

TORI IN THE CREMONA GROUPS

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ABSTRACT. We classify up to conjugacy the subgroups of certain types in the full, in the affine, and in the special affine Cremona groups. We prove that the normalizers of these subgroups are algebraic. As an application, we obtain new results in the Linearization Problem generalizing to disconnected group Białynicki-Birula's results of 1966–67. We prove “fusion theorems” for n -dimensional tori in the affine and in the special affine Cremona groups of rank n . In the final section we introduce and discuss the notions of Jordan decomposition and torsion primes for the Cremona groups.

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1. INTRODUCTION

This work arose from the attempt to solve the problem posed in [Po05₁], [Po05₂]. In these papers have been introduced the notions of root α and root vector D of an affine algebraic variety X with respect to an algebraic torus $T \subseteq \text{Aut } X$. Namely, D is a locally-nilpotent derivation of the coordinate algebra of the variety X , and α is a character of the torus T such that $t^* \circ D \circ t^{*-1} = \alpha(t)D$ for all $t \in T$. These definitions are inspired by the natural analogy with the classical definitions of the theory of algebraic groups and

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purport an attempt to apply to, in general, infinite dimensional group $\text{Aut } X$ the techniques important in the theory of usual algebraic groups¹.

In [Po05₁], [Po05₂] the following two problems related to the classical case where $X = \mathbf{A}^n$ and $T = D_n^*$ is the maximal diagonal torus that leaves fixed the standard volume form (see below (9)) have been posed:

- (R) Find all the roots and root vectors of the variety \mathbf{A}^n with respect to D_n^* .
- (W) Describe the normalizer and centralizer of the torus D_n^* in the group $\text{Aut}^* \mathbf{A}^n$ of automorphisms of the space \mathbf{A}^n that leave fixed the standard volume form.

Problem (R) has been solved by Liendo in [Li11]. The answer is the following. Let x_1, \dots, x_n be the standard coordinate functions on \mathbf{A}^n and let $\varepsilon_1, \dots, \varepsilon_n$ be the “coordinate” characters of the standard n -dimensional diagonal torus D_n in $\text{Aut } \mathbf{A}^n$ (see below (8) and (10)). Then, up to multiplication by a nonzero constant, the root vectors are precisely all the derivations D of the form

$$x_1^{l_1} \cdots x_n^{l_n} (\partial / \partial x_i), \quad (1)$$

where l_1, \dots, l_n are nonnegative integers and $l_i = 0$. The root α corresponding to root vector (1) is the restriction to D_n^* of the character

$$\varepsilon_i^{-1} \prod_{j=1}^n \varepsilon_j^{l_j}.$$

The problem mentioned in the beginning of this introduction is Problem (W). It is clear that it is aimed at getting a description of the “Weyl group” of the root system from Problem (R). We solve it in the present paper. Namely, we prove (Theorem 14) that the normalizer (centralizer) of the torus D_n^* in $\text{Aut}^* \mathbf{A}^n$ coincides with its normalizer (centralizer) in SL_n , so that the Weyl group of D_n^* in $\text{Aut}^* \mathbf{A}^n$ is the same as that of D_n^* in SL_n — it is the group of all permutations of the characters $\varepsilon_1, \dots, \varepsilon_n$.

In fact, this result is only one special case of the series of general results that we obtain here. Namely, D_n^* is only one of the infinitely many nonconjugate diagonalizable algebraic subgroups G of dimension $\geq n - 1$ in the group $\text{Aut } \mathbf{A}^n$. We prove that the normalizer of G in $\text{Aut } \mathbf{A}^n$ always is an *algebraic* subgroup in $\text{Aut } \mathbf{A}^n$ (Theorem 16). It is the characteristic property of the specified dimensions: in general, for the diagonalizable subgroups of dimension $\leq n - 2$ it does not hold. Moreover, in the case when nonconstant G -invariant polynomial functions on \mathbf{A}^n exist, we explicitly describe the normalizer of G in $\text{Aut } \mathbf{A}^n$, in particular, we show that in all but one cases it coincides with the normalizer of G in a group conjugated to GL_n (Theorem 6).

Using the found information, we obtain the new results in the Linearization Problem. In 1966–67 Białynicki-Birula proved [B-B66], [B-B67] that every algebraic action on \mathbf{A}^n of an algebraic torus of dimension $\geq n - 1$ is equivalent to a linear action. We extend this statement to disconnected

¹In [Po05₁], [Po05₂] was considered the case where $X = \mathbf{A}^n$ and T is the maximal diagonal torus that leaves fixed the standard volume form, but this restriction plays no role in the definition.

groups proving that every algebraic action on \mathbf{A}^n of either an n -dimensional algebraic group whose connected component of identity is a torus, or an $(n - 1)$ -dimensional diagonalizable group is equivalent to a linear action (Theorems 13 and 15).

We also obtain the following classifications:

- (i) the classification of diagonalizable subgroups of the group Aff_n of affine transformations (see below (6)) up to conjugacy in the full Cremona group $\text{Cr}_n = \text{Bir } \mathbf{A}^n$ (Theorem 1);
- (ii) the classification of n -dimensional diagonalizable subgroups of $\text{Aut } \mathbf{A}^n$ up to conjugacy in $\text{Aut } \mathbf{A}^n$ (Theorem 11);
- (iii) the classification of $(n - 1)$ -dimensional diagonalizable subgroups of $\text{Aut}^* \mathbf{A}^n$ up to conjugacy in $\text{Aut}^* \mathbf{A}^n$ (Theorem 12);
- (iv) the classification, up to conjugacy in $\text{Aut } \mathbf{A}^n$, of maximal n -dimensional algebraic subgroups G in $\text{Aut } \mathbf{A}^n$ such that G^0 is a torus (Theorem 11);
- (v) the classification, up to conjugacy in $\text{Aut}^* \mathbf{A}^n$, of maximal $(n - 1)$ -dimensional algebraic subgroups G in $\text{Aut}^* \mathbf{A}^n$ such that G^0 is a torus (Theorem 14);
- (vi) the classification of $(n - 1)$ -dimensional diagonalizable subgroups of $\text{Aut } \mathbf{A}^n$ up to conjugacy in $\text{Aut } \mathbf{A}^n$ (Theorem 15);
- (vii) the classification of diagonalizable subgroups in $\text{Aut } \mathbf{A}^n$ of dimension $\geq n - 1$ up to conjugacy in Cr_n (Theorems 11 and 17);
- (viii) the classification of one-dimensional tori of $\text{Aut } \mathbf{A}^3$ up to conjugacy in $\text{Aut } \mathbf{A}^3$ (Theorem 18).

For instance, we prove that the set of classes of $(n - 1)$ -dimensional diagonalizable subgroups of $\text{Aut } \mathbf{A}^n$ that are conjugate in $\text{Aut } \mathbf{A}^n$ is bijectively parametrized by the set of nonzero nondecreasing sequences

$$(l_1, \dots, l_n) \in \mathbf{Z}^n, \quad (2)$$

such that $(l_1, \dots, l_n) \leq (-l_n, \dots, -l_1)$ with respect to the lexicographic order. Under this parametrization, to sequence (2) corresponds the class of the subgroup $\ker \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}$.

Another example: we show that diagonalizable subgroups of Aff_n are conjugate in Cr_n if and only if they are isomorphic and we specify their canonical representatives. In particular (see Corollary 5), every two isomorphic finite Abelian subgroups of Aff_n are conjugate in Cr_n (for finite cyclic subgroups this has been proved in [Bl06₁]).

In [Se10] Serre proved “fusion theorem” for the torus D_n in Cr_n . We prove and use “fusion theorems” for n -dimensional tori in $\text{Aut } \mathbf{A}^n$ and in $\text{Aut}^* \mathbf{A}^{n+1}$ (Theorem 9).

In the final section, developing further the theme of analogies between the Cremona groups and algebraic groups, we introduce and discuss the notions of Jordan decomposition and torsion primes for the Cremona groups. In the course of discussion, we formulate some open questions.

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Notation and conventions.

In the sequel, “variety” means “algebraic variety over the fixed algebraically closed field k of characteristic zero” in the sense of Serre’s FAC [Se55]. Apart from the standard notation and conventions of [Bo91] and [PoVi94] used without reminders we also use the following:

- $\text{Mat}_{m \times n}(R)$ is the set of all matrices with m rows, n columns, and the coefficients in R .
- $N_H(S)$ and $Z_H(S)$ are, respectively, the normalizer and centralizer of the subgroup S of the group H .
- μ_d is the subgroups of order d in \mathbf{G}_m .
- $X(D)$ is the group of rational characters of the diagonalizable algebraic group D .
- $\chi(X)$ is the Euler characteristic of the variety X relative to the l -adic cohomology (for $k = \mathbf{C}$, by [La81] it coincides with the Euler characteristic relative to the usual cohomology with compact supports (cf. also [KrPo85, Appendix])).
- If the group G acts on a set M and $\varphi: G \times M \rightarrow M$ is the map defining this action, then for the subsets $S \subseteq G$ and $X \subseteq M$, the subset $\varphi(S \times X) \subseteq M$ is denoted by $S \cdot X$ (each time it is clear from the context what φ is meant). In particular, by $G \cdot a$ is denoted the G -orbit of the point a . By G_a is denoted the G -stabilizer of the point a .
- x_1, \dots, x_n are the standard coordinate functions on \mathbf{A}^n :

$$x_i(a) := a_i, \quad a := (a_1, \dots, a_n) \in \mathbf{A}^n.$$

In the sequel it is assumed that all considered algebraic groups are affine and all their homomorphisms are algebraic. Below all tori and diagonalizable groups are algebraic.

An action of a group G on a vector space V is called *locally finite*, if for every vector $v \in V$ the linear span of the orbit $G \cdot v$ is finite dimensional.

The group $\text{Cr}_n := \text{Bir } \mathbf{A}^n$ is called *the Cremona group of rank n* . The map $\varphi \mapsto (\varphi^*)^{-1}$ identifies it with $\text{Aut}_k k(x_1, \dots, x_n)$. Every birational isomorphism $X \dashrightarrow \mathbf{A}^n$ identifies Cr_n with $\text{Bir } X$. For every $g \in \text{Cr}_n$ the functions

$$g_i = g^*(x_i) \in k(\mathbf{A}^n) \tag{3}$$

determine g by the formula

$$g(a) = (g_1(a), \dots, g_n(a)) \text{ if } g \text{ is defined at } a \in \mathbf{A}^n; \tag{4}$$

we use the notation

$$(g_1, \dots, g_n) := g. \tag{5}$$

By means of the notion of “algebraic family” $S \rightarrow \text{Cr}_n$ (see [Ra64]) the group Cr_n is endowed with the Zariski topology (see [Se10], [Bl10]). If a homomorphism $G \rightarrow \text{Cr}_n$ of an algebraic group G is an algebraic family, then its image is called an *algebraic subgroup* of Cr_n (see [Po12]).

The subgroup

$$\text{Aut } \mathbf{A}^n := \{(g_1, \dots, g_n) \in \text{Cr}_n \mid g_1, \dots, g_n \in k[\mathbf{A}^n] = k[x_1, \dots, x_n]\}$$

is called the *affine Cremona group of rank n* .

It contains the algebraic subgroup of affine transformations

$$\text{Aff}_n = \{(g_1, \dots, g_n) \in \text{Aut } \mathbf{A}^n \mid \deg g_1 = \dots = \deg g_n = 1\}, \quad (6)$$

and Aff_n , in turn, contains the algebraic subgroup of linear transformations

$$\text{GL}_n = \{g \in \text{Aff}_n \mid g(0) = 0\}.$$

If $g = (g_1, \dots, g_n) \in \text{Aut } \mathbf{A}^n$ (see (5)), then we put

$$\text{Jac}(g) := \det(\partial g_i / \partial x_j).$$

Since $g \in \text{Aut } \mathbf{A}^n$, we have $\text{Jac}(g) \in k \setminus \{0\}$. Therefore, $g \mapsto \text{Jac}(g)$ is the homomorphism of $\text{Aut } \mathbf{A}^n$ in the multiplicative group of the field k . Its kernel

$$\text{Aut}^* \mathbf{A}^n := \{f \in \text{Aut } \mathbf{A}^n \mid \text{Jac}(f) = 1\}$$

consists of the automorphisms of \mathbf{A}^n that leaves fixed the standard volume form; it is called *the special affine Cremona group of rank $n - 1$* (regarding the ranks in these names see Theorems 2(i) and 4(i) below). The latter group contains the algebraic subgroup

$$\text{SL}_n := \text{GL}_n \cap \text{Aut}^* \mathbf{A}^n.$$

The embeddings $\text{Cr}_n \hookrightarrow \text{Cr}_{n+1}$, $(g_1, \dots, g_n) \mapsto (g_1, \dots, g_n, x_{n+1})$ are arranged the tower $\text{Cr}_1 \hookrightarrow \text{Cr}_2 \hookrightarrow \dots \hookrightarrow \text{Cr}_n \hookrightarrow \dots$. Its direct limit Cr_∞ is called *the Cremona group of infinite rank*, see [Po12, Sect. 1].

In GL_n is distinguished the “standard” maximal torus

$$D_n := \{(t_1 x_1, \dots, t_n x_n) \mid t_1, \dots, t_n \in k\} \subset \text{GL}_n. \quad (7)$$

Its normalizer in GL_n is the group of all monomial transformations in GL_n :

$$N_{\text{GL}_n}(D_n) = \{(t_1 x_{\sigma(1)}, \dots, t_n x_{\sigma(n)}) \mid \sigma \in S_n, t_1, \dots, t_n \in k\} \subset \text{GL}_n, \quad (8)$$

where S_n is the symmetric group of degree n . In $\text{Aut}^* \mathbf{A}^n$ is contained the torus

$$D_n^* := D_n \cap \text{Aut}^* \mathbf{A}^n = \{(t_1 x_1, \dots, t_n x_n) \mid t_1, \dots, t_n \in k, t_1 \cdots t_n = 1\}. \quad (9)$$

The “coordinate” characters $\varepsilon_1, \dots, \varepsilon_n$ of the torus D_n defined by

$$\varepsilon_i: D_n \rightarrow \mathbf{G}_m, \quad (t_1 x_1, \dots, t_n x_n) \mapsto t_i, \quad (10)$$

constitute a base of the (free Abelian) group $X(D_n)$.

2. SOME SUBGROUPS OF Cr_n

In the sequel we consider the elements of the group \mathbf{Z}^n as rows of length n . Then the rows of any matrix $A = (a_{ij}) \in \text{Mat}_{m \times n}(\mathbf{Z})$ become the elements of this group and we use the following notation:

$$\mathcal{R}_A := \text{the subgroup of } \mathbf{Z}^n \text{ generated by the rows of matrix } A, \quad (11)$$

$$D_n(A) = \bigcap_{i=1}^m \ker \lambda_i, \quad \text{where } \lambda_i := \varepsilon_1^{a_{i,1}} \cdots \varepsilon_n^{a_{i,n}}. \quad (12)$$

If $m = 1$, then in place of $D_n((l_1 \dots l_n))$ we write $D_n(l_1, \dots, l_n)$. In particular,

$$D_n(0, \dots, 0) = D_n. \quad (13)$$

Clearly, $D_n(A)$ is a closed subgroup of D_n and (see (11))

$$D_n(A) = \bigcap_{(l_1, \dots, l_n) \in \mathcal{R}(A)} \ker \varepsilon_1^{l_1} \dots \varepsilon_n^{l_n}. \quad (14)$$

Recall the terminology used below (see, e.g., [Vi03] and [MaMi64]).

Every finite Abelian group G decomposes as a direct sum of cyclic subgroups of orders d_1, \dots, d_m , where d_i divides d_{i+1} for $i = 1, \dots, m-1$, and $d_1 > 1$ if $|G| > 1$. The numbers d_1, \dots, d_s are uniquely determined by G and are called the *invariant factors* of G .

Every nonzero integer matrix A can be transformed by means of elementary transformations of its rows and columns into a matrix $S = (s_{ij})$ such that only the coefficients s_{ii} for $i = 1, \dots, r$ are nonzero and s_{ii} divides $s_{i+1, i+1}$ for $i = 1, \dots, r-1$. The integers s_{11}, \dots, s_{rr} are uniquely determined by A ($s_{ii} = f_i/f_{i-1}$, where f_i is gcd of the minors of order i of the matrix A and $f_0 := 1$) and are called the *invariant factors of matrix* A . The matrix S is called the *Smith normal form of matrix* A .

Lemma 1. *If B is obtained from $A \in \text{Mat}_{m \times n}(\mathbf{Z})$ by means of the elementary transformations of rows and columns, then the subgroups $D_n(A)$ and $D_n(B)$ are conjugate in Cr_n .*

Proof. Let τ_1, \dots, τ_n be some base of the group $X(D_n)$. Then (see (12)) $\lambda_i = \tau_1^{c_{i,1}} \dots \tau_n^{c_{i,n}}$ for some $c_{ij} \in \mathbf{Z}$ and

$$D_n(A) = \bigcap_{i=1}^m \ker \tau_1^{c_{i,1}} \dots \tau_n^{c_{i,n}}. \quad (15)$$

The group $\text{Aut}_{\text{gr}} D_n$ of automorphisms of the algebraic group D_n is naturally identified with $\text{GL}_n(\mathbf{Z})$. Its natural action on the set of bases of the group $X(D_n)$ is transitive. Therefore, there is an automorphism

$$\varphi \in \text{Aut}_{\text{gr}} D_n, \quad (16)$$

such that $\tau_i \circ \varphi = \varepsilon_i$ for each i . From (15) it then follows that

$$\varphi^{-1}(D_n(A)) = D_n(C), \quad C = (c_{ij}) \in \text{Mat}_{m \times n}(\mathbf{Z}). \quad (17)$$

Since the map of varieties $D_n \rightarrow \mathbf{A}^n$, $(t_1 x_1, \dots, t_n x_n) \mapsto (t_1, \dots, t_n)$, is a birational isomorphism, by means of it we can identify the group $\text{Cr}_n = \text{Bir } \mathbf{A}^n$ with the group of birational automorphisms of the underlying variety of torus D_n . Then φ becomes an element of the group Cr_n and from (16) and (17) it is easy to deduce that in this group we have the equality

$$\varphi^{-1} D_n(A) \varphi = D_n(C).$$

Further, notice that if the base τ_1, \dots, τ_n is obtained from the base $\varepsilon_1, \dots, \varepsilon_n$ by an elementary transformation, then the matrix C is obtained from A by an elementary transformation of columns, and every elementary transformation of columns of A is realizable in this way. Also, notice that if the

sequence $\varphi_1, \dots, \varphi_m \in X(D_n)$ is obtained by an elementary transformation from the sequence $\lambda_1, \dots, \lambda_m$, then $D_n(A) = \bigcap_{i=1}^m \ker \varphi_i$, the matrix (c_{ij}) defined by the equalities $\varphi_i = \varepsilon_1^{c_{i,1}} \cdots \varepsilon_n^{c_{i,n}}$ is obtained from A by an elementary transformation of rows, and every elementary transformation of rows of A is realizable in this way.

Clearly, the said implies the claim of lemma. \square

Corollary 1. *If S is the Smith normal form of matrix A , then the subgroups $D_n(A)$ and $D_n(S)$ are conjugate in Cr_n .*

Lemma 2.

- (i) *Let $q_1 \leq \dots \leq q_r$ be all the invariant factors of matrix $A \in \text{Mat}_{m \times n}(\mathbf{Z})$. Then the group $D_n(A)$ is isomorphic to*

$$\mu_{q_1} \times \cdots \times \mu_{q_r} \times \mathbf{G}_m^{n-r}. \quad (18)$$

- (ii) *The closed $(n - m)$ -dimensional subgroups of D_n are every possible subgroups $D_n(A)$, where $A \in \text{Mat}_{m \times n}(\mathbf{Z})$, $\text{rk } A = m$, and only they.*
- (iii) $\mathcal{R}_A = \{(l_1, \dots, l_n) \in \mathbf{Z}^n \mid D_n(A) \subseteq \ker \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}\}$ for every $A \in \text{Mat}_{m \times n}(\mathbf{Z})$.
- (iv) *If $A \in \text{Mat}_{s \times n}(\mathbf{Z})$, $B \in \text{Mat}_{t \times n}(\mathbf{Z})$, then*
- (a) $D_n(A) = D_n(B)$ if and only if $\mathcal{R}_A = \mathcal{R}_B$;
 - (b) *The following properties are equivalent:*
 - (b₁) $D_n(A)$ and $D_n(B)$ are conjugate in GL_n ;
 - (b₂) $D_n(A)$ and $D_n(B)$ are conjugate in $N_{\text{GL}_n}(D_n)$;
 - (b₂) *by a permutation of columns, it is possible to transform B into a matrix C such that $\mathcal{R}_A = \mathcal{R}_C$.*

Proof. (i) Let $S = (s_{ij})$ be the normal Smith form of matrix A . Then $s_{11} = q_1, \dots, s_{rr} = q_r$ and $s_{ij} = 0$ in the other cases. Hence $D_n(S)$ is isomorphic to group (18). But $D_n(A)$ is isomorphic to $D_n(S)$ by Corollary 1. This proves (i).

- (ii) From (i) we deduce that

$$\dim D_n(A) = n - \text{rk } A; \quad (19)$$

therefore, $\dim D_n(A) = n - m$, where $\text{rk } A = m$. Conversely, let H be a closed subgroup of D_n such that $\dim H = n - m$. Then D_n/H is an m -dimensional torus [Bo91, p. 114] and therefore, there is an isomorphism $\alpha: D_n/H \rightarrow \mathbf{G}_m^m$. Let $\lambda_i \in X(D_n)$ be the composition of homomorphisms

$$D_n \xrightarrow{\pi} D_n/H \xrightarrow{\alpha} \mathbf{G}_m^m \xrightarrow{\text{pr}_i} \mathbf{G}_m,$$

where π is the canonical projection and pr_i is the projection to the i th factor. Then $H = \bigcap_{i=1}^m \ker \lambda_i$. From (12) we then deduce that $H = D_n(A)$ and from (19) that $\text{rk } A = m$. This proves (ii).

(iii) From (11), (12) it follows that the left-hand side of the equality under proof is contained in the right-hand side. Proving the inverse inclusion consider a character $\lambda = \varepsilon_1^{l_1} \cdots \varepsilon_n^{l_n}$ whose kernel contains $D_n(A)$. Without changing $D_n(A)$ and \mathcal{R}_A , we can leave in A only the rows that form a base of the group \mathcal{R}_A removing the other rows, i.e., we can reduce our considerations

to the case where $\text{rk } A = m$. Consider then the characters $\lambda_1, \dots, \lambda_m$ defined by formula (12), and the homomorphism

$$\varphi: D_n \rightarrow \mathbf{G}_m^m, \quad g \mapsto (\lambda_1(g), \dots, \lambda_m(g)).$$

In view of (12), we have $\ker \varphi = D_n(A)$. From this and (19) it follows that $\dim \varphi(D_n) = m$. Therefore, φ is a surjection. Hence \mathbf{G}_m^m is the quotient group of D_n by $D_n(A)$ and φ is the canonical homomorphism to it. Since λ is constant on the fibers of φ , the universal property of quotient implies the existence of character $\mu: \mathbf{G}_m^m \rightarrow \mathbf{G}_m$ such that $\lambda = \mu \circ \varphi$. Hence $\lambda = \lambda_1^{c_1} \cdots \lambda_m^{c_m}$ for some $c_1, \dots, c_m \in \mathbf{Z}$, and this means that $(l_1, \dots, l_n) \in \mathcal{R}_A$. This proves (iii).

(iv)(a) If $D_n(A) = D_n(B)$, then $\mathcal{R}_A = \mathcal{R}_B$ because of (iii). Conversely, if $\mathcal{R}_A = \mathcal{R}_B$, then $D_n(A) = D_n(B)$ because of (14). This proves (iv)(a).

(iv)(b) By fusion theorem [Se00, 1.1.1] the subgroups $D_n(A)$ and $D_n(B)$ are conjugate in GL_n if and only if they are conjugate in $N_{\text{GL}_n}(D_n)$. But (8) and (12) imply that $D_n(A)$ and $D_n(B)$ are conjugate in $N_{\text{GL}_n}(D_n)$ if and only if by a permutation of columns one can obtain from B a matrix C such that $D_n(A) = D_n(C)$. Because of (iii)(a), the latter equality is equivalent to the equality $\mathcal{R}_C = \mathcal{R}_A$. This proves (iv)(b). \square

Corollary 2.

- (i) Let $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and $d := \gcd(l_1, \dots, l_n)$. Then $D_n(l_1, \dots, l_n)$ is isomorphic to $\mu_d \times \mathbf{G}_m^{n-1}$. In particular, the group $D_n(l_1, \dots, l_n)$ is connected (i.e., is a torus) if and only if $d = 1$.
- (ii) The closed $(n-1)$ -dimensional subgroups of D_n are every possible subgroups $D_n(l_1, \dots, l_n)$ with $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and only they.
- (iii) $D_n(l_1, \dots, l_n) = D_n(l'_1, \dots, l'_n)$ if and only if

$$(l_1, \dots, l_n) = \pm(l'_1, \dots, l'_n).$$

(iv) The following properties are equivalent:

- (iv₁) $D_n(l_1, \dots, l_n)$ and $D_n(l'_1, \dots, l'_n)$ are conjugate in GL_n ;
- (iv₂) $D_n(l_1, \dots, l_n)$ and $D_n(l'_1, \dots, l'_n)$ are conjugate in $N_{\text{GL}_n}(D_n)$;
- (iv₃) there is a permutation $\sigma \in S_n$ such that

$$(l_1, \dots, l_n) = \pm(l'_{\sigma(1)}, \dots, l'_{\sigma(n)}).$$

The following Lemma 3 gives an effective numerical criterion for the equality $\mathcal{R}_A = \mathcal{R}_B$ from Lemma 2(iii).

Let $A \in \text{Mat}_{m \times n}(\mathbf{Z})$, $\text{rk } A = m$. For every strictly increasing sequence of m integers i_1, \dots, i_m taken from the interval $[1, n]$, put

$$p_{i_1, \dots, i_m}(A) := \det A_{i_1, \dots, i_m}, \quad (20)$$

where A_{i_1, \dots, i_m} is the submatrix of matrix A obtained by intersecting rows with numbers $1, \dots, m$ and columns with numbers i_1, \dots, i_m (it is natural to call the $p_{i_1, \dots, i_m}(A)$'s the *Plücker coordinates of matrix A*).

Lemma 3. For every two matrices A and $B \in \text{Mat}_{m \times n}(\mathbf{Z})$ of rank m the following properties are equivalent:

- (i) $\mathcal{R}_A = \mathcal{R}_B$.

- (ii) *Two conditions hold:*
- (a) *either $p_{i_1, \dots, i_m}(A) = p_{i_1, \dots, i_m}(B)$ for all i_1, \dots, i_m , or $p_{i_1, \dots, i_m}(A) = -p_{i_1, \dots, i_m}(B)$ for all i_1, \dots, i_m ;*
 - (b) *for every sequence i_1, \dots, i_m such that $p_{i_1, \dots, i_m}(A) \neq 0$, the following inclusion holds:*

$$B_{i_1, \dots, i_m}(A_{i_1, \dots, i_m})^{-1} \in \text{Mat}_{m \times m}(\mathbf{Z}). \quad (21)$$

Proof. Whereas $\text{rk } A = \text{rk } B = m$, the rows of A and B form the bases in \mathcal{R}_A and \mathcal{R}_B respectively. Therefore, $\mathcal{R}_A = \mathcal{R}_B$ if and only if when there is a matrix $Q \in \text{GL}_m(\mathbf{Z})$ such that

$$A = QB. \quad (22)$$

Let $\mathcal{R}_A = \mathcal{R}_B$. Then (22) implies that

$$A_{i_1, \dots, i_m} = QB_{i_1, \dots, i_m} \quad (23)$$

for every i_1, \dots, i_m , and therefore, $p_{i_1, \dots, i_m}(A) = \det Q p_{i_1, \dots, i_m}(B)$ because of (20). Since $Q \in \text{GL}_m(\mathbf{Z})$, we have $\det Q = \pm 1$. Hence condition (ii)(a) holds. If $p_{i_1, \dots, i_m}(A) \neq 0$, then $p_{i_1, \dots, i_m}(B) \neq 0$ as well, hence B_{i_1, \dots, i_m} is non-degenerate and (23) implies that $Q = A_{i_1, \dots, i_m}(B_{i_1, \dots, i_m})^{-1}$. Hence condition (ii)(b) holds. This proves (i) \Rightarrow (ii).

Proving the inverse implication, consider \mathbf{Z}^n as a subset of the coordinate vectors space (of rows) \mathbf{Q}^n . Condition (ii)(a) shows that the \mathbf{Q} -linear spans of subsets \mathcal{R}_A and \mathcal{R}_B of \mathbf{Q}^n have the same Plücker coordinates. Hence these spans are one and the same linear subspace L (see, e.g., [ShRe09, Theorem 10.1]). Since the rows of A and rows of B form two bases in L , there exists a matrix $P \in \text{GL}_m(\mathbf{Q})$ such that $A = PB$. Therefore, $A_{i_1, \dots, i_m} = PB_{i_1, \dots, i_m}$ for every i_1, \dots, i_m and hence $P = A_{i_1, \dots, i_m}(B_{i_1, \dots, i_m})^{-1}$ if $p_{i_1, \dots, i_m}(B) \neq 0$. Then (ii)(b) implies that $P \in \text{GL}_m(\mathbf{Z})$. Therefore, $\mathcal{R}_A = \mathcal{R}_B$. This proves (ii) \Rightarrow (i). \square

Remark 1.

1. In the proof of Lemma 3 it is established that, in fact, (ii)(a) implies that the matrix $B_{i_1, \dots, i_m}(A_{i_1, \dots, i_m})^{-1}$ is independent of the choice of a sequence i_1, \dots, i_m for which $p_{i_1, \dots, i_m}(A) \neq 0$. Therefore, (ii)(b) follows from (ii)(a) and feasibility of (21) for any *one* such a sequence.

2. If $m=1$, then (ii)(b) follows from (ii)(a) (but for $m > 1$ this is not so).

Theorem 1 (Classification of diagonalizable subgroups of Aff_n up to conjugacy in Cr_n).

- (i) *Two diagonalizable subgroups of the group Aff_n are conjugate in Cr_n if and only if they are isomorphic.*
- (ii) *Any diagonalizable subgroup G of the group Aff_n is conjugate in Cr_n to a unique closed subgroup of the torus D_n that has the form*

$$\ker \varepsilon_{r+1}^{d_1} \cap \dots \cap \ker \varepsilon_{r+s}^{d_s} \cap \ker \varepsilon_{r+s+1} \cap \dots \cap \ker \varepsilon_n \quad (24)$$

where $0 \leq r \leq n$, $0 \leq s \leq n$, $r + s \leq n$, $2 \leq d_1$ and d_i divides d_{i+1} for every $i < s$. The integers determining subgroup (24) have the following meaning: $r = \dim G$ and d_1, \dots, d_s are all the invariant factors of the finite Abelian group G/G^0 .

Proof. Since maximal reductive subgroups of algebraic group are conjugate (see [BoSe64, 5.1]) and GL_n is one of them in Aff_n , every diagonalizable subgroup of Aff_n is conjugate to a subgroup of GL_n . In turn, every diagonalizable subgroup of GL_n is conjugate to a subgroup of the torus D_n (see [Bo91, I.4.6]). In view of Lemma 2(ii) this shows that it suffices to prove (i) for the subgroups $D_n(A)$ and $D_n(B)$ of the torus D_n . Further, appending, if necessary, zero rows we may assume that A and B have the same number of rows. Let now $D_n(A)$ and $D_n(B)$ be isomorphic. Then their dimensions are equal and the groups $D_n(A)/D_n(A)^0$ and $D_n(B)/D_n(B)^0$ have the same invariant factors. This and Lemma 2(i) imply that the matrices A and B have the same invariant factors (because the latter are obtained by appending the same number of 1's to the invariant factors of the specified groups). Hence the normal Smith forms of A and B are equal. By Corollary 1 this implies that $D_n(A)$ and $D_n(B)$ are conjugate in Cr_n . This proves (i).

It is clear that the integers determining subgroup (24) have the meaning specified in (ii). Since every diagonalizable group is a direct product of a finite Abelian group and a torus, it is uniquely, up to isomorphism, determined by its dimension and the invariant factors of the group of connected components. This and (i) implies (ii). \square

Corollary 3. *$D_n(A)$ and $D_n(B)$ are conjugate in Cr_n if and only if A and B have the same invariant factors.*

Corollary 4. *Every torus T in Aff_n is conjugate in Cr_n to the torus D_r , $r = \dim T$.*

Proof. This follows from Theorem 1(ii). \square

Corollary 5. *Every two isomorphic finite Abelian subgroups of Aff_n are conjugate in Cr_n .*

Proof. Since $\mathrm{char} k = 0$, every element in Aff_n of finite order is semisimple. Hence every finite Abelian subgroup of Aff_n is reductive, and therefore, is conjugate in Aff_n to a subgroups of GL_n (see the proof of Theorem 1). But every commutative subgroup of GL_n consisting only of semisimple elements is diagonalizable (see [Bo91, Prop. 4.6(b)]). The claim now follows from Theorem 1(i). \square

Corollary 6 ([Bl06₁, Thm. 1]). *Every two elements in Aff_n of the same finite order are conjugate in Cr_n .*

3. TORI IN Cr_n , $\mathrm{Aut} \mathbf{A}^n$, AND $\mathrm{Aut}^* \mathbf{A}^n$

Theorem 2 (Tori in Cr_n).

- (i) *In Cr_n there are no tori of dimension $> n$.*
- (ii) *In Cr_n every r -dimensional torus for $r = n, n - 1, n - 2$ is conjugate to the torus D_r .*
- (iii) *If $n \geq 5$, then in Cr_n there are $(n - 3)$ -dimensional tori that are not conjugate to subtori of the torus D_n .*
- (iv) *Every r -dimensional torus in Cr_n is conjugate in Cr_{n+r} to the torus D_r .*
- (v) *In Cr_∞ every r -dimensional torus is conjugate to the torus D_r .*

Proof. (i) This is proved, e.g., in [De70], see also [Po12, Cor. 2.2].

(ii) According to [B-B66, Cor. 2] (see also [Po12, Cor. 2.4(b)]), every r -dimensional torus in Cr_n for $r = n, n-1, n-2$ is conjugate to a subtorus of the torus D_n . Therefore, the claim follows from Corollary 4.

(iii) This is proved in [Po12, Cor. 2.5].

(iv) According to [Po12, Thm. 2.6], every r -dimensional torus in Cr_r is conjugate in Cr_{n+r} to a subtorus of the torus D_{n+r} . Therefore, the claim follows from Corollary 4.

(v) This follows from (iv). \square

Corollary 7.

- (i) In Cr_n every n -dimensional torus is maximal.
- (ii) In Cr_n there are no maximal $(n-1)$ - and $(n-2)$ -dimensional tori.
- (iii) For $n \geq 5$, in Cr_n there are maximal $(n-3)$ -dimensional tori.

Remark 2. For $n \leq 3$, Theorem 2 gives the classification of tori in Cr_n up to conjugacy: the classes of conjugate nontrivial tori are exhausted by that of the tori D_1, \dots, D_n .

Let $(l_1, \dots, l_n) \in \mathbf{Z}^n$ be a nonzero element with $\gcd(l_1, \dots, l_n) = 1$. Clearly, the homomorphism

$$\mathbf{G}_m \rightarrow D_n, \quad t \mapsto (t^{l_1} x_1, \dots, t^{l_n} x_n), \quad (25)$$

is an embedding and every embedding $\mathbf{G}_m \hookrightarrow D_n$ is of this form. Denote by $T(l_1, \dots, l_n)$ the image of embedding (25). It is a one-dimensional torus in D_n and every one-dimensional torus in D_n is of this form.

Lemma 4. *The following properties are equivalent:*

- (i) $T(l_1, \dots, l_n) = T(l'_1, \dots, l'_n)$.
- (ii) $(l_1, \dots, l_n) = \pm(l'_1, \dots, l'_n)$.

Proof. (i) \Rightarrow (ii) This is clear.

(ii) \Rightarrow (i) Assume that (i) holds. Take an element $t \in \mathbf{G}_m$ of infinite order. From (i) and the definition of $T(l_1, \dots, l_n)$ we deduce that there is an element $s \in \mathbf{G}_m$ such that $t^{l_i} = s^{l'_i}$ for every $i = 1, \dots, n$. Hence $t^{l_i l'_j} = s^{l'_i l'_j} = t^{l_j l'_i}$ for every two nonequal integers i and j taken from the interval $[1, n]$. Since the order of t is infinite, this implies that $l_i l'_j - l_j l'_i = 0$. Hence

$$\text{rk} \begin{pmatrix} l_1 & \cdots & l_n \\ l'_1 & \cdots & l'_n \end{pmatrix} = 1.$$

Therefore, $(l_1, \dots, l_n) = \gamma(l'_1, \dots, l'_n)$ for some $\gamma \in \mathbf{Q}$, or, equivalently, $p(l_1, \dots, l_n) = q(l'_1, \dots, l'_n)$, where $p, q \in \mathbf{Z}$, $\gcd(p, q) = 1$. Hence p divides each of l'_1, \dots, l'_n , and q divides each of l_1, \dots, l_n . Since the integers in each of these two sets are coprime, this implies that $\gamma = \pm 1$, i.e., (ii) holds. \square

Theorem 3 (Tori in $\text{Aut } \mathbf{A}^n$).

- (i) In $\text{Aut } \mathbf{A}^n$ every n -dimensional torus is conjugate to the torus D_n .

- (ii) In $\text{Aut } \mathbf{A}^n$ all $(n - 1)$ -dimensional tori are exhausted, up to conjugacy in $\text{Aut } \mathbf{A}^n$, by all the groups of the form $D_n(l_1, \dots, l_n)$ with $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and $\gcd(l_1, \dots, l_n) = 1$.
- (iii) In $\text{Aut } \mathbf{A}^3$ all one-dimensional tori are exhausted, up to conjugacy, by all the groups of the form $T(l_1, l_2, l_3)$.

Proof. According to [B-B66] and [B-B67], every r -dimensional torus in the group $\text{Aut } \mathbf{A}^n$ for $r = n$ and $n - 1$, respectively, is conjugate to a subtorus of the torus D_n . This and Corollary 2 imply (i) and (ii).

According to [KKM-LR97], every one-dimensional torus in $\text{Aut } \mathbf{A}^3$ is conjugate to subtorus of the torus D_3 . This implies (iii). \square

Theorem 4 (Tori in $\text{Aut}^* \mathbf{A}^n$).

- (i) In $\text{Aut}^* \mathbf{A}^n$ there are no tori of dimension $> n - 1$.
- (ii) In $\text{Aut}^* \mathbf{A}^n$ every $(n - 1)$ -dimensional torus is maximal and conjugate to the torus D_n^* (see (9)).

Proof. (i) If $\text{Aut}^* \mathbf{A}^n$ contains an n -dimensional torus T , then by Theorem 3(i) there is an element $g \in \text{Aut } \mathbf{A}^n$ such that

$$T = gD_n g^{-1}. \quad (26)$$

Replacing g by gz , where $z \in D_n$ is an element such that $\text{Jac}(z) = \det z = 1/\text{Jac}(g)$, we may assume that $g \in \text{Aut}^* \mathbf{A}^n$. This and (26) imply that $D_n \subset \text{Aut}^* \mathbf{A}^n$, — a contradiction. This proves (i).

(ii) Let S be an $(n - 1)$ -dimensional torus in $\text{Aut}^* \mathbf{A}^n$. By Theorem 3(ii) and Corollary (ii) of Lemma 2, there are $g \in \text{Aut } \mathbf{A}^n$ and $(l_1, \dots, l_n) \in \mathbf{Z}^n$ such that

$$S = gD_n(l_1, \dots, l_n)g^{-1}. \quad (27)$$

As in the proof of (i), we may assume that $g \in \text{Aut}^* \mathbf{A}^n$. From (27) it then follows that $D_n(l_1, \dots, l_n) \subset \text{Aut}^* \mathbf{A}^n$. This and $D_n \cap \ker \text{Jac} = D_n^* := D_n(1, \dots, 1)$ imply the inclusion $D_n(l_1, \dots, l_n) \subseteq D_n^*$. By Corollary (i) of Lemma 2, both sides of this inclusion are $(n - 1)$ -dimensional tori. Hence it is the equality. This proves (ii) \square

Remark 3. For $\text{Aut } \mathbf{A}^n$ and $\text{Aut}^* \mathbf{A}^n$, unlike for Cr_n (see Corollary 7(iii)), at present nothing is known on the existence of tori of nonmaximal dimension. This problem is intimately related to the *Cancellation Problem*: Is there an affine variety X not isomorphic to \mathbf{A}^m , $m = \dim X$, such that $X \times \mathbf{A}^d$ is isomorphic to \mathbf{A}^{m+d} for some d ? If the answer is positive, then in $\text{Aut } \mathbf{A}^n$ and $\text{Aut}^* \mathbf{A}^n$ with $n = m + d$ there exists a maximal torus T of a nonmaximal dimension. Indeed, multiplying, if necessary, $X \times \mathbf{A}^d$ by \mathbf{A}^1 , one may assume that $d \geq 2$. Let λ be the character $t \mapsto t$ of the torus \mathbf{G}_m . Consider the linear action of \mathbf{G}_m on \mathbf{A}^d with precisely two isotypic components: one is $(d - 1)$ -dimensional of type λ , another is one-dimensional of type λ^{1-d} . It determines the action of \mathbf{G}_m on $X \times \mathbf{A}^d$ via the second factor and hence, an action of \mathbf{G}_m on \mathbf{A}^n . Consider in $\text{Aut } \mathbf{A}^n$ the torus that is the image of \mathbf{G}_m under the homomorphism determined by this action. The construction implies that this torus lies in $\text{Aut}^* \mathbf{A}^n$. Let T (respectively, T') be a maximal torus of $\text{Aut } \mathbf{A}^n$ (respectively, of $\text{Aut}^* \mathbf{A}^n$) that contains this image. If T is

n -dimensional (respectively, T' is $(n-1)$ -dimensional), then by Theorem 3(i) (respectively, by Theorem 4) it is conjugate to a subtorus of the torus D_n and, therefore, the specified action of T (respectively, T') on \mathbf{A}^n is equivalent to a linear one. Hence the action of \mathbf{G}_m on \mathbf{A}^n is equivalent to a linear one as well, and therefore, the set F of its fixed points is isomorphic to an affine space. But, by the construction, in \mathbf{A}^d there is a single \mathbf{G}_m -fixed point, consequently, F is isomorphic to X — a contradiction.

4. ORBITS AND STABILIZERS OF THE ACTION OF $D_n(l_1, \dots, l_n)$ ON \mathbf{A}^n

Here we establish some properties, needed for what follows, of orbits and stabilizers of the natural action on \mathbf{A}^n of the group

$$G := D_n(l_1, \dots, l_n). \quad (28)$$

Clearly, every coordinate hyperplane

$$H_i := \{a \in \mathbf{A}^n \mid x_i(a) = 0\} \quad (29)$$

is G -invariant.

Lemma 5. *The G -stabilizer of every point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is trivial.*

- (i) *If $(l_1, \dots, l_n) = (0, \dots, 0)$, then $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is a G -orbit, and dimension of the G -stabilizer of every point of $\bigcup_{i=1}^n H_i$ is positive.*
- (ii) *If $(l_1, \dots, l_n) \neq (0, \dots, 0)$, then $\dim G \cdot a = n - 1$ for every point $a \in \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$.*

Proof. This immediately follows from (28), (12), (4) and Lemma 2. \square

We now consider the case

$$(l_1, \dots, l_n) \neq (0, \dots, 0).$$

Lemma 6. *If $l_i \neq 0$, then the open subset $\mathcal{O}_i := H_i \setminus \bigcup_{j \neq i} H_j$ in H_i is a G -orbit.*

Proof. By virtue of G -invariance of \mathcal{O}_i , it suffices to show that \mathcal{O}_i is contained in a G -orbit. Since $l_i \neq 0$, the equation $x^i = \alpha$ has a solution for any $\alpha \in k$, $\alpha \neq 0$. This and (28), (12) imply that for any two points

$$b = (b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_n), \quad c = (c_1, \dots, c_{i-1}, 0, c_{i+1}, \dots, c_n) \in H_i \setminus \bigcup_{j \neq i} H_j$$

there exists an element $g = (t_1 x_1, \dots, t_n x_n) \in G$ such that $t_j = b_j^{-1} c_j$ for every $j \neq i$. From (4) we then obtain that $g \cdot b = c$, as required. \square

Lemma 7. *The following properties are equivalent:*

- (i) *all the numbers l_1, \dots, l_n are nonzero and have the same sign;*
- (ii) *G -orbit of every point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is closed in \mathbf{A}^n ;*
- (iii) *G -orbit of some point of $\mathbf{A}^n \setminus \bigcup_{i=1}^n H_i$ is closed in \mathbf{A}^n .*

Proof. Consider a point

$$a = (a_1, \dots, a_n) \in \mathbf{A}^n \setminus \bigcup_{i=1}^n H_i.$$

Assume that (i) holds. Suppose that the G -orbit of the point a is not closed in \mathbf{A}^n . Then its boundary can be accessed by a one-parameter subgroup, i.e., there is homomorphism $\varphi: \mathbf{G}_m \rightarrow G$ such that there is a limit

$$\lim_{t \rightarrow 0} \varphi(t) \cdot a = b \in \overline{G \cdot a} \setminus G \cdot a \quad (30)$$

(see [PoVi94, Thm. 6.9]); the latter means that the morphism $\mathbf{G}_m = \mathbf{A}^1 \setminus \{0\} \rightarrow \mathbf{A}^n$, $t \mapsto \varphi(t) \cdot a$, extends to a morphism $\mathbf{A}^1 \rightarrow \mathbf{A}^n$ that maps 0 to the point b . As φ is algebraic, there is a vector $(d_1, \dots, d_n) \in \mathbf{Z}^n$ such that $\varphi(t) = (t^{d_1} x_1, \dots, t^{d_n} x_n)$ for every $t \in \mathbf{G}_m$. Since $\varphi(t) \cdot a = (t^{d_1} a_1, \dots, t^{d_n} a_n)$ and $a_i \neq 0$ for every i , the existence of the specified limit means that

$$d_1 \geq 0, \dots, d_n \geq 0. \quad (31)$$

On the other hand, it follows from $\varphi(t) \in G$ and (28), (12) that $t^{d_1 l_1 + \dots + d_n l_n} = 1$ for every t , i.e.,

$$d_1 l_1 + \dots + d_n l_n = 0. \quad (32)$$

But (31), (32) and condition (i) imply that $d_1 = \dots = d_n = 0$. Hence $b = a$ contrary to (30). This contradiction proves (i) \Rightarrow (ii).

Conversely, assume that (i) is not fulfilled, i.e., among l_1, \dots, l_n there are either two nonzero numbers with different signs, or one number equal to zero. In the first case, let, for instance, be $l_1 > 0, l_2 < 0$. Then (12) implies that the image of the homomorphism $\varphi: \mathbf{G}_m \rightarrow D_n$, $t \mapsto (t^{-l_2} x_1, t^{l_1} x_2, x_3, \dots, x_n)$, lies in G . Since

$$\lim_{t \rightarrow 0} \varphi(t) \cdot a = (0, 0, a_3, \dots, a_n) \notin G \cdot a,$$

this shows that the orbit $G \cdot a$ is not closed. In the second case, let, say, $a_1 = 0$. Then G contains the image of the homomorphism $\varphi: \mathbf{G}_m \rightarrow D_n$, $t \mapsto (t x_1, x_2, \dots, x_n)$ and, since

$$\lim_{t \rightarrow 0} \varphi(t) \cdot a = (0, a_2, \dots, a_n) \notin G \cdot a,$$

the orbit $G \cdot a$ is not closed. This proves (iii) \Rightarrow (i). \square

From Lemmas 5, 6, and 7 we infer

Corollary 8. *If all the numbers l_1, \dots, l_n are nonzero and have the same sign, then in \mathbf{A}^n there are precisely n nonclosed $(n-1)$ -dimensional G -orbits — these are the orbits $\mathcal{O}_1, \dots, \mathcal{O}_n$ from Lemma 6.*

Remark 4. Recall from [Po70] that an action of an algebraic group on an algebraic variety is called *stable*, if orbits of points in general position are closed. Lemma 7 shows that the following properties are equivalent:

- (i) all the numbers l_1, \dots, l_n are nonzero and have the same sign;
- (ii) the action G on \mathbf{A}^n is stable.

Lemma 8. *Assume that none of the numbers l_1, \dots, l_n are equal to ± 1 . Then the following properties of the point $a = (a_1, \dots, a_n) \in \mathbf{A}^n$ are equivalent:*

- (i) a has a nontrivial G -stabilizer;
- (ii) $a \in \bigcup_{i=1}^n H_i$.

Proof. Lemma 5 implies (i) \Rightarrow (ii). Now assume that (ii) holds. Then there are the indices i_1, \dots, i_s , where $s \geq 1$, such that $a_j = 0$ for $j = i_1, \dots, i_s$ and $a_j \neq 0$ for other j 's. It follows from (4), (28), and (12) that an element $(t_1 x_1, \dots, t_n x_n) \in D_n$ lies in the G -stabilizer of the point a if and only if $t_j = 1$ for $j \neq i_1, \dots, i_s$ and

$$t_{i_1}^{l_1} \dots t_{i_s}^{l_s} = 1. \quad (33)$$

Since there is no ± 1 among the numbers l_1, \dots, l_n and k is an algebraically closed field of characteristic zero, equality (33), considered as the equation in t_{i_1}, \dots, t_{i_s} , has at least two solutions. Hence the G -stabilizer of the point a is nontrivial. This proves (ii) \Rightarrow (i). \square

Lemma 9. *If among l_1, \dots, l_n there are two nonzero numbers with different signs, then every G -orbit contains $(0, \dots, 0)$ in its closure.*

Proof. For definiteness, let, for instance, be

$$l_1 > 0, l_2 \geq 0, \dots, l_s \geq 0, l_{s+1} < 0, l_{s+2} \leq 0, \dots, l_n \leq 0.$$

Choose a positive integer d so large that

$$q := l_2 + \dots + l_s + dl_{s+1} + l_{s+2} + \dots + l_n < 0.$$

Since $-ql_1 + l_1 l_2 + \dots + l_1 l_s + l_1 dl_{s+1} + l_1 l_{s+2} + \dots + l_1 l_n = 0$, from (28) and (12) we infer that the image of the homomorphism

$$\varphi: \mathbf{G}_m \rightarrow D_n, \quad t \mapsto (t^{-q} x_1, t^{l_1} x_2, \dots, t^{l_1} x_s, t^{l_1 d} x_{s+1}, t^{l_1} x_{s+2}, \dots, t^{l_1} x_n) \quad (34)$$

lies in G . On the other hand, since the numbers $-q, l_1$ and d are positive, (4) implies that for every point $a \in \mathbf{A}^n$ the limit $\lim_{t \rightarrow 0} \varphi(t) \cdot a$ exists and is equal to $(0, \dots, 0)$. \square

Now consider the case where 0 and ± 1 are contained among l_1, \dots, l_n , there are at least two nonzero l_i 's, and all of them have the same sign. By (12), without loss of generality we may assume that this sign is positive. Up to replacing the group G by its conjugate by means of an element of $N_{\text{GL}_n}(D_n)$, we may then assume that

$$l_1 = \dots = l_p = 1, l_{p+1} \geq 2, \dots, l_q \geq 2, l_{q+1} = \dots = l_n = 0, \quad (35)$$

where $p \geq 1, n > q \geq p$ and $q \geq 2$.

Lemma 10. *Assume that (35) holds. Take a point $a = (a_1, \dots, a_n) \in \mathbf{A}^n$.*

- (i) *Let $a \notin \bigcup_{i=1}^n H_i$. Then the orbit $G \cdot b$, where $b = (a_1, \dots, a_q, 0, \dots, 0)$, lies in the closure of the orbit $G \cdot a$, is closed, and $\dim G \cdot b = q - 1$.*
- (ii) *Let $a \in H_i$. Then the group G_a is*
 - (a) *trivial if $1 \leq i \leq p$ and $a \in \mathcal{O}_i$ (see Lemma 6);*
 - (b) *nontrivial and finite if $p + 1 \leq i \leq q$ and $a \in \mathcal{O}_i$;*
 - (c) *has positive dimension if $i > q$.*

Proof. From (28), (12), and (35) it follows that the image of the homomorphism

$$\varphi: \mathbf{G}_m \rightarrow D_n, \quad t \mapsto (x_1, \dots, x_q, tx_{q+1}, \dots, tx_n),$$

lies in G . Hence the point

$$\lim_{t \rightarrow 0} \varphi(t) \cdot a = (a_1, \dots, a_q, 0, \dots, 0) = b \quad (36)$$

lies in the closure of the orbit $G \cdot a$.

Now let $a \notin \bigcup_{i=1}^n H_i$. If the orbit $G \cdot b$ is not closed, then, as in the proof of Lemma 7, there exist a homomorphism

$$\psi: \mathbf{G}_m \rightarrow G, \quad t \mapsto (t^{d_1} x_1, \dots, t^{d_n} x_n), \quad (37)$$

such that $c := \lim_{t \rightarrow 0} \psi(t) \cdot b \in \overline{G \cdot b} \setminus G \cdot b$. From (36) it follows that $d_1 \geq 0, \dots, d_q \geq 0$, and from (35), (28), and (12) it follows that $d_1 l_1 + \dots + d_q l_q = 0$. Since l_1, \dots, l_q are positive, this yields $d_1 = \dots = d_q = 0$. In view of (36), from this we infer that $\psi(t) \cdot b = b$ for every t , and therefore, $c = b$ —a contradiction. Thus, $G \cdot b$ is closed. Since a_1, \dots, a_q are nonzero, (28), (12), (4), (35), and (36) imply that an element $(t_1 x_1, \dots, t_n x_n) \in D_n$ lies in G_b if and only if $t_1 = \dots = t_q = 1$. This proves (i).

The arguments analogous to that used in the proof of Lemma 8 yield (ii).

□

Finally, consider the case where one of the numbers l_1, \dots, l_n is equal to ± 1 (in view of (12), without loss of generality we may assume that it is equal to 1), and all the others are equal to 0.

Lemma 11. *Let $l_i = 1$ and let $l_j = 0$ for $j \neq i$. For any $s \in k$ denote by $H(s)$ the hyperplane in \mathbf{A}^n defined by the equation $x_i + s = 0$. Then:*

- (i) $\bigcup_{j \neq i} H_j$ is the set of points with nontrivial G -stabilizer (that automatically has positive dimension).
- (ii) The open subset $H(s) \setminus \bigcup_{j \neq i} H_j$ of $H(s)$ is an $(n-1)$ -dimensional G -orbit and every $(n-1)$ -dimensional G -orbit is of this form.

Proof. Part (i) immediately follows from (4), (28), and (12), and part (ii) follows from (i), the invariance of $H(\alpha)$, and the equality $\dim G = \dim H(\alpha) = n-1$. □

5. THE GROUP $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$

First, we shall prove several general statements about normalizers for the actions on arbitrary affine varieties.

Lemma 12. *Let X be an irreducible affine variety and let G be an algebraic subgroup of $\text{Aut } X$. Then the following properties are equivalent:*

- (i) $N_{\text{Aut } X}(G)$ is an algebraic subgroup of $\text{Aut } X$.
- (ii) The natural action of $N_{\text{Aut } X}(G)$ on $k[X]$ is locally finite.

Proof. (i) \Rightarrow (ii) This follows from the fact that the natural action on $k[X]$ of every algebraic subgroup of $\text{Aut } X$ is locally finite (see [Bo91, Prop. 1.9]).

(ii) \Rightarrow (i) Assume that (ii) holds. Then in $k[X]$ there is an $N_{\text{Aut } X}(G)$ -invariant finite dimensional k -linear subspace V containing a system of generators of the k -algebra $k[X]$. Hence the homomorphism

$$\rho: N_{\text{Aut } X}(G) \rightarrow \text{GL}(V^*)$$

determined by the action of $N_{\text{Aut } X}(G)$ on V is an embedding. Consider the $N_{\text{Aut } X}(G)$ -equivariant map

$$\iota: X \rightarrow V^*, \quad \iota(x)(f) := f(x) \quad \text{for every } x \in X, f \in V.$$

The standard argument (see [Bo91, Prop. 1.12]) shows that $\rho|_G$ is a morphism of algebraic groups, and ι is a closed embedding. Identify X with $\iota(X)$ by means of ι , and $N_{\text{Aut } X}(G)$ with $\rho(N_{\text{Aut } X}(G))$ by means of ρ . Then X is a closed subvariety of V^* , and $N_{\text{Aut } X}(G)$ and G are the subgroups of $\text{GL}(V^*)$; besides, G is closed, and

$$N_{\text{Aut } X}(G) \subset N_{\text{GL}(V^*)}(G) \cap \text{Tran}_{\text{GL}(V^*)}(X, X), \quad \text{where} \quad (38)$$

$$\text{Tran}_{\text{GL}(V^*)}(X, X) := \{g \in \text{GL}(V^*) \mid g \cdot X \subset X\}. \quad (39)$$

In fact, in the right-hand side of (39) the equality $g \cdot X = X$ automatically holds: indeed, X is irreducible and closed in V^* , and $g \in \text{Aut } V^*$ implies that $g \cdot X$ is a closed subset of X of the same dimension as X . Hence $\text{Tran}_{\text{GL}(V^*)}(X, X)$, as well as $N_{\text{GL}(V^*)}(G)$, is a subgroup of $\text{GL}(V^*)$, and therefore, the right-hand side of (38) is a subgroup of $\text{GL}(V^*)$. Its elements normalize G and, being restricted to X , are the automorphisms of X ; whence, they lie in $N_{\text{Aut } X}(G)$. Therefore,

$$N_{\text{Aut } X}(G) = N_{\text{GL}(V^*)}(G) \cap \text{Tran}_{\text{GL}(V^*)}(X, X). \quad (40)$$

From closedness of G in $\text{GL}(V^*)$ and that of X in V^* we deduce, respectively, that $N_{\text{GL}(V^*)}(G)$ and $\text{Tran}_{\text{GL}(V^*)}(X, X)$ are closed in $\text{GL}(V^*)$ (see [Bo91, Prop. 1.7]). This and (40) imply that $N_{\text{Aut } X}(G)$ is closed in $\text{GL}(V^*)$. Hence $N_{\text{Aut } X}(G)$ is an algebraic subgroup of $\text{Aut } X$. \square

Theorem 5. *Let X be an irreducible affine variety and let G be a reductive algebraic subgroup in $\text{Aut } X$ such that*

$$k[X]^G = k. \quad (41)$$

In either of the following cases $N_{\text{Aut } X}(G)$ is the algebraic subgroup of $\text{Aut } X$:

- (i) G has a fixed point in X .
- (ii) G^0 is semisimple.

Proof. Take $f \in k[X]$. We shall prove that in each of cases (i) and (ii) the k -linear span of the orbit $N_{\text{Aut } X}(G) \cdot f$ is finite dimensional. The claim of the theorem will then follow from Lemma 12.

Let $\mathcal{M}(G)$ be the set of isomorphism classes of algebraic simple G -modules. If L is an algebraic G -module, denote by L_μ its isotypic component of type $\mu \in \mathcal{M}(G)$.

Since G is reductive, we have (see [PoVi94, 3.13])

$$k[X] = \bigoplus_{\mu \in \mathcal{M}(G)} k[X]_\mu \quad (42)$$

The group $N_{\text{Aut } X}(G)$ permutes the isotypic components of the G -module $k[X]$.

Since $k[X]_\mu$ is a finitely generated $k[X]^G$ -module (see [PoVi94, Thm. 3.24]), (41) implies that

$$\dim_k k[X]_\mu < \infty \quad \text{for every } \mu. \quad (43)$$

In view of (42), there are elements $\mu_1, \dots, \mu_s \in \mathcal{M}(G)$ such that

$$k[X]_{\mu_i} \neq 0 \quad \text{for all } i, \quad \text{and} \quad f \in k[X]_{\mu_1} \oplus \dots \oplus k[X]_{\mu_s}. \quad (44)$$

(i) Assume that in X there is a G -fixed point a . Since closed orbits are separated by G -invariant regular functions (see [PoVi94, Thm. 4.7]), it follows from (41) that there are no other G -fixed points in X . Hence a is fixed by $N_{\text{Aut } X}(G)$ as well. Therefore, the ideal

$$\mathfrak{m}_a := \{f \in k[X] \mid f(a) = 0\}$$

is $N_{\text{Aut } X}(G)$ -invariant. Hence every member of the decreasing filtration

$$\mathfrak{m}_a \supset \dots \supset \mathfrak{m}_a^d \supset \mathfrak{m}_a^{d+1} \supset \dots \quad (45)$$

is $N_{\text{Aut } X}(G)$ -invariant. This filtration has the property (see [AtMa69, Cor. 10.18]) that

$$\bigcap_d \mathfrak{m}_a^d = 0. \quad (46)$$

In view of (45), we have a decreasing system of nested linear subspaces $\{k[X]_\mu \cap \mathfrak{m}_a^d \mid d = 1, 2, \dots\}$. Since they are finite dimensional (see (43)), there is d_μ such that $k[X]_\mu \cap \mathfrak{m}_a^d = k[X]_\mu \cap \mathfrak{m}_a^{d+1}$ for every $d \geq d_\mu$. From (46) it then follows that, in fact,

$$k[X]_\mu \cap \mathfrak{m}_a^d = 0 \quad \text{for every } d \geq d_\mu. \quad (47)$$

Let $l \in \mathbf{Z}$, $l \geq \max\{d_{\mu_1}, \dots, d_{\mu_s}\}$. From (47) it follows that

$$k[X]_{\mu_i} \cap \mathfrak{m}_a^l = 0 \quad \text{for every } i = 1, \dots, s. \quad (48)$$

Since the natural projection $\pi: k[X] \rightarrow k[X]/\mathfrak{m}_a^l$ is an epimorphism of G -modules, we have $\pi(k[X]_\mu) = (k[X]/\mathfrak{m}_a^l)_\mu$ for every $\mu \in \mathcal{M}(G)$. Whereas

$$\dim_k k[X]/\mathfrak{m}_a^l < \infty,$$

(see [AtMa69, Prop. 11.4]), this implies finiteness of the set of $\mu \in \mathcal{M}(G)$ such that

$$k[X]_\mu \neq 0 \quad k[X]_\mu \cap \mathfrak{m}_a^l = 0. \quad (49)$$

Let $\{\lambda_1, \dots, \lambda_t\}$ be this set. Since (49) holds for $\mu = \mu_1, \dots, \mu_s$ (see (44), (48)), we may assume that

$$\lambda_i = \mu_i \quad i = 1, \dots, s. \quad (50)$$

Whereas the group $N_{\text{Aut } X}(G)$ permutes the isotypic components of the G -module $k[X]$ and sends \mathfrak{m}_a^l into itself, it permutes $k[X]_{\lambda_1}, \dots, k[X]_{\lambda_t}$. Hence $k[X]_{\lambda_1} \oplus \dots \oplus k[X]_{\lambda_t}$ is an $N_{\text{Aut } X}(G)$ -invariant subspace of $k[X]$. In view of (43), (44), and (50), it is finite dimensional and contains f . Hence the k -linear span of the orbit $N_{\text{Aut } X}(G) \cdot f$ is finite dimensional, as claimed.

(ii) Let G^0 be semisimple. From the Weyl formula for dimension of simple G^0 -module and finiteness of the index $[G:G^0]$ it follows that, up to isomorphism, there are only finitely many algebraic simple G -modules whose

dimension does not exceed any preassigned constant. This implies finiteness of the set of $\mu \in \mathcal{M}(G)$ such that

$$k[X]_{\mu} \neq 0 \quad \dim_k k[X]_{\mu} \leq \max_i \dim_k k[X]_{\mu_i}.$$

Let $\{\lambda_1, \dots, \lambda_t\}$ be this set. We may assume that (50) holds. Since $N_{\text{Aut } X}(G)$, permuting isotypic components, preserves their dimensions, $k[X]_{\lambda_1} \oplus \dots \oplus k[X]_{\lambda_t}$ is invariant with respect to $N_{\text{Aut } X}(G)$. The proof can be now completed as in case (i). \square

Corollary 9. *Let X and G be the same as in Theorem 5 and let $k = \mathbf{C}$. Assume that the variety X is simply connected and smooth, and $\chi(X) = 1$. Then $N_{\text{Aut } X}(G)$ is an algebraic subgroup of $\text{Aut } X$.*

Proof. By corollary of theorem on étale slice (see [Lu73, III, Cor.2] and [PoVi94, Thm.6.7]), condition (41) and smoothness of X imply that X is a homogeneous vector bundle over the unique closed G -orbit \mathcal{O} in X . Hence \mathcal{O} is simply connected and $\chi(X) = \chi(\mathcal{O})$. Being affine, \mathcal{O} is isomorphic to G/H for some reductive subgroup H (see [PoVi94, Thm. 4.17]). The conditions that G/H is simply connected and $\chi(G/H) = 1$ imply that $G = H$ (see [KrPo85, 5.1]). Hence \mathcal{O} is a fixed point. The claim now follows from Theorem 5(i). \square

Corollary 10. *Let G be an algebraic subgroup of $\text{Aut } \mathbf{A}^n$ and $k[\mathbf{A}^n]^G = k$. Then $N_{\text{Aut } \mathbf{A}^n}(G)$ is the algebraic subgroup of $\text{Aut } \mathbf{A}^n$.*

Proof. In view of $\text{char } k = 0$, by the Lefschetz principle [Ha77, 15.1] we may assume that $k = \mathbf{C}$. Since \mathbf{A}^n is simply connected and smooth, and $\chi(\mathbf{A}^n) = 1$, the claim follows from Corollary 9. \square

Remark 5. The following example shows that condition (41) alone (for irreducible affine X and reductive G) does not, in general, imply that $N_{\text{Aut } X}(G)$ is algebraic.

Example 1. Let G be an algebraic torus of dimension $n \geq 2$. Take as X the underlying variety of the algebraic group G . Its group of automorphisms $\text{Aut}_{\text{gr}} G$ is embedded in $\text{Aut } X$ and is isomorphic to $\text{GL}_n(\mathbf{Z})$. The action of G on X by left translations embeds G in $\text{Aut } X$. These two subgroups generate $\text{Aut } X$, more precisely, $\text{Aut } X = \text{Aut}_{\text{gr}} G \rtimes G$. Therefore, $N_{\text{Aut } X}(G) = \text{Aut } X$. Let $g \in \text{Aut}_{\text{gr}} G$ be an element of infinite order and let $f_1, \dots, f_n \in k[X]$ be a base of $X(G)$. Then $g^d \cdot f_i \in X(G)$ for every $d \in \mathbf{Z}$ and $i = 1, \dots, n$, and the set $C_i := \{g^d \cdot f_i \mid i \in \mathbf{Z}\}$ is finite if and only if the stabilizer of f_i with respect to the cyclic group generated by g is nontrivial. Assume that all C_1, \dots, C_n are finite. Then there is $d \in \mathbf{Z}$, $d \neq 0$, such that $g^d \cdot f_i = f_i$ for every $i = 1, \dots, n$. Since f_1, \dots, f_n is a base in $X(G)$, this means that the automorphism g^d is trivial contrary to the assumption that the order of g is infinite and $d \neq 0$. Hence C_i is infinite for some i . Since different characters are linear independent over k (see [Bo91, Lemma 8.1]), this implies that the k -linear span of the set C_i (and hence of the orbit $N_{\text{Aut } X}(G) \cdot f_i$) is infinite dimensional.

We shall now use the obtained information in the proof of algebraicity of the groups $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$.

Theorem 6. $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ for every l_1, \dots, l_n is an algebraic subgroup of $\text{Aut } \mathbf{A}^n$. Moreover,

- (i) $N_{\text{Aut } \mathbf{A}^n}(D_n) = N_{\text{GL}_n}(D_n)$.
- (ii) If $(l_1, \dots, l_n) \neq (0, \dots, 0)$, then

$$N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n)) \subseteq N_{\text{GL}_n}(D_n) \quad (51)$$

in either of the following cases:

- (a) all the numbers l_1, \dots, l_n are nonzero and have the same sign;
 - (b) none of the numbers l_1, \dots, l_n is equal to ± 1 ;
 - (c) the numbers l_1, \dots, l_n contain 0 and ± 1 , at least two of them are nonzero, and all nonzero of them have the same sign.
- (iii) If $l_i = 1$ and $l_j = 0$ for $j \neq i$, then $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ is isomorphic to $N_{\text{GL}_{n-1}}(D_{n-1}) \times \text{Aff}_1$ and consists of all $(g_1, \dots, g_n) \in \text{Aff}_n$ of the form

$$g_j = \begin{cases} t_j x_{\sigma(j)} & j \neq i; \\ t_j x_j + s & j = i, \end{cases} \quad (52)$$

where $t_1, \dots, t_n, s \in k$ and σ is a permutation of the set $\{1, \dots, i-1, i+1, \dots, n\}$.

Proof. If G is a subgroup of $\text{Aut } \mathbf{A}^n$ and $g \in N_{\text{Aut } \mathbf{A}^n}(G)$, $a \in \mathbf{A}^n$, then

$$g(G \cdot a) = G \cdot g(a), \quad gG_ag^{-1} = G_{g(a)}, \quad g(\overline{G \cdot a}) = \overline{g(G \cdot a)}, \quad \text{and} \quad (53)$$

if G is algebraic, $\dim G \cdot a = \dim g(G \cdot a)$.

Let $G = D_n$. Lemma 5(i) implies that $\bigcup_{i=1}^n H_i$ is the set of points whose G -stabilizer has positive dimension. By (53), it is g -invariant. Since the restriction of g to the variety $\bigcup_{i=1}^n H_i$ is its automorphism, g permutes its irreducible components H_1, \dots, H_n , i.e., there is a permutation $\sigma \in S_n$ such that

$$g(H_i) = H_{\sigma(i)} \quad \text{for every } i = 1, \dots, n. \quad (54)$$

Since the ideal in $k[\mathbf{A}^n]$ determined by H_i is generated by x_i , this shows that the polynomial $g^*(x_i)$ divides $x_{\sigma(i)}$, hence $g^*(x_i) = t_i x_{\sigma(i)}$ for some nonzero $t_i \in k$. Therefore, $g \in N_{\text{GL}_n}(D_n)$ (see (3), (8)). This proves (i).

Further, let $G = D_n(l_1, \dots, l_n)$, $(l_1, \dots, l_n) \neq (0, \dots, 0)$.

Assume that condition (a) holds. Then (53) and Corollary of Lemma 7 imply that g permutes the orbits $\mathcal{O}_1, \dots, \mathcal{O}_n$, i.e., there exists a permutation $\sigma \in S_n$ such that $g(\mathcal{O}_i) = \mathcal{O}_{\sigma(i)}$ for every i . In view of Lemma 6 and (53), this implies that g has property (54), and as is shown above, the latter implies that $g \in N_{\text{GL}_n}(D_n)$. Thus, (51) is proved in the case when condition (a) holds.

Assume that condition (b) holds. Then g -invariance of $\bigcup_{i=1}^n H_i$ follows from (53) and Lemma 8. Now the same argument as for $G = D_n$ completes the proof of (51) in the case when condition (b) holds.

Assume that condition (c) holds. Proving (51), we may replace G by the group conjugate to G by means of an appropriate element of $N_{\text{GL}_n}(D_n)$

and assume that (35) holds. The set $\{a \in \mathbf{A}^n \mid \dim G_a > 0\}$ is closed (see [PoVi94, 1.4]). By virtue of (53), it is g -invariant, and Lemmas 5 and 10(ii) imply that its $(n-1)$ -dimensional irreducible components are H_{q+1}, \dots, H_n . Hence g permutes H_{q+1}, \dots, H_n . Further, it follows from Lemmas (5) and (10)(ii) that $\mathcal{O}_{p+1}, \dots, \mathcal{O}_q$ are all the G -orbits \mathcal{O} in \mathbf{A}^n such that G_a is finite and nontrivial for $a \in \mathcal{O}$. From (53) it then follows that g permutes the orbits $\mathcal{O}_{p+1}, \dots, \mathcal{O}_q$ and, therefore, permutes their closures H_{p+1}, \dots, H_q . Finally, in view of Lemmas 5 and 10(ii), all the G -orbits \mathcal{O} in \mathbf{A}^n such that G_a is trivial for $a \in \mathcal{O}$ are exhausted by the orbits $\mathcal{O}_1, \dots, \mathcal{O}_p$ and $G \cdot a$, where $a \notin \bigcup_{i=1}^n H_i$. Since G is reductive, every G -orbit in \mathbf{A}^n contains in its closure a unique closed G -orbit (see [PoVi94, Cor. p. 189]). By Lemma 6, for each of the orbits $\mathcal{O}_1, \dots, \mathcal{O}_p$ this closed orbit is the fixed point $(0, \dots, 0)$. On the other hand, by Lemma 10(i), the closed orbit lying in $\overline{G \cdot a}$ for $a \notin \bigcup_{i=1}^n H_i$ has dimension $q-1 \geq 1$ and, therefore, is not the fixed point. Hence g permutes $\mathcal{O}_1, \dots, \mathcal{O}_p$, and therefore, permutes their closures H_1, \dots, H_p . Thereby, it is proved that (54) holds for a certain permutation $\sigma \in S_n$. As above, this allows to conclude that (51) holds in case (c). Thus, (ii) is proved.

Assume now that $l_i = 1$ and $l_j = 0$ for $j \neq i$. From (53) and Lemma 11(i) it follows that the closed set $\bigcup_{j \neq i} H_j$ is invariant with respect to g . Hence g permutes its irreducible components $H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_n$. As above, from here we conclude that for $j \neq i$ the equality $g_j = t_j x_{\sigma(j)}$ holds for some $t_j \in k$ and some permutation σ of the set $\{1, \dots, i-1, i+1, \dots, n\}$. Further, (53) and Lemmas 11(i), 11(ii) imply that $g(\overline{\mathcal{O}_i})$ is an orbit open in a certain hyperplane $H(c)$ (see Lemma 11(ii)). Since $\overline{\mathcal{O}_i} = H_i$, and x_i and $x_i + c$ are, respectively, the generators of the ideals of hyperplanes H_i and $H(c)$, we conclude that $g_i = g^*(x_i)$ differs from $x_i + c$ only by a nonzero constant factor: $g_i = t_i x_i + s$ for some $t_i, s \in k$, $t_i \neq 0$. Thus, g is of the form (52). Conversely, clearly, every element $g \in \text{Aut } \mathbf{A}^n$ of the form (52) normalizes G . This proves (iii).

Finally, let us prove the first claim of this theorem. According to (i), (ii), and (iii), the group $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ is algebraic if either $(l_1, \dots, l_n) = (0, \dots, 0)$, or $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and any of conditions (a), (b), (c) of statement (ii) or condition of statement (iii) hold. The only case not covered by these conditions is that when among l_1, \dots, l_n there are numbers with different signs. However, it follows from Lemma 9 that in this case there are nonconstant $D_n(l_1, \dots, l_n)$ -invariant regular functions on \mathbf{A}^n . But then by Corollary 2 of Theorem 5 we conclude that $N_{\text{Aut } \mathbf{A}^n}(D_n(l_1, \dots, l_n))$ is algebraic. \square

6. FUSION THEOREMS FOR TORI IN $\text{Aut } \mathbf{A}^n$ AND $\text{Aut}^* \mathbf{A}^n$

Fusion theorems describe subgroups that control fusion of subsets under conjugation. Namely, let G be an (abstract) groups and let H be its subgroup. We say that $N_G(H)$ controls fusion of subsets in H under conjugation by the elements of G , if the following property holds:

for any subset $S \subseteq H$ and an element $g \in G$ such that $gSg^{-1} \subseteq H$, there is an element $w \in N_G(H)$, such that $gsg^{-1} = wsw^{-1}$ for every element $s \in S$. (F)

If property (F) holds for a pair (G, H) , then we say that for H in G fusion theorem holds. Notice that the action of $N_G(H)$ on H by conjugation boils down to the action of the ‘‘Weyl group’’ $N_G(H)/Z_G(H)$.

Examples 7. Fusion theorem for H in G holds in the following cases:

1. G is a finite group and H is its Abelian Sylow p -subgroup. It is the classical Burnside’s result.

2. G is an affine algebraic group and H is its maximal torus. It is the classical result of the theory of algebraic groups, see, e.g., [Se00, 1.1.1].

3. $(G, H) = (\text{Cr}_n, D_n)$. It is Serre’s result [Se10, Thm. 1.1]. Since every n -dimensional torus in Cr_n is maximal and conjugate to D_n (see Theorem 2(i),(ii)), here one can replace D_n by any n -dimensional torus. Recall that for $n \geq 5$ in Cr_n there exist $(n - 3)$ -dimensional maximal tori (see Corollary 7(iii)).

Question 8. Does fusion theorem hold if D_n is replaced by such a torus?

We shall now prove that fusion theorem holds for n -dimensional tori in $\text{Aut } \mathbf{A}^n$ and $(n - 1)$ -dimensional tori in $\text{Aut}^* \mathbf{A}^n$.

Lemma 13. For every element $g = (g_1, \dots, g_n) \in \text{Aut } \mathbf{A}^n$ there is an element $g' \in \text{SL}_n$ such that if $s, gsg^{-1} \in \text{GL}_n$, then

$$gsg^{-1} = g'sg'^{-1}. \quad (55)$$

Proof. Let $g_i = g_i^{(0)} + g_i^{(1)} + \dots$, where $g_i^{(s)}$ is a form of degree s in x_1, \dots, x_n . Since $\text{Jac}(g) \in k$, we have $\text{Jac}(g) = \det(\partial g_i / \partial x_j |_{x_1=\dots=x_n=0})$. But the right-hand side of this equality is equal to $\det(\partial g_i^{(1)} / \partial x_j)$. Hence

$$g^{(1)} := (g_1^{(1)}, \dots, g_n^{(1)}) \in \text{GL}_n. \quad (56)$$

The automorphism $g^{(1)}$ is the differential of the automorphism g at the point $(0, \dots, 0)$. From (56) it follows that

$$\begin{aligned} \text{(i)} \quad & g = g^{(1)} \quad \text{if } g \in \text{GL}_n; \\ \text{(ii)} \quad & (ga)^{(1)} = g^{(1)}a \quad \text{and} \quad (ag)^{(1)} = ag^{(1)} \quad \text{if } a \in \text{GL}_n. \end{aligned} \quad (57)$$

Now let s and $t := gsg^{-1}$ be the elements of GL_n . In view of (56), we have $g^{(1)} \in \text{GL}_n$, and $gs = tg$ and (57) imply that $g^{(1)}s = tg^{(1)}$. Therefore, the product of $g^{(1)}$ by a constant $\alpha \in k$ such that $\alpha^n \det g^{(1)} = 1$ can be taken as g' . \square

Theorem 9. The following pairs (G, H) have property (F):

(i) (Fusion theorem for n -dimensional tori in $\text{Aut } \mathbf{A}^n$)

$$(G, H) = (\text{Aut } \mathbf{A}^n, \text{an } n\text{-dimensional torus}).$$

(ii) (Fusion theorem for $(n - 1)$ -dimensional tori in $\text{Aut}^* \mathbf{A}^n$)

$$(G, H) = (\text{Aut}^* \mathbf{A}^n, \text{an } (n - 1)\text{-dimensional torus}).$$

Proof. (i) Let $G = \text{Aut } \mathbf{A}^n$, let H be a torus, and $\dim H = n$. Then, by Theorem 3 we may assume that H is a maximal torus in GL_n . Maintain the notation of the formulation of property (F). By Lemma 13 there exists an element $g' \in \text{GL}_n$ such that for every element $s \in S$ equality (55) holds. Therefore, $g'Sg'^{-1} = gSg^{-1} \subseteq H$. Since $g' \in \text{GL}_n$, this shows that

$$H' := g'^{-1}Hg' \quad (58)$$

is another maximal torus in GL_n containing S . The tori H and H' lie in the (closed [Bo91, I.1.7]) subgroup $Z_{\text{GL}_n}(S)$ of the group GL_n and therefore, are maximal tori of $Z_{\text{GL}_n}(S)$. In view of conjugacy of maximal tori in any affine algebraic group [Bo91, IV.11.3], there is an element

$$z \in Z_{\text{GL}_n}(S), \quad (59)$$

such that

$$H' = zHz^{-1}. \quad (60)$$

From (58) and (60) it follows that $w := g'z \in N_{\text{GL}_n}(H)$, and from (59) and (55) that $gsg^{-1} = wsw^{-1}$ for every $s \in S$. This proves (i).

(ii) The same argument applies in the case where $G = \text{Aut}^* \mathbf{A}^n$, H is a torus, and $\dim H = n - 1$: one only has to replace GL_n by SL_n , the reference to Theorem 3 by the reference to Theorem 4, and notice that by Lemma 13 the element g' may be taken from SL_n . \square

7. APPLICATIONS: THE CLASSIFICATIONS OF CLASSES OF CONJUGATE SUBGROUPS

In this section, using the obtained results, we derive the classifications specified in the Introduction and, in particular, prove the generalizations to disconnected groups of the Białynicki-Birula's theorems on linearization of actions on \mathbf{A}^n of tori of dimension $\geq n - 1$.

Theorem 10. *Let G be an algebraic subgroups of $\text{Aut } \mathbf{A}^n$ such that G^0 is a torus.*

- (i) *If $\dim G = n$ or $n - 1$, then there is an element $g \in \text{Aut } \mathbf{A}^n$ such that $gGg^{-1} \subset \text{GL}_n$ and $gG^0g^{-1} \subset D_n$.*
- (ii) *If $G \subset \text{Aut}^* \mathbf{A}^n$ and $\dim G = n - 1$, then there is an element $g \in \text{Aut}^* \mathbf{A}^n$ such that $gGg^{-1} \subset \text{SL}_n$ and $gG^0g^{-1} = D_n^*$ (see (9)).*

Proof. (i) By virtue of conjugacy of maximal tori in GL_n , it suffices to prove the existence of $g \in \text{Aut } \mathbf{A}^n$ such that $gGg^{-1} \subset \text{GL}_n$.

The group G is reductive. Therefore, by Corollary of Theorem on étale slice [Lu73, Cor. 2, p. 98] the claim holds if $k[\mathbf{A}^n]^G = k$ (here only reductivity of G is used, not the stronger condition that G^0 is a torus). Therefore, in what follows we may assume that $k[\mathbf{A}^n]^G \neq k$. The latter is equivalent to the condition

$$k[\mathbf{A}^n]^{G^0} \neq k. \quad (61)$$

In view of Theorem 3 and equality (13), replacing G by a conjugate group, we may assume that

$$G^0 = D_n(l_1, \dots, l_n) \subset \text{GL}_n. \quad (62)$$

The claim on the existence of g will be proved if we show that (62) and (61) imply the inclusion

$$G \subset \mathrm{GL}_n. \quad (63)$$

Let F be a finite subgroup of G that intersects every connected component of this group — such a subgroup exists, see [BoSe64, Lemme 5.11]. Then

$$G = FG^0. \quad (64)$$

Since G^0 is a normal subgroup of G , we have $F \subset N_{\mathrm{Aut} \mathbf{A}^n}(D_n(l_1, \dots, l_n))$. In view of Theorem 6 this shows that

$$F \subset N_{\mathrm{GL}_n}(D_n) \subset \mathrm{GL}_n, \quad (65)$$

if either $(l_1, \dots, l_n) = (0, \dots, 0)$, or $(l_1, \dots, l_n) \neq (0, \dots, 0)$ and any of conditions (a), (b), (c) of statement (ii) of Theorem 6 holds. By virtue of (62) and (64), this proves (63) for the specified (l_1, \dots, l_n) 's. From Lemma 9 and (61) it now follows that it remains to consider the last possibility for (l_1, \dots, l_n) , namely, that where $l_i = 1$ and $l_j = 0$ if $j \neq i$ for certain i .

Turning to its consideration, take some element $g \in F$. Then, by Theorem 6(iii), equalities (52) hold. Since F is finite, the order of g is finite. This implies that in (52) we have $s = 0$. This and (8) mean that in the case under consideration (65) holds as well, and therefore, (63), too. This proves (i).

(ii) The proof is the same as that of (i) if $\mathrm{Aut} \mathbf{A}^n$ is replaced by $\mathrm{Aut}^* \mathbf{A}^n$, GL_n by $\mathrm{SL}_n = \mathrm{GL}_n \cap \mathrm{Aut}^* \mathbf{A}^n$, $D_n(l_1, \dots, l_n)$ by D_n^* , the reference to Theorem 3 by the reference to Theorem 4, and it is taken into account that D_n^* is the maximal torus of SL_n . \square

Theorem 11 (Classification of n -dimensional diagonalizable subgroups of $\mathrm{Aut} \mathbf{A}^n$ up to conjugacy in $\mathrm{Aut} \mathbf{A}^n$). *Up to conjugacy in $\mathrm{Aut} \mathbf{A}^n$, the torus D_n is the unique n -dimensional diagonalizable subgroup of $\mathrm{Aut} \mathbf{A}^n$.*

Proof. By virtue of Theorem 10 this follows from the fact that in GL_n every diagonalizable subgroup is conjugate to a subgroup of the torus D_n (see [Bo91, p. 112, Prop.(d)]). \square

Theorem 12 (Classification of $(n - 1)$ -dimensional diagonalizable subgroups of $\mathrm{Aut}^* \mathbf{A}^n$ up to conjugacy in $\mathrm{Aut}^* \mathbf{A}^n$). *Up to conjugacy in $\mathrm{Aut}^* \mathbf{A}^n$, the torus D_n^* is the unique $(n - 1)$ -dimensional diagonalizable subgroup of $\mathrm{Aut}^* \mathbf{A}^n$.*

Proof. By virtue of Theorem 10, this follows from the fact that in SL_n every diagonalizable subgroup is conjugate to a subgroup of the torus D_n^* (this easily follows from [Bo91, p. 112, Prop.(d)]). \square

Theorem 13 (Classification up to conjugacy in $\mathrm{Aut} \mathbf{A}^n$ of maximal n -dimensional algebraic subgroups G of $\mathrm{Aut} \mathbf{A}^n$ such that G^0 is a torus). *Up to conjugacy in $\mathrm{Aut} \mathbf{A}^n$, the group $N_{\mathrm{GL}_n}(D_n)$ (see (8)) is the unique maximal algebraic subgroup of $\mathrm{Aut} \mathbf{A}^n$ whose connected component of identity is an n -dimensional torus.*

Proof. By virtue of Theorem 10, this follows from the fact that $D_n = N_{\mathrm{GL}_n}(D_n)^0$. \square

Theorem 14 (Classification up to conjugacy in $\text{Aut}^* \mathbf{A}^n$ of maximal $(n-1)$ -dimensional algebraic subgroups G in $\text{Aut}^* \mathbf{A}^n$ such that G^0 is a torus). *Up to conjugacy in $\text{Aut}^* \mathbf{A}^n$,*

$$N_{\text{SL}_n}(D_n^*) = N_{\text{GL}_n}(D_n) \cap \text{SL}_n$$

is the unique maximal algebraic subgroup of $\text{Aut}^ \mathbf{A}^n$ whose connected component of identity is an $(n-1)$ -dimensional torus.*

Proof. By virtue of Theorem 10, this follows from the fact that $D_n^* = N_{\text{SL}_n}(D_n^*)^0$. \square

Denote by \mathcal{L}_n the additive monoid of all $(l_1, \dots, l_n) \in \mathbf{Z}^n$ such that

- (a) $l_1 \leq \dots \leq l_n$;
- (b) $(l_1, \dots, l_n) \leq (-l_n, \dots, -l_1)$ with respect to the lexicographical order on \mathbf{Z}^n .

Theorem 15 (Classification of $(n-1)$ -dimensional diagonalizable subgroups in $\text{Aut} \mathbf{A}^n$ up to conjugacy in $\text{Aut} \mathbf{A}^n$).

- (i) *Every $(n-1)$ -dimensional diagonalizable subgroup of the group $\text{Aut} \mathbf{A}^n$ is conjugate in $\text{Aut} \mathbf{A}^n$ to a unique subgroup of the form*

$$D_n(l_1, \dots, l_n), \quad (l_1, \dots, l_n) \in \mathcal{L}_n \setminus \{(0, \dots, 0)\}. \quad (66)$$

- (ii) *Every subgroup (66) is an $(n-1)$ -dimensional diagonalizable group.*

Proof. Let G be an $(n-1)$ -dimensional diagonalizable subgroup of $\text{Aut} \mathbf{A}^n$. Taking into account Theorem 10(i), conjugacy in GL_n of every diagonalizable subgroup of this group to a subgroup of the torus D_n , and Corollary 2(ii), we conclude that G is conjugate in $\text{Aut} \mathbf{A}^n$ to a certain subgroup $D_n(l_1, \dots, l_n)$ with $(l_1, \dots, l_n) \neq (0, \dots, 0)$. It follows from the definition of \mathcal{L}_n and Corollary 2(iv) that we may presume that $(l_1, \dots, l_n) \in \mathcal{L}_n$. Assume that G is also conjugate in $\text{Aut} \mathbf{A}^n$ to a subgroup $D_n(l'_1, \dots, l'_n)$ with $(l'_1, \dots, l'_n) \in \mathcal{L}_n$. Then $D_n(l_1, \dots, l_n)$ and $D_n(l'_1, \dots, l'_n)$ are conjugate in $\text{Aut} \mathbf{A}^n$, and therefore, Theorem 9(i) implies that they are conjugate in $N_{\text{GL}_n}(D_n)$. In view of Corollary 2(iv) and Definition \mathcal{L}_n , this implies that $(l_1, \dots, l_n) = (l'_1, \dots, l'_n)$. This proves (i).

Statement (ii) follows from Corollary 2(ii). \square

From Theorems 6, 11, and 15 we deduce

Theorem 16. *If G is a diagonalizable subgroup of $\text{Aut} \mathbf{A}^n$ of dimension $\geq n-1$, then $N_{\text{Aut} \mathbf{A}^n}(G)$ is an algebraic subgroup of $\text{Aut} \mathbf{A}^n$.*

Remark 6. It is easy to see that $n-1$ in Theorem 16 cannot be replaced by a smaller integer.

Theorem 17 (Classification of $(n-1)$ -dimensional diagonalizable subgroups of $\text{Aut} \mathbf{A}^n$ up to conjugacy in Cr_n).

- (i) *Two diagonalizable $(n-1)$ -dimensional subgroups of the group $\text{Aut} \mathbf{A}^n$ are conjugate in Cr_n if and only if they are isomorphic.*

- (ii) *Every $(n-1)$ -dimensional diagonalizable subgroup of the group $\text{Aut } \mathbf{A}^n$ is conjugate in Cr_n to a unique closed subgroup of D_n of the form*

$$\ker \varepsilon_n^d, \quad d \in \mathbf{Z}.$$

Proof. This follows from Theorems 15 and 1 and Corollary 2(i). \square

For $n \leq 3$, Theorems 2(i), 11, and 15 yield the classification of all tori in $\text{Aut } \mathbf{A}^n$ up to conjugacy in $\text{Aut } \mathbf{A}^n$ except for one-dimensional tori in $\text{Aut } \mathbf{A}^3$. The classification of the latter is given below in Theorem 18.

Theorem 18 (Classification of one-dimensional tori in $\text{Aut } \mathbf{A}^3$ up to conjugacy in $\text{Aut } \mathbf{A}^3$). *Every one-dimensional torus in $\text{Aut } \mathbf{A}^3$ is conjugate in $\text{Aut } \mathbf{A}^3$ to a unique torus of the form $T(l_1, l_2, l_3)$, where $(l_1, l_2, l_3) \in \mathcal{L}_3$.*

Proof. Let G be a one-dimensional torus in $\text{Aut } \mathbf{A}^3$. By Theorem 3 it is conjugate in $\text{Aut } \mathbf{A}^3$ to some torus $T(l_1, l_2, l_3)$. In view of (8) and the equality $T(l_1, l_2, l_3) = T(-l_1, -l_2, -l_3)$, we may presume that $(l_1, l_2, l_3) \in \mathcal{L}_3$. Assume that G is conjugate in $\text{Aut } \mathbf{A}^3$ to another torus $T(l'_1, l'_2, l'_3)$ with $(l'_1, l'_2, l'_3) \in \mathcal{L}_3$. Then $T(l_1, l_2, l_3)$ and $T(l'_1, l'_2, l'_3)$ are conjugate in $\text{Aut } \mathbf{A}^3$, and therefore, by Theorem 9(i), also in $N_{\text{GL}_3}(D_3)$. From this, Corollary 2(iv), the definition of \mathcal{L}_n , and Lemma 4, it is then not difficult to deduce that $(l_1, l_2, l_3) = (l'_1, l'_2, l'_3)$. \square

8. JORDAN DECOMPOSITION IN Cr_n . TORSION PRIMES FOR THE CREMONA GROUPS

Although the Cremona groups are infinite dimensional (this has a precise meaning, see [Ra64]), the analogies between them and algebraic groups strike the eye: they have the Zariski topology, algebraic subgroups, tori, roots, the Weyl groups, ... In [Se10, 1.2] Serre writes about the analogy

“groupe de Cremona de rang $n \leftrightarrow$ groupe semi-simple de rang n ”.

Below we briefly touch upon two topics demonstrating that these analogies extend further. The second of them is intimately related to tori in the Cremona groups.

Jordan decomposition in Cr_n .

Let X be an algebraic variety. “Algebraic families” endow $\text{Bir } X$ with the Zariski topology [Ra64], [Bl10, 2], [Se10, 1.6]: a subset of $\text{Bir } X$ is closed if and only if its inverse image for every algebraic family $S \rightarrow \text{Bir } X$ is closed. For every algebraic subgroup G in $\text{Bir } X$ and its subset Z , the closures of Z in this topology and in the Zariski topology of the group G coincide. In particular, G is closed in $\text{Bir } X$.

Let us call an element $g \in \text{Bir } X$ *algebraic* if in Cr_n there is an algebraic subgroup G containing g . This is equivalent to the property that the closure of the cyclic group generated by g is an algebraic group. If G is affine, then for g the Jordan decomposition in G is defined, see [Bo91, Chap. I, §4]:

$$g = g_s g_n \tag{67}$$

In fact, g_s and g_n depend only of g , not of the choice of G . Indeed, let G' be another affine algebraic subgroup of $\text{Bir } X$ containing g , and let $g = g'_s g'_n$ be the Jordan decomposition in G' . Since $G \cap G'$ is a closed subgroup of G and of G' , there exists the Jordan decomposition $g = g''_s g''_n$ in $G \cap G'$. Applying theorem on behaviour of the Jordan decompositions under homomorphisms (see [Bo91, Thm. 4.4(4)]) to the embeddings $G \leftarrow G \cap G' \hookrightarrow G'$, we obtain $g_s = g''_s, g_n = g''_n, g'_s = g''_s, g'_n = g''_n$. Hence, if we call (67) the *Jordan decomposition in $\text{Bir } X$* , we get the well-defined notion.

According to [Ma63], every algebraic subgroup of Cr_n is affine. Therefore, in Cr_n every algebraic element admits the Jordan decomposition.

Jordan decompositions in algebraic groups have several known properties [St65], [Bo91], for instance:

- (a) Every semisimple element of a connected group lies in its torus.
- (b) The set of all unipotent elements is closed.
- (c) The conjugacy class of every semisimple element of a connected reductive group is closed.
- (d) The closure of the conjugacy class of every element g of a connected reductive group contains g_s .

Recall that Cr_n is connected and, for $n = 1, 2$, simple [Bl10], and that $\text{Aut } \mathbf{A}^n, \text{Aut}^* \mathbf{A}^n$ are connected and $\text{Aut}^* \mathbf{A}^n$ is simple [Sh66], [Sh82]. It is natural to ask:

Question 19. *Are there analogues or modifications of the mentioned properties for the groups $\text{Cr}_n, \text{Aut } \mathbf{A}^n$, and $\text{Aut}^* \mathbf{A}^n$?*

For instance, property (a) for Cr_n holds if $n = 1$, but does not hold if $n > 1$. Indeed, if $n > 2$, then by [Po12, Thm. 4.3] in Cr_n there is a semisimple element of order two that is not contained in any connected algebraic subgroup of the group Cr_n . If $n = 2$, then by virtue of Theorem 2(ii) and Corollary 5, all the elements of order $d < \infty$ in Cr_n that can be included in tori constitute a single conjugacy class, while, for instance, for even d , the set of all elements of order d is the union of infinitely many conjugacy classes [Bl07]. On the other hand, in the groups $\text{Aut } \mathbf{A}^2$ and $\text{Aut}^* \mathbf{A}^2$ property (a) holds, because action of every finite group on \mathbf{A}^2 is linearizable [Ig77].

Torsion primes for the Cremona groups.

Let G be a connected reductive algebraic group and let p be a prime integer. Recall (see [Se00, 1.3] and references therein) that p is called *torsion prime for the group G* if in G there is a finite Abelian p -subgroup not contained in any torus of the group G . The set $\text{Tors}(G)$ of all torsion primes for the group G admits various interpretations. For instance, $p \in \text{Tors}(G)$ for $k = \mathbf{C}$ if and only if $\bigoplus_i H_i(G, \mathbf{Z})$ contains an element of order p (this explains the name). Finding $\text{Tors}(G)$ is reduced to the case where G is simply connected. For every simple G the set $\text{Tors}(G)$ is explicitly described.

The possibility to speak about tori in groups of the form $\text{Bir } X$ where X is an algebraic variety, makes it possible to replace G in the above definition by $\text{Bir } X$ or its subgroup thereby obtaining the well-defined notion. In particular, we consider what is obtained when G is replaced by $\text{Cr}_n, \text{Aut } \mathbf{A}^n$, or

$\text{Aut}^* \mathbf{A}^n$ as the definitions of torsion primes for these groups. Denote the sets of these primes respectively by $\text{Tors}(\text{Cr}_n)$, $\text{Tors}(\text{Aut } \mathbf{A}^n)$, and $\text{Tors}(\text{Aut}^* \mathbf{A}^n)$. It is natural to ask

Question 20. *What are, explicitly, the sets $\text{Tors}(\text{Cr}_n)$, $\text{Tors}(\text{Aut } \mathbf{A}^n)$, and $\text{Tors}(\text{Aut}^* \mathbf{A}^n)$?*

(About $\text{Tors}(\text{Cr}_n)$ this question has been formulated and discussed in talk [Po12].)

Since $\text{Cr}_1 = \text{PGL}_2$, we have

$$\text{Tors}(\text{Cr}_1) = \{2\}. \quad (68)$$

According to [Bl07], for every $d = 2, 3$, and 5 , in Cr_2 there are infinitely many conjugacy classes of elements of order d . On the other hand, as is explained at the end of section on Jordan decomposition, in Cr_2 there is the unique conjugacy class of elements of order d contained in tori. Hence $2, 3$, and 5 are torsion primes for the group Cr_2 . Consider a prime integer $p > 5$. According to [Fe04, Thm. E], every element $g \in \text{Cr}_2$ of order p lies in a subgroup isomorphic to $\text{Aut } \mathbf{P}^2 = \text{PGL}_3$. Hence g lies in a torus. Finally, according to [Bl06₂, Thm. B, p. 146], in Cr_2 there is a unique up to conjugacy noncyclic finite Abelian p -group (it is denoted by $0.mn$), and this group is contained in a maximal torus of a subgroup isomorphic to $(\text{Aut}(\mathbf{P}^1 \times \mathbf{P}^1))^0 = \text{PGL}_2 \times \text{PGL}_2$. Thus we conclude that the torsion primes for Cr_2 are the same as that for the exceptional simple algebraic group E_8 :

$$\text{Tors}(\text{Cr}_2) = \{2, 3, 5\}. \quad (69)$$

From $\text{Aut } \mathbf{A}^1 = \text{Aff}_1$ and [Ig77] it follows that every finite subgroup of $\text{Aut } \mathbf{A}^n$ (respectively, $\text{Aut}^* \mathbf{A}^n$) for $n \leq 2$ is contained in a subgroup isomorphic to GL_n (respectively, SL_n). Therefore,

$$\text{Tors}(\text{Aut } \mathbf{A}^n) = \text{Tors}(\text{Aut}^* \mathbf{A}^n) = \{\emptyset\} \text{ for } n \leq 2.$$

For $n \geq 3$, there is no comprehensive information about the sets $\text{Tors}(\text{Cr}_n)$, $\text{Tors}(\text{Aut } \mathbf{A}^n)$, and $\text{Tors}(\text{Aut}^* \mathbf{A}^n)$. From [Po12, Thm. 4.3], (68), and (69) it follows that

$$\text{Tors}(\text{Cr}_n) = \{2, \dots\} \text{ for every } n. \quad (70)$$

According to [Be07], in Cr_2 there is a 3-elementary Abelian subgroups of rank 3 (i.e., isomorphic to $(\mu_d)^3$). Since Cr_1 contains a cyclic subgroup of order 3, and the direct product of Cr_2 and $n - 2$ copies of Cr_1 can be embedded in Cr_n for $n \geq 3$, this implies that in Cr_n there is a 3-elementary Abelian subgroup G of rank $n + 1$. But from Lemma (2)(i) it follows that for every prime integer p , the rank of any elementary Abelian p -subgroup of an r -dimensional torus is at most r . From this and Theorem (2)(i) it follows that G is not contained in a torus of Cr_n . In view of (69) and (70), this yields

$$\text{Tors}(\text{Cr}_n) = \{2, 3, \dots\} \text{ for any } n \geq 2.$$

Question 21. *What is the minimal n such that 7 lies in one of the sets $\text{Tors}(\text{Cr}_n)$, $\text{Tors}(\text{Aut } \mathbf{A}^n)$, and $\text{Tors}(\text{Aut}^* \mathbf{A}^n)$?*

Question 22. *Are these sets finite?*

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