

UPPER BOUNDS FOR THE ESSENTIAL DIMENSION OF E_7

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ABSTRACT. This paper gives a new upper bound for the essential dimension and the essential 2-dimension of the split simply connected group of type E_7 over a field of characteristic not 2 or 3. In particular, $\text{ed}(E_7) \leq 29$, and $\text{ed}(E_7; 2) \leq 27$.

The essential dimension of an algebraic group G is a numerical invariant which measures the complexity of its G -torsors. The essential dimensions of the exceptional algebraic groups remain largely open (see [Re10] for a recent survey). For example, even if the base field is the complex numbers we only know that $9 \leq \text{ed}(E_8) \leq 231$ ([CS06], [Le04]).

We will say that a G -variety X is *generically free* if there is an open subset $U \subset X(\bar{k})$ on which $G(\bar{k})$ acts with trivial stabilizers, where \bar{k} is an algebraic closure of the base field k . We will define the *essential dimension* of G to be the minimum value of $\dim(X) - \dim(G)$, where X is a generically free G -variety and there exists a G -equivariant dominant map $V \dashrightarrow X$ (also called a *compression*), such that V is a generically free linear representation of G .

We assume throughout this paper that the characteristic of the base field k is not 2. Let E_n denote the split simply connected group of that type. We have the lower bound $7 \leq \text{ed}(E_7)$ from [CS06] (for characteristic zero, see [RY00]), and over the complex numbers, we know that three copies of E_7 's faithful 56 dimensional representation V_{56} is generically free [Po86, 13], and so is $V_{56} \times \mathbb{P}(V_{56}) \times \mathbb{P}(V_{56})$. Therefore $\text{ed}(E_7) \leq (56 + 55 + 55) - 133 = 33$, which is the best known upper bound. In this paper we will improve this bound, and only assume the base field has characteristic not 2 or 3.

For a prime p , we are also interested in the “localized” notion of essential p -dimension, which is often easier to compute. For primes p not equal to 2 or 3, we know $\text{ed}(E_7; p) = 0$. Also, $\text{ed}(E_7; 3) = 3$ ([GR09] or [Ga09]). For $p = 2$ we have $7 \leq \text{ed}(E_7; 2) \leq 33$ ([RY00], [CS06], [Mac11]). In this paper we improve the upper bound.

We will take the subgroup $E_6 \rtimes \mu_4 \subset E_7$ as defined in [Ga01, 3.5]. For every field extension l/k , inclusion induces a surjection $H^1(l, E_6 \rtimes \mu_4) \rightarrow H^1(l, E_7)$ (see also [Ga09, 12.13]); in other words, we can reduce the structure of E_7 -torsors. In particular, this implies

$$\text{ed}(E_7) \leq \text{ed}(E_6 \rtimes \mu_4) \quad \text{and} \quad \text{ed}(E_7; 2) \leq \text{ed}(E_6 \rtimes \mu_4; 2).$$

The new upper bounds will be achieved by using these inequalities. As always, we have $\text{ed}(E_7; 2) \leq \text{ed}(E_7)$, but we do not know if this inequality is strict.

Consider E_7 's fundamental faithful representation of dimension 56 (also known as its minuscule representation). As an E_6 representation it decomposes as $k \oplus k \oplus V \oplus V^*$, where V and V^* are E_6 's two fundamental representations of dimension 27 (also known as its minuscule representations), and k denotes a one-dimensional trivial representation. Let $w \in \mu_4$ be defined as in [Ga01, 3.5], sending an element $(\alpha_1, \alpha_2, v_1, v_2)$ to $(i\alpha_2, i\alpha_1, iv_2, iv_1)$, where i is a square root of -1 . Conjugation by w is an outer automorphism of E_6 , it preserves a split maximal torus $T \subset E_6$, and it fixes a subgroup isomorphic to F_4 [Ga01, 2.4 and 3.5]. So $V \oplus V^*$ is faithful and irreducible as an $E_6 \rtimes \mu_4$ representation, and also as an $N_{E_6}(T) \rtimes \mu_4$ representation. Also notice that w^2 is in

the centre of E_7 . We will denote the set of T -weights of $V \oplus V^*$ as Λ , which consists of all weights in \hat{T} of length $4/3$.

1. ESSENTIAL DIMENSION

Definition 1.1. [PV89, 7] Let k be an algebraically closed field, and let G be an algebraic group acting on an irreducible variety X over k . We say that $H \subset G$ is a *stabilizer in general position* (SGP) if there is a dense open subset $U \subset X$ such that $\text{Stab}_G(x)$ is conjugate to H for every $x \in U$.

Lemma 1.2 (Popov). *Let X and Y be irreducible G -varieties. Assume H_1 is an SGP for the G actions on X , and H_2 is an SGP for the H_1 action on Y . Then H_2 is an SGP for the G action on $X \times Y$.*

Proof. Assume $\text{Stab}_G(x) = g^{-1}H_1g$ and $\text{Stab}_{H_1}(y) = h^{-1}H_2h$. Then $\text{Stab}_G(x, y) = \text{Stab}_{\text{Stab}_G(x)}(y) = \text{Stab}_{g^{-1}H_1g}(y) = gh^{-1}H_2hg^{-1}$. \square

There is a series of subgroups as follows.

$$\{1\} \subset \text{SL}_2 \subset \text{SL}_3 \subset G_2 \subset \text{Spin}_7 \subset \text{Spin}_8 \subset F_4 \subset E_6 \subset E_7.$$

We will abuse notation slightly and use the above symbols to refer to specific subgroups of these types, as described in the proofs of Propositions 1.3 and 1.4.

Several of the results in the following two propositions may be found in [Bo11, 1.5] for the complex numbers, and the arguments there are analogous to the ones used here. For the rest of this section we will assume the characteristic of k is not 2 or 3.

Proposition 1.3. *Assume that $k = \bar{k}$.*

- (1) F_4 is an SGP of the E_6 action on V ,
- (2) Spin_8 is an SGP of the F_4 action on V (or V^*),
- (3) Spin_8 is an SGP of the $E_6 \rtimes \mu_4$ action on $V \oplus V^*$,
- (4) $\text{Spin}_8 \times \mu_6$ is an SGP of the $E_6 \rtimes \mu_4$ action on $\mathbb{P}(V \oplus V^*)$.

Proof. Recall the $E_6 \rtimes \mu_4$ action on the 54 dimensional faithful representation $V \oplus V^*$ described above. As in [Ga01, 3.5], each of V and V^* come equipped with isomorphic Jordan algebra structures, which include a cubic form called the norm. E_6 preserves both of those cubic forms, but as E_6 representations V and V^* are not isomorphic. The subgroup of E_6 which fixes the identity element e in V (equivalently, in V^*), is the automorphism group of the Jordan algebra structure, and is of type F_4 (see also [SV00, 5.9.4 and Section 7]). The subgroup of F_4 which fixes a chosen triple (e_1, e_2, e_3) of pairwise orthogonal primitive idempotents in V , is isomorphic to Spin_8 [Ja68, p.378].

(1) Two elements with non-zero norms in V are in the same E_6 -orbit iff their norms are equal ([Ja68, Chap. IX.6, exer. 6], or [SV00, Proof of Thm. 7.3.2]).

(2) The subgroup $F_4 \subset E_6$ is the group of automorphisms of the Jordan algebra structure on V (which is the same as that on V^*). By Jacobson [Ja68, p.381], any element can be “diagonalized”, in other words, the F_4 -orbit of any element non-trivially intersects the subspace spanned by e_1, e_2 , and e_3 . If y is in that subspace and its e_i coefficients are all distinct (an open condition), then $\text{Stab}_{F_4}(y) \cong \text{Spin}_8$.

(3) We have $e \in V$, and choose $y \in V^*$ as above, such that the norm of y is not a fourth root of unity. Then (e, y) can be considered to be in general position; by Lemma 1.2, we see that Spin_8 is the SGP of E_6 on $V \oplus V^*$. For any $g \in E_6$, and $i = 1, 2, 3$, the assumption on the norm of y , together with the fact that E_6 is norm preserving, tells us that $gw^i(e, y) \neq (e, y)$. The result follows.

(4) For e and y as above, consider the element $[e, y] \in \mathbb{P}(V \oplus V^*)$. Notice that $\text{Stab}_{F_4}([e, y]) \subset N_{F_4}(\text{Stab}_{F_4}(e, y)) = \text{Spin}_8 \rtimes S_3$. Here the symmetric group S_3 permutes the diagonal coefficients of e_i in the Jordan algebra V (and V^*). Since the e_i coefficients of y are distinct, we see $\text{Stab}_{F_4}([e, y]) = \text{Spin}_8$. Also $\text{Stab}_{E_6}([e]) = F_4 \times \mu_3$ [Ga01, 3.4] (see also [Ga09, 9.12]), so $\text{Stab}_{E_6}([e, y]) = \text{Spin}_8 \times \mu_3$. Finally, if $g \in E_6$ such that gw fixes $[e, y]$, then y would have to be a scalar multiple of e , which is not the case. Therefore $\text{Stab}_{E_6 \times \mu_4}([e, y]) = \text{Spin}_8 \times \mu_6$, where μ_6 is the product of the centres of E_6 and E_7 . \square

Spin_8 has three inequivalent 8 dimensional irreducible representations. We will choose one and call it the vector representation, and denote it by Σ . Then the other two, $\mathbb{O}^+, \mathbb{O}^-$, will be called the half-spin representations. Also, \mathbb{O} will denote the 8 dimensional spin representation of Spin_7 .

Proposition 1.4. *Assume $k = \bar{k}$.*

- (1) Spin_7 is an SGP of the Spin_8 action on Σ ,
- (2) G_2 is an SGP of the Spin_7 action on \mathbb{O} ,
- (3) SL_3 is an SGP of the G_2 action on \mathbb{O} ,
- (4) SL_2 is an SGP of the SL_3 action on \mathbb{O} ,
- (5) SL_2 acts generically freely on \mathbb{O} .

Proof. (1) The stabilizer of a point with non-zero norm in the vector representation is a subgroup of Spin_8 isomorphic to Spin_7 . This is the copy of $\text{Spin}_7 \subset \text{Spin}_8$ that we will use below.

(2) As Spin_7 representations, \mathbb{O}^+ and \mathbb{O}^- are isomorphic to \mathbb{O} . The claim follows from [Ig70, Prop. 4]; here G_2 is the automorphism group of an octonion algebra structure on \mathbb{O} .

(3) As G_2 representations, Σ and \mathbb{O} are isomorphic. In the 7-dimensional space of elements orthogonal to the identity (also called *pure octonions*), G_2 acts transitively on norm one elements, and the stabilizer of such an element x_1 is SL_3 (see also [KMRT, §36 Exer. 6]).

(4) SL_3 acts transitively on norm one elements orthogonal to both the identity and x_1 . The stabilizer of such an element x_2 is SL_2 (see also [KMRT, §36 Exer. 7]).

(5) SL_2 acts transitively on norm one elements orthogonal to all of $1, x_1, x_2$, and x_1x_2 . The stabilizer of such an element x_3 is trivial, because an automorphism of the octonions is determined by its action on the “basic triple” (x_1, x_2, x_3) , and therefore SL_2 acts generically freely on \mathbb{O} . \square

Remark 1.5. As noted in [Ig70, Prop. 1], if H is an SGP of Spin_8 acting on the half-spin representation \mathbb{O}^+ , then H is isomorphic (but not conjugate) to Spin_7 by a triality automorphism. And the Spin_8 representation Σ restricted to H is the spin representation (rather than the vector Spin_7 -representation).

Theorem 1.6. $\text{ed}(E_7) \leq 29$.

Proof. From the introduction, we know that $\text{ed}(E_7) \leq \text{ed}(E_6 \times \mu_4)$. Consider the rational map

$$(V \oplus V^*)^2 \dashrightarrow W := V \oplus V^* \times \mathbb{P}(V \oplus V^*).$$

It is dominant and $E_6 \times \mu_4$ -equivariant. To show the upper bound on essential dimension, we just need to show that W is generically free, and for this we may assume that $k = \bar{k}$.

By Lemma 1.2 and Proposition 1.3, we see that an SGP of the $E_6 \times \mu_4$ action on W is equal to an SGP of the Spin_8 action on $V \oplus V^*$. As a Spin_8 representation this

decomposes as the sum of six trivial representations, together with two copies each of Σ , \mathbb{O}^+ , and \mathbb{O}^- . Therefore, by combining Lemma 1.2 with Proposition 1.4, we see the SGP of this Spin_8 action is trivial, and hence the action of $E_6 \rtimes \mu_4$ on W is generically free. Therefore $\text{ed}(E_7) \leq \text{ed}(E_6 \rtimes \mu_4) \leq 54 + 53 - 78 = 29$. \square

2. ESSENTIAL 2-DIMENSION

Lemma 2.1. [MR09, Lemma 3.3] *Assume we have algebraic groups $1 \rightarrow T \rightarrow N \rightarrow F \rightarrow 1$, such that T is a split torus, and F is finite. Let V be a faithful N -representation which decomposes into a sum of one-dimensional weight spaces for T , and let Λ be the associated set of weights. Consider the N -invariant open subset $U = (\mathbb{A}^1 - 0)^\Lambda \subset \mathbb{A}^\Lambda = V$.*

- (1) *If $f \in F$ acts non-trivially on $\ker(\mathbb{Z}[\Lambda] \rightarrow \hat{T})$, then f acts non-trivially U/T .*
- (2) *N/T acts faithfully on U/T if and only if N acts generically freely on V .*

Proof. For the first statement, assume f acts trivially on U/T and choose an arbitrary $\Sigma c_\lambda \in \ker(\mathbb{Z}[\Lambda] \rightarrow \hat{T})$. Choose an isomorphism between the varieties $U \cong (\mathbb{G}_m)^\Lambda$, inducing a multiplicative structure on U (which is not necessarily compatible with the action of N). Consider the functions $U \rightarrow k^*$ defined in terms of the λ -coordinates u_λ of an arbitrary $u \in U$, as

$$\phi_1(u) = \prod_{\lambda \in \Lambda} (u_\lambda)^{c_\lambda}, \quad \phi_2(u) = \prod_{\lambda \in \Lambda} (u_{f \cdot \lambda})^{c_\lambda}.$$

These functions are invariant under the action of T on U . Now choose $n_f \in N$ in the preimage of f . Since n_f acts linearly on V , there are constants b_λ and c such that $\phi_1(u) = \phi_1(n_f \cdot u) = \prod (b_\lambda u_{f \cdot \lambda})^{c_\lambda} = c \phi_2(u)$. Since $u \in (\mathbb{G}_m)^\Lambda$ was arbitrary, the only way ϕ_1 and ϕ_2 can be constant multiples of each other is if $c_\lambda = f \cdot c_\lambda$ for every $\lambda \in \Lambda$; in other words, f fixes $\Sigma c_\lambda \lambda$.

For the second statement, assume N/T acts faithfully on U/T . For each non-trivial $f \in N/T$, choose a representative $n_f \in N$. The action of f on U/T is non-trivial, and therefore there is an open dense set $U_f \subset U$ on which $n_f t$ has no fixed points for any $t \in T$. The finite intersection $\cap U_f$ is open and dense in U , and N acts with trivial stabilizers on it. In other words, N acts generically freely on U , and hence on V . For the converse, we just need to notice that a faithful representation of a finite group is always generically free. \square

Remark 2.2. The above Lemma was proved in [MR09] when the exact sequence is split, in which case the converse of the first statement is also true; the standard representation of $N_{\text{SL}_2}(T)$ is a counter-example to the converse of the first statement.

Theorem 2.3. $\text{ed}(E_7; 2) \leq 27$.

Proof. This upper bound will be proved using the following sequence of inequalities:

$$\text{ed}(E_7; 2) \leq \text{ed}(E_6 \rtimes \mu_4; 2) \leq \text{ed}(N_{E_6}(T) \rtimes \mu_4; 2) \leq 33 - 6 = 27.$$

Since conjugation by μ_4 preserves T ([Ga01, 2.4]), we have $N_{E_6 \rtimes \mu_4}(T) = N_{E_6}(T) \rtimes \mu_4$. Now the middle inequality follows from [Se02, III.4.3, Lemma 6].

Choose a Sylow 2-subgroup $W^{(2)} \subset W(E_6)$, and let $N_2 \subset N_{E_6}(T) \rtimes \mu_4$ be the preimage of $F := W^{(2)} \rtimes \mu_4$. Let σ be the square of a generator of μ_4 ; it is the non-trivial element in the centre of E_7 . One can verify the following facts using a computer algebra program such as Magma.

- The F orbits of Λ are of size 2, 4, 16 and 32.
- The weights in the largest orbit Λ_{32} additively generate the weight lattice

- The kernel of the F action on $\ker(\mathbb{Z}[\Lambda_{32}] \rightarrow \hat{T})$ is $\langle \sigma \rangle$.

So the subspace $V_{32} \subset V \oplus V^*$ generated by the weight spaces of Λ_{32} is an irreducible N_2 -representation of dimension 32. Let V_1 be the one-dimensional non-trivial representation of μ_4 , which we also consider as an N_2 representation via projection to μ_4 . We claim that N_2 acts generically freely on $V_{32} \times V_1$.

For $U \subset V_{32} \times V_1$ as in the above Lemma, the third point implies that the F action on U/T has kernel contained in $\langle \sigma \rangle$. But σ acts non-trivially on V_1 , and hence on U/T , since T acts trivially on V_1 . So F acts faithfully on U/T , and hence by Lemma 2.1, the N_2 action on $V_{32} \times V_1$ is generically free. Now we have the upper bound $\text{ed}(N_{E_6}(T) \rtimes \mu_4; 2) \leq 33 - 6 = 27$, and the result follows. \square

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