RATIONALITY OF CYCLES ON FUNCTION FIELD OF EXCEPTIONAL PROJECTIVE HOMOGENEOUS VARIETIES

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Abstract. In this article we prove a result comparing rationality of algebraic cycles over the function field of a projective homogeneous variety under a linear algebraic group of type F_4 or E_8 and over the base field, which can be of any characteristic.

Keywords: Chow groups and motives, exceptional algebraic groups, projective homogeneous varieties.

1. Introduction

Let G be a linear algebraic group of type F_4 or E_8 over a field F and let X be a projective homogeneous G-variety. We write Ch for the Chow group with coefficient in $\mathbb{Z}/p\mathbb{Z}$, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8. The purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles on function field of such a projective homogeneous G-variety.

Theorem 1.1. *For any equidimensional variety Y, the change of field homomorphism*

$$Ch(Y) \to Ch(Y_{F(X)})$$

is surjective in codimension $< p+1$. It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin deg\, Ch_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

The proof is given in section 3.

In previous papers ([2], [3], after the so-called Main Tool Lemma by A. Vishik, cf [16], [17]), similar issues about rationality of cycles, with quadrics instead of exceptional projective homogeneous varieties, have been treated. The above statement is to put in relation with [10, Theorem 4.3], where *generic splitting varieties* have been considered. Also, Theorem 1.1 is contained in [10, Theorem 4.3] if $\mathrm{char}(F) = 0$.

On the one hand, our method of proof is basically the method used to prove [10, Theorem 4.3]. On the other hand, our method mainly relies on a motivic decomposition result for projective homogeneous varieties due to V. Petrov, N. Semenov and K. Zainoulline (cf [14, Theorem 5.17]). It also relies on a linkage between the γ-filtration and Chow groups, in the spirit of [5]. Our method works in any characteristic and is particularly suitable for groups of type F_4 and E_8 mainly because the latter have an opportune J-invariant.

In the aftermath of Theorem 1.1, we get the following statement dealing with integral Chow groups (see [10, Theorem 4.5]).

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Corollary 1.2. *If $p \in deg\, CH_0(X)$ then for any equidimensional variety Y, the change of field homomorphism*

$$CH(Y) \to CH(Y_{F(X)})$$

is surjective in codimension $< p+1$. It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin deg\, Ch_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

Remark 1.3. Our method of proof for Theorem 1.1 works for groups of type G_2 as well (with p=2). However, the case of G_2 can be treated in a more elementary way if $\mathrm{char}(F) = 0$.

Indeed, it is known that to each group G of type G_2 one can associate a 3-fold Pfister quadratic form ρ such that, by denoting X_ρ the Pfister quadric associated with ρ, the variety X has a rational point over $F(X_\rho)$ and vice-versa. Thus, for any equidimensional variety Y, one has the commutative diagram

$$\begin{array}{ccc} Ch(Y) & \longrightarrow & Ch(Y_{F(X)}) \\ \downarrow & & \downarrow \\ Ch(Y_{F(X_\rho)}) & \longrightarrow & Ch(Y_{F(X_\rho \times X)}) \end{array}$$

where the right and the bottom maps are isomorphisms. Furthermore, as suggested in [17, Remark on Page 665] (where the assumption $\mathrm{char}(F) = 0$ is required), the change of field homomorphism $Ch(Y) \to Ch(Y_{F(Q)})$ is surjective in codimension < 3.

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2. Filtrations on projective homogeneous varieties

In this section, we prove two propositions which play a crucial role in the proof of Theorem 1.1.

First of all, we recall that for any smooth projective variety X over a field E, one can consider two particular filtrations on the Grothendieck ring $K(X)$ (see [5, §1.A]), i.e the γ-filtration and the topological filtration, whose respective terms of codimension i are given by

$$\gamma^i(X) = \langle c_{n_1}(a_1) \cdots c_{n_m}(a_m) \,|\, n_1 + \cdots + n_m \geq i \text{ and } a_1, \ldots, a_m \in K(X) \rangle$$

and

$$\tau^i(X) = \langle [\mathcal{O}_Z] \,|\, Z \hookrightarrow X \text{ and } \mathrm{codim}(Z) \geq i \rangle,$$

where c_n is the n-th Chern Class with values in $K(X)$ and $[\mathcal{O}_Z]$ is the class of the structure sheaf of a closed subvariety Z. We write $\gamma^{i/i+1}(X)$ and $\tau^{i/i+1}(X)$ for the respective quotients. For any i, one has $\gamma^i(X) \subset \tau^i(X)$ and one even has $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. We denote by pr the canonical surjection

$$\begin{array}{ccc} CH^i(X) & \twoheadrightarrow & \tau^{i/i+1}(X) \\ {[Z]} & \longmapsto & [\mathcal{O}_Z] \end{array},$$

where CH stands for the integral Chow group.

The method of proof of the following proposition is largely inspired by the proof of [9, Theorem 6.4 (2)].

Proposition 2.1. *Let G_0 be a split semisimple linear algebraic group over a field F and let B be a Borel subgroup of G_0. There exist an extension E/F and a cocycle $\xi \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration coincide on $K({}_\xi(G_0/B))$.*

Proof. Let n be an integer such that $G_0 \subset \mathbf{GL}_n$ and let us set $S := \mathbf{GL}_n$ and $E := F(S/G_0)$. We denote by $\mathbf{T}$ the E-variety $S \times_{S/G_0} \mathrm{Spec}(E)$ given by the generic fiber of the projection $S \to S/G_0$. Note that since $\mathbf{T}$ is clearly a G_0-torsor over E, there exists a cocycle $\xi \in H^1(E, G_0)$ such that the smooth projective variety $X := \mathbf{T}/B_E$ is isomorphic to ${}_\xi(G_0/B)$. We claim that the Chow ring $CH(X)$ is generated by Chern classes.

Indeed, the morphism $h : X \to S/B$ induced by the canonical G_0-equivariant morphism $\mathbf{T} \to S$ being a localisation, the associated pull-back

$$h^* : CH(S/B) \longrightarrow CH(X)$$

is surjective. Furthermore, the ring $CH(S/B)$ itself is generated by Chern classes: by [9, §6,7] there exist a morphism

$$\mathbb{S}(T^*) \longrightarrow CH(S/B), \tag{2.2}$$

(where $\mathbb{S}(T^*)$ is the symmetric algebra of the group of characters T^* of a split maximal torus $T \subset B$) with its image generated by Chern classes. Moreover, the morphism (2.2) is surjective by [9, Proposition 6.2]. Since h^* is surjective and Chern classes commute with pull-backs, the claim is proved.

We show now that the two filtrations coincide on $K(X)$ by induction on dimension. Let $i \geq 0$ and assume that $\tau^{i+1}(X) = \gamma^{i+1}(X)$. Since for any $j \geq 0$, one has $\gamma^j(X) \subset \tau^j(X)$, the induction hypothesis implies that

$$\gamma^{i/i+1}(X) \subset \tau^{i/i+1}(X).$$

Thus, the ring $CH(X)$ being generated by Chern classes, one has $\gamma^{i/i+1}(X) = \tau^{i/i+1}(X)$ by [8, Lemma 2.16]. Therefore one has $\tau^i(X) = \gamma^i(X)$ and the proposition is proved. □

Note that this result remains true when one consider a *special* parabolique subgroup P instead of B.

Now, we prove a result which will be used in section 3 to get the second conclusion of Theorem 1.1.

We recall that for any smooth projective variety X over a field and for any $i < p+1$, the canonical surjection $pr : Ch^i(X) \twoheadrightarrow \tau^{i/i+1}(X)$ with $\mathbb{Z}/p\mathbb{Z}$-coefficient is an isomorphism (cf [5, §1.A] for example). The following proposition extends this fact to $i = p+1$ provided that X is a projective homogeneous variety under a linear algebraic group G of type F_4 or E_8.

Proposition 2.3. *Let X be a projective homogeneous variety under a group G of type F_4 or E_8, then the canonical surjection*

$$pr : Ch^{p+1}(X) \twoheadrightarrow \tau^{p+1/p+2}(X)$$

is injective.

Proof. The epimorphism $pr : Ch^{p+1}(X) \twoheadrightarrow \tau^{p+1/p+2}(X)$ coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_2^{p+1,-p-1}(X) \Rightarrow K(X)$, i.e $E_r^{p+1,-p-1}(X)$ stabilizes for $r >> 0$ with $E_\infty^{p+1,-p-1}(X) = \tau^{p+1/p+2}(X)$, and for any $r \geq 2$ the differential $E_r^{p+1,-p-1}(X) \to E_r^{p+1+r,-p-r}(X)$ is zero, so that the epimorphism pr coincides with the composition

$$Ch^{p+1}(X) \simeq E_2^{p+1,-p-1}(X) \twoheadrightarrow E_3^{p+1,-p-1}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p+1,-p-1}(X) = \tau^{p+1/p+2}(X).$$

Now, it is equivalent in order to prove the proposition to prove that for any $r \geq 2$, the differential $E_r^{p+1-r,-p-2+r}(X) \to E_r^{p+1,-p-1}(X)$ is zero.

First of all, since we work with $\mathbb{Z}/p\mathbb{Z}$-coefficient, by [12,Theorem 3.6], the differential $E_r^{p+1-r,-p-2+r}(X) \to E_r^{p+1,-p-1}(X)$ is zero for any $r \geq 2$ with $r \neq p$. Hence, one only has to show that the differential $E_p^{1,-2}(X) \to E_p^{p+1,-p-1}(X)$ is zero.

Let us consider the following composition given by the BGQ-structure

$$E_\infty^{1,-2}(X) \hookrightarrow \cdots \hookrightarrow E_3^{1,-2}(X) \hookrightarrow E_2^{1,-2}(X).$$

Note that one has $E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X)$ if and only if for any $r \geq 2$ the differential $E_r^{1,-2}(X) \to E_r^{1+r,-2-r+1}(X)$ is zero. Therefore it is sufficient to prove that $E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X)$ to get that the differential $E_p^{1,-2}(X) \to E_p^{p+1,-p-1}(X)$ is zero.

On the one hand, by the very defintion, the group $E_\infty^{1,-2}(X)$ is the first quotient $K_1^{(1/2)}(X)$ of the topological filtration on $K_1(X)$. On the other hand, one has $E_2^{1,-2}(X) \simeq H^1(X, K_2)$ (for any integers p and q, one has $E_2^{p,q}(X) \simeq H^p(X, K_{-q})$).

Let us now consider the commutative diagram (cf [7,§4])

$$\begin{array}{ccc} K_1^{(1/2)}(X) & \hookrightarrow & H^1(X, K_2) \\ & \nwarrow \quad \nearrow & \\ & H^0(X, K_1) \otimes Ch^1(X) & \end{array}$$

We claim that the natural map $H^0(X, K_1) \otimes Ch^1(X) \to H^1(X, K_2)$ is an isomorphism. Indeed since G is of type F_4 or E_8, it has only trivial Tits algebras, and therefore, by [11, Theorem], one has

$$H^1(X, K_2) \simeq H^1(X_{\mathrm{sep}}, K_2)^\Gamma,$$

where Γ is the absolute Galois group of F. Moreover, since the variety X_{sep} is cellular, by [11, Proposition 1], one has

$$H^1(X_{\mathrm{sep}}, K_2) \simeq K_1 F_{\mathrm{sep}} \otimes Ch^1(X_{\mathrm{sep}}).$$

Thus, since the Picard group of any homogeneous projective variety under a group of type F_4 or E_8 is rational (cf [15, Example 4.1.1]) and since $(K_1 F_{\mathrm{sep}})^\Gamma = K_1 F = H^0(X, K_1)$, one has

$$H^1(X, K_2) \simeq K_1 F \otimes Ch^1(X) \simeq H^0(X, K_1) \otimes Ch^1(X),$$

and the claim is proved. Therefore, one has $E_\infty^{1,-2}(X) \simeq E_2^{1,-2}(X)$ and the proposition is proved. □

Remark 2.4. Assume that G_0 of *strongly inner* type (e.g F_4 and E_8) and consider an extension E/F and a cocycle $\xi \in H^1(E, G_0)$. By [13, Theorem 2.2.(2)], the change of field homomorphism

$$K({}_{\xi}(G_0/B)_E) \to K({}_{\xi}(G_0/B)_{\overline{E}}) \simeq K(G_0/B)$$

is an isomorphism, where $\overline{E}$ denotes an algebraic closure of E. Therefore, since the γ-filtration is defined in terms of Chern classes and the latter commute with pull-backs, the quotients of the γ-filtration on $K({}_{\xi}(G_0/B)_E)$ do not depend nor on the extension E/F neither on the choice of $\xi \in H^1(E, G_0)$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

First of all, note that the F-variety X is A-trivial in the sense of [10, Definition 2.3] (see [10, Example 2.5]), i.e for any extension L/F with $X(L) \neq \emptyset$, the degree homomorphism $\deg : Ch_0(X_L) \to \mathbb{Z}/p\mathbb{Z}$ is an isomorphism. Therefore, by [10, Lemma 2.9], the change of field homomorphism $Ch(Y) \to Ch(Y_{F(X)})$ is an isomorphism (in any codimension) if $1 \in \deg Ch_0(X)$. Hence, one can assume that $1 \notin \deg Ch_0(X)$.

Now, we know from [14, Table 4.13] that the J-invariant $J_p(G)$ of G is equal to (1) or (0). However, the assumption $J_p(G) = (0)$ implies that there exists a splitting field K/F of degree coprime to p (see [14, Corollary 6.7]), and in that case one has $Ch_0(X) \simeq Ch_0(X_K)$ and $1 \in \deg Ch_0(X_K)$ by A-triviality of X. Thus, under the assumption $1 \notin \deg Ch_0(X)$, one necessarly has $J_p(G) = (1)$ and that is why we can assume $J_p(G) = (1)$ in the sequel.

Since X is A-trivial, one can use the following proposition (cf [10, Proposition 2.8]).

Proposition 3.1 (Karpenko, Merkurjev)**.** *Given an equidimensional F-variety Y and an integer m such that for any i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$Ch^{m-i}(X) \to Ch^{m-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$Ch^m(Y) \to Ch^m(Y_{F(X)})$$

is also surjective.

Consequently, it is sufficient in order to prove the first conclusion of Theorem 1.1 to show that for any extension L/F, the change of field homomorphism

$$Ch(X) \longrightarrow Ch(X_L) \tag{3.2}$$

is surjective in codimension $< p+1$.

Moreover, the F-variety being generically split (see [14, Example 3.6]), one can apply the motivic decomposition result [14, Theorem 5.17] to X and get that the motive $\mathcal{M}(X, \mathbb{Z}/p\mathbb{Z})$ decomposes as a sum of twists of an indecomposable motive $\mathcal{R}_p(G)$ (in the same way as (3.5)). Note that the quantity and the value of those twists do not depend on the base field. In particular, we get that for any extension L/F and any integer k, the group $Ch^k(X_L)$ is isomorphic to a direct sum of groups $Ch^{k-i}(\mathcal{R}_p(G)_L)$ with $0 \leq i \leq k$.

Therefore, the surjectivity of (3.2) in codimension $< p+1$ is a consequence of the following proposition.

Proposition 3.3. *For any extension L/F, the change of field*

$$Ch(\mathcal{R}_p(G)) \longrightarrow Ch(\mathcal{R}_p(G)_L) \tag{3.4}$$

is surjective in codimension $< p+1$.

Proof. Let G_0 be a split linear algebraic group of the same type of the type of G and let $\xi \in H^1(F, G_0)$ be a cocycle such that G is isogenic to the twisted form ${}_\xi G_0$. We write $\mathfrak{B}$ for the Borel variety of G (i.e $\mathfrak{B} = {}_\xi(G_0/B)$, where B is a Borel subgroup of G_0).

By [14, Theorem 5.17], one has the motivic decomposition

$$\mathcal{M}(\mathfrak{B}, \mathbb{Z}/p\mathbb{Z}) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{3.5}$$

where $\Sigma_{i\geq 0} a_i t^i = P(CH(\overline{\mathfrak{B}}), t)/P(CH(\overline{\mathcal{R}_p(G)}), t)$, with $P(-,t)$ the *Poincaré polynomial.* Thus, for any integer k,we get the following decomposition concerning Chow groups

$$Ch^k(\mathfrak{B}_L) \simeq \bigoplus_{i\geq 0} Ch^{k-i}(\mathcal{R}_p(G)_L)^{\oplus a_i} \tag{3.6}$$

First of all, the homomorphism (3.4) is clearly surjective in codimension 0 since one has $Ch^0(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ for any extension L/F. Then, $Ch^1(\overline{\mathfrak{B}})$ is identified with the Picard group $\mathrm{Pic}(\overline{\mathfrak{B}})$ and is rational (see [15, Example 4.1.1]). Furthermore, thanks to the Solomon Theorem for example (see [15, §2.5]), one can compute the coefficients a_i's: we get $a_0 = 1$ and $a_1 = \mathrm{rank}(G) = \mathrm{rank}(Ch^1(\overline{\mathfrak{B}}))$. Thus, the isomorphism (3.6) implies that $Ch^1(\mathcal{R}_p(G)_L) = 0$ for any extension L/F.

We have already shown that the homomorphism (3.4) is surjective in codimension 0 and 1. The following lemma implies the surjectivity in codimension 2 and 3 (and therefore proves the first conclusion of Theorem 1.1 if G is of type F_4).

Lemma 3.7. *Under the assumption $J_p(G) = (1)$, one has*

$$Ch^2(\mathcal{R}_p(G)) = \mathbb{Z}/p\mathbb{Z} \ \text{ and } \ Ch^3(\mathcal{R}_p(G)) = 0$$

Proof. Since $J_p(G) = (1)$, by [6, Example 5.3], the cocycle $\xi \in H^1(F, G_0)$ match with a *generic* G_0-torsor in the sense of [6]. Thus, by [5, Proposition 3.2] and [4, pp. 31, 133], one has $\mathrm{Tors}_p CH^2(\mathfrak{B}) \neq 0$ (note that since an algebraic group of type F_4 or E_8 is simply connected, it is of strictly inner type, and we can use material from [5, §3]). The conclusion is given by [5, Proposition 5.4]. □

Let us fix an extension L/F. We now prove the surjectivity of (3.4) in codimension 2 and 3. By [14, Example 4.7], one has $J_p(G_L) = (0)$ or $J_p(G_L) = (1)$.

If $J_p(G_L) = (0)$ then one has $\mathcal{R}_p(G_L) = \mathbb{Z}/p\mathbb{Z}$ by [14, Corollary 6.7], and on the other hand the motivic decomposition given in [14, Proposition 5.18 (i)] implies the following decomposition on Chow groups for any integer k

$$Ch^k(\mathcal{R}_p(G)_L) \simeq \bigoplus_{i=0}^{p-1} Ch^{k-i(p+1)}(\mathcal{R}_p(G_L)). \tag{3.8}$$

In particular, one has $Ch^k(\mathcal{R}_p(G)_L) = 0$ for $k = 2$ or 3 and the conclusion follows.

If $J_p(G_L) = (1)$ then by Lemma 3.7 one has $Ch^2(\mathcal{R}_p(G_L)) = \mathbb{Z}/p\mathbb{Z}$ and $Ch^3(\mathcal{R}_p(G_L)) = 0$. Moreover, since $J_p(G_L) = J_p(G)$, one has $\mathcal{R}_p(G_L) \simeq \mathcal{R}_p(G)_L$ (see [14, Proposition 5.18 (i)]). Therefore, the homomorphism (3.4) is clearly surjective in codimension 3.

We claim that it is also surjective in codimension 2. By (3.6) it suffices to show that the change of field $Ch^2(\mathfrak{B}) \to Ch^2(\mathfrak{B}_L)$ is an isomorphism. We use material and notation introduced in section 2. Since $J_p(G) = J_p(G_L) = (1)$, the cocycles ξ and ξ_L match with generic G_0-torsors and one consequently has $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ and $\gamma^3(\mathfrak{B}_L) = \tau^3(\mathfrak{B}_L)$ (see [5, Theorem 3.1(ii)]). It follows that

$$\gamma^{2/3}(\mathfrak{B}) = \tau^{2/3}(\mathfrak{B}) \text{ and } \gamma^{2/3}(\mathfrak{B}_L) = \tau^{2/3}(\mathfrak{B}_L).$$

Therefore, since $2 < p+1$, the homomorphsim $Ch^2(\mathfrak{B}) \to Ch^2(\mathfrak{B}_L)$ coincides with

$$Ch^2(\mathfrak{B}) \simeq \gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_L) \simeq Ch^2(\mathfrak{B}_L)$$

and the center arrow is an isomorphism by Remark 2.4.

The surjectivity of (3.4) in codimension 4 and 5 is a direct consequence of the following statement, where G is of type E_8 and $p = 5$. Consequently, Lemma 3.9 completes the proof of the first conclusion of Theorem 1.1 for G of type E_8.

Lemma 3.9. *For any extension L/F, one has*

$$Ch^4(\mathcal{R}_5(G)_L) = 0 \text{ and } Ch^5(\mathcal{R}_5(G)_L) = 0$$

Proof. Since $J_5(G) = (1)$, we know that $J_5(G_L) = (1)$ or (0). If $J_5(G_L) = (0)$ then one has $R_5(G_L) = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism (3.8) implies that $Ch^4(\mathcal{R}_5(G)_L) = Ch^5(\mathcal{R}_5(G)_L) = 0$. Thus, one can assume $L = F$ and we have to prove that $Ch^4(\mathcal{R}_5(G)) = Ch^5(\mathcal{R}_5(G)) = 0$.

By Proposition 2.1 there exist an extension E/F and a cocycle $\xi' \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration coincide on $K(\mathfrak{B}')$, with $\mathfrak{B}' = {}_{\xi'}(G_0/B)$. Let us denote G' the variety ${}_{\xi'}G_0$.

We claim that $J_5(G') = (1)$. Indeed, assume that $J_5(G') = (0)$. In that case, one has $R_5(G') = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism (3.6) gives that $Ch^2(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$. Since $2 < p+1$, it implies that $\gamma^{2/3}(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$, and consecutively $\gamma^{2/3}(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ by Remark 2.4. However, we have $\gamma^{2/3}(\mathfrak{B}) \simeq \tau^{2/3}(\mathfrak{B})$ (because $\gamma^3(\mathfrak{B}) \simeq \tau^3(\mathfrak{B})$ since $\xi \in H^1(F, G_0)$ is generic). Thus, we have $Ch^2(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ which contradicts $Ch^2(\mathcal{R}_5(G)) = \mathbb{Z}/5\mathbb{Z}$ and the claim is proved (we recall that for any $i < 6 = p+1$, one has $\tau^{i/i+1}(X) \simeq Ch^i(X)$).

We now compute the groups $\gamma^{i/i+1}(\mathfrak{B}')$ for $i = 3, 4, 5$. Note that since $K(\mathfrak{B}') \simeq K(G_0/B)$ and since the description of the free group $K(G_0/B)$ in terms of generators does not depend on the characteristic char(E) of E (see [1, Lemma 13.3(4)]), we can assume that char$(E) = 0$ in order to compute those groups.

In that case, since $J_5(G') \neq (0)$, the isomorphism (3.6) combined with the following theorem (adapted from [10, Theorem RM.10] to our situation)

Theorem 3.10 (Karpenko, Merkurjev). *Let H be a semisimple linear algebraic group of inner type over a field of characteristic 0 and let p be a torsion prime of H. If $J_p(H) \neq (0)$ then*

$$Ch^j(\mathcal{R}_p(H)) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & if\ j = 0\ \ or\ j = k(p+1) - p + 1,\ 1 \leq k \leq p-1 \\ 0 & otherwise \end{cases}$$

gives that

$$\gamma^{i/i+1}(\mathfrak{B}') \simeq Ch^i(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for } i = 3, 4, 5$$

(where the first isomorphism is due to $i < p + 1$). Therefore, we get

$$\gamma^{i/i+1}(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for } i = 3, 4, 5$$

(with no particular assumption on $\text{char}(F)$). Thus, since $\tau^{3/4}(\mathfrak{B}) \simeq Ch^3(\mathfrak{B})$, the isomorphism (3.6) for $k = 3$ gives that $\tau^{3/4}(\mathfrak{B}) \simeq \gamma^{3/4}(\mathfrak{B})$. Since the γ-filtration is contained in the topological one, we get

$$\tau^4(\mathfrak{B}) = \gamma^4(\mathfrak{B}),$$

which implies the existence of an exact sequence

$$0 \to (\tau_5(\mathfrak{B})/\gamma_5(\mathfrak{B})) \to \gamma^{4/5}(\mathfrak{B}) \to \tau^{4/5}(\mathfrak{B}) \to 0.$$

Thus, since $\tau^{4/5}(\mathfrak{B}) \simeq Ch^4(\mathfrak{B})$, by applying the isomorphism (3.6) for $k = 4$, we get a surjection

$$\mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)} \twoheadrightarrow Ch^4(\mathcal{R}_5(G)) \oplus \mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)},$$

which implies that $Ch^4(\mathcal{R}_5(G)) = 0$.

We prove that $Ch^5(\mathcal{R}_5(G)) = 0$ by proceeding in exactly the same way. □

Consequently, Proposition 3.3 is proved. □

Finally, we want to prove the second conclusion of Theorem 1.1 ($p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8). First of all, since for any generic point ζ of Y, one has

$$1 \notin \deg Ch_0(X_{F(\zeta)}) \Leftrightarrow J_p(G_{F(\zeta)}) = (1),$$

by Proposition 3.1 and in view of what has already been done, it is sufficient to prove the following lemma to get the second conclusion.

Lemma 3.11. *Under the assumption $J_p(G) = (1)$, one has $Ch^{p+1}(\mathcal{R}_p(G)) = 0$.*

Proof. Thanks to Proposition 2.3, one can prove the lemma by proceeding in exactly the same way Lemma 3.9 has been proved. □

This concludes the proof of Theorem 1.1.

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