DECOMPOSABLE AND INDECOMPOSABLE ALGEBRAS OF DEGREE 8 AND EXPONENT 2

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Abstract. We study the decomposition of central simple algebras of exponent 2 into tensor products of quaternion algebras. We consider in particular decompositions in which one of the quaternion algebras contains a given quadratic extension. Let B be a biquaternion algebra over $F(\sqrt{a})$ with trivial corestriction. A degree 3 cohomological invariant is defined and we show that it determines whether B has a descent to F. This invariant is used to give examples of indecomposable algebras of degree 8 and exponent 2 over a field of 2-cohomological dimension 3 and over a field $\mathbb{M}(t)$ where the u-invariant of $\mathbb{M}$ is 8 and t is an indeterminate. The construction of these indecomposable algebras uses Chow group computations provided by A. S. Merkurjev in Appendix.

1. Introduction

Let A be a central simple algebra over a field F. We say that A is *decomposable* if $A \simeq A_1 \otimes_F A_2$ for two central simple F-algebras A_1 and A_2 both non isomorphic to F; otherwise A is called *indecomposable*. Let $K = F(\sqrt{a})$ be a quadratic separable extension of F. We say that A admits a *decomposition adapted* to K if K is in a quaternion subalgebra of A, that is, $A \simeq (a, a') \otimes_F A'$ for some $a' \in F$ and some subalgebra $A' \subset A$. If A is isomorphic to a tensor product of quaternion algebras, we will say that A is *totally decomposable*. Let B be a central simple algebra over K. The algebra B has a *descent* to F if there exists an F-algebra B' such that $B \simeq B' \otimes_F K$. It is clear that the algebra A admits a decomposition adapted to K if and only if the centralizer $C_A K$ of K in A has a descent to F.

The first example of a non-trivial indecomposable central simple algebra of exponent 2 was given by Amitsur-Rowen-Tignol. More precisely, an explicit central division algebra of degree 8 and exponent 2 which has no quaternion subalgebra is constructed in [2]. Other examples of such algebras were given by Karpenko (see [11] and [12]) by computing torsion in Chow groups. In fact, 8 is the smallest possible degree for such an algebra by a well-known theorem of Albert which asserts that every algebra of exponent 2 and degree 4 is decomposable.

The decomposability question depends on the cohomological dimension of the ground field. Indeed, for obvious reasons there is no indecomposable algebra of exponent 2 over a field of cohomological dimension 0 or 1 (since the Brauer group is trivial in these cases). It follows from a result of Merkurjev (Theorem 3.1) that over a field of cohomological dimension 2 any central simple algebra of exponent

2 is isomorphic to a tensor product of quaternion algebras. On the other hand, the known examples of indecomposable algebras of exponent 2 are constructed over fields of cohomological dimension greater than or equal to 5.

The main goal of this article is to extend the existence of indecomposable algebras of exponent 2 to some fields of cohomological dimension smaller than or equal to 4. We also give an example over a field of rational functions in one variable over a field of u-invariant 8. The problem will be addressed through the study of the decomposability adapted to a quadratic extension of the ground field. More precisely, let $K = F(\sqrt{a}) \subset A$ be a quadratic extension field. We first prove (Section 3) that if $cd_2(F) \leq 2$ then K lies in a quaternion subalgebra of A. If $\deg A = 8$, we define a degree 3 cohomological invariant, depending only on the centralizer $C_A K$ of K in A, which determines whether A admits a decomposition adapted to K, that is, whether $C_A K$ has a descent to F (see Section 4). This invariant is used to give examples of indecomposable algebras of exponent 2 (Theorem 1.2 and Theorem 1.3). Although our invariant depends only on $C_K A$, it is nothing but a refinement of the invariant $\Delta(A)$ defined by Garibaldi-Parimala-Tignol [8, §11] (see Remark 4.3). Through Remark 4.10 and the proof of Theorem 1.2, our invariant provides an example of indecomposable algebra A of degree 8 and exponent 2 such that $\Delta(A)$ is nonzero, this is an answer to Garibaldi-Parimala-Tignol's question [8, Question 11.2].

For the proofs of Theorem 1.2 and Theorem 1.3 we need some results — injections (5.1) and (5.2) — on Chow groups of cycles of codimension 2. These results follow by Theorem A.9 and Theorem A.7 provided by A. S. Merkurjev in Appendix.

Statement of main results. Standard examples of fields of cohomological dimension 3 are $k(t_1, t_2, t_3)$ or $k((t_1))((t_2))((t_3))$ where k is an algebraically closed field and t_1, t_2, t_3 are independent indeterminates over k. Recall also that the u-invariant of such fields is 8 by Tsen-Lang (see for instance [9, Theorem 1]). The following theorem shows that there is no indecomposable algebra of degree 8 and exponent 2 over these standard fields. We recall that the *u-invariant* of a field F is defined as

$$u(F) = \max\{\dim \varphi \,|\, \varphi \text{ anisotropic form over } F\}.$$

If no such maximum exists, $u(F)$ is defined to be ∞.

Theorem 1.1. *There exists no indecomposable algebra of degree* 8 *and exponent* 2 *over a field of u-invariant smaller than or equal to* 8.

On the other hand, examples of indecomposable algebras do exist over a function field of one variable over a suitable field of u-invariant 8:

Theorem 1.2. *Let A be a central simple algebra of degree* 8 *and exponent* 2 *over a field F and let $K = F(\sqrt{a})$ be a quadratic field extension of F contained in A. If K is not in a quaternion subalgebra of A then there exists an extension $\mathbb{M}$ of F with $u(\mathbb{M}) = 8$ such that the division algebra Brauer equivalent to*

$$A_{\mathbb{M}} \otimes_{\mathbb{M}} (a, t)_{\mathbb{M}(t)}$$

is an indecomposable algebra of degree 8 *and exponent* 2 *over $\mathbb{M}(t)$, where t is an indeterminate.*

To produce explicit examples, one may take for A any indecomposable algebra, as those constructed in [2] or [12]. Alternately, we give in [3] an example of a decomposable algebra satisfying the hypothesis of Theorem 1.2.

The following theorem shows that there exists an indecomposable algebra of degree 8 and exponent 2 over some field of 2-cohomological dimension 3. Let us recall that $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}}$ is the torsion in the Chow group of cycles of codimension 2 over the Severi-Brauer variety $\mathrm{SB}(A)$ of A modulo rational equivalence. If A is of prime exponent p and index p^n (except the case $p = 2 = n$) Karpenko showed in [12, Proposition 5.3] that if $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \neq 0$ then A is indecomposable. Examples of such indecomposable algebras are given in [12, Corollary 5.4].

Theorem 1.3. *Let A be a central simple algebra of degree 8 and exponent 2 such that $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \neq 0$. Then there exists an extension $\mathbb{M}$ of F with $cd_2(\mathbb{M}) = 3$ such that $A_{\mathbb{M}}$ is indecomposable.*

2. Notations and preliminaries

Throughout this paper the characteristic of the base field F is assumed to be different from 2 and all algebras are associative and finite-dimensional over F. The main tools in this paper are central simple algebras, quadratic forms and Galois cohomology. Pierce's book [21] is a reference for the general theory of central simple algebras, references for quadratic form theory over fields are [5], [14] and [22].

Let F_s be a separable closure of F. For any integer $n \geq 0$, $H^n(F, \mu_2)$ denotes the Galois cohomology group

$$H^n(F, \mu_2) := H^n(\mathrm{Gal}(F_s/F), \mu_2)$$

where $\mu_2 = \{\pm 1\}$. The group $H^1(F, \mu_2)$ is identified with $F^\times/F^{\times 2}$ by Kummer theory. For any $a \in F^\times$, we write (a) for the element of $H^1(F, \mu_2)$ corresponding to $aF^{\times 2}$. The group $H^2(F, \mu_2)$ is identified with the 2-torsion $\mathrm{Br}_2(F)$ in the Brauer group $\mathrm{Br}(F)$ of F, and we write $[A]$ for the element of $H^2(F, \mu_2)$ corresponding to a central simple algebra A of exponent 2. For more details on Galois cohomology the reader can consult Serre's book [24].

For any quadratic form q, we denote by $C(q)$ and $d(q)$ the Clifford algebra and the discriminant of q. If q has dimension $2m + 2$, it is easy to see that $C(q)$ is a tensor product of $m + 1$ quaternion algebras. Conversely, any tensor product of quaternion algebras is a Clifford algebra. Let I^nF be the n-th power of the fundamental ideal IF of the Witt ring WF. Abusing notations, we write $q \in I^nF$ if the Witt class of q lies in I^nF. We shall use frequently the following property: if $q \in I^2F$ (i.e, $\dim q$ is even and its discriminant is trivial) the Clifford algebra of q has the form $C(q) \simeq M_2(E(q))$ for some central simple algebra $E(q)$ which is totally decomposable.

Now, let E/F be an extension of F and let q be a quadratic form defined over F. We say that E/F is *excellent* for q if the anisotropic part $(q_E)_{\mathrm{an}}$ of q_E is defined over F. If E/F is excellent for every quadratic form defined over F, the extension E/F is called *excellent*.

Generally, the biquadratic extensions are not excellent (see for instance [6, §5]) but we have the following result:

Lemma 2.1. *Assume that $I^3F = 0$ and let $L = F(\sqrt{a_1}, \sqrt{a_2})$ be a biquadratic extension of F. Then, L/F is excellent for the quadratic forms $q \in I^2F$. More precisely, for all $q \in I^2F$ there exists $q_0 \in I^2F$ such that $(q_L)_{an} \simeq (q_0)_L$.*

Proof. Let $s : K = F(\sqrt{a_1}) \to F$ be the F-linear map such that $s(1) = 0$ and $s(\sqrt{a_1}) = 1$. The corresponding Scharlau's transfer will be denoted by s_*. Notice that $I^3K = 0$ because of $I^3F = 0$ and the exactness of the sequence (see [5, Theorem 40.3])

$$\langle 1, -a_1\rangle I^2F \longrightarrow I^3F \longrightarrow I^3K \xrightarrow{s_*} I^3F.$$

Let $q \in I^2F$ be an anisotropic form such that q_L is isotropic. We first show that there exists a quadratic form q_0 defined over F such that $(q_0)_L \simeq (q_L)_{\mathrm{an}}$. If q_L is hyperbolic, there is nothing to show. We suppose q_L is not hyperbolic. It is well-known (see e.g. [14, Theorem 3.1, p.197]) that $(q_K)_{\mathrm{an}}$ is defined over F. If $(q_K)_{\mathrm{an}}$ remains anisotropic over L, we take $q_0 = (q_K)_{\mathrm{an}}$. Otherwise, one has

$$(q_K)_{\mathrm{an}} = \langle 1, -a_2\rangle \otimes \langle \gamma_1, \dots, \gamma_r\rangle \perp q'$$

for some $\gamma_1, \dots, \gamma_r \in K$ and some subform q' of $(q_K)_{\mathrm{an}}$ defined over K with q'_L anisotropic (see for instance [10, Proposition 3.2.1]). On the other hand, the form $(q_K)_{\mathrm{an}}$ being in I^2K and $I^3K = 0$, one has $(q_K)_{\mathrm{an}} \otimes \langle 1, -\gamma_1\rangle = 0$, that is, $(q_K)_{\mathrm{an}} \simeq \gamma_1(q_K)_{\mathrm{an}}$. Thus, we may suppose $\gamma_1 = 1$.

We claim that the dimension of the form $\langle 1, \gamma_2 \dots, \gamma_r\rangle$ is 1. Indeed, assume that $r \geq 2$ and write $(q_K)_{\mathrm{an}} = \langle 1, -a_2\rangle \otimes \langle 1, \gamma_2\rangle \perp \varphi$ for some φ over K. Since φ_L is nonzero, it is clear that φ is nonzero. Let $\alpha \in K^\times$ be represented by φ. The form $\langle 1, -a_2\rangle \otimes \langle 1, \gamma_2\rangle \otimes \langle 1, \alpha\rangle$ is hyperbolic since $I^3K = 0$, hence $\langle 1, -a_2\rangle \otimes \langle 1, \gamma_2\rangle$ represents $-\alpha$. It follows that $(q_K)_{\mathrm{an}}$ is isotropic, a contradiction. Hence $r = 1$ and so $(q_K)_{\mathrm{an}} \simeq \langle 1, -a_2\rangle \perp q'$.

Now, we are going to show that $q'_L \simeq (q_0)_L$ for some q_0 defined over F. The Scharlau transfers $s_*((q_K)_{\mathrm{an}})$ and $s_*(\langle 1, -a_2\rangle)$ are both hyperbolic because $(q_K)_{\mathrm{an}}$ and $\langle 1, -a_2\rangle$ are defined over F. This implies that $s_*(q')$ is hyperbolic, so q' is Witt equivalent to $(q_0)_K$ for some q_0 defined over F. We deduce from the excellence of K/L that $q' \simeq (q_0)_K$. Whence $(q_L)_{\mathrm{an}} \simeq (q_0)_L$.

It remains to prove that q_0 may be chosen to be in I^2F. Since $(q_0)_L \in I^2L$, the discriminant $d(q_0) \in \{F^{\times 2}, a_1.F^{\times 2}, a_2.F^{\times 2}, a_1a_2.F^{\times 2}\}$. If $d(q_0) = 1$, we are done. Otherwise, assume for example $d(q_0) = a_1$ and write $q_0 \simeq \langle c_1, \dots, c_m\rangle$. The quadratic form $q_1 \simeq \langle a_1c_1, c_2, \dots, c_m\rangle$ is such that $(q_1)_L \simeq (q_0)_L$ and $d(q_1) = 1$; that is $q_1 \in I^2F$. It suffices to replace q_0 by q_1. This concludes the proof. □

3. Adapted decomposition under $cd_2(F) \leq 2$

In this section we assume that the 2-cohomological dimension $cd_2(F) \leq 2$. If the characteristic of F is different from 2, Merkurjev proved that division algebras of exponent 2 over F are totally decomposable. We use this observation to show that

any quadratic field extension of F in a central simple algebra over F of exponent 2 lies in a quaternion subalgebra. We also prove a related result for a biquadratic extension of F. First, let us recall the following result:

Theorem 3.1 (Merkurjev)**.** *Assume that $cd_2(F) \leq 2$. Then*

(1) *Any central division algebra over F whose class is in $\mathrm{Br}_2(F)$ is totally decomposable.*
(2) *A quadratic form $q \in I^2F$ is anisotropic if and only if $E(q)$ is a division algebra.*

Proof. See [9, Theorem 3]. □

Let A be a 2-power dimensional central simple algebra over F and let $K \subset A$ be a quadratic extension of F. If A is not a division algebra, K is in any split quaternion algebra. We have the following general result in the division case:

Proposition 3.2. *Assume that $cd_2(F) \leq 2$. Let A be a central division algebra of exponent 2 over F. Every square-central element of A lies in a quaternion subalgebra.*

Proof. Assume that $\deg A = 2^m$. By Theorem 3.1, the algebra A decomposes into a tensor product of m quaternion algebras. We may associate with $M_2(A)$ a $(2m+2)$-dimensional quadratic form $q \in I^2F$ in such a way that $C(q) \simeq M_2(A)$. This form q is anisotropic by Theorem 3.1 (2). The form q being in I^2F, the center of the Clifford algebra $C_0(q)$ is $F \times F$ and $C_0(q) \simeq C_+(q) \times C_-(q)$ with $A \simeq C_+(q) \simeq C_-(q)$. Let $x \in A - F$ be such that $x^2 = a \in F^\times$. The element $z = x^{-1}\sqrt{a}$, which is different from 1 and -1, is such that $z^2 = 1$; hence $A_{F(\sqrt{a})}$ is not a division algebra. Then Theorem 3.1 and [14, Theorem 3.1, p.197] indicate that the form q has a subform $\alpha\langle 1, -a\rangle$ for some $\alpha \in F^\times$. Write $q \simeq \alpha\langle 1, -a\rangle \perp q_1$ and let V_q be the underlying space of q. Let $e_1, \ldots, e_{2m+2}$ be an orthogonal basis of V_q which corresponds to this diagonalization. One has:

$$\begin{cases} e_1e_2, e_1e_3 \in C_0(q) \\ (e_1e_2)^2 = \alpha^2 a,\ (e_1e_3)^2 = -\alpha^2 q(e_3) \\ (e_1e_2)(e_1e_3) = -(e_1e_3)(e_1e_2). \end{cases}$$

Let $z = e_1 \ldots e_{2m+2}$. Since $q \in I^2F$, we have $z^2 \in F^{\times 2}$. Let $z^2 = \lambda^2$ with $\lambda \in F^\times$ and set $z_+ = \frac{1}{2}(1+\lambda^{-1}z)$ and $z_- = \frac{1}{2}(1-\lambda^{-1}z)$. So, z_+, z_- are the primitive central idempotents in $C_0(q)$. The elements $(e_1e_2)z_+$ and $(e_1e_3)z_+$ generate a quaternion subalgebra isomorphic to (a, a') in $C_+(q) \simeq A$ where $a' = (e_1e_3)^2$. Hence, A contains some element y such that $y^2 = a$ and y lies in a quaternion subalgebra of A. By the Skolem-Noether Theorem x and y are conjugated. Therefore, x is in a quaternion subalgebra. □

Recall that if $cd_2(F) \leq 2$ then $I^3F = 0$ ([15, Theorem A5']). This fact allows to use Lemma 2.1 in the proof of the following results.

Theorem 3.3. *Assume that $cd_2(F) \leq 2$. Let A be a central division algebra of degree 2^n and exponent 2 over F. Let $L = F(\sqrt{a_1}, \sqrt{a_2})$ be a biquadratic extension of F*

contained in A. There exist quaternion algebras Q_1, Q_2 and a subalgebra $A' \subset A$ over F such that $F(\sqrt{a_i}) \subset Q_i$ $(i = 1, 2)$ and $A \simeq Q_1 \otimes Q_2 \otimes A'$.

Proof. Let $q \in I^2F$ be an anisotropic quadratic form such that $C(q) \simeq M_2(A)$ (Theorem 3.1). Notice that the index of A_L is 2^{n-2}. Since $cd_2(L) = 2$, it follows by [25, Lemma 8] that

$$\dim(q_L)_{\mathrm{an}} = 2\log_2 \mathrm{ind}(A_L) + 2 = 2n - 2.$$

By Lemma 2.1, there is $q' \in I^2F$ such that $(q_L)_{\mathrm{an}} \simeq q'_L$. Put $\psi = q - q'$. The form ψ_L being hyperbolic it follows by [14, Theorem 4.3, p.444] that there exist quadratic forms φ_1, φ_2 such that

$$\psi = \langle 1, -a_1\rangle \otimes \varphi_1 \perp \langle 1, -a_2\rangle \otimes \varphi_2 \ \text{ in } \ WF.$$

Since $\psi \in I^2F$, we must have $\varphi_1, \varphi_2 \in IF$. Taking the Witt-Clifford invariant of each side of the identity above, we obtain

$$A \otimes A' \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2))$$

where A' is such that $M_2(A') = C(q')$. This yields that

$$A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes A'.$$

□

If $\deg A = 8$, the above theorem holds without the division condition on A as shows the following:

Proposition 3.4. *Assume that $cd_2(F) \leq 2$. Let A be a central simple algebra of degree 8 and exponent 2 over F. Let $L = F(\sqrt{a_1}, \sqrt{a_2})$ be a biquadratic extension of F contained in A. There exist quaternion algebras Q_1, Q_2 and Q over F such that $F(\sqrt{a_i}) \subset Q_i$ $(i = 1, 2)$ and $A \simeq Q_1 \otimes Q_2 \otimes Q$.*

Proof. We are going to argue on the index $\mathrm{ind}(A)$ of A: if $\mathrm{ind}(A) \leq 2$, the statement is clear since L lies in any split algebra of degree 4. If $\mathrm{ind}(A) = 8$, the result follows from Theorem 3.3.

Suppose $\mathrm{ind}(A) = 4$. Let $q \in I^2F$ be an anisotropic quadratic form such that $C(q) \sim M_2(A)$. Such a form exists by Merkurjev's Theorem. Since A_L is split or of index 2, the form q_L is either hyperbolic or equivalent to a multiple of the anisotropic norm form, n_Q, of some quaternion algebra Q defined over L. Notice that every multiple of n_Q is isometric to n_Q. We may consider n_Q as being defined over F because of Lemma 2.1; so Q is defined over F. Moreover, $(q - n_Q)_L$ is hyperbolic. Put $\psi = q$ if $\mathrm{ind}(A_L) = 1$ and $\psi = q - n_Q$ if $\mathrm{ind}(A_L) = 2$; the form ψ_L is hyperbolic. As in the proof of Theorem 3.3 there exist quadratic forms $\varphi_1, \varphi_2 \in IF$ such that

$$A \otimes Q \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \ \text{ if } \ \mathrm{ind}(A_L) = 2,$$

and

$$A \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \ \text{ if } \ \mathrm{ind}(A_L) = 1.$$

This yields that

$$A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes M_2(F) \quad \text{or} \quad A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes Q.$$

□

Remark 3.5. For central simple algebras of degree 8 and exponent 2 over F with $cd_2(F) = 2$, Proposition 4.9 cannot be generalized to the triquadratic extensions of F. For example, let $F = \mathbb{C}(x, y)$ where $\mathbb{C}$ is the field of complex numbers and x, y are independent indeterminates over $\mathbb{C}$. Notice that $cd_2(F) = 2$. Since all 5-dimensional quadratic forms over F are isotropic by Tsen-Lang (see for instance [14, p.376]), Theorem 3.1 yields that index equals 2 for every division algebra of exponent 2 over F. Let $a_1 = x^2(1 + y)^2 - 4xy$, $a_2 = 1 - x$, $a_3 = y$ and $M = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$. It is shown in [6, (5.8)] that there exists an algebra A of exponent 2 and degree 8 over F containing M such that A admits no decomposition of the form $(a_1, a_1') \otimes (a_2, a_2') \otimes (a_3, a_3')$ with $a_1', a_2', a_3' \in F$.

4. Degree 3 invariant

In this section, we assume no condition on the 2-cohomological dimension of F. Let $K = F(\sqrt{a})$ be a quadratic field extension of F. Throughout this section B is a biquaternion algebra over K such that $\mathrm{cor}_{K/F}(B)$ is trivial. In fact, this condition on $\mathrm{cor}_{K/F}(B)$ means that B has a descent to F up to Brauer equivalence because of the exactness of the sequence (see for instance [1, (4.6)])

$$\mathrm{Br}_2(F) \xrightarrow{r_{K/F}} \mathrm{Br}_2(K) \xrightarrow{\mathrm{cor}_{K/F}} \mathrm{Br}_2(F).$$

We are going to attach to B a cohomological invariant which determines whether B has a descent to F. This descent problem is another way of studying the question of adapted decomposition. Indeed, let A be a central simple algebra over F of degree greater than or equal to 8 whose restriction is B. As explained in the introduction, B has a descent to F if and only if A admits a decomposition adapted to K.

Let φ be an Albert form of B, that is, φ is a 6-dimensional form in I^2K such that $C_0(\varphi) \simeq C_+(\varphi) \times C_-(\varphi)$ with $C_+(\varphi) \simeq C_-(\varphi) \simeq B$. Recall that an Albert form is unique up to a scalar. We have the following lemma where e_3 denotes the Arason invariant $I^3F \to H^3(F, \mu_2)$:

Lemma 4.1. *We keep the above notations. One has the following statements:*

(1) *Scharlau's transfer $s_*(\varphi)$ lies in I^3F.*
(2) *For any $\lambda \in K^\times$, we have $e_3(s_*(\varphi)) = e_3(s_*(\langle\lambda\rangle \otimes \varphi)) + \mathrm{cor}_{K/F}((\lambda) \cdot [B])$.*
(3) *For $\lambda \in K^\times$, the following are equivalent:*
   (i) $s_*(\langle\lambda\rangle \otimes \varphi) = 0$.
   (ii) $e_3(s_*(\varphi)) = \mathrm{cor}_{K/F}((\lambda) \cdot [B])$.

Proof. (1) Since the diagram

$$\begin{tikzcd} I^2K \arrow[r, "s_*"] \arrow[d, "e_2"'] & I^2F \arrow[d, "e_2"] \\ \mathrm{Br}_2(K) \arrow[r, "\mathrm{cor}_{K/F}"] & \mathrm{Br}_2(F) \end{tikzcd}$$

commutes (see [1, (4.18)]) and $\mathrm{cor}_{K/F}(B) = 0$, we have $e_2(s_*(\varphi)) = 0$. Hence,

$$s_*(\varphi) \in \mathrm{Ker}(e_2 : I^2F \to \mathrm{Br}_2(F)) = I^3F.$$

(2) From the identity

$$s_*(\varphi) = s_*(\langle\lambda\rangle \otimes \varphi + \langle 1, -\lambda\rangle \otimes \varphi) = s_*(\langle\lambda\rangle \otimes \varphi) + s_*(\langle 1, -\lambda\rangle \otimes \varphi)$$

we have

$$e_3(s_*(\varphi)) = e_3(s_*(\langle\lambda\rangle \otimes \varphi)) + e_3(s_*(\langle 1, -\lambda\rangle \otimes \varphi)). \tag{4.1}$$

On the other hand, since the diagram

$$\begin{tikzcd} I^3K \arrow[r, "s_*"] \arrow[d, "e_3"'] & I^3F \arrow[d, "e_3"] \\ H^3(K, \mu_2) \arrow[r, "\mathrm{cor}_{K/F}"] & H^3(F, \mu_2) \end{tikzcd}$$

commutes ([1, (5.7)]), one has

$$\begin{aligned} e_3(s_*(\langle 1, -\lambda\rangle \otimes \varphi)) &= \mathrm{cor}_{K/F}(e_3(\langle 1, -\lambda\rangle \otimes \varphi)) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot e_2(\varphi)) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot [C_+(\varphi)]) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot [B]). \end{aligned}$$

Therefore the relation (4.1) becomes

$$e_3(s_*(\varphi)) = e_3(s_*(\langle\lambda\rangle \otimes \varphi)) + \mathrm{cor}_{K/F}((\lambda) \cdot [B])$$

as desired.
(3) This point follows immediately from (2) because $s_*(\langle\lambda\rangle \otimes \varphi) = 0$ if and only if $e_3(s_*(\langle\lambda\rangle \otimes \varphi) = 0$ by the Arason-Pfister Hauptsatz (see for instance [5, Chap. 4]) □

It follows from Lemma 4.1 (2) that $e_3(s_*(\varphi))$ modulo $\mathrm{cor}_{K/F}((K^\times) \cdot [B])$ does not depend on the choice of the Albert form φ. Therefore we may define an invariant:

Definition 4.2. Let B be a biquaternion algebra over K such that $\mathrm{cor}_{K/F}(B) = 0$. The invariant

$$\delta_{K/F}(B) \in \frac{H^3(F, \mu_2)}{\mathrm{cor}_{K/F}((K^\times) \cdot [B])}$$

is the class of $e_3(s_*(\varphi))$.

Remark 4.3. If A is a central simple F-algebra of degree 8 containing K, and if $B = C_A K$, then $\delta_{K/F}(B)$ is the image in $H^3(F,\mu_2)/\operatorname{cor}_{K/F}((K^\times)\cdot[B])$ of the discriminant $\Delta(A,\sigma) \in H^3(F,\mu_2)$ of any symplectic involution σ on A which leaves K elementwise fixed. This follows by comparing the definition of $\delta_{K/F}(B)$ with Garibaldi-Parimala-Tignol [8, Proposition 8.1]. Therefore, the image of $\delta_{K/F}(B)$ in $H^3(F,\mu_2)/(F^\times)\cdot[A]$ is the invariant $\Delta(A)$ defined in [8, §11] (Note that $[B] = \operatorname{res}_{K/F}[A]$, so by the projection formula $\operatorname{cor}_{K/F}((K^\times)\cdot[B]) = (N_{K/F}(K^\times))\cdot[A] \subset (F^\times)\cdot[A]$).

The main property of the invariant is the following result:

Proposition 4.4. *The algebra B has a descent to F if and only if $\delta_{K/F}(B) = 0$.*

Proof. It follows from the definition that $\delta_{K/F}(B) = 0$ if and only if there exists $\lambda \in K^\times$ such that the equivalent conditions (i) and (ii) of Lemma 4.1 hold. We are going to prove that B has a descent to F if and only if (i) holds. We know that B has a descent to F if and only if there is $\varphi_0 \in I^2F$ with $\dim\varphi_0 = 6$ such that

$$B \simeq C_+(\varphi) \simeq C_+(\varphi_0)\otimes K \simeq C_+((\varphi_0)_K).$$

In other words, B has a descent to F if and only if there exists $\lambda \in K^\times$ such that $\varphi \simeq \langle\lambda\rangle \otimes (\varphi_0)_K$ because of the uniqueness of the Albert form up to similarity. Therefore to get the statement, it suffices to show for given $\lambda \in K^\times$, there exists $\varphi_0 \in I^2F$ with $\dim\varphi_0 = 6$ and $\varphi \simeq \langle\lambda\rangle\otimes(\varphi_0)_K$ if and only if $s_*(\langle\lambda\rangle\otimes\varphi) = 0$.

Suppose B has a descent to F, that is, $\varphi \simeq \langle\lambda\rangle\otimes(\varphi_0)_K$. We have automatically $s_*(\langle\lambda\rangle\otimes\varphi) = 0$. Conversely, assume that $s_*(\langle\lambda\rangle\otimes\varphi) = 0$. Then, as in the proof of Lemma 2.1 there exists a quadratic form $\varphi_0 \in I^2F$ with $\langle\lambda\rangle\otimes\varphi \simeq (\varphi_0)_K$. That concludes the proof. □

Now, let us denote by X the Weil transfer of $\mathrm{SB}(B)$ over F. Such a transfer exists (see for instance [4, (2.8)] or [23, Chapter 4]) and is projective ([13, Corollary 2.4]) since $\mathrm{SB}(B)$ is a projective variety. Moreover, we have

$$X_K \simeq \mathrm{SB}(B)\times\mathrm{SB}(B)$$

(see [4, (2.8)]). We denote by $F(X)$ the function field of X. Notice that $F(X)$ splits B; that also means φ is hyperbolic over $F(X)$.

For any integer $d \geq 1$, let $\mathbb{Q}/\mathbb{Z}(d-1) = \varinjlim \mu_n^{\otimes(d-1)}$, where μ_n is the group of n-th roots of unity in F_s. We let (see [7, Appendix A]) $H^d(F,\mathbb{Q}/\mathbb{Z}(d-1))$ be the Galois cohomology group with coefficients in $\mathbb{Q}/\mathbb{Z}(d-1)$. By definition, one has the canonical map $H^3(F,\mu_{2n}^{\otimes 2}) \to H^3(F,\mathbb{Q}/\mathbb{Z}(2))$. On the other hand, using the surjectivity of the map $H^3(F,\mu_{2n}^{\otimes 2}) \to H^3(F,\mu_n^{\otimes 2})$ (see [18]), the infinite long exact sequence in cohomology induced by the natural exact sequence of Galois modules

$$1 \to \mu_2 \to \mu_{2n}^{\otimes 2} \to \mu_n^{\otimes 2} \to 1$$

shows that the canonical map $H^3(F,\mu_2) \to H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ is injective.

Since φ is hyperbolic over $F(X)$, it is clear that

$$e_3(s_*(\varphi)) \in \operatorname{Ker}\Big(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\Big).$$

On the other hand, since $F(X)$ splits B, we have

$$\mathrm{cor}_{K/F}((K^\times)\cdot[B]) \subset \mathrm{Ker}\Big(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\Big).$$

We have just proved the following consequence:

Corollary 4.5. *The invariant $\delta_{K/F}(B)$ is in the quotient group*

$$\frac{\mathrm{Ker}\Big(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\Big)}{\mathrm{cor}_{K/F}((K^\times)\cdot[B])}. \tag{4.2}$$

Remark 4.6. The quotient (4.2) is canonically identified with the torsion $\mathrm{CH}^2(X)_{\mathrm{tors}}$ (see [20] or [19]). Therefore, we may view $\delta_{K/F}(B)$ as belonging to this group.

We now study the behavior of $\delta_{K/F}(B)$ under an odd degree extension. Let $\mathbb{F}$ be an odd degree extension of F. We denote by $K\mathbb{F}$ the quadratic extension $\mathbb{F}(\sqrt{a})$. The following result shows that if B has no descent to F, then the same holds for $B_{\mathbb{F}}$.

Proposition 4.7. *The scalar extension map*

$$\frac{\mathrm{Ker}\Big(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\Big)}{\mathrm{cor}_{K/F}((K^\times)\cdot[B])} \longrightarrow \frac{\mathrm{Ker}\Big(H^3(\mathbb{F},\mathbb{Q}/\mathbb{Z}(2)) \to H^3(\mathbb{F}(X_{\mathbb{F}}),\mathbb{Q}/\mathbb{Z}(2))\Big)}{\mathrm{cor}_{K\mathbb{F}/\mathbb{F}}(((K\mathbb{F})^\times)\cdot[B_{K\mathbb{F}}])}$$

is an injection.

Proof. Let $M \supset K \supset F$ be a separable biquadratic extension of K in B which splits B; so $[M:F]=8$. Since M splits B, we have

$$X_M \simeq \mathrm{SB}(B)_M \times \mathrm{SB}(B)_M \simeq \mathbf{P}^3_M \times \mathbf{P}^3_M$$

where $\mathbf{P}^3_M$ denotes the projective 3-space considered as variety. The function field $M(\mathbf{P}^3_M)$ of $\mathbf{P}^3_M$ being a purely transcendental extension of M, the function field $M(X_M)$ is a purely transcendental extension of M. So, we have

$$H^3(M,\mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow H^3(M(X_M),\mathbb{Q}/\mathbb{Z}(2)).$$

Let $\xi \in \mathrm{Ker}\Big(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\Big)$ and denote by $\xi_{\mathbb{F}}$ its image in $H^3(\mathbb{F},\mathbb{Q}/\mathbb{Z}(2))$ under the scalar extension. Suppose $\xi_{\mathbb{F}} = \mathrm{cor}_{K\mathbb{F}/\mathbb{F}}((\lambda)\cdot[B_{K\mathbb{F}}])$ for some $\lambda \in (K\mathbb{F})^\times$. Let A be a central simple F-algebra such that $r_{K/F}(A) = B$. Note that such an algebra A exists because the sequence

$$\mathrm{Br}_2(F) \xrightarrow{r_{K/F}} \mathrm{Br}_2(K) \xrightarrow{\mathrm{cor}_{K/F}} \mathrm{Br}_2(F)$$

is exact. One has

$$\begin{aligned}
[\mathbb{F}:F]\xi = \mathrm{cor}_{\mathbb{F}/F}(\xi_{\mathbb{F}}) &= \mathrm{cor}_{\mathbb{F}/F}\Big(\mathrm{cor}_{K\mathbb{F}/\mathbb{F}}((\lambda)\cdot[B_{K\mathbb{F}}])\Big)\\
&= \mathrm{cor}_{\mathbb{F}/F}\Big(N_{K\mathbb{F}/\mathbb{F}}(\lambda)\cdot[A_{\mathbb{F}}]\Big)\\
&= N_{\mathbb{F}/F}\Big(N_{K\mathbb{F}/\mathbb{F}}(\lambda)\Big)\cdot[A]\\
&= N_{K/F}\Big(N_{K\mathbb{F}/K}(\lambda)\Big)\cdot[A]\\
&= \mathrm{cor}_{K/F}\left(N_{K\mathbb{F}/K}(\lambda)\cdot[A_K]\right)\in \mathrm{cor}_{K/F}((K^\times)\cdot[B]).
\end{aligned}$$

This implies that the order of ξ is odd in

$$\frac{\mathrm{Ker}\left(H^3(F,\mathbb{Q}/\mathbb{Z}(2))\to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\right)}{\mathrm{cor}_{K/F}((K^\times)\cdot[B])}$$

since $[\mathbb{F}:F]$ is odd. On the other hand, consider the following commutative diagram where the vertical maps are given by scalar extension and the horizontal maps are restriction and corestriction maps

$$\begin{tikzcd}
H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \arrow[r] \arrow[d] & H^3(M,\mathbb{Q}/\mathbb{Z}(2)) \arrow[r] \arrow[d, hook] & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d] \\
H^3(F(X),\mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & H^3(M(X_M),\mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}$$

The image of ξ by the top row is $[M:F]\xi = 8\xi$. Moreover, a diagram chase shows that 8ξ is trivial in $H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ because ξ is in the kernel of the left vertical map, and the central vertical map is injective. So, the order of ξ is then prime to $[\mathbb{F}:F]$. It follows that $\xi\in \mathrm{cor}_{K/F}((K^\times)\cdot[B])$. The proof is complete. □

Now, assume that A a central simple algebra of degree 8 and exponent 2 over F. Let $K=F(\sqrt{a})$ be a quadratic field extension of F contained in A and let $\mathbb{F}$ be an odd degree extension of F. Denote by $B=C_AK$ the centralizer of K in A.

Remark 4.8. (1) The algebra A admits a decomposition adapted to K if and only if B has a descent to F. In other words, A admits a decomposition adapted to K if and only if $\delta_{K/F}(B)=0$. So, $\delta_{K/F}(B)\neq 0$ if A is indecomposable.
(2) One deduces from Proposition 4.7 that if K is not in a quaternion subalgebra of A, then the same holds over $\mathbb{F}$.
(3) Let Ker(res) be the kernel of the restriction map

$$H^3(F,\mathbb{Q}/\mathbb{Z}(2))\longrightarrow H^3(F(\mathrm{SB}(A)),\mathbb{Q}/\mathbb{Z}(2)).$$

The quotient group

$$\mathrm{Ker(res)}/[A]\cdot H^1(F,\mathbb{Q}/\mathbb{Z}(2))$$

is identified with $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}}$ in [19]. Arguing as in the proof of Proposition 4.7, we may see that the scalar extension map

$$\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{F}})_{\mathrm{tors}}$$

is injective. This injection may be also deduced from [12, Corollary 1.2 and Proposition 1.3].

Let A' be the division algebra Brauer equivalent to

$$A' \sim A \otimes_F (a,t)_{F(t)}$$

where t is an indeterminate over F. Note that

$$A'_K \sim A_K \otimes K(t) \sim B_{K(t)}.$$

The following proposition shows that if $\delta_{K/F}(B)$ is nonzero then the invariant $\Delta(A')$, defined in [8, §11], is nonzero.

Proposition 4.9. *The scalar extension map*

$$\frac{H^3(F,\mu_2)}{\mathrm{cor}_{K/F}((K^\times)\cdot[B])} \longrightarrow \frac{H^3(F(t),\mu_2)}{(F(t)^\times)\cdot[A']} \tag{4.3}$$

is an injection.

Proof. The map is clearly well-defined because

$$\mathrm{cor}_{K/F}(K^\times\cdot[B]) \subset \mathrm{cor}_{K(t)/F(t)}(K(t)^\times\cdot[B_{K(t)}]) \subset (F(t)^\times)\cdot[A'].$$

Let $\xi \in H^3(F,\mu_2)$. Suppose $\xi = f[A'] = f([A]+[(a,t)])$ for some $f \in F(t)^\times$. Consider the residue map

$$\partial_t : H^3(F(t),\mu_2) \longrightarrow H^2(F,\mu_2)$$

where $F(t)$ is equipped with its t-adic valuation v_t (see for instance [7, §7]). We have

$$\partial_t(\xi) = \partial_t(f[A]+f[(a,t)]) = 0.$$

Since f is taken modulo $F(t)^{\times 2}$, we may assume that $v_t(f)$ is either 0 or 1.

If $v_t(f)=1$, that is, $f = tf_0$ for some t-adic unit f_0, then

$$\begin{aligned}\partial_t(\xi) = \partial_t(f[A]+f[(a,t)]) &= [A]+\partial_t((tf_0,a,t))\\ &= [A]+\partial_t((-f_0,a,t))\\ &= [A]+[(-f_0(0),a)] = 0.\end{aligned}$$

Therefore A is Brauer equivalent to $(-f_0(0),a)$. It follows that the two quotients of the map (4.3) are trivial. In this case, there is nothing to show.

Suppose $v_t(f)=0$. One has

$$\partial_t(f[A]+f[(a,t)]) = [(f(0),a)] = 0.$$

On the other hand, $\xi = f(0)[A]$ by the specialization map $\ker\partial_t \to H^3(F,\mu_2)$ associated with t at 0. Since $(f(0),a)=0$, we deduce that $\xi \in (N_{K/F}(K^\times))\cdot[A] = \mathrm{cor}_{K/F}((K^\times)\cdot[B])$. The proof is complete. □

Remark 4.10. According to Remark 4.3, the image of $\delta_{K/F}(B)$ by the scalar extension map (4.3) is $\Delta(A')$. The proof of Theorem 1.2 provides an example of indecomposable algebra A' of degree 8 and exponent 2 such that $\Delta(A')$ is nonzero.

5. Proofs of the main statements

5.1. **Proof of Theorem 1.1.** We start out by the following lemma (a part of [9, Theorem 2]):

Lemma 5.1. *Every central simple F-algebra of exponent 2 is Brauer equivalent to a tensor product of at most $\frac{1}{2}u(F)-1$ quaternion algebras.*

Proof. Let A be a central simple F-algebra of exponent 2. Let $\varphi \in I^2F$ be an anisotropic quadratic form such that $C(\varphi) \sim A$ (by Merkurjev's Theorem [16]). The form φ being anisotropic, $\dim\varphi \leq u(F)$. Let φ' be a subform of φ of codimension 1. The algebra $C(\varphi)$ is Brauer equivalent to $C_0(\varphi')$ which is a tensor product of $\frac{1}{2}(\dim(\varphi')-1) = \frac{1}{2}(\dim\varphi)-1$ quaternion algebras. □

Now, let A be a degree 8 and exponent 2 algebra over a field of u-invariant smaller than or equal to 8. By Lemma 5.1, A is Brauer equivalent to a tensor product of three quaternion algebras. Therefore, this equivalence is an isomorphism by dimension count; this proves that A is decomposable. This concludes the proof of Theorem 1.1.

5.2. **A consequence.** Let U and V be smooth complete geometrically irreducible varieties over F. In the appendix, Merkurjev gives conditions under which the scalar extension map $\mathrm{CH}(V) \longrightarrow \mathrm{CH}(V_{F(U)})$ is injective. For instance, let A be central simple F-algebra of degree 8 and exponent 2 and let K be a quadratic separable extension of F contained in A. Denote by X the Weil transfer of $\mathrm{SB}(C_AK)$ over F. Let q be a quadratic form over F with $\dim q \geq 9$. It follows from Theorem A.9 that the scalar extension map

$$\mathrm{CH}^2(X)_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(X_{F(q)})_{\mathrm{tors}} \tag{5.1}$$

is injective, where $F(q)$ is the function field of the projective quadric defined by $q=0$. This property does not hold anymore if one replaces X by $\mathrm{SB}(A)$. Indeed, Theorem 1.1 implies the following:

Corollary 5.2. *Let A be a central simple algebra of degree 8 and exponent 2 over F. Assume that $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \neq 0$. Then there exist an extension F' of F and a 9-dimensional quadratic form q defined over F_ℓ such that the scalar extension map*

$$\mathrm{CH}^2(\mathrm{SB}(A)_{F'})_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(\mathrm{SB}(A)_{F'(q)})_{\mathrm{tors}}$$

is not injective.

For the proof we need the following particular class of fields of u-invariant at most 8: starting with any field F over which there is an anisotropic quadratic form

of dimension 8, one defines a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_\infty = \bigcup_i F_i =: \mathrm{M}_8^u(F)$$

inductively as follows: if F_{i-1} is already given, the field F_i is the composite of all function fields $F_{i-1}(\varphi)$, where φ ranges over (the isometry classes of) all 9-dimensional forms over F_{i-1}. Clearly, any 9-dimensional form over $\mathbb{M}$ is isotropic. Therefore $u(\mathrm{M}_8^u(F)) \leq 8$. Such a construction is due to Merkujev (see [17]).

Proof of Corollary 5.2. Let $\mathbb{M} = \mathrm{M}_8^u(F)$ be an extension as above; recall that $u(\mathbb{M}) \leq 8$. The algebra $A_{\mathbb{M}}$ being decomposable by Theorem 1.1, we have $\mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{M}})_{\mathrm{tors}} = 0$. Since $\mathbb{M} = \bigcup_i F_i$, there exists ℓ such that $\mathrm{CH}^2(\mathrm{SB}(A)_{F_\ell})_{\mathrm{tors}} \neq 0$ and $\mathrm{CH}^2(\mathrm{SB}(A)_{F_{\ell+1}})_{\mathrm{tors}} = 0$. By definition, $F_{\ell+1}$ is the composite of all function fields $F_\ell(q)$, where q ranges over all 9-dimensional forms over F_ℓ. Hence, there exists an extension F' of F_ℓ and a 9-dimensional form q over F_ℓ such that $\mathrm{CH}^2(\mathrm{SB}(A)_{F'})_{\mathrm{tors}} \neq 0$ and $\mathrm{CH}^2(\mathrm{SB}(A)_{F'(q)})_{\mathrm{tors}} = 0$ as was to be shown. □

5.3. **Proof of Theorem 1.2.** Put $\mathbb{M} = \mathrm{M}_8^u(F)$ and $B = C_A K$. As in the previous section we denote by A' be the division algebra Brauer equivalent to $A \otimes_F (a,t)_{F(t)}$.

First proof. Recall that $\delta_{K/F}(C_A K)$ is in $\mathrm{CH}^2(X)_{\mathrm{tors}}$ by Remark 4.6, where X is the Weil transfer of $\mathrm{SB}(C_A K)$. Since $\delta_{K/F}(C_A K) \neq 0$ (see Remark 4.8), it follows by injection (5.1) that the extension $\mathbb{M}(\sqrt{a})$ is not in a quaternion subalgebra of $A_{\mathbb{M}}$. By [26, Proposition 2.10] (or [3]) the algebra $A'_{\mathbb{M}}$ is an indecomposable algebra of degree 8 and exponent 2. This concludes the proof. □

Second proof. Consider the following injections

$$\frac{H^3(F,\mu_2)}{\mathrm{cor}_{K/F}((K^\times)\cdot[B])} \hookrightarrow \frac{H^3(\mathbb{M},\mu_2)}{\mathrm{cor}_{\mathbb{M}K/\mathbb{M}}(((\mathbb{M}K)^\times)\cdot[B_{\mathbb{M}K}])} \hookrightarrow \frac{H^3(\mathbb{M}(t),\mu_2)}{(\mathbb{M}(t)^\times)\cdot[A'_{\mathbb{M}}]}$$

where the first is due to Merkurjev (injection (5.1)) and the second by Proposition 4.9. Since $\delta_{K/F}(B) \neq 0$ and its image by the composite of these above injections is $\Delta(A'_{\mathbb{M}})$, we have $\Delta(A'_{\mathbb{M}}) \neq 0$. This also means $A'_{\mathbb{M}}$ is indecomposable by Garibaldi-Parimala-Tignol [8, §11]. □

5.4. **Proof of Theorem 1.3.** Here, we also need a particular class of fields of cohomological dimension at most 3: starting with any field F over which there is an anisotropic 3-fold Pfister form, one defines a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_\infty = \bigcup_i F_i =: \mathrm{M}_3^{cd}(F)$$

inductively as follows: the field F_{2i+1} is the maximal odd degree extension of F_{2i}; the field F_{2i+2} is the composite of all the function fields $F_{2i+1}(\pi)$, where π ranges over all 4-fold Pfister forms over F_{2i+1}. The arguments used by Merkurjev in [17], show that $cd_2(\mathrm{M}_3^{cd}(F)) \leq 3$. Merkurjev used such a technique for constructing fields of cohomological dimension 2 over which there exist anisotropic quadratic forms

of dimension $2d$ for an arbitrary integer d, i.e, counterexamples to Kaplansky's conjecture in the theory of quadratic forms.

Now, let A be a central simple algebra of degree 8 and exponent 2 such that $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \neq 0$. Put $\mathbb{M} = \mathrm{M}_3^{cd}(F)$ and let $\mathbb{F}$ be an odd degree extension of F. The scalar extension map

$$\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{F}})_{\mathrm{tors}}$$

is injective by Remark 4.8. On the other hand, let π be a 4-fold Pfister form over $\mathbb{F}$. It follows from Theorem A.7 that the scalar extension map

$$\mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{F}})_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{F}(\pi)})_{\mathrm{tors}} \tag{5.2}$$

is injective. We deduce from these two latter injections that $\mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{M}})_{\mathrm{tors}} \neq 0$; and so $A_{\mathbb{M}}$ is indecomposable. The algebra $A_{\mathbb{M}}$ being indecomposable, we must have $cd_2(\mathbb{M}) > 2$ by Theorem 3.1. Hence, $cd_2(\mathbb{M}) = 3$. This completes the proof.

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Appendix A.

On the Chow Group of Cycles of Codimension 2

by Alexander S. Merkurjev

Let X be an algebraic variety over F. We write $A^i(X, K_n)$ for the homology group of the complex

$$\coprod_{x\in X^{(i-1)}} K_{n-i+1}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i)}} K_{n-i}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i+1)}} K_{n-i-1}\big(F(x)\big),$$

where K_j are the Milnor K-groups and $X^{(i)}$ is the set of points in X of codimension i (see [7, §5]). In particular, $A^i(X, K_i) = \mathrm{CH}^i(X)$ is the Chow group of classes of codimension i algebraic cycles on X.

Let X and Y be smooth complete geometrically irreducible varieties over F.

Proposition A.1. *Suppose that for every field extension K/F we have:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism of torsion free groups,*
(2) *The product map $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ is an isomorphism.*

Then the natural sequence

$$0 \longrightarrow \big(\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)\big) \oplus \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)})$$

is exact.

Proof. Consider the spectral sequence

$$E_1^{p,q} = \coprod_{y \in Y^{(p)}} A^q(X_{F(y)}, K_{2-p}) \Longrightarrow A^{p+q}(X \times Y, K_2)$$

for the projection $X \times Y \longrightarrow Y$ (see [7, Cor. 8.2]). The nonzero terms of the first page are the following:

$$\begin{array}{l} \mathrm{CH}^2(X_{F(Y)}) \\ \\ A^1(X_{F(Y)}, K_2) \longrightarrow \coprod_{y \in Y^{(1)}} \mathrm{CH}^1(X_{F(y)}) \\ \\ A^0(X_{F(Y)}, K_2) \longrightarrow \coprod_{y \in Y^{(1)}} A^0(X_{F(y)}, K_1) \longrightarrow \coprod_{y \in Y^{(2)}} \mathrm{CH}^0(X_{F(y)}). \end{array}$$

Then $E_1^{2,0} = \coprod_{y \in Y^{(2)}} \mathbb{Z}$ is the group of cycles on X of codimension 2 and $E_1^{1,0} = \coprod_{y \in Y^{(1)}} F(y)^\times$ as X is complete. It follows that $E_2^{2,0} = \mathrm{CH}^2(Y)$.

By assumption, the differential $E_1^{0,1} \longrightarrow E_1^{1,1}$ is identified with the map

$$\mathrm{CH}^1(X) \otimes \Big(F(Y)^\times \longrightarrow \coprod_{y \in Y^{(1)}} \mathbb{Z}\Big).$$

Since Y is complete and $\mathrm{CH}^1(X)$ is torsion free, we have $E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$ and $E_\infty^{1,1} = E_2^{1,1} = \mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$.

The edge map

$$A^1(X \times Y, K_2) \longrightarrow E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$$

is split by the product map

$$\mathrm{CH}^1(X) \otimes F^\times = A^1(X, K_1) \otimes A^0(Y, K_1) \longrightarrow A^1(X \times Y, K_2),$$

hence the edge map is surjective. Therefore, the differential $E_2^{0,1} \longrightarrow E_2^{2,0}$ is trivial and hence $E_\infty^{2,0} = E_2^{2,0} = \mathrm{CH}^2(Y)$. Thus, the natural homomorphism

$$\mathrm{CH}^2(Y) \longrightarrow \mathrm{Ker}\big(\mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)})\big)$$

is injective and its cokernel is isomorphic to $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$. The statement follows. $\square$

Example A.2. Let X be a projective homogeneous variety of a semisimple algebraic group over F. There exist an étale F-algebra E and an Azumaya E-algebra A such that for $i = 0$ and 1, we have an exact sequence

$$0 \longrightarrow A^1(X, K_{i+1}) \longrightarrow K_i(E) \xrightarrow{\rho} H^{i+2}\big(F, \mathbb{Q}/\mathbb{Z}(i+1)\big),$$

where $\rho(x) = N_{E/F}\big((x) \cup [A]\big)$ (see [4] and [5]). If the algebras E and A are split, then ρ is trivial and for every field extension K/F,

$$\mathrm{CH}^1(X) \simeq K_0(E) \simeq K_0(E \otimes K) \simeq \mathrm{CH}^1(X_K),$$

$$A^1(X_K, K_2) \simeq K_1(E \otimes K) \simeq K_0(E) \otimes K^\times \simeq \mathrm{CH}^1(X_K) \otimes K^\times.$$

Therefore, the condition (1) and (2) in Proposition A.1 hold. For example, if X is a smooth projective quadric of dimension at least 3, then $E = F$ and A is split.

Now consider the natural complex

$$\mathrm{CH}^2(X) \oplus \big(\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)\big) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)}). \tag{A.1}$$

Proposition A.3. *Suppose that*

(1) *The Grothendieck group $K_0(Y)$ is torsion-free,*
(2) *The product map $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$ is an isomorphism.*

Then the sequence (A.1) is exact.

Proof. It follows from the assumptions that the map $K_0(Y) \longrightarrow K_0(Y_{F(X)})$ is injective and the kernel of the natural homomorphism $K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})$ coincides with

$$I_0(X) \otimes K_0(Y),$$

where $I_0(X)$ is the kernel of the rank homomorphism $K_0(X) \longrightarrow \mathbb{Z}$.

The kernel of the second homomorphism in the sequence (A.1) is generated by the classes of closed integral subschemes $Z \subset X \times Y$ that are not dominant over X. By Riemann-Roch (see [2]), we have $[Z] = -c_2\big([O_Z]\big)$ in $\mathrm{CH}^2(X \times Y)$, where $c_i : K_0(X \times Y) \longrightarrow \mathrm{CH}^i(X \times Y)$ is the i-th Chern class map. As

$$[O_Z] \in \mathrm{Ker}(K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})\big) = I_0(X) \otimes K_0(Y),$$

it suffices to to show that $c_2\big(I_0(X) \otimes K_0(Y)\big)$ is contained in the image M of the first map in the sequence (A.1).

The formula $c_2(x + y) = c_2(x) + c_1(x)c_1(y) + c_2(y)$ shows that it suffices to prove that for all $a, a' \in I_0(X)$ and $b, b' \in K_0(Y)$, the elements $c_1(ab) \cdot c_1\big(a'b'\big)$ and $c_2(ab)$ are contained in M. This follows from the formulas (see [1, Remark 3.2.3 and Example 14.5.2]): $c_1(ab) = mc_1(a) + nc_1(b)$ and

$$c_2(ab) = \frac{m^2 - m}{2} c_1(a)^2 + mc_2(a) + (nm - 1)c_1(a)c_1(b) + \frac{n^2 - n}{2} c_1(b)^2 + nc_2(b),$$

where $n = \mathrm{rank}(a)$ and $m = \mathrm{rank}(b)$. □

Example A.4. If Y is a projective homogeneous variety, then the condition (1) holds by [6]. If X is a projective homogeneous variety of a semisimple algebraic group G over F and the Tits algebras of G are split, then it follows from [6] that the condition (2) also holds for any Y. For example, if the even Clifford algebra of a nondegenerate quadratic form is split, then the corresponding projective quadric X satisfies (2) for any Y.

For any field extension K/F, let K^s denote the subfield of elements that are algebraic and separable over F.

Proposition A.5. *Suppose that for every field extension K/F we have:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism,*
(2) *The natural map $\mathrm{CH}^1(Y_{K^s}) \to \mathrm{CH}^1(Y_K)$ is an isomorphism.*

Then the sequence (A.1) is exact.

Proof. Consider the spectral sequence

$$E_1^{p,q}(F) = \coprod_{x \in X^{(p)}} A^q(Y_{F(x)}, K_{2-p}) \Longrightarrow A^{p+q}(X \times Y, K_2) \tag{A.2}$$

for the projection $X \times Y \longrightarrow X$. The nonzero terms of the first page are the following:

$$\begin{array}{l}
\mathrm{CH}^2(Y_{F(X)}) \\[2ex]
A^1(Y_{F(X)}, K_2) \longrightarrow \coprod_{x \in X^{(1)}} \mathrm{CH}^1(Y_{F(x)}) \\[2ex]
A^0(Y_{F(X)}, K_2) \longrightarrow \coprod_{x \in X^{(1)}} A^0(Y_{F(x)}, K_1) \longrightarrow \coprod_{x \in X^{(2)}} \mathrm{CH}^0(Y_{F(x)}).
\end{array}$$

As in the proof of Proposition A.1, we have $E_2^{2,0}(F) = \mathrm{CH}^2(X)$. For a field extension K/F, write $C(K)$ for the factor group

$$\mathrm{Ker}\big(\mathrm{CH}^2(X_K \times Y_K) \longrightarrow \mathrm{CH}^2(Y_{K(X)})\big) / \mathrm{Im}\big(\mathrm{CH}^2(X_K) \longrightarrow \mathrm{CH}^2(X_K \times Y_K)\big).$$

The spectral sequence (A.2) for the varieties X_K and Y_K over K yields an isomorphism $C(K) \simeq E_2^{1,1}(K)$. We have a natural composition

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \longrightarrow E_1^{1,1}(K) \longrightarrow E_2^{1,1}(K) \simeq C(K).$$

We claim that the group $C(F)$ is generated by images of the compositions

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \longrightarrow C(K) \stackrel{N_{K/F}}{\longrightarrow} C(F)$$

over all finite separable field extensions K/F (here $N_{K/F}$ is the norm map for the extension K/F).

The group $C(F)$ is generated by images of the maps

$$\varphi_x : \mathrm{CH}^1(Y_{F(x)}) \longrightarrow E_2^{1,1}(F) \simeq C(F)$$

over all points $x \in X^{(1)}$. Pick such a point x and let $K := F(x)^s$ be the subfield of elements that are separable over F. Then K/F is a finite separable field extension. Let $x' \in X_K^{(1)}$ be a point over x such that $K(x') \simeq F(x)$. Then φ_x coincides with the composition

$$\mathrm{CH}^1(Y_{K(x')}) \longrightarrow C(K) \stackrel{N_{K/F}}{\longrightarrow} C(F).$$

By assumption, the map $\mathrm{CH}^1(Y_K) \longrightarrow \mathrm{CH}^1(Y_{K(x')})$ is an isomorphism, hence the image of φ_x coincides with the image of

$$[x'] \otimes \mathrm{CH}^1(Y_K) \longrightarrow C(K) \stackrel{N_{K/F}}{\longrightarrow} C(F),$$

whence the claim.

As $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism for every field extension K/F, the projection formula shows that the map $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow C(F)$ is surjective. The statement follows. □

Example A.6. Let Y be a projective homogeneous variety with the F-algebras E and A as in Example A.2. If A is split, then $\mathrm{CH}^1(Y_K) = K_0(E \otimes K)$ for every field extension K/F. As K^s is separably closed in K, the natural map $K_0(E \otimes K^s) \longrightarrow K_0(E \otimes K)$ is an isomorphism, therefore, the condition (2) holds.

Write $\widetilde{\mathrm{CH}}^2(X \times Y)$ for the cokernel of the product map $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow \mathrm{CH}^2(X \times Y)$. We have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{CH}^2(X) \arrow[d] \arrow[dr] & \\
0 \arrow[r] & \mathrm{CH}^2(Y) \arrow[r] \arrow[dr] & \widetilde{\mathrm{CH}}^2(X \times Y) \arrow[r] \arrow[d] & \mathrm{CH}^2(X_{F(Y)}) \\
 & & \mathrm{CH}^2(Y_{F(X)}) &
\end{tikzcd}$$

Proposition A.1 gives conditions for the exactness of the row in the diagram and Propositions A.3 and A.5 - for the exactness of the column in the diagram.

A diagram chase yields together with Propositions A.1, A.3 and A.5 yields the following statements.

Theorem A.7. *Let X and Y be smooth complete geometrically irreducible varieties such that for every field extension K/F:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism of torsion free groups,*
(2) *The natural map $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$ is injective,*

(3) *The product map* $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ *is an isomorphism,*
(4) *The Grothendieck group* $K_0(Y)$ *is torsion-free,*
(5) *The product map* $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$ *is an isomorphism.*

Then the natural map $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$ *is injective.*

Remark A.8. The conditions (1) – (3) hold for a smooth projective quadric X of dimension at least 7 by [3, Theorem 6.1] and Example A.2. By Example A.4, the conditions (4) and (5) hold if the even Clifford algebra of X is split and Y is a projective homogeneous variety.

Theorem A.9. *Let* X *and* Y *be smooth complete geometrically irreducible varieties such that for every field extension* K/F*:*

(1) *The natural map* $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ *is an isomorphism of torsion free groups,*
(2) *The natural map* $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$ *is injective,*
(3) *The product map* $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ *is an isomorphism,*
(4) *The natural homomorphism* $\mathrm{CH}^1(Y_{K^s}) \to \mathrm{CH}^1(Y_K)$ *is an isomorphism.*

Then the natural map $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$ *is injective.*

Remark A.10. The conditions (1) – (3) hold for a smooth projective quadric X of dimension at least 7 and a projective homogeneous variety Y with the split Azumaya algebra by Remark A.8 and Example A.6.

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