RATIONALITY AND THE FML INVARIANT

VLADIMIR L. POPOV*

To C. S. Seshadri on his 80*th birthday*

Abstract. We construct counterexamples to the rationality conjecture regarding the new version of the Makar-Limanov invariant introduced in [Li_2].

Let k be an algebraically closed field. Below variety means algebraic variety over k in the sense of Serre (so algebraic group means algebraic group over k). We use standard notation and conventions of [Bo] and [Sp]. In particular, given a variety X, we denote by $k[X]$ and $k(X)$ respectively the algebra of regular functions and the field of rational functions on X. Action of algebraic group on variety means algebraic action.

Recall that the *Makar-Limanov invariant* of a variety X is the subalgebra

$$\mathrm{ML}(X) := \bigcap\nolimits_H k[X]^H \tag{1}$$

of $k[X]$, where H in (1) runs over the images of all algebraic homomorphisms $\mathbf{G}_{\mathrm{a}} \to \mathrm{Aut}(X)$, see [Fr, Chap. 9]. The usefulness of the ML invariant in applications to geometric problems has been amply demonstrated over the last two decades. The highlight is its role in proving that the Koras–Russell threefold is not isomorphic to $\mathbf{C}^3$ that, in turn, is crucial in proving the Linearization Problem for $\mathbf{C}^*$-actions on $\mathbf{C}^3$. The ML invariant serves for distinguishing some affine varieties from the affine space $\mathbf{A}^n$ whose ML invariant is trivial, i.e., $\mathrm{ML}(\mathbf{A}^n) = k$. However, there are nonrational affine varieties with trivial ML invariant: such singular varieties are constructed in [Li_1, Sect. 4.2] and smooth in [Po, Example 1.22]. By [Li_2, Thm. 4.2], if $\mathrm{char}\, k = 0$ and X is an irreducible affine variety of dimension $\geqslant 2$, then

$$\mathrm{ML}(X) = k \tag{2}$$

implies that X is birationally isomorphic to $Y \times \mathbf{A}^2$ for some variety Y and, conversely, for any irreducible variety Y there is an affine variety X birationally isomorphic to $Y \times \mathbf{A}^2$ such that (2) holds. So the ML invariant does not serve for distinguishing birational types of varieties.

With a view of eliminating this "pathology", in [Li_2, Sect. 5] a generalization of the ML invariant for irreducible varieties, called the FML *invariant*, was proposed. By definition, if X is an irreducible vatiety, then

$$\mathrm{FML}(X) := \bigcap\nolimits_H \mathrm{Frac}\,(k[X]^H) \tag{3}$$

* Supported by grants РФФИ 11-01-00185-а, НШ–5139.2012.1, and the program *Contemporary Problems of Theoretical Mathematics* of the Russian Academy of Sciences, Branch of Mathematics.

where H in (3) runs over the images of all algebraic homomorphisms $\mathbf{G}_{\rm a} \to \operatorname{Aut}(X)$ and $\operatorname{Frac}(k[X]^H)$ denotes the subfield of $k(X)$ generated by $k[X]^H$. In fact, since H is unipotent,

$$\operatorname{Frac}(k[X]^H) = (\operatorname{Frac}(k[X]))^H,$$

see [PV, Thm. 3.3]. This and (3) imply that if $k(X) = \operatorname{Frac}(k[X])$, then

$$\operatorname{FML}(X) = \textstyle\bigcap_H k(X)^H; \tag{4}$$

in particular, (4) holds for affine X.

In [Li$_2$, 5.3] is put forward the following

Conjecture 1. *Let* $\operatorname{char} k = 0$ *and let* X *be an affine variety. If* $\operatorname{FML}(X) = k$*, then* X *is rational.*

According to [Li$_2$, Thm. 5.6, Lemma 5.4], the statement of Conjecture 1 holds in either of the cases:

(i) $\dim X \leqslant 3$;

(ii) X is birationally isomorphic to $C \times \mathbf{A}^n$ where C is an algebraic curve.

The aim of the present note is to show that, in general, Conjecture 1 is *false*. Our counterexamples, valid for k of any characteristic, are based on the ideas sketched in [Po, Example 1.22] for constructing nonrational affine varieties X with $\operatorname{ML}(X) = k$. These counterexamples are described in Theorem 2 below. To prove it we first need to prove some statements about stable rationality of homogeneous spaces and fields of invariants of linear actions.

Recall from [Se, Subsect. 4.1] that an algebraic group G is called *special* if for any field extension K/k, the Galois cohomology set $H^1(K, G)$ is reduced to one point. By [Se, Sect. 4], such a group is automatically connected and affine, any extension of a special group by a special group is special, and any connected solvable group is special. A semisimple group G is special if and only if it is isomorphic to a direct product

$$\mathbf{SL}_{n_1} \times \cdots \times \mathbf{SL}_{n_r} \times \mathbf{Sp}_{2m_1} \times \cdots \times \mathbf{Sp}_{2m_s} \tag{5}$$

for some integers $r, s, n_1, \ldots, n_r, m_1, \ldots, m_s$. That groups (5) are special is proved in [Se, Subsect. 4.4], that only these semisimple groups are is proved in [Gr, Thm. 3].

An action of an algebraic group G on a variety X is called *locally free* if there is a dense open subset U of X such that the G-stabilizer of every point of U is trivial. For every affine algebraic group G, there exists a locally free linear action of G on a finite dimensional vector space V over k. Indeed, we may assume that G is a closed subgroup of some $\mathbf{GL}_n$ (see [Bo, Sect. I.1.10]) and then take as V the space Mat_n of $n \times n$ matrices over k with the action of G by left multiplication.

Let $\pi\colon E \to X$ be an algebraic vector bundle over X. Assume that E and X are endowed with the actions of an affine algebraic group G such that π is G-equivariant and, for every elements $g \in G$, $x \in X$, the map $\pi^{-1}(x) \to \pi^{-1}(g \cdot x)$ determined by g is k-linear. The proof of the following useful result in arbitrary characteristic, together with a brief historical account and the relevant references can be found in [RV, Sect. 2] (see also [CS, Sect. 3.2]).

Lemma 1 (No-name Lemma)**.** *In the above notation, let $d = \dim E - \dim X$. Consider the action of G on $X \times \mathbf{A}^d$ via the first factor and let $\pi_1\colon X \times \mathbf{A}^d \to X$ be the natural projection. If the action of G on X is locally free, then there exists a G-equivariant birational isomorphism $\varphi\colon E \dashrightarrow X \times \mathbf{A}^d$ such that $\pi = \pi_1 \circ \varphi$.*

Corollary 1. *Let V_1 and V_2 be finite dimensional linear spaces over k endowed with locally free linear actions of an affine algebraic group G. Then*

(i) *V_1 and V_2 are birationally stably G-isomorphic*;

(ii) *the invariant fields $k(V_1)^G$ and $k(V_2)^G$ are stably k-isomorphic.*

Proof. Lemma 1 applied to the projections $V_1 \leftarrow V_1 \times V_2 \to V_2$ yields that both $V_1 \times \mathbf{A}^{\dim V_2}$ and $V_2 \times \mathbf{A}^{\dim V_1}$ endowed with the natural G-actions via the first factors are birationally G-isomorphic to $V_1 \times V_2$. This proves (i) and hence (ii). □

For the proof of Theorem 1 below we need, apart from Lemma 1, also the following fact:

Lemma 2. *The underlying variety of any connected affine algebraic group G is rational.*

As we do not know a reference containing a proof in arbitrary characteristic (in characteristic zero it is [Ch, Cor. 2]), here is a short proof.

Proof of Lemma 2. Let B be a Borel subgroup of G. Since B is connected solvable, the underlying variety of B is rational by [Gr, Cor. of Prop. 2] and that of G is birationally isomorphic to $B \times (G/B)$ by [Bo, Cor. 15.8]. Since B is parabolic, G/B is rational by [Bo, Thm. 21.20(ii)]; whence the claim. □

Theorem 1. *The following properties of an affine algebraic group G are equivalent:*

(i) *there is a finite dimensional vector space V over k endowed with a locally free linear action of G such that $k(V)^G$ is stably rational over k*;

(ii) *there are an integer $n > 0$ and a closed embedding $G \hookrightarrow \mathbf{GL}_n$ such that the variety $\mathbf{GL}_n/G$ is stably rational*;

(iii) *there are a special algebraic group H and a closed embedding $G \hookrightarrow H$ such that the variety H/G is stably rational.*

If these properties hold, then

(a) *for every finite dimensional vector space U over k endowed with a locally free linear action of G, the field $k(U)^G$ is stably rational over k*;

(b) *for every closed embeddings $G \hookrightarrow S \hookrightarrow H$ where H is a connected affine algebraic group and S is a special algebraic group, the variety H/G is stably rational.*

Proof. First, (i) $\Rightarrow$ (a) by Corollary 1.

Further, assume that (i) holds. Since the action of G on V is faithful, G may be identified with a closed subgroup of $\mathrm{GL}(V)$. In turn, fixing a basis in V, we may identify $\mathrm{GL}(V)$ with $\mathbf{GL}_n$, $n = \dim V$. Consider then the linear action of G on Mat_n by left multiplication. Since it is locally free,

(a) implies that $k(\mathrm{Mat}_n)^G$ is stably rational over k. But $\mathbf{GL}_n$ is a G-stable open subset of Mat_n, hence $k(\mathrm{Mat}_n)^G$ is k-isomorphic to $k(\mathbf{GL}_n)^G$. Since the latter is k-isomorphic to $k(\mathbf{GL}_n/G)$ (see [Bo, Sect. II.6]), we conclude that $\mathbf{GL}_n/G$ is stably rational. This proves (i) $\Rightarrow$ (ii).

Assume that (ii) holds. Then we may consider the action of G on Mat_n by left multiplication. It is locally free and linear. The same argument as above shows that $k(\mathrm{Mat}_n)^G$ is k-isomorphic to $k(\mathbf{GL}_n/G)$ and hence is stably rational over k. This proves (ii) $\Rightarrow$ (i).

Since $\mathbf{GL}_n$ is special, we have (ii) $\Rightarrow$ (iii).

Assume that (iii) holds. Then H is connected and affine. The latter implies that we may assume that H is a closed subgroup of some $\mathbf{GL}_n$. Consider the action of H on $\mathbf{GL}_n$ by left multiplication. Since H is special, by [Se, Thm. 2] there is an H-equivariant birational isomorphisms between the varieties $\mathbf{GL}_n$ and $H \times (\mathbf{GL}_n/H)$ where H acts on the latter via left multiplication of the first factor. Since by Lemma 2 the varieties $\mathbf{GL}_n$ and H are rational, this shows that $\mathbf{GL}_n/H$ is stably rational. On the other hand, this also shows that $\mathbf{GL}_n/G$ is birationally isomorphic to $(H/G) \times (\mathbf{GL}_n/H)$. Since both H/G and $\mathbf{GL}_n/H$ are stably rational, we then conclude that $\mathbf{GL}_n/G$ is stably rational as well. This proves (iii) $\Rightarrow$ (ii).

Assume now that (i)–(iii) and hence (a) hold for G. Let $G \hookrightarrow S \hookrightarrow H$ be closed embeddings such that H is a connected affine algebraic group and S is a special algebraic group. We may consider S as a closed subgroup of some $\mathbf{GL}_m$. Arguing as in the proof of (iii) $\Rightarrow$ (ii), we then infer that $\mathbf{GL}_m/S$ is stably rational and $\mathbf{GL}_m/G$ is birationally isomorphic to $(S/G) \times (\mathbf{GL}_m/S)$. This implies that stable rationality of S/G is equivalent to that of $\mathbf{GL}_m/G$. To prove the latter, consider the action of G on Mat_m by left multiplication. It is locally free, so by (a) the field $k(\mathrm{Mat}_m)^G$ is stably rational over k. As is explained above, this implies stable rationality of $\mathbf{GL}_m/G$. Thus, S/G is stably rational. On the other hand, applying again the same argument as in the proof of (iii) $\Rightarrow$ (ii), we infer that H/S is stably rational and H/G is birationally isomorphic to $H/S \times S/G$. Since both factors are stably rational, we then conclude that H/G is stably rational as well. This proves (b). □

Corollary 2. *If G is a special algebraic group, then*

(i) *for every finite dimensional vector space U over k endowed with a locally free linear action of G, the field $k(U)^G$ is stably rational over k;*
(ii) *for every connected affine algebraic group H and every closed embedding $G \hookrightarrow H$, the variety H/G is stably rational.*

Theorem 2. *Let $d := p^9$ where p is a prime integer, $p \neq \operatorname{char} k$. Let G be a group of type* (5) *such that either $n_i > d$ for some i or $m_j \geqslant d$ for some j. Then G contains a finite subgroup F of order d such that the variety $X := G/F$ has the following properties*:

(i) *X is affine and smooth*;
(ii) *X is not stably rational*;
(iii) $\mathrm{FML}(X) = k$.

Proof. Let V be a d-dimensional vector space over k. By [Sa, Thm. 3.6], there is a finite subgroup F of $\mathrm{GL}(V)$ such that for the natural action of F on V the field $k(V)^F$ is not stably rational. Since F is finite, this action is locally free. Fixing a basis in V, we may identify $\mathrm{GL}(V)$ with $\mathbf{GL}_d$. Since there are embeddings of $\mathbf{GL}_d$ in $\mathbf{SL}_n$ for any $n > d$ and in $\mathbf{Sp}_{2m}$ for any $m \geqslant d$, the assumptions on n_i's and m_j's in the formulation of Theorem 2 imply that there is an embedding of F in G. As G is special, we then deduce from Theorem 1 that (ii) holds.

Consider the natural action of G on X. By (4), if H is a one-dimensional unipotent subgroup of G, then $\mathrm{FML}(X) \subseteq k(X)^H$. But semisimplicity of G implies that it is generated by one-dimensional unipotent subgroups, see [Sp, Thm. 8.1.5(i)] and [Po, Lemma 1.1]. From this we infer that $\mathrm{FML}(X) \subseteq k(X)^G$. On the other hand, since G acts on X transitively, $k(X)^G = k$. This proves (iii).

As F is reductive, X is affine by Matsushima's criterion [Ri, Thm. A]. Since G acts on X transitively, X is smooth. This proves (i). □

Remark. Further development of the approach of [Sa] has led, at least in zero characteristic, to finding finite linear groups K of order $< p^9$ whose fields of invariants are not stably rational over k, see [Sh, Sect. 4°], [CS, Sect. 7] and references therein. The proof of Theorem 2 is applicable to them without any change and gives an embedding of K in a group H of type (5) such that the variety $X := H/K$ has properties (i), (ii), and (iii).

References

[Bo] A. Borel, *Linear Algebraic Groups*, 2nd ed., Springer-Verlag, New York, 1991.

[Ch] C. Chevalley, *On algebraic group varieties*, J. Math. Soc. Japan **6** (1954), nos. 3–4, 303–324.

[CS] J.-L. Colliot-Thélène, J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, in: *Algebraic Groups and Homogeneous Spaces*, Mumbai 2004, Narosa, 2007, pp. 113–186.

[Fr] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences, Vol. 136, Subseries *Invariant Theory and Algebraic Transformation Groups*, Vol. VII, Springer, Berlin, 2006.

[Gr] A. Grothendieck, *Torsion homologique et sections rationnelles*, in *Anneaux de Chow et Applications*, Séminaire C. Chevalley, 1958, exposé 5.

[Li_1] A. Liendo, *Affine $\mathbb{T}$-varieties of complexity one and locally nilpotent derivations*, Transform. Groups **15** (2010), no. 2, 389–425.

[Li_2] A. Liendo, *$\mathbf{G}_{\mathrm{a}}$-actions of fiber type on affine $\mathbf{T}$-varieties*, J. Algebra **324** (2010), 3653–3665.

[Po] V. L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, in: *Affine Algebraic Geometry: The Russell Festschrift*, CRM Proceedings and Lecture Notes, Vol. 54, Amer. Math. Soc., 2011, pp. 289–311.

[PV] V. L. Popov, E. B. Vinberg, *Invariant theory*, in: *Algebraic Geometry* IV, Encyclopaedia of Mathematical Sciences, Vol. 55, Springer-Verlag, Berlin, 1994, pp. 123–284.

[RV] Z. Reichstein, A. Vistoli, *Birational isomorphisms between twisted group actions*, J. Lie Theory **16** (2006), no. 4, 791–802.

[Ri] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), 38–41.

[Sa] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), 71–84.

[Se] J.-P. Serre, *Espaces fibrés algébriques*, in: *Anneaux de Chow et Applications*, Séminaire C. Chevalley, 1958, exposé 1.

[Sh] I. R. Shafarevich, *On Lüroth's problem*, Proc. Steklov Inst. Math. **183** (1991), 241–246.

[Sp] T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Progress in Mathematics, Vol. 9, Birkhäuser, Boston, 1998.

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow 119991, Russia

Higher School of Economics, Myasnitskaya 20, Moscow 101000, Russia

E-mail address: `popovvl@mi.ras.ru`