L(\mathbb{Q}_2) = \{1,2,3,4\}$. We have $\mathrm{s}_{\langle -2 \rangle}(\mathbb{Q}_2) = 2$, $\mathrm{s}_{\langle 2 \rangle}(\mathbb{Q}_2) = 3$ and $\mathrm{s}_{\langle 1 \rangle}(\mathbb{Q}_2) = 4$.

The fields $L_n = K(X_1, \cdots, X_n)$ and $M_n = K((X_1)) \cdots ((X_n))$: if $\mathrm{s}(K) = 2^m$ then $p(L_n) = p(M_n) = 2^m + 1$ ([26, Ch. 7, Proposition 1.5]), hence $L(L_n) = L(M_n) = \{1, \cdots, 2^m + 1\}$. Recall that $p(K) \in \{2^m, 2^m + 1\}$.

Each value in $\{1, \cdots, p(K)\}$ is attained as a q-level over K and $\mathrm{s}_q(K) = \mathrm{s}_q(L_n) = \mathrm{s}_q(M_n)$ by Proposition 3.6. If $p(K) = \mathrm{s}(K) = 2^m$, let $q = \langle X_n \rangle$ over L_n. As M_n is a local field with residue field $K((X_1)) \cdots ((X_{n-1}))$ of level 2^m and uniformizer X_n, we have $\mathrm{s}_q(L_n) = \mathrm{s}_q(M_n) = 2^m + 1$ by Springer's theorem.

Non formally real global fields: if K is a number field, we have $p(K) = \min(\mathrm{s}(K)+1, 4)$ by Hasse-Minkowski principle (see [26, Examples 1.4 (2)]) and $\mathrm{s}(K) = 1, 2$ or 4 ([14, Theorem XI.1.4]). For example, if $K = \mathbb{Q}(\sqrt{-7})$ then $\mathrm{s}_{\langle -5 \rangle}(K) = 2$, $\mathrm{s}_{\langle -6 \rangle}(K) = 3$ and $p(K) = \mathrm{s}(K) = \mathrm{s}_{\langle 1 \rangle}(K) = 4$.

3.4.2 Formally real fields

Real closed fields: over such a field, a quadratic form q has q-level $+\infty$ if and only if q is positive definite, otherwise it has q-level one.

Formally real global fields: we have $p(K) = 4$ (resp. $p(K) = 3$) if K has a dyadic place P such that $[K_P : \mathbb{Q}]$ is odd (resp. otherwise) by [26, Ch. 7, Examples 1.4 (3)].

In particular, $L(\mathbb{Q}) = \{1, 2, 3, 4\}$. By Hasse-Minkowski principle, any indefinite quadratic form q of dimension $\geqslant 5$ is isotropic hence has q-level one, see [14, Ch. VI.3]. Any positive definite quadratic form q has an infinite q-level. A quadratic form q that is not positive definite has a finite q-level. For example $\mathrm{s}_{\langle -5\rangle}(\mathbb{Q}) = 2$, $\mathrm{s}_{\langle -6\rangle}(\mathbb{Q}) = 3$ and $\mathrm{s}_{\langle -7\rangle}(\mathbb{Q}) = 4$ as 6 (resp. 7) is a sum of three squares (resp. four squares) but not a sum of two squares (resp. three squares) in $\mathbb{Q}$.

The field $\mathbb{R}(X_1, \cdots, X_n)$: its Pythagoras number is 2 if $n = 1$, is 4 if $n = 2$ and is in the interval $[n + 2; 2^n]$ for $n \geqslant 3$ (the two last results are due to J. W. S. Cassels, W. J Ellison and A. Pfister, see [3] or [14, Examples XI.5.9 (4)]).

If $n = 1$, we thus have $L(\mathbb{R}(X)) = \{1, 2\}$. By a Theorem due to E. Witt, we know that any totally indefinite quadratic form over $\mathbb{R}(X)$ of dimension $\geqslant 3$ is isotropic hence has q level one. Consider $q = \langle -(1 + X^2)\rangle$. As $1 + X^2$ is not a square in $\mathbb{R}(X)$, q does not represent -1. But $\langle 1\rangle \perp 2 \cdot q$ is totally indefinite as 1 is totally positive whereas $-(1 + X^2)$ is totally negative, hence it is isotropic. This proves that $\mathrm{s}_q(\mathbb{R}(X)) = 2$.

The field $\mathbb{Q}(X_1, \cdots, X_n)$: its Pythagoras number is 5 if $n = 1$ (by a result due to Y. Pourchet, see [27]). For $n \geqslant 2$, we only know that $p(\mathbb{Q}(X_1, X_2)) \leqslant 8$, $p(\mathbb{Q}(X_1, X_2, X_3)) \leqslant 16$ and $p(\mathbb{Q}(X_1, \cdots, X_n)) \leqslant 2^{n+2}$. The first two results are due to J.-L. Colliot-Thélène and U. Jannsen in [4] and the last one is due to J. K. Arason.

If $n = 1$, the study done for $\mathbb{Q}$ together with Proposition 3.6 show that $\mathrm{s}_{\langle -5\rangle}(\mathbb{Q}(X)) = 2$, $\mathrm{s}_{\langle -6\rangle}(\mathbb{Q}(X)) = 3$ and $\mathrm{s}_{\langle -7\rangle}(\mathbb{Q}(X)) = 4$. By a result of Y. Pourchet (see [27, Proposition 10]), for $d \in \mathbb{Z}$, the polynomial $X^2 + d$ is a sum of exactly five squares of $\mathbb{Q}[X]$ if and only if $\mathrm{s}(\mathbb{Q}(\sqrt{-d})) = 4$ which in turn is equivalent to $d > 0$ and $d \equiv -1 (\mathrm{mod}\ 8)$ (see [14, Remark XI.2.10]). For $q = \langle -(X^2 + 7)\rangle$, this readily implies that $\mathrm{s}_q(\mathbb{Q}(X)) = 5$.

3.4.3 Appendix

Suppose that K is a non formally real field. In all the above examples in which the values of $\mathrm{s}(K)$, $u(K)$ and $p(K)$ are known, we have $p(K) = \min(\mathrm{s}(K) + 1, u(K))$. We always have $p(K) \leqslant \min(\mathrm{s}(K) + 1, u(K))$ but there is no equality in general.

To see this, we use the construction of fields with prescribed even u-invariant due to A. S. Merkurjev (see [21] or [8, Section 5]).

Theorem 3.16 (Merkurjev)**.** *Let m be an even number and E be a field. There exists a non formally real field F over E such that $u(F) = m$ and $I^3(F) = 0$.*

Put $m = 2n+2$. The proof of Merkurjev's result is based upon a construction of an infinite tower of fields F_i. More precisely $F_0 = E(X_1, Y_1 \cdots, X_n, Y_n)$ and if F_i is constructed then F_{i+1} is the free compositum over F_i of all function fields $F_i(\psi)$ where ψ ranges over (1) all quadratic forms in $I^3(F_i)$ (2) all quadratic forms of dimension $2n + 3$ over F_i. Then $F = \cup_{i=0}^{\infty} F_i$ is the desired field.

Choose a field E with $\mathrm{s}(E) = 4$. Then there exists a field F over E such that $u(F) = 2n + 2$ and $I^3(F) = 0$. If we consider the 2-fold anisotropic Pfister form $\varphi = \langle 1, 1, 1, 1\rangle$ over E, the form φ stays anisotropic over any F_i of the tower as it cannot become hyperbolic over the function field of a quadratic form of dimension strictly greater than 4 by [14, Theorem X.4.5], hence $\mathrm{s}(F) = 4$. Now, choose $n = 2$ so that $\min(\mathrm{s}(F)+1, u(F)) = 5$. Let $x \in K^\times$ and consider the quadratic form $\varphi \perp \langle -x \rangle$: it is isotropic since it is the Pfister neighbor of $\langle\langle -1, -1, x \rangle\rangle$ which is hyperbolic as $I^3(F) = 0$. As $\mathrm{s}(F) = 4$, this means that φ is anisotropic over F, hence every $x \in K^\times$ is a sum of at most 4 squares in K. Now, -1 is a sum of 4 squares and is not a sum of three squares which shows that $p(F) = 4 < \min(\mathrm{s}(F)+1, u(F)) = 5$.

4 The case of Pfister forms

4.1 On the equality of q-level and q-sublevel

In §3 we presented some results concerning the relations between the q-level and the q-sublevel of a field K. Here we consider a case where these invariants coincide. For this purpose and for the sequel, it seems also relevant to recall the notions of multiplicative, round and group forms. Let q be a quadratic form over K. The form q is said to be:
-*multiplicative* if $q(X) \cdot q(Y) \in D_{K(X,Y)}(q)$ where $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ are sets of independent indeterminates over K and $n = \dim(q)$;
-*round* if $D_K(q) = G_K(q)$ where $G_K(q) = \{a \in K^\times \mid a \cdot q \simeq q\}$ is the group of similarity factors of q;
-a *group form* if $D_K(q)$ is a subgroup of $K^\times$.
Any Pfister form is multiplicative and any multiplicative form is a round form (and a fortiori a group form)[1]. We now come to the problem of relating the q-level and the q-sublevel of a field.

Proposition 4.1. *Let (V, q) be a quadratic form over K. If q is an anisotropic group form over K then we have $\mathrm{s}_q(K) = \underline{\mathrm{s}}_q(K)$.*

Proof. One may assume that $\underline{\mathrm{s}}_q(K) = s < +\infty$. Then, there exists $v_1, \cdots, v_{s+1} \in V \setminus \{0\}$ such that $\sum_{i=1}^{s+1} q(v_i) = 0$. As q is anisotropic, we deduce that

$$\sum_{i=1}^{s} \frac{q(v_i)}{q(v_{n+1})} = -1.$$

As q is a group form, $\dfrac{q(v_i)}{q(v_{n+1})} \in D_K(q)$ which implies that $\mathrm{s}_q(K) \leqslant s$. The result now follows from Lemma 3.1 (8). $\square$

Remark 4.2. In general, we have $\mathrm{s}_q(K) \neq \underline{\mathrm{s}}_q(K)$ even if q represents 1 (or is a Pfister neighbor). Take K to be a local field with $|\overline{K}| \equiv 3 \mod 4$ and let π be an uniformizer of K. If $q = \langle 1, \pi, \pi \rangle$ then $\mathrm{s}_q(K) = 2$ and $\underline{\mathrm{s}}_q(K) = 1$. >From this example, we easily derive that both values can occur in the assertion (8) of Lemme 3.1.

[1] More precisely, the relation between these notions can be made explicit by a characterization due to Pfister: an anisotropic quadratic form over K is a Pfister form if and only if it is multiplicative if and only it is round over any field extension of K if and only if it is a group form over any field extension of K (see [14, Theorem X.2.8] and [29, Theorem 4.4, p.153]).

4.2 Values of the q-level: the case of Pfister forms

Proposition 4.3. *Let φ be a round form over a field K. Then $\mathrm{s}_\varphi(K)$ is either infinite or a power of two.*

Proof. One may assume that $s = \mathrm{s}_\varphi(K) < \infty$. Suppose that $2^n \leqslant s < 2^{n+1}$. As $1 \in D_K(\varphi)$, it follows that $\langle 1\rangle \perp \sigma_{s,\varphi}$ is a subform of $\sigma_{2^{n+1},\varphi}$. The form $\sigma_{2^{n+1},\varphi}$ is thus isotropic which implies that there exists $x \in D_K(\sigma_{2^n,\varphi}) \cap D_K(\sigma_{2^n,-\varphi})$. By the Round Form Theorem due to Witt (see [14, Theorem X.1.14]), the form $\sigma_{2^n,\varphi}$ is round which implies that $-x \cdot x \in D_K(\sigma_{2^n,\varphi})$. We conclude that $-1 \in D_K(\sigma_{2^n,\varphi})$ and $\mathrm{s}_\varphi(K) \leqslant 2^n$ hence the result. □

Theorem 4.4. *If φ is a Pfister form over K, then $\mathrm{s}_\varphi(K)$ is a 2-power or infinite. Moreover $L_\varphi(K) = \{1, 2, \cdots, 2^i, \cdots, \mathrm{s}_\varphi(K)\}$.*

Proof. The direct inclusion follows from Proposition 4.3 together with Lemma 3.1 (3). To prove the converse, first note that the two numbers 1 and $\mathrm{s}_\varphi(K)$ are respectively attained, as the φ-level, over $K(\varphi)$ and K. Let $n > 0$ be such that $2^n < \mathrm{s}_\varphi(K)$. For an integer k, put $\varphi_k = \sigma_{2^k,\varphi}$. Then φ_k is a Pfister form for any k. Put $K' = K(\varphi_{n+1})$. As $2^n < \mathrm{s}_\varphi(K)$, φ_n is not hyperbolic over K, the Cassels-Pfister subform Theorem shows that $(\varphi_n)_{K'}$ is anisotropic which implies that $\mathrm{s}_\varphi(K') > 2^{n-1}$. Also $\mathrm{s}_\varphi(K')$ is a 2-power by the previous proposition. Moreover, the form $\psi_n = \langle 1\rangle \perp \varphi_n$ is a Pfister neighbor of φ_{n+1}. As φ_{n+1} is hyperbolic over K', ψ_n is isotropic over K' so $\mathrm{s}_\varphi(K') = 2^n$, hence the result. □

Remark 4.5. In view of Proposition 4.3, it is natural to ask if the above theorem is still true when φ is only supposed to be a round form. Our impression is that the answer is less likely to be affirmative as a form that is round over a field K does not necessarily stay round over L for a field extension L/K and a form that is not round over K can give rise to a round form by passing to a particular field extension. To see this, consider the form $q = \langle 1, 1, 1\rangle$. Then $G_K(q) = (K^\times)^2$ (by using the discriminant) and thus q is round over $\mathbb{R}$. But q is neither round over $\mathbb{Q}$ nor over $\mathbb{R}(T)$ as $T^2 + 1 \in D_{\mathbb{R}(T)}(q) \setminus G_{\mathbb{R}(T)}(q)$.

Corollary 4.6. *Let φ be a n-fold Pfister form over K and q be a subform of φ such that $\dim(q) > 2^{n-1}$. Then $\mathrm{s}_\varphi(K) \leqslant \mathrm{s}_q(K) \leqslant 2\,\mathrm{s}_\varphi(K)$.*

Proof. If $\mathrm{s}_\varphi(K) = +\infty$, we have $\mathrm{s}_q(K) = +\infty$ by Lemma 3.1 (2) so suppose that $\mathrm{s}_\varphi(K) < +\infty$. By Proposition 4.3 we have $\mathrm{s}_\varphi(K) = 2^r$ where r is an integer and we obtain $\mathrm{s}_\varphi(K) = 2^r \leqslant \mathrm{s}_q(K)$ by Lemma 3.1 (2). If $n = 1$, $q = \varphi$ and the result is clear. Suppose $n \geqslant 2$. By the definition of the φ-level the quadratic form $\langle 1\rangle \perp \sigma_{2^r,\varphi}$ is isotropic. This form is a Pfister neighbor of $\sigma_{2^{r+1},\varphi}$ which is thus hyperbolic. Finally, $\sigma_{2^{r+1},q}$ is isotropic being a Pfister neighbor of $\sigma_{2^{r+1},\varphi}$. Hence -1 is represented by $\sigma_{2^{r+1},q}$ and thus $\mathrm{s}_q(K) \leqslant 2^{r+1} = 2\,\mathrm{s}_\varphi(K)$. □

Example 4.7. The upper bound given in Corollary 4.6 is sharp. Let $p \neq 2$ be a prime number and $K = \mathbb{Q}_p$. If φ is the unique 4-dimensional anisotropic form over K and q is the pure subform of φ, we have $\mathrm{s}_q(K) = 2$ and $\mathrm{s}_\varphi(K) = 1$.

4.3 Behavior under quadratic extensions

We now investigate the behavior of the q-level under quadratic extensions in the case of Pfister forms.

Lemma 4.8. *Let K be a field, (V,φ) a Pfister form over K and let $L = K(\sqrt{d})$ be a quadratic field extension of K. Then we have $\ell_\varphi(-d) \leqslant 2s$ where $s = \mathrm{s}_\varphi(L)$.*

Proof. By hypothesis, there exist $2s$ vectors $v_1, w_1, \cdots, v_s, w_s$ in V such that

$$-1 = \varphi(v_1 \otimes 1 + w_1 \otimes \sqrt{d}) + \cdots + \varphi(v_s \otimes 1 + w_s \otimes \sqrt{d}). \tag{1}$$

Denote by b_φ the bilinear form associated to φ. Then for $v,\ w \in V$ we have $\varphi(v \otimes 1 + w \otimes \sqrt{d}) = \varphi(v) + d\varphi(w) + 2b_\varphi(v,w)\sqrt{d}$. >From equation (1) we obtain the following equation

$$-1 = (\varphi(v_1) + \cdots + \varphi(v_s)) + d(\varphi(w_1) + \cdots + d\varphi(w_s)), \tag{2}$$

Thus $-d(\varphi(w_1) + \cdots + \varphi(w_s))^2$ is equal to

$$(\varphi(v_1) + \cdots + \varphi(v_s))(\varphi(w_1) + \cdots + \varphi(w_s)) + (\varphi(w_1) + \cdots + \varphi(w_s)). \tag{3}$$

As $\sigma_{s,\varphi}$ is a Pfister form, it is multiplicative, hence the first term of the expression (3) is represented by $\sigma_{s,\varphi}$. The expression (3) can therefore be represented by the form $\sigma_{2s,\varphi}$, hence the result. □

Example 4.9. Note that the bound obtained in the previous lemma is optimal. Take $K = \mathbb{Q}$, $d = -3$ and $L = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$. Let $\varphi = \langle 1, 1\rangle$. As $-1 = (\frac{1+\sqrt{-3}}{2})^2 + (\frac{1-\sqrt{-3}}{2})^2$, the element -1 is represented by φ, hence $\mathrm{s}_\varphi(L) = 1$. As $-d = 3$ is represented by $\sigma_{2,\varphi}$ but it is not represented by φ we have $\ell_\varphi(3) = 2$.

Proposition 4.10. *Let φ be a Pfister form over a field K. Let $d \in K$ be an element such that $\ell_\varphi(-d) = n$. If $L = K(\sqrt{d})$, we have $\mathrm{s}_\varphi(L) = 2^k$ or 2^{k-1} where k is determined by $2^k \leqslant n < 2^{k+1}$.*

Proof. As $\ell_\varphi(-d) = n$, there exist vectors $v_1, \cdots, v_n$ such that $-d = \varphi(v_1) + \cdots + \varphi(v_n)$. We thus have

$$-1 = \varphi(v_1 \otimes \frac{1}{\sqrt{d}}) + \cdots + \varphi(v_n \otimes \frac{1}{\sqrt{d}}),$$

so $\mathrm{s}_\varphi(L) \leqslant n$ and $\mathrm{s}_\varphi(L) \leqslant 2^k$ by Proposition 4.3. It suffices to prove that the case $\mathrm{s}_\varphi(L) \leqslant 2^{k-2}$ cannot occur. If $\mathrm{s}_\varphi(L) \leqslant 2^{k-2}$ the previous lemma implies that $\ell_\varphi(-d) \leqslant 2^{k-1} < n$, which is a contradiction. □

Remark 4.11. In Proposition 4.10, if $\varphi = \langle 1\rangle$ the possibility $\mathrm{s}_\varphi(K) = 2^{k-1}$ is ruled out, see [29, Ch. 4, Thm. 4.3] or [15, Prop. 3.3]. In general, both values 2^k and 2^{k-1} can happen as we now show. For instance, by taking $K = \mathbb{Q}$, $d = -3$ and $\varphi = \langle 1, 1\rangle$ we obtain $n = \ell_\varphi(3) = 2$ and so $k = 1$. In this case $\mathrm{s}_\varphi(L) = 1 = 2^{k-1}$. Now take $d = -1$ and $\varphi = \langle 1\rangle$. We obtain $n = \ell_\varphi(1) = \ell(1) = 1$ and so $k = 0$. In this case we have $\mathrm{s}_\varphi(L) = \mathrm{s}(\mathbb{Q}(i)) = 1 = 2^k$.

4.4 Values represented by a Pfister form

Recall that if φ is a Pfister form over K, the set $D_K(\varphi)$ of non-zero values represented by φ is a subgroup of $K^\times$. We first state some useful facts concerning $\mathrm{s}_\varphi(K)$ and $p_\varphi(K)$. The proof is adapted from [24, Satz 18].

Proposition 4.12. *Let (V,φ) be a Pfister form over K with $\mathrm{s}_\varphi(K) < +\infty$. Then:*
(1) *For every $a \in K$, there exists an integer n such that $a \in D_K(\sigma_{n,\varphi})$.*
(2) *We have $p_\varphi(K) \in \{\mathrm{s}_\varphi(K), \mathrm{s}_\varphi(K)+1\}$.*
(3) *If $t = p_\varphi(K)$ then $D_K(\sigma_{t,\varphi}) = K^\times$.*
(4) *If K is non formally real and $\mathrm{s}_\varphi(K) < p_\varphi(K)$ then $2\,\mathrm{s}_\varphi(K)\dim(\varphi) \leqslant u(K)$.*

Proof. In the proof, we will use the notations $s = \mathrm{s}_\varphi(K)$ and $p = p_\varphi(K)$.

(1) Note that $a = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2$ and that the Pfister form $\sigma_{s,\varphi}$ represents -1. As Pfister forms are multiplicative (see [14, Theorem X.2.8]), $\sigma_{s,\varphi}$ represents $(-1)(\frac{a-1}{2})^2$, hence $\sigma_{s+1,\varphi}$ represents $(\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2 = a$.

(2) comes from the fact that $s = \ell_\varphi(-1) \leqslant p$ and from (1) as $\sigma_{s+1,\varphi}$ is universal and (3) is a consequence of (1).

(4) One may assume that $u(K) < +\infty$. As $s < p$, the quadratic form $\sigma_{s,\varphi}$ is not universal (otherwise, we would have $s = p$). So there exists an element $-a \in K^\times$ which is not represented by this form. Define $\psi = \langle 1, a\rangle \otimes \sigma_{s,\varphi}$. We claim that ψ is anisotropic. If it is isotropic, as $\sigma_{s,\varphi}$ is anisotropic, there exist elements $b,\ c \in K^\times$ both represented by $\sigma_{s,\varphi}$ such that $b = -ac$. As $\sigma_{s,\varphi}$ is multiplicative, it represents bc hence $-a$ which is a contradiction. The form ψ is thus anisotropic with dimension $2s \times \dim(\varphi)$, hence the result. □

Remark 4.13. If $K = \mathbb{Q}$, $q = \langle -5\rangle$ then by subsection 3.4, $\mathrm{s}_q(K) = 2$ and $p_q(K) = 4$ (as $\ell_q(-35) = 4$, $p(\mathbb{Q}) = 4$ and $D_{\mathbb{Q}}(\sigma_{n,q}) = -5\,\mathbb{Q}^+$ if $n \geqslant 4$). Hence statements (1), (2), (3) in Proposition 4.12 are not valid if φ is not a Pfister form. Consider a field F with $u(F) = s(F) = 4$ (given by the construction in §3.4.3 for example) and take $q = \langle -1,-1,-1\rangle$. Then $\mathrm{s}_q(F) = 1$, $p_q(F) = 2$ (as q does not represent 1 but $\sigma_{2,q}$ is universal) and $2\,\mathrm{s}_q(F)\dim(q) > u(F) = 4$, hence statement (4) is also false in general.

Theorem 4.14. *Let φ be a Pfister form over a field K whose φ-level is 2^n. Then $|K^\times/D_K(\varphi)| \geqslant 2^{n(n+1)/2}$.*

Proof. To prove this result, we adapt Pfister's proof of the fact that $|K^\times/K^{\times^2}| \geqslant 2^{n(n+1)/2}$ whenever $\mathrm{s}(K) = 2^n$. In the sequel, we set $G_j := D_K(\sigma_{j,\varphi}) \subset K^\times$ for any integer j and write s for $\mathrm{s}_\varphi(K)$. We have:

$$-1 = \varphi(e_1) + \cdots + \varphi(e_s) \tag{4}$$

Note that the conclusion is clear if $n = 0$ and is true if $n = 1$ (in this case, $-1 \notin D_K(\varphi)$, hence $|K^\times/D_K(\varphi)| \geqslant 2$). One may assume that $n \geqslant 2$. Let $j = 2^i$ where $0 \leqslant i < n$. We claim that the elements $a_1 = \varphi(e_1) + \cdots + \varphi(e_{2j})$, $a_2 = \varphi(e_{2j+1}) + \cdots + \varphi(e_{4j})$, $\cdots$ are pairwise non congruent modulo G_j. Indeed, if we would have $\varphi(e_{2j+1}) + \cdots + \varphi(e_{4j}) = c(\varphi(e_1) + \cdots + \varphi(e_{2j}))$ for some $c \in G_j$ then we would obtain $\varphi(e_1) + \cdots + \varphi(e_{4j}) = (1+c)(\varphi(e_1) + \cdots + \varphi(e_{2j})) \in G_{2j}$ which would contradict the minimality of s in (4), hence the claim.

The elements $a_1, a_2, \cdots$ are not in G_j (otherwise, this would also contradict the minimality in (4)), hence there are at least $1 + \frac{s}{2j} = 1 + 2^{n-i-1}$ elements in

G_{2j}/G_j (the $\frac{n}{2j}$ elements a_i together with the element 1). Now, G_{2j}/G_j injects in $K^\times/D_K(\varphi)$. If G_{2j}/G_j has infinite order, there is nothing to prove. Otherwise, G_{2j}/G_j is a 2-group and what we have done above shows that $[G_{2j} : G_j] \geqslant 2^{n-i}$. We have the following sequence of inclusions

$$D_K(\varphi) = G_1 \subset G_2 \subset \cdots \subset G_{2^{n-1}} \subset G_{2^n} \subset K^\times.$$

We thus obtain $|K^\times/D_K(\varphi)| \geqslant |G_{2^n}/G_1|$ and

$$|G_{2^n}/G_1| = |G_{2^n}/G_{2^{n-1}}| \times \cdots \times |G_2/G_1| \geqslant \prod_{i=0}^{n-1} 2^{n-i} = 2^{\frac{n(n+1)}{2}}.$$

□

Remarks 4.15. (1) The lower bound indicated in Theorem 4.14 is attained as we see by taking $K = \mathbb{Q}_p$, $p \neq 2$ and φ the unique 4-dimensional anisotropic form over K since $|K^\times/D_K(\varphi)| = 1$ and $\mathrm{s}_\varphi(K) = 1$.
(2) For a field K of level 2^n, we have already pointed out that the lower bound $|\frac{K^\times}{K^{\times 2}}| \geqslant 2^{n(n+1)/2}$ is best possible for $n \leqslant 2$. For higher n, however, a result due to D. Z. Djoković and refined by D. B. Leep shows that $|\frac{K^\times}{K^{\times 2}}| \geqslant 2^{2^n+2-n}$, see [5] and [1, Remark 5.2]. In the case where $n = 3$ and $n = 4$, this bound has in turn been refined by K. J. Becher: if $s(K) = 8$ (resp. $s(K) = 16$) then $|\frac{K^\times}{K^{\times 2}}| \geqslant 512$ (resp. 2^{15}) (see [1, Theorem 5.3]).

Corollary 4.16. *Let K be a field with level 2^n and $t = 2^m \geqslant 0$. Consider the subgroup $G = \{a_1^2 + \cdots + a_t^2 \neq 0 \mid a_i \in K\} \subset K^\times$. If $t > \mathrm{s}(K)$ then we have $G = K^\times$. If $t \leqslant \mathrm{s}(K)$ then we have $|K^\times/G| \geqslant 2^{k(k+1)/2}$ where $k = n - m$.*

Proof. Consider the Pfister form $\varphi = \sigma_t$. We have $G = D_K(\varphi)$. If $t > \mathrm{s}(K)$, φ is isotropic, hence $G = K^\times$. If $t \leqslant \mathrm{s}(K)$ then by Lemma 3.1 (5), we have $\mathrm{s}_\varphi(K) = \left\lceil \frac{\mathrm{s}(K)}{t} \right\rceil = 2^{n-m} = 2^k$ and the result follows from Theorem 4.14. □

Corollary 4.17. *Let $L = K(\sqrt{d})$ be a quadratic extension of K where $d \in K^\times \setminus K^{\times^2}$ and $\varphi = \langle 1, -d\rangle$. If $\mathrm{s}_\varphi(K) = 2^n$ then $|K^\times/N_{L/K}(L^\times)| \geqslant 2^{n(n+1)/2}$.*

Proof. As $N_{L/K}(L^\times) = D_K(\varphi)$, the result follows from Theorem 4.14. □

Remark 4.18. It is relevant to mention the following formula obtained by D. W. Lewis in [16, Cor. after Prop. 3]: if L/K is a quadratic field extension,

$$|L^\times/L^{\times 2}||K^\times/N_{L/K}(L^\times)| = \frac{1}{2}|K^\times/K^{\times 2}|^2.$$

The below result has first been proved by A. Pfister for the form $\varphi = \langle 1\rangle$ in [23]. Our reformulation is taken from [1].

Proposition 4.19. *Let φ be a Pfister form over a field K.*
(1) *For every $x,\ y \in K^\times$ we have $\ell_\varphi(xy) \leqslant \ell_\varphi(x) + \ell_\varphi(y) - 1$.*
(2) *For every $x \in K^\times$ we have $\mathrm{s}_\varphi(K) \leqslant \ell_\varphi(x) + \ell_\varphi(-x) - 1$.*

Proof. First note that (2) is a consequence of (1) as $\mathrm{s}_\varphi(K) = \ell_\varphi(-x^2)$. Pfister's original proof of (1) in the case $\varphi = \langle 1\rangle$ can be used more or less *verbatim* to prove (1) in general so this proof is left to the reader. □

Remark 4.20. The two bounds given in the previous proposition are sharp for any Pfister form φ: choose $x = 1$, $y = -1$ so that $\ell_\varphi(1) = 1$ and $\ell_\varphi(-1) = \mathrm{s}_\varphi(K)$.

The first inequality does not hold in general. In the formally real case, choose $\varphi = \langle -1 \rangle$ and $x = y = -1$: then $\ell_\varphi(-1) = 1$ but $\ell_\varphi((-1) \times (-1)) = +\infty$. In the non formally real case, take $K = \mathbb{Q}_p$ (p odd) and choose $x = u$ where u is a unit such that $\overline{u} \notin \overline{K}^2$ and $y = \pi$ where π is a uniformizer. Choose $\varphi = \langle u, \pi \rangle$ over K. Then $\ell_\varphi(u) = \ell_\varphi(\pi) = 1$. Now $\ell_\varphi(u\pi) = 1$ if and only if the quadratic form $\langle -u, -\pi, u\pi \rangle$ is isotropic. As this latter form is a Pfister neighbor of $\langle\langle u, \pi \rangle\rangle$ which is anisotropic, it follows that $\ell_\varphi(u\pi) = 2 > \ell_\varphi(u) + \ell_\varphi(\pi) - 1 = 1$.

4.5 Pythagoras q-number and field extensions

If (V, φ) is a Pfister form over a field K then for every $x,\ y \in V$ there exists $z \in V$ such that $\varphi(x) \cdot \varphi(y) = \varphi(z)$. As Pfister observed, z can be chosen in such a way that its first component is of the form $b_\varphi(x, y)$ where b_φ is the bilinear form associated to φ, see [26, Ch. 2, Cor. 2.3] or [25, Satz 1]. For the case where $\varphi = \sigma_{2^r}$ ($r \geqslant 0$), this implies in particular that for every $x_1, \cdots, x_{2^r}, y_1, \cdots, y_{2^r} \in K$ there exists an identity like

$$(x_1^2 + \cdots + x_{2^r}^2) \cdot (y_1^2 + \cdots + y_{2^r}^2) = (x_1 y_1 + \cdots + x_{2^r} y_{2^r})^2 + z_2^2 + \cdots + z_{2^r}^2,$$

where $z_2, \cdots, z_{2^r}$ are suitable elements of K.

The following result was proved for the form $\varphi = \langle 1 \rangle$ in [26, Ch. 7, 1.12].

Proposition 4.21. *Let (V, φ) be a Pfister form over a field K. Suppose that $\mathrm{s}_\varphi(K) = +\infty$. Let $f(x) \in K[x]$ be such that $\ell_{\varphi|_{K(x)}}(f(x)) < +\infty$. Then $\deg(f(x)) = 2n$ is even and $\ell_{\varphi|_{K(x)}}(f(x)) \leqslant p_\varphi(K)(n + 1)$.*

Proof. If $p_\varphi(K) = +\infty$ the result is trivial. Assume now that $p_\varphi(K) < +\infty$ and consider the smallest positive integer m such that $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K(x)$. Using the first representation Theorem of Cassels-Pfister (see [14, Theorem IX.1.3]) we obtain that $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K[x]$. This implies that the degree of $f(x)$ can not be odd, otherwise $\sigma_{m,\varphi}$ would be isotropic over K which contradicts the hypothesis $\mathrm{s}_\varphi(K) = +\infty$. So the degree of $f(x)$ is even and we may suppose that $\deg(f(x)) = 2n$. We proceed by induction on n. If $n = 0$, then $f(x) = c \in K$ is a constant polynomial. As $f(x) = c$ is a represented by $\sigma_{m,\varphi}$ over $K(x)$, it is represented by $\sigma_{m,\varphi}$ over K by Substitution Principle (see [26, Ch. 1, 3.1]). It follows that $\ell_{K(x)}(f(x)) \leqslant p_\varphi(K)$, hence the result. Suppose now that $n \geqslant 1$. Take $f(x) = a_{2n} x^{2n} + \cdots + a_0$ where $a_{2n}, \cdots, a_0 \in K$. As $f(x)$ is represented by $\sigma_{m,\varphi}$ over $K(x)$ it follows that a_{2n} is represented by $\sigma_{m,\varphi}$ over K. Thus the polynomial $\frac{f(x)}{a_{2n}}$ is also represented by $\sigma_{m,\varphi}$ over $K[x]$. Let k be a positive integer such that $m \leqslant 2^k$. The polynomial $\frac{f(x)}{a_{2n}}$ is also represented by $\sigma_{2^k,\varphi}$. It follows that

$$\frac{f(x)}{a_{2n}} = g_1(x) + \cdots + g_{2^k}(x) \tag{5}$$

where the polynomials $g_1(x), \cdots, g_{2^k}(x)$ are represented by the form φ over $K[x]$. Let $v_1(x), \cdots, v_{2^k}(x) \in V[x]$ be the elements such that $\varphi(v_i(x)) = g_i(x)$ for every $i = 1, \cdots, 2^k$. By comparing the leading coefficients of the equation (5) we obtain a relation

$$1 = b_1 + \cdots + b_{2^k} \tag{6}$$

where $b_1, \cdots, b_{2^k} \in K$ are represented by φ. Let $w_1, \cdots, w_{2^k} \in V$ be the elements such that $\varphi(w_i) = b_i$ for every $i = 1, \cdots, 2^k$. By multiplying the relations of the equations (5) and (6) and taking into account the preliminary observation before the statement of this result we obtain:

$$\frac{f(x)}{a_{2n}} = \mathrm{s}(x) + r(x) \tag{7}$$

where $\mathrm{s}(x) = (b_\varphi(v_1(x), w_1) + \cdots + b_\varphi(v_{2^k}(x), w_{2^k}))^2$ and $r(x)$ is represented by $\sigma_{m,\varphi}$ over $K[x]$. Note that $\mathrm{s}(x)$ is a monic polynomial with the same degree as $f(x)$. It follows that $r(x)$ is a polynomial whose degree satisfies $\deg(r(x)) < 2n$. As $r(x)$ is represented by $\sigma_{m,\varphi}$, the degree of $r(x)$ should be even, otherwise $\sigma_{m,\varphi}$ would be isotropic over K which is a contradiction. We then have $\deg(r(x)) \leqslant 2n-2$. The relation (7) implies that $f(x) = a_{2n}\,\mathrm{s}(x) + a_{2n}r(x)$. As a_{2n} is represented by $\sigma_{m,\varphi}$ over K, $a_{2n}r(x)$ is also represented by $\sigma_{m,\varphi}$ over $K[x]$. By the induction hypothesis we obtain $\ell_{\varphi|K[x]}(a_{2n}r(x)) \leqslant p_\varphi(K)(n)$. We obviously have $\ell_{\varphi|K[x]}(a_{2n}\,\mathrm{s}(x)) = \ell_\varphi(a_{2n}) \leqslant p_\varphi(K)$. We so obtain $\ell_{\varphi|_{K[x]}}(f(x)) \leqslant p_\varphi(K)(n) + p_\varphi(K) = p_\varphi(K)(n+1)$. □

Proposition 4.22. (1) *(Pfister) Let K be a real field and let L/K be a field extension of finite degree. Then $p(L) \leqslant p(K)[L:K]$.*
(2) *Let L/K be a field extension of finite degree. Let φ be a Pfister form over K. Suppose that $\mathrm{s}_\varphi(K) < +\infty$. Then $p_\varphi(L) \leqslant p_\varphi(K)[L:K]$.*

Proof. The statement (1) is proved in [26, Ch. 7, 1.13]. It is clear that (1) is a particular case of (2) by taking $\varphi = \langle 1 \rangle$. To prove (2), it suffices to prove the result for the case where $L = K(\alpha)$ is a simple extension. Let V be the underlying vector space of φ. Suppose that $[L:K] = n$. If $p_\varphi(K) = +\infty$ the conclusion is trivial. Assume that $p_\varphi(K) < +\infty$. Let $\beta \in L$ be an element such that $r := \ell_{\varphi|_L}(\beta) < +\infty$. In order to prove the result we have to show that $r \leqslant p_\varphi(K)[L:K]$. Every element of the vector space $V \otimes_K L$ can be written as $v_0 \otimes 1 + v_1 \otimes \alpha + \cdots + v_{n-1} \otimes \alpha^{n-1}$ where $v_i \in V$ for every $i = 1, \cdots, n$. There exist $w_1, \cdots, w_r \in V \otimes L$ such that $\beta = \varphi(w_1) + \cdots + \varphi(w_r)$. Let $w_j = \sum_{i=0}^{n-1} v_{ij} \otimes \alpha^i$ where $v_{ij} \in V$ and $j = 1, \cdots, r$. Put $w_j(x) = \sum_{i=0}^{n-1} v_{ij} \otimes x^i \in V[x] = V \otimes_K K[x]$. We so have $w_j = w_j(\alpha)$ for every $j = 1, \cdots, r$. Consider the polynomial $f(x) = \varphi(w_1(x)) + \cdots + \varphi(w_r(x))$. The degree of $f(x)$ is even and satisfies $\deg(f(x)) \leqslant 2(n-1)$. According to Proposition 4.21, we have $\ell_{\varphi|_{K[x]}} \leqslant p_\varphi(K)n$. By substituting $x := \alpha$, we obtain $\ell_\varphi(\beta) \leqslant p_\varphi(K)n$. □

Remark 4.23. Note that Proposition 4.12, Theorem 4.14 and Proposition 4.19 stay true if we only suppose that the considered form is round but it seems that Theorem 4.4, Lemma 4.8 and Proposition 4.10 are not true under this weaker hypothesis (as the class of round forms is not stable by field extension, see Remark 4.5) although we do not have any counterexample. Finally, we do not know if Proposition 4.21 and Proposition 4.22 are true for round forms.

5 Some results on the finiteness of the q-level

5.1 The case of Pfister forms

The purpose of this subsection is to characterize the finiteness of the q-level of a Pfister form q in terms of usual notions in quadratic form theory. In the following result, we denote $\Sigma_q K = \Sigma_q K^\bullet \cup \{0\}$.

Proposition 5.1. *Let q be a Pfister form over a formally real field K. Then the following are equivalent:*
(1) $\mathrm{s}_q(K) = +\infty$.
(2) *$\Sigma_q K$ is a preordering of K.*
(3) *There exists an ordering P of K containing the preordering $\Sigma_q K$.*
(4) $\underline{\mathrm{s}}_q(K) = +\infty$.
(5) *q is not torsion.*
(6) *There exists an ordering P of K for which $\mathrm{sgn}_P(q) = \dim(q)$.*

Proof. First note that we easily have (3) $\Rightarrow$ (2) $\Rightarrow$ (1) (as a preordering does not contain -1) and (1) $\iff$ (4) by Proposition 4.1. Note also that (5) $\iff$ (6) by Pfister's local global principle and the fact that the signature of a Pfister form q at any ordering is 0 or $\dim(q)$.

Suppose now that (1) holds. Then $-1 \notin \Sigma_q K$, $\Sigma_q K + \Sigma_q K \subset \Sigma_q K$ and $\Sigma_q K \cdot \Sigma_q K \subset \Sigma_q K$ as q is multiplicative. As q represents 1, we also have $\Sigma K \subset \Sigma_q K$ which implies (2).

If (2) holds then there exists a maximal preordering P containing $\Sigma_q K$ by Zorn's lemma. A usual argument then shows that P is in fact an ordering of K (that is $P \cup -P = K$ and $P \cap -P = \{0\}$) and (3) follows.

If q is torsion then there exists an integer l for which $\sigma_{2^l,q}$ is hyperbolic, hence $\mathrm{s}_q(K) \leqslant 2^l$ and (1) $\Rightarrow$ (5) holds. Suppose that $\underline{\mathrm{s}}_q(K) < +\infty$ and let m be such that $\sigma_{m,q}$ is isotropic. Take k such that $2^k \geqslant m$. Then $\langle 1\rangle \perp \sigma_{2^k,q}$ is isotropic and is a Pfister neighbor of $\sigma_{2^{k+1},q}$ which is thus hyperbolic and q is torsion which shows that (5) $\Rightarrow$ (4) and concludes the proof. □

Remarks 5.2. (1) When q is a Pfister form, Proposition 4.12 (1) shows that $\Sigma_q K = K$ if and only if $\mathrm{s}_q(K) < +\infty$.
(2) The equivalences (1) $\iff$ (2) $\iff$ (3) in the above proposition generalizes Artin-Schreier's Theorem which says that $\mathrm{s}(K)$ is infinite if and only if ΣK is a preordering of K if and only if K has an ordering.

5.2 The general case

Lemma 5.3. *Let q be a quadratic form over a formally real field K. If there exists an ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$, then $s_q(K)$ is infinite.*

Proof. In fact the condition $\mathrm{sgn}_P(q) = \dim(q)$ implies that q is positive definite with respect to P, thus for every positive integer n, the form $\sigma_{n,q}$ is also positive definite and it does not represent -1. □

Remark 5.4. If $K = \mathbb{R}((X))((Y))$ and $q = \langle X, Y, -XY\rangle$ then using a theorem due to T. A. Springer (see [29, Ch. 6, 2.6]), $\mathrm{s}_q(K)$ is infinite. On the other hand for any ordering P of K, the signature is never 3, so the converse of 5.3 does not hold in general (although it holds for Pfister form by Proposition 5.1). Nonetheless, in the case where $\dim q = 1$ or 2, or over number fields, the converse is true as the following propositions show.

Proposition 5.5. *Let K be a number field and let q be a quadratic form over K. Then $\mathrm{s}_q(K)$ is infinite if and only if there exists an ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$.*

Proof. Regarding to Lemma 5.3, it is enough to show that if $\mathrm{s}_q(K)$ is infinite, then there exists an ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$. If it is not the case, then for every ordering P of K we have $\mathrm{sgn}_P(q) < \dim(q)$. Put $\varphi := \langle 1\rangle \perp \sigma_{4,q}$, we have $\mathrm{sgn}_P(\varphi) < 1 + 4\dim(q)$. Furthermore, as φ is not negative definite, we have $\mathrm{sgn}_P(\varphi) > -\dim(\varphi)$. The form φ is therefore a totally indefinite form of dimension $\geqslant 5$ and it is isotropic by Hasse-Minkowski theorem. Thus $\mathrm{s}_q(K) \leqslant 4$, which yields a contradiction. □

Proposition 5.6. (1) *Consider the one-dimensional quadratic form $q = \langle a\rangle$ over a field K. Then $\mathrm{s}_q(K)$ is infinite if and only if there exists an ordering P of K such that* $\mathrm{sgn}_P(\langle a\rangle) = 1$.
(2) *Consider the two-dimensional quadratic form $q = \langle a, b\rangle$ over a field K. Then $\mathrm{s}_q(K)$ is infinite if and only if there exists an ordering P of K such that* $\mathrm{sgn}_P(q) = 2$.

Proof. (1) In fact $\mathrm{s}_q(K)$ is infinite if and only if $-a$ is not a sum of squares if and only if $-a$ is not totally positive.

(2) If the conclusion does not hold then for every ordering P of K we have $\mathrm{sgn}_P(q) = 0$ or $\mathrm{sgn}_P(q) = -2$, which means that the elements a and b are either both negative or one of them is positive and the other one is negative. In both cases the signature of the form $\langle 1, a, b, ab\rangle$ with respect to P is zero. Pfister's local-global principle so implies that there exists a positive integer n such that $n\times\langle 1, a, b, ab\rangle$ is hyperbolic. We then obtain $n\times\langle a, b\rangle \simeq n\times\langle -1, -ab\rangle$. The form $n\times\langle a, b\rangle$ therefore represents -1, so $\mathrm{s}_q(K) \leqslant n$ which yields a contradiction. □

Recall that a quadratic form q over K is called *weakly isotropic* (resp. *weakly hyperbolic*) when there exists a positive integer m such that $\sigma_{m,q}$ is isotropic (resp. hyperbolic). Of course, a form is weakly isotropic if and only if $\underline{\mathrm{s}}_q(K) < +\infty$.

Corollary 5.7. *Let q be a quadratic form over a field K such that $\varphi = \langle 1\rangle \perp q$ is weakly isotropic, then $\mathrm{s}_q(K) < \infty$.*

Proof. As φ is weakly isotropic, there exists a positive integer n such that $\sigma_{n,\varphi}$ is isotropic. There exists so an element $a \in K^\times$ which is simultaneously represented by σ_n and $\sigma_{n,-q}$. The form $\sigma_{n,q}$ represents therefore the totally negative element $-a$. Lemma 3.1 (3) implies that $\mathrm{s}_{n\times q}(K) \leqslant \mathrm{s}_{\langle -a\rangle}(K)$. Proposition 5.6 (1), implies that $\mathrm{s}_{\langle -a\rangle}(K) < \infty$. Lemma 3.1 (5) concludes the proof. □

Remark 5.8. Recall that a field K satisfies the *Strong Approximation Property* (SAP) if for any disjoint closed subsets A and B of the space ordering of K (endowed with the Harrison topology), there exists $a \in K^\times$ such that a is positive (resp. negative) with respect to every ordering in A (resp. B). This notion was introduced by M. Knebusch, A. Rosenberg and R. Ware [11]. A characterization of the fields for which every totally indefinite form is weakly isotropic was given by A. Prestel [28]. It turned out that these are exactly the SAP-fields or equivalently the fields for which every quadratic form of the shape $\langle 1, a, b, -ab\rangle$ is weakly isotropic. Formally real number fields and more generally every formally real algebraic extension of $\mathbb{Q}$, every formally real algebraic extension of $\mathbb{R}(X)$ and $\mathbb{Q}((t))$ are examples of SAP-fields.

In the following result we characterize the fields for which the converse of Lemma 5.3 holds.

Theorem 5.9. *Let K be a formally real field. The fields K for which for every quadratic form q the relation $\mathrm{s}_q(K) = \infty$ would imply the existence of an ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$ are exactly the fields for which the strong approximation property holds.*

Proof. First suppose that K is a SAP-field. Let q be a quadratic form over K with $\mathrm{s}_q(K) = \infty$. If there is no ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$, then q is not totally positive. It follows that the form $\varphi = \langle 1 \rangle \perp q$ is totally indefinite. By Prestel's theorem [28, Satz. 3.1], φ is weakly isotropic and Corollary 5.7 yields a contradiction.

Conversely suppose that for every quadratic form q the relation $\mathrm{s}_q(K) = \infty$ implies that there exists an ordering P of K such that $\mathrm{sgn}_P(q) = \dim(q)$. In order to prove that K is a SAP-field, it is enough to show that the form $\langle 1, a, b, -ab \rangle$ is weakly isotropic. Consider the form $q = \langle a, b, -ab \rangle$. For any ordering P of K, the elements a, b and $-ab$ can not be simultaneously positive with respect to P. It follows that there is no ordering P for which $\mathrm{sgn}_P(q) = \dim(q)$, thus $\mathrm{s}_q(K) < \infty$. There exists so a positive integer n such that $\langle 1 \rangle \perp \sigma_{n,q}$ is isotropic. It follows that $n \times \langle 1, a, b, -ab \rangle$ is isotropic as well. □

Lemma 5.10. *Let K be a formally real field and let q be an n-dimensional quadratic form over K. Then the following statement are equivalent:*
(1) *For any ordering P of K on has $\mathrm{sgn}_P(q) \leqslant 2 - \dim(q)$.*
(2) *The form $\psi := n(n-2) \times \langle 1 \rangle \perp (2n-2) \times q \perp q \otimes q$ is weakly hyperbolic.*

Proof. It is easy to verify that $\mathrm{sgn}_P(\psi) = 0$ if and only if $\mathrm{sgn}_P(q) \leqslant 2 - \dim(q)$. The conclusion then follows from Pfister's local-global principle. □

Proposition 5.11. *Let K be a formally real field and let q be a quadratic form of dimension $n > 2$ over K. If for every ordering P of K on has $\mathrm{sgn}_P(q) \leqslant 2 - \dim(q)$ then $s_q(K) < \infty$.*

Proof. Put $\varphi_1 := (2n-2) \times q$ and $\varphi_2 := n(n-2) \times \langle 1 \rangle \perp q \otimes q$. According to 5.10, the form $\varphi_1 \perp \varphi_2$ is weakly hyperbolic. There exists so a positive integer m such that $m \times (\varphi_1 \perp \varphi_2)$ is hyperbolic. As $\dim(\varphi_1) = \dim(\varphi_2)$, we obtain $\sigma_{m,\varphi_1} \simeq \sigma_{m,-\varphi_2}$, therefore $m(2n-2) \times q \simeq m \times \langle -1, \cdots \rangle$, this implies that $s_q(K) \leqslant m(2n-2)$. □

6 Some related questions

Our first question concerns the converse of Corollary 3.14.

Question 6.1. Let q be a quadratic form of dimension 1 or 2 (resp. 3) such that $\mathrm{s}_q(K) = +\infty$. Are all the elements of the set $L_q(K)$ of the form 2^k (resp. of the form $\frac{2^{2k}+2}{3}$ or $\frac{2^{2k+1}+1}{3}$) where $k \in \mathbb{N}$?

The assertion of Proposition 4.1 gives a sufficient condition for the q-level and the q-sublevel to coincide. This leads us to pose the following:

Question 6.2. Is it possible to characterize the quadratic forms q for which $\mathrm{s}_q(K) = \underline{\mathrm{s}}_q(K)$?

In the following question, we ask whether or not Proposition 4.3 is best possible in some sense.

Question 6.3. Is it possible to find an example of a group form q such that $\mathrm{s}_q(K)$ is not a 2-power ?

Finally, in relation with Proposition 5.11, it would be interesting to know an answer to the following:

Question 6.4. Characterize all fields K such that the infiniteness of $s_q(K)$ for every q is equivalent to the existence of an ordering P of K with $\mathrm{sgn}_P(q) > 2 - \dim(q)$.

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