A rational construction of Lie algebras of type E_7

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Abstract

We give an explicit construction of Lie algebras of type E_7 out of a Lie algebra of type D_6 with some restrictions. Up to odd degree extensions, every Lie algebra of type E_7 arises this way. Some applications to Tits algebras and Rost invariant are mentioned.

1 Introduction

In [13] Jacques Tits wrote the following: "It might be worthwile trying to develop a similar theory for strongly inner groups of type E_7. For instance, can one give a general construction of such groups showing that there exist anisotropic strongly inner K-groups of type E_7 as soon as there exist central division associative 16-dimensional K-algebras of order 4 in $\operatorname{Br} K$ whose reduced norm is not surjective?"

The goal of the present paper is to give such (and much more general) construction. We deal with Lie algebras; of course, the corresponding group is just the automorphism group of its Lie algebra. By *rational* constructions we mean those not appealing to the Galois descent, that is involving only terms defined over the base field.

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Let us recall several milestones in the theory. Freudenthal in [5] gave an elegant explicit construction of the *split* Lie algebra of type E_7. On the language of maximal Lie subalgebras it is a particular case of A_7-construction. Another approach was proposed by Brown in [3] (see also [6] for a recent exposition); this is an E_6-construction. It gives only isotropic Lie algebras. In full generality A_7-construction was described by Allison and Faulkner in [1] as a particular case of a Cayley-Dickson doubling; generically it produces anisotropic Lie algebras of type E_7. Another construction with this property was discovered by Tits in [13]; in our terms it is an $A_3+A_3+A_1$-construction. On the other hand, some Lie algebras of type E_7 can be obtained via the Freudenthal magic square, see [11] (or [7] for a particular case).

Our strategy is to define a Lie triple system structure on the (64-dimensional over F) simple module of the even Clifford algebra of a central simple algebra of degree 12 with an orthogonal involution under some restrictions. Then the embedding Lie algebra is of type E_7. Our construction is of type $D_6 + A_1$.

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2 Lie triple systems and quaternionic gifts

Let F be a field of characteristic not 2. Recall that a *Lie triple system* is a vector space W over F together with a trilinear map

$$\begin{aligned} &W \times W \times W \to W \\ &\quad (u, v, w) \mapsto [u, v, w] = D(u, v)w \end{aligned}$$

satisfying the following axioms:

$$\begin{aligned} &D(u, u) = 0 \\ &D(u, v)w + D(v, w)u + D(w, u)v = 0 \\ &D(u, v)[x, y, z] = [D(u, v)x, y, z] + [x, D(u, v)y, z] + [x, y, D(u, v)z]. \end{aligned}$$

A *derivation* is a linear map $D\colon W \to W$ such that

$$D[x, y, z] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz].$$

The vector space of all derivations form a Lie algebra $\mathrm{Der}\,(W)$ under the usual commutator map.

The vector space $\mathrm{Der}\,(W) \oplus W$ under the map

$$[D+u, E+v] = [D,E] + D(u,v) + Dv - Eu$$

form a $\mathbb{Z}/2$-graded Lie algebra called the *embedding* Lie algebra of W. Conversely, degree 1 component of any $\mathbb{Z}/2$-graded Lie algebra is a Lie triple system under the triple commutator map.

Consider a $\mathbb{Z}$-graded Lie algebra

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

with one-dimensional components $L_{-2} = Ff$, $L_2 = Fe$, such that each L_i is an eigenspace of the map $[[e,f],\cdot]$ with the eigenvalue i. Then e, f and $[e,f]$ form an $\mathfrak{sl}_2$-triple, and the maps $[e,\cdot]$ and $[f,\cdot]$ are mutualy inverse isompor-phisms of L_{-1} and L_1. Moreover, maps $[x,\cdot]$ with x from $\langle e, f, [e,f]\rangle = \mathfrak{sl}_2$ defines a structure of left $\mathrm{M}_2(F)$-module on $L_1 \oplus L_{-1}$, that by inspection coincides with the usual structure on $F^2 \otimes L_1$ (after identification of L_{-1} and L_1 mentioned above).

Now L defines two kind of structures: one is a Lie triple structure on $L_1 \oplus L_{-1}$, and the other is a ternary system considered by Faulkner in [4] on L_1 (roughly speaking, it is an asymmetric version of a Freudenthal triple system). Namely, define maps $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot,\cdot\rangle$ by formulas

$$\begin{aligned}
&[u,v] = \langle u,v\rangle e \\
&\langle u,v,w\rangle = [[[f,u],v],w].
\end{aligned}$$

Note that $\langle\cdot,\cdot\rangle$ allows to identify the dual L_1^* with L_1, and so the map

$$L_1 \otimes L_1 \to \mathrm{End}\,(L_1)$$

corresponding to $\langle\cdot,\cdot,\cdot\rangle$ produces a linear map

$$\pi\colon\ \mathrm{End}\,(L_1) \to \mathrm{End}\,(L_1),$$

namely

$$\pi(\langle\cdot,u\rangle v) = \langle u,v,w\rangle.$$

By the Morita equivalence, we can consider π as a map

$$\mathrm{End}_{\,\mathrm{M}_2(F)}(F^2 \otimes L_1) \to \mathrm{End}_{\,\mathrm{M}_2(F)}(F^2 \otimes L_1).$$

Also, the same equivalence gives rise to a Hermitian (with respect to the canonical symplectic involution on $\mathrm{M}_2(F)$) form

$$\phi\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)=\begin{pmatrix}\langle u_1,v_2\rangle & -\langle u_1,v_1\rangle\\ \langle u_2,v_2\rangle & -\langle u_2,v_1\rangle\end{pmatrix}.$$

Now we want to relate the two structures on $V_1\oplus V_{-1}\simeq F^2\otimes V_1$. Direct calculation shows that

$$D(u,v)=\frac{1}{2}\big(\pi(\phi(\cdot,u)v-\phi(\cdot,v)u)+\phi(v,u)-\phi(u,v)\big). \qquad (*)$$

This description admits a Galois descent. Namely, let Q be a quaternion algebra over F, W be a left Q-module equipped with a Hermitian (with respect to the canonical involution on Q) form ϕ and a linear map

$$\pi\colon\ \mathrm{End}_Q(W)\to\mathrm{End}_Q(W).$$

Assume that ϕ and π become maps as above over a splitting field of Q. In terms of [7] this means that $\mathrm{End}_Q(W)$ together with π and the symplectic involution adjoint to ϕ form a *gift* (an abbreviation for a *generalized Freudenthal (or Faulkner) triple*); one can state the conditions on π and ϕ as a list of axioms not appealing to the descent (Garibaldi assumes that W is of dimension 28 over Q, but this really doesn't matter, at least under some additional restrictions on the characteristic of F). Then equation (*) defines on W a structure of a Lie triple system, hence the embedded Lie algebra $\mathrm{Der}(W)\oplus W$.

3 D_6+A_1-construction

We say that a map of functors $A\to B$ from fields to sets is *surjective at* 2 if for any field F and $b\in B(F)$ there exists an odd degree separable extension E/F and $a\in A(E)$ such that the images of a and b in $B(E)$ coincide.

We enumerate simple roots as in [2]. Erasing vertex 1 from the extended Dynkin diagram of E_7 we see that the simply connected split group E_7^{sc} contains a subgroup of type D_6+A_1, namely $(\mathrm{Spin}_{12}\times\mathrm{SL}_2)/\mu_2$. Its image in the adjoint group E_7^{ad} is $(\mathrm{HSpin}_{12}\times\mathrm{SL}_2)/\mu_2$, which we denote by H for brevity.

Theorem 1. *The map* $\mathrm{H}^1(F,H)\to\mathrm{H}^1(F,E_7^{ad})$ *is surjective at* 2.

Proof. Note that $W(D_6 + A_1)$ and $W(E_7)$ has the same Sylow 2-subgroup. Then the result follows by repeating the argument from the proof of Proposition 14.7, Step 1 in [8] (this argument is a kind of folklore). $\square$

The long exact sequence

$$\mathrm{H}^1(F, \mu_2) \to \mathrm{H}^1(F, H) \to \mathrm{H}^1(\mathrm{PGO}_{12}^+ \times \mathrm{PGL}_2) \to \mathrm{H}^2(F, \mu_2)$$

shows that the orbits of $\mathrm{H}^1(F, H)$ under the action of $\mathrm{H}^1(F, \mu_2)$ are the isometry classes of central simple algebras of degree 12 with orthogonal involutions (A, σ) and fixed isomorphism $\mathrm{Cent}(\mathrm{C}_0(A, \sigma)) \simeq F \times F$, with $[\mathrm{C}_0^+(A, \sigma)] = [Q]$ in $\mathrm{Br}(F)$ for some quaternion algebra Q, where C_0 stands for the Clifford algebra and C_0^+ for its first component (see [9, § 8] for definitions). Now

$$\mathrm{C}_0^+(A, \sigma) \simeq \mathrm{End}_Q(W)$$

for a 16-dimensional space W over Q, and the canonical involution on $\mathrm{C}_0^+(A, \sigma)$ induces a Hermitian form ϕ on W up to a scalar factor. It is not hard to see that $\mathrm{H}^1(F, H)$ parametrizes all the mentioned data together with ϕ (and not only its similarity class), and $\mathrm{H}^1(F, \mu_2)$ multiplies ϕ by the respective constant. Over a splitting field of Q the 32-dimensional half-spin representation carries a structure of Faulkner ternary system, so we are in the situation of Section 2. The resulting embedding $66 + 3 + 64 = 133$-dimensional Lie algebra $\mathrm{Der}(W) \oplus W$ is the twist of the split Lie algebra of type E_7 obtained by a cocycle representing the image in $\mathrm{H}^1(F, E_7^{ad})$. Theorem 1 shows that any Lie algebra of type E_7 over F arises this way up to an odd degree extension.

4 Tits algebras and Rost invariant

Recall that the class of Tits algebra of a cocycle class from $\mathrm{H}^1(F, E_7^{ad})$ is its image under the connecting map of the long exact sequence

$$\mathrm{H}^1(F, E_7^{sc}) \to \mathrm{H}^1(F, E_7^{ad}) \to \mathrm{H}^2(F, \mu_2).$$

The sequence fits in the following diagram:

$$\begin{tikzcd}
\mathrm{H}^1(F, (\mathrm{Spin}_{12} \times \mathrm{SL}_2)/\mu_2) \arrow[r] \arrow[d] & \mathrm{H}^1(F, H) \arrow[r] \arrow[d] & \mathrm{H}^2(F, \mu_2) \arrow[d, equal] \\
\mathrm{H}^1(F, E_7^{sc}) \arrow[r] & \mathrm{H}^1(F, E_7^{ad}) \arrow[r] & \mathrm{H}^2(F, \mu_2).
\end{tikzcd}$$

Since the middle vertical arrow is surjective at 2, we obtain the following result:

Theorem 2. *The class of the Tits algebra in* $\mathrm{Br}(F)$ *of the class in* $\mathrm{H}^1(F, E_7^{ad})$ *corresponding to* $\mathrm{Der}\,(W) \oplus W$ *is* $[A] + [Q]$. *For any cocycle class from* $\mathrm{H}^1(F, E_7^{ad})$ *there is an odd degree extension* E/F *such that the class of the Tits algebra in* $\mathrm{Br}(E)$ *is a sum of three symbols.*

Proof. Indeed, by the fundamental relation for groups of type D_6 (see [9, 9.14]) the image of a cocycle class from $\mathrm{H}^1(F, H)$ in $\mathrm{H}^2(F, \mu_2)$ is $[A] + [Q]$. Here Q is a quaternion algebra, and A over an odd degree extension is Brauer-equivalent to an algebra of degree 4 and exponent 2, that is to a biquaternion algebra. The second claim follows from Theorem 1. $\square$

Now we reproduce a construction from [13] in our terms. Let D be an algebra of degree 4 and μ be a constant from $F^\times$. By the exceptional isomorphism $A_3 = D_3$ the group $\mathrm{PGL}_1(D)$ defines a 3-dimensional anti-Hermitian form h over Q up to a constant, where $[Q] = 2[D]$ in $\mathrm{Br}(F)$. Consider the algebra $\mathrm{M}_6(Q)$ with the orthogonal involution σ adjoint to the 6-dimensional form $h \perp -\mu h$. One of the component of $\mathrm{C}_0(\mathrm{M}_6(Q), \sigma)$ is trivial in $\mathrm{Br}(F)$ and the other is Brauer-equivalent to Q. Choose ϕ on $W = Q^1 6$; by Theorem 2 the class of the Tits algebra of the corresponding cocycle class in $\mathrm{H}^1(F, E_7^{ad})$ is trivial, so the cocycle class comes from some $\xi \in \mathrm{H}^1(F, E_7^{sc})$. Let us compute the Rost invariant of ξ (see [8] or [9, § 31] for definitions).

Theorem 3. *For* D *and* μ *as above, there is a cocycle from* $\mathrm{H}^1(F, E_7^{sc})$ *whose Rost invariant is* $(\mu) \cup [D]$. *In particular, if this element cannot be written as a sum of two symbols from* $\mathrm{H}^3(F, \mathbb{Z}/2)$ *with a common slot, then there is a strongly inner anisotropic group of type* E_7 *over* F.

Proof. Consider ξ as above. Over the function field $F(\mathrm{SB}(Q))$ the image of the Rost invariant of ξ equals to the Arason invariant of the 12-dimensional quadratic form Morita-equivalent to $h - \mu h$. Explicitly, over $F(\mathrm{SB}(Q))$ the algebra D becomes a biquaternion algebra $(a, b) \otimes (c, d)$, and the quadratic form is Witt equivalent to $\langle\langle \mu \rangle\rangle(\langle\langle a, b\rangle\rangle - \langle\langle c, d\rangle\rangle)$, so the Arason invariant equals the image of $(\mu) \cup [D]$ over $F(\mathrm{SB}(Q))$. It follows that the Rost invariant is $(\mu) \cup [D] + (\lambda) \cup [Q]$ for some Q. Changing ϕ to $\lambda\phi$ adds $(\lambda) \cup [Q]$ (cf. [9, p. 441]), so there is a cocycle class from $\mathrm{H}^1(F, E_7^{sc})$ whose Rost invariant is $(\mu) \cup [D]$. The last claim follows from the easy computation of the Rost invariant of cocycles corresponding to isotropic groups of type E_7 (cf. [8, Appendix A, Proposition]). $\square$

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