MOTIVIC DECOMPOSITIONS OF TWISTED FLAG VARIETIES AND REPRESENTATIONS OF HECKE-TYPE ALGEBRAS

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Abstract. Motivated by the motivic Galois group and the Kostant-Kumar results on equivariant cohomology of flag varieties, we provide a uniform description of motivic (direct sum) decompositions with integer coefficients of versal flag varieties in terms of integer representations of the associated affine nil-Hecke algebra H.

More generally, we establish an equivalence between the h-motivic subcategory generated by the motive of E/B and the category of projective modules of the associated rational algebra D of push-pull operators, where E is a torsor for a split semisimple linear algebraic group G over a field k, B is a Borel subgroup of G, h is an algebraic oriented cohomology theory in the sense of Levine-Morel (e.g. Chow ring CH or an algebraic cobordism Ω). The algebra D can be think of as an integer-analogue of the 'Hopf-algebra of the h-motivic Galois group of E/B.

As an application, taking $h = CH$ and specializing the coefficients to the finite field $\mathbb{F}_p$ we obtain that p-modular projective representations of $D = H$ are generated by an irreducible H-module corresponding to the generalized Rost-Voevodsky motive for (G, p).

Contents

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1. INTRODUCTION

Let G be a split semisimple linear algebraic group over a field k, let E be a G-torsor over K/k. Consider a twisted form E/B over K of the variety of Borel subgroups G/B of G, e.g., a variety of (complete) flags of ideals in a central division algebra over K. In general, such a variety neither have a K-rational point nor any (relative) cellular filtration over K.

Consider the pseudo-abelian tensor category of Grothendieck-Chow motives of smooth projective varieties over K with coefficients in a ring $\mathbf{R}$. The main result of [25] says that the motive of E/B with finite coefficients ($\mathbf{R} = \mathbb{F}_p$) is always a direct sum of Tate twists of some indecomposable motive $\mathcal{R}_{E,p}$, a generalization of the Rost-Voevodsky motive. In other words, the tensor subcategory $\langle E/B\rangle_{\mathbb{F}_p}$ generated by all direct summands of E/B is, indeed, generated by $\mathcal{R}_{E,p}$, i.e.,

$$\langle E/B\rangle_{\mathbb{F}_p} = \langle \mathcal{R}_{E,p}\rangle.$$

The motive $\mathcal{R}_{E,p}$ has several remarkable properties. For instance, it is closely related to various cohomological invariants of G-torsors. If p is not a torsion prime of G or if the coefficient ring $\mathbf{R}$ has characteristic 0, then $\mathcal{R}_{E,p}$ coincides with the motive of a point, so $\langle E/B\rangle_{\mathbb{F}_p}$ gives the subcategory of Tate motives. While being indecomposable over k, the motive $\mathcal{R}_{E,p}$ becomes isomorphic to a direct sum of Tate motives over the splitting field $\bar{k}$ of E (as $\bar{k}$ one can always take the algebraic closure of k or the function field of E/B). Moreover, the generating function of $\mathcal{R}_{E,p}$ over $\bar{k}$ (counting the number of Tate motives in each dimension) is given by an explicit cyclotomic polynomial involving the p-exceptional degrees of V.Kac [19]. For example, if E is a G-torsor, where G is an exceptional group of type F_4 and E splits by a cubic field extension, then $\mathcal{R}_{E_3}$ corresponds to the Rost-Serre cohomological invariant and $\mathcal{R}_{E,3}|_{\bar{k}} \simeq \mathbb{F}_3 \oplus \mathbb{F}_3(4) \oplus \mathbb{F}_3(8)$.

As for integer coefficients ($\mathbf{R} = \mathbb{Z}$) only very few facts are known concerning the category $\langle E/B\rangle_{\mathbb{Z}}$. An integer version of the motive $\mathcal{R}_E$ was introduced and discussed in [26]; in [5], [10] it was shown that $\langle E/B\rangle_{\mathbb{Z}}$ is not Krull-Schmidt (the uniqueness of a direct sum decomposition fails).

In the present paper we cover all the mentioned cases ($\mathbf{R} = \mathbb{Z}, \mathbb{F}_p$). More generally, we consider the category of $\mathtt{h}$-correspondences with coefficients in $\mathbf{R}$, where $\mathtt{h}$ is any algebraic oriented Borel-Moore homology of [23] (e.g. Chow groups, connective K-theory, elliptic cohomology, algebraic cobordism Ω of Levine-Morel) and $\mathbf{R} = \mathtt{h}(K)$ is its coefficient ring. Let $\langle E/B\rangle_{\mathtt{h}}$ denote the respective tensor subcategory generated by indecomposable summands of the $\mathtt{h}$-motive of E/B. Our main result (Theorem 8.1) establishes an equivalence between the motivic category $\langle E/B\rangle_{\mathtt{h}}$ and certain category of finitely generated projective $\overline{\mathbf{D}}_F$-modules

$$\langle E/B\rangle_{\mathtt{h}} \simeq \mathit{Proj}\ \overline{\mathbf{D}}_F, \tag{1}$$

where $\overline{\mathbf{D}}_F$ is the $\mathbf{R}$-algebra defined using the formal push-pull operators for the group G and the theory $\mathtt{h}$. So it provides a direct link between integer/modular $\mathtt{h}$-motivic decompositions of twisted flag varieties and integer/modular representations of Hecke-type algebras $\overline{\mathbf{D}}_F$.

If E is a versal (generic) torsor, then $\overline{\mathbf{D}}_F$ can be replaced by the formal affine Demazure algebra $\mathbf{D}_F$. The theory of such algebras and formal push-pull operators has been recently developed in [6], [18], [7], [8], [9] motivated by Bernstein-Gelfand-Gelfand [2], Demazure [11], [12], Bressler-Evens [3], [4] and Kostant-Kumar [21], [20] results. The key properties of $\mathbf{D}_F$ are

- it is a free module over the T-equivariant oriented cohomology ring $\mathbf{S} = \mathtt{h}_T(K)$ of a point, where T is a split maximal torus in G;
- its $\mathbf{S}$-dual $\mathbf{D}_F^\star = Hom_{\mathbf{S}}(\mathbf{D}_F, \mathbf{S})$ is isomorphic to the T-equivariant oriented cohomology $\mathtt{h}_T(G/B)$ of G/B [9] and
- its structure (generators and relations) is very close to those of the affine Hecke algebra [18].

For example, if $\mathtt{h} = CH$ (Chow groups) and $\mathbf{R} = \mathbb{F}_p$ as before, then $\mathbf{D}_F = \mathbf{H}_{nil,p}$ is the affine nil-Hecke algebra (in the notation of Ginzburg [16, §12]) over $\mathbb{F}_p$ which is a free module of rank $|W|$ over the polynomial ring $\mathbf{S} = \mathbb{F}_p[x_1, \ldots, x_n]$, where n is the rank of G and W is the Weyl group, and $\mathbf{D}_F^\star \simeq CH_T(G/B; \mathbb{F}_p)$ is the T-equivariant Chow groups. For a versal torsor E the equivalence (1) then turns into

$$\langle \mathcal{R}_{E,p} \rangle \simeq \mathit{Proj}\ \mathbf{H}_{nil,p}$$

meaning that all indecomposable projective $\mathbf{H}_{nil,p}$-modules are isomorphic to each other (up to a shift). Moreover, their ranks over $\mathbf{S}$ equal to the p-part of the product of p-exceptional degrees of the group G.

Roughly speaking, the algebra $\overline{\mathbf{D}}_F$ can be viewed as an integral analogue of the Hopf-algebra of the motivic Galois group of E/B (see e.g. [1]). Indeed, if taken with $\mathbb{Q}$-coeffcients (or if E is split), the algebra $\overline{\mathbf{D}}_F$ becomes isomorphic to $\mathrm{End}_{\mathbf{R}}\, \mathtt{h}(G/B) \simeq M_{|W|}(\mathbf{R})$ and, hence, the category $\mathit{Proj}\, \overline{\mathbf{D}}_F$ can be identified with the category of representations $\mathit{Proj}\, \mathbb{Q}[\mathbb{G}_m] = \mathit{Rep}\, \mathbb{G}_m$ with $\mathbb{G}_m$ known to be the motivic Galois group of $\langle E/B \rangle_{\mathbb{Q}}$. Observe that in general, $\overline{\mathbf{D}}_F$ is not a matrix algebra over $\mathbf{R}$.

In the paper we restrict ourselves to varieties E/B of Borel subgroups only. However, by [5] we have $\langle E/B \rangle_{\mathtt{h}} = \langle E/P \rangle_{\mathtt{h}}$ for any special parabolic subgroup P. Hence, B can be replaced by any such P without affecting the equivalence (1). For instance, for $G = PGL_{p^n}$, $\mathtt{h} = CH$, $\mathbf{R} = \mathbb{Z}$ and E corresponding to a generic central division algebra A of degree p^n we get

$$\langle SB(A) \rangle_{\mathbb{Z}} \simeq \mathit{Proj}\, \mathbf{H}_{nil,\mathbb{Z}},$$

where $SB(A)$ is the Severi-Brauer variety of A and $\mathbf{H}_{nil,\mathbb{Z}}$ is the affine nil-Hecke algebra with integer coefficients.

The paper is organized as follows. In section 2 we recall definitions and basic facts concerning Borel-Moore homology $\mathtt{h}$ and the respective category of $\mathtt{h}$-motives. We state a version of the Künneth isomorphism for cellular spaces. In the next section we generalize it to the equivariant setting. In section 4 we introduce the convolution product on the equivariant cohomology of group powers and study its properties. In the next section we identify this equivariant cohomology with the endomorphism ring on equivariant cohomology of G/B and then in section 6 with the formal affine Demazure algebra. In section 7 we introduce the notion of a rational algebra of push-pull operators $\overline{\mathbf{D}}_F$ and identify it with the subring of rational endomorhisms. In the last section we prove the equivalence (1) and provide some applications and examples.

2. Oriented (co-)homology

We recall definitions of an algebraic oriented Borel-Moore homology and of the respective category of correspondences. We also recall a version of the Künneth isomorphism for cellular spaces (Lemmas 2.4 and 2.5).

Fix a smooth scheme S over a field k. Let Sch_S denote the category of finite type quasi-projective separated S-schemes and let Sm_S denote its full subcategory consisting of smooth quasi-projective S-schemes.

Following [23, Def. 5.1.3] consider an oriented graded Borel-Moore homology theory $\mathtt{h}_\bullet$ defined on some admissible [23, (1.1)] subcategory $\mathcal{V}$ of Sch_S. So that there are pull-backs $f^*\colon \mathtt{h}_\bullet(X) \to \mathtt{h}_{\bullet+d}(Y)$ for l.c.i. morphisms $f\colon Y \to X$ in $\mathcal{V}$ of relative dimension d and push-forwards $f_*\colon \mathtt{h}_\bullet(Y) \to \mathtt{h}_\bullet(X)$ for projective morphisms $f\colon X \to Y$ in $\mathcal{V}$. According to [23, Prop. 5.2.1] the Borel-Moore homology $\mathtt{h}_\bullet$ restricted to Sm_S defines an algebraic oriented cohomology theory $\mathtt{h}^\bullet$ (with values in the category of graded commutative rings with unit) in the sense of [23, Def. 1.1.2] by

$$\mathtt{h}^{\dim_S X-\bullet}(X) := \mathtt{h}_\bullet(X), \quad X \in Sm_S.$$

If the (co-)dimension is clear from the context we will write simply $\mathtt{h}(X)$.

Following [26, §2] (see also [14, §63]) we define the category of $\mathtt{h}$-correspondences $\mathtt{h}\text{-}Corr_S$ over S. The objects are pairs $([X \to S], i)$, where $[X \to S]$ is an isomorphism class of a smooth projective map $X \to S$ and $i \in \mathbb{Z}$. The morphisms are defined by

$$Hom_{\mathtt{h}\text{-}Corr_S}(([Y \to S], i), ([X \to S], j)) := \bigoplus_l Hom_{i-j}([Y_l \to S], [X \to S]),$$

taken over all connected components Y_l of Y, where

$$Hom_\bullet([Y_l \to S], [X \to S]) := \mathtt{h}_{\dim_S Y_l+\bullet}(Y_l \times_S X).$$

The composition of morphisms is given by the correspondence product. Namely, if $p_i\colon X_1 \times_S X_2 \times_S X_3 \to X_j \times_S X_{j'}$ denotes the projection obtained by removing the i-th coordinate, then given $\alpha \in \mathtt{h}(X_1 \times_S X_2)$ and $\beta \in \mathtt{h}(X_2 \times_S X_3)$ we set

$$\beta \circ \alpha := (p_2)_*(p_1^*(\beta) \cdot p_3^*(\alpha)) \in \mathtt{h}(X_1 \times_S X_3). \tag{2}$$

Let $\mathtt{h}\text{-}Corr_S^+$ denote the additive completion of $\mathtt{h}\text{-}Corr_S$. We simply write X for the respective class in $\mathtt{h}\text{-}Corr_S^+$.

Definition 2.1. (cf. [23, (CD')]) Let X be smooth projective over S. Suppose that there is a filtration by proper closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

such that

- each irreducible component X_{ij} of $X_i \setminus X_{i-1}$ is a locally trivial affine fibration over S of rank d_{ij}, and
- the closure of X_{ij} in X admits a resolution of singularities $\widetilde{X}_{ij} \to \overline{X}_{ij}$ over S; we set $g_{ij}\colon \widetilde{X}_{ij} \to \overline{X}_{ij} \hookrightarrow X$ and, therefore, $(g_{ij})_*(1_{\widetilde{X}_{ij}}) \in \mathtt{h}_{d_{ij}}(X)$.

We call such X (together with the filtration) a cellular space over S.

Definition 2.2. We say that the theory $\mathtt{h}$ satisfies the cellular decomposition (CD) property if given a cellular space X over S the respective elements $(g_{ij})_*(1_{\widetilde{X}_{ij}})$ form a $\mathtt{h}(S)$-basis of $\mathtt{h}(X)$.

Example 2.3. The property (CD) holds for any oriented Borel-Moore homology $\mathtt{h}$ over a field k of characteristic 0.

Indeed, the same reasoning as in [14, Thm. 66.2] shows that for every $Z \in Sm_S$ there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z \colon \bigoplus_{ij} CH_{\bullet - d_{ij}}(Z) \to CH_{\bullet}(Z \times_S X).$$

By the Yoneda lemma (cf. [14, Lemma 63.9]) the latter induces an isomorphism in the category $CH\text{-}Corr_S^+$ (cf. [14, Cor. 66.4]).

Following [29, §2] consider the specialization functor $\Omega\text{-}Corr_S^+ \to CH\text{-}Corr_S^+$, $[f \colon Y \to X] \mapsto f_*(1_Y)$. It is surjective on the classes of objects and morphisms. Moreover, for every X the kernel of

$$\Omega_{\dim_S X}(X \times_S X) \longrightarrow CH_{\dim_S X}(X \times_S X)$$

is $\Omega_{\geqslant 1}(k) \cdot \Omega_{\bullet}(X \times_S X)$ by [23, Rem.4.5.6]. Hence for every y in this kernel

$$y^{\circ(\dim_S X+1)} \in \Omega_{\dim_S X}(X \times_S X) \cap (\Omega_{\geqslant (dim_S X+1)}(k) \cdot \Omega_{\bullet}(X \times_S X)).$$

So $y = 0$ since $\Omega_{<0}(Y) = 0$. Therefore, the kernel of

$$\mathrm{End}_{\Omega\text{-}Corr_S^+}(X, i) \to \mathrm{End}_{CH\text{-}Corr_S^+}(X, i)$$

consists of nilpotents.

Finally, by [29, Lemma 2.1] the isomorphism $\sum_{ij} (g_{ij})_*(1)$ in $CH\text{-}Corr_S^+$ can be lifted to an isomorphism in the category $\Omega\text{-}Corr_S^+$. Specializing it via $\Omega \to \mathtt{h}$ we obtain the desired isomorphism.

From this point on we assume that $\mathtt{h}$ satisfies the property (CD).

Lemma 2.4. *Let X be a cellular space over S. Then there is an isomorphism in $\mathtt{h}\text{-}Corr_S^+$*

$$\sum_{ij} (g_{ij})_*(1_{\widetilde{X}_{ij}}) \colon \bigoplus_{ij} (S, d_{ij}) \to X,$$

where $(g_{ij})_*(1_{\widetilde{X}_{ij}}) \in \mathtt{h}_{d_{ij}}(X) = Hom_{\mathtt{h}\text{-}Corr_S^+}((S, d_{ij}), X)$.

Proof. Transversal base change implies that there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z \colon \bigoplus_{ij} \mathtt{h}_{\bullet - d_{ij}}(Z \times_S S) \to \mathtt{h}_{\bullet}(Z \times_S X)$$

for any Z smooth projective over S. So by the Yoneda lemma (cf. [14, Lemma 63.9]) it induces an isomorphism in $\mathtt{h}\text{-}Corr_S^+$ (cf. [14, Cor. 66.4]). ☐

Lemma 2.5. *The pairing $(\cdot,\cdot) \colon \mathtt{h}(X) \otimes_{\mathtt{h}(S)} \mathtt{h}(X) \to \mathtt{h}(S)$ given by $(a, b) = p_*(ab)$ is non-degenerate and the map*

$$f \colon (\mathtt{h}(X \times_S X), \circ) \to \mathrm{End}_{\mathtt{h}(S)}\, \mathtt{h}(X) \quad \text{given by } a \mapsto f_a,\ f_a(x) = (p_2)_*(p_1^*(x) \cdot a)$$

is an $\mathtt{h}(S)$-linear isomorphism of graded rings. In particular, it gives an $\mathtt{h}(S)$-linear isomorphism

$$(\mathtt{h}_{\dim_S X}(X \times_S X), \circ) \simeq \mathrm{End}_{\mathtt{h}\text{-}Corr_S^+}(X).$$

Observe that the endomorphism ring of $\mathtt{h}(S)$-linear operators $\mathrm{End}_{\mathtt{h}(S)}(\mathtt{h}(X))$ is a graded ring. Its n-th graded component consists of operators increasing the codimension by n. By definition the subring of degree-0 operators (preserving the codimension) coincides with $\mathrm{End}_{\mathtt{h}\text{-}Corr_S^+}(X)$.

Proof. By the previous lemma there is an isomorphism

$$\bigoplus_{ij}\mathtt{h}(S)=\bigoplus_{k=-\infty}^{\infty} Hom((S,k),\oplus_{ij}(S,d_{ij}))\overset{\simeq}{\to}\bigoplus_{k=-\infty}^{\infty} Hom((S,k),X)=\mathtt{h}(X),$$

where each component is given by $x\mapsto x\cdot(g_{ij})_*(1)$. Let $\sum_{ij}a_{ij}\colon X\to\oplus_{ij}(S,d_{ij})$ be the inverse isomorphism in $\mathtt{h}\text{-}Corr_S^+$. Observe that

$$a_{ij}\in Hom(X,(S,d_{ij}))=\mathtt{h}_{\dim(X/S)-d_{ij}}(X).$$

Since $a_{ij}\circ(g_{ij})_*(1)=p_*(a_{ij}\cdot(g_{ij})_*(1))=\delta_{i,j}$, the pairing $(\cdot,\cdot)$ is non-degenerate.

The pairing $(\cdot,\cdot)$ gives an isomorphism $\mathtt{h}(X)\to Hom_{\mathtt{h}(S)}(\mathtt{h}(X),\mathtt{h}(S))$ and, hence, an isomorphism $\mathrm{End}_{\mathtt{h}(S)}\mathtt{h}(X)\overset{\simeq}{\to}\mathtt{h}(X)\otimes_{\mathtt{h}(S)}\mathtt{h}(X)$. Consider the composition

$$\rho\colon\mathtt{h}(X\times_S X)\xrightarrow{f}\mathrm{End}_{\mathtt{h}(S)}\mathtt{h}(X)\overset{\simeq}{\to}\mathtt{h}(X)\otimes_{\mathtt{h}(S)}\mathtt{h}(X)$$

and a map $\pi\colon\mathtt{h}(X)\otimes\mathtt{h}(X)\to\mathtt{h}(X\times_S X)$ given by $\pi(a\otimes b)=p_1^*(a)\cdot p_2^*(b)$.

By definition, we have

$$f_{p_1^*(a)p_2^*(b)}(x)=(p_2)_*(p_1^*(x)p_1^*(a)p_2^*(b))=(x,a)b.$$

Hence, $\rho(\pi(a\otimes b))=a\otimes b$ and the map ρ is surjective. By the property (CD) for $X\times_S X\to X$, $\mathtt{h}(X\times_S X)$ is a free $\mathtt{h}(X)$-module of rank $rk_{\mathtt{h}(S)}\mathtt{h}(X)$. Thus, ρ is a surjective homomorphism between free modules of the same rank, hence, it is an isomorphism. □

3. THE EQUIVARIANT KÜNNETH ISOMORPHISM

In the present section we introduce an equivariant Borel-Moore homology following [7, §2] and [17]. We provide an equivariant analogue of the Künneth isomorphism (Lemma 3.7).

Let G be a smooth group scheme over S. Consider an admissible subcategory $\mathcal{V}^G$ of the category of G-varieties $X\in Sch_S$ with G-equivariant morphisms. By a G-equivariant oriented (graded) Borel-Moore homology theory we will call an additive functor $\mathtt{h}_\bullet^G$ from $\mathcal{V}^G$ to graded abelian groups such that

1. There are pull-backs for l.c.i. maps and push-forwards for projective maps that satisfy

(TS) (l.c.i. base change) For a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where f (hence f') is l.c.i. and g (hence g') is projective, we have $f^*g_*=g'_*(f')^*$.

(Loc) (localization) If $U\subset X$ is an open G-equivariant embedding with $Z=X\setminus U$, then there is a right exact sequence:

$$\mathtt{h}_\bullet^G(Z)\to\mathtt{h}_\bullet^G(X)\to\mathtt{h}_\bullet^G(U)\to 0.$$

2. The functor $\mathtt{h}_\bullet^G$ restricted to Sm_S defines a graded G-equivariant oriented cohomology theory $\mathtt{h}_G^\bullet$ in the sense of [9] (we refer to [9, §2, A1-9] for the precise definition) by

$$\mathtt{h}_G^{\dim_S X-\bullet}(X):=\mathtt{h}_\bullet^G(X),\quad X\in Sm_S.$$

In addition to the axioms of [9, §2] we require that $\mathtt{h}_G$ satisfies the following stronger version of the homotopy invariance axiom:

(HI) (extended homotopy invariance) Let $p\colon Y\to X$ be a G-equivariant torsor of a vector bundle of rank r over X, then the pull-back induced by projection

$$p^*\colon \mathtt{h}_G^\bullet(X)\to \mathtt{h}_G^\bullet(Y)$$

is an isomorphism.

If a variety is smooth we will always use the cohomology notation.

Example 3.1. Given a linear algebraic group G over a field k of characteristic zero an example of such G-equivariant Borel-Moore homology theory $\mathtt{h}_\bullet^G$ was constructed in [17] as follows.

Consider a system of G-representations V_i and its open subsets $U_i\subseteq V_i$ such that

- G acts freely on U_i and the quotient U_i/G exists as a scheme over k,
- $V_{i+1}=V_i\oplus W_i$ for some representation W_i,
- $U_i\subseteq U_i\oplus W_i\subseteq U_{i+1}$, and $U_i\oplus W_i\to U_{i+1}$ is an open inclusion, and
- $\operatorname{codim}(V_i\setminus U_i)$ strictly increases.

Such a system is called a good system of representations of G.

Let $X\in Sch_k$ be a G-variety. Following [17, §3 and §5] the inverse limit induced by pull-backs

$$\varprojlim_i \mathtt{h}_{\bullet-\dim G+\dim U_i}(X\times^G U_i),\quad X\times^G U_i=(X\times_k U_i)/G,$$

does not depend on the choice of the system (V_i,U_i) and, hence, defines the G-equivariant oriented homology group $\mathtt{h}_\bullet^G(X)$.

In the present paper we will extensively use the following property (cf. [9, §2, A6]) of an equivariant theory

(Tor) Let $X\to X/G$ be a G-torsor over S and a G'-equivariant map for some group scheme G' over S. Then there is an isomorphism

$$\mathtt{h}_{G\times G'}^\bullet(X)\xrightarrow{\simeq}\mathtt{h}_{G'}^\bullet(X/G).$$

that is natural with respect to the maps of pairs

$$(\phi,\gamma)\colon (X,G\times G')\to(X_1,G_1\times G_1'),\quad \phi(x\cdot(g,g'))=\phi(x)\cdot\gamma(g,g').$$

Observe that the theory of Example 3.1 satisfies this property by [17, Prop. 27].

We have the following equivariant analogues of Definitions 2.1 and 2.2

Definition 3.2. Let $X\in\mathcal{V}^G$. Suppose that there is a filtration by G-equivariant proper closed subschemes

$$\emptyset=X_{-1}\subset X_0\subset X_1\subset\ldots\subset X_n=X$$

such that

- each irreducible component X_{ij} of $X_i\setminus X_{i-1}$ is a G-equivariant (locally trivial) affine fibration over S of rank d_{ij}, and

- the closure of X_{ij} in X admits a G-equivariant resolution of singularities $g_{ij}\colon \widetilde{X}_{ij} \to \overline{X}_{ij}$ over S.

We call such X (together with the filtration) a G-equivariant cellular space over S.

Definition 3.3. We say that the equivariant theory $\mathtt{h}^G$ satisfies the cellular decomposition (CD) property if given a G-equivariant cellular space X over S the respective elements $(g_{ij})_*(1_{\widetilde{X}_{ij}})$ form a $\mathtt{h}^G(S)$-basis of $\mathtt{h}^G(X)$.

Lemma 3.4. *Suppose a morphism $f\colon X \to Y$ in Sm_k factors as $f\colon X \xrightarrow{z} L \xrightarrow{j} Y$ where $p\colon L \to X$ is a vector bundle, $z\colon X \to L$ is a zero section and j is an open embedding.*

Then for every projective map $a\colon Y' \to Y$ and $X' = X \times_Y Y'$ the following diagram of pull-back and push-forward maps commutes (we omit the grading)

$$\begin{tikzcd}
\mathtt{h}(X') \arrow[r, "a'_*"] & \mathtt{h}(X) \\
\mathtt{h}(Y') \arrow[r, "a_*"] \arrow[u, "f'^*"] & \mathtt{h}(Y) \arrow[u, "f^*"']
\end{tikzcd}$$

Proof. Observe that the map $f'\colon X' \to Y'$ factors as $X' \xrightarrow{z'} L \times_Y Y' \xrightarrow{j'} Y'$ where z' is the zero section of the vector bundle $p'\colon L' = L \times_Y Y' \to X'$ and j' is an open embedding. Let b denote the canonical map $L' \to L$. Since j and j' are flat, we have $j^*a_* = b_*j'^*$ by the l.c.i. base change for oriented theories. Note that by the homotopy invariance $z^* = (p^*)^{-1}$ and $z'^* = (p'^*)^{-1}$. Since p and p' are flat, $p^*a'_* = b_*p'^*$. Then $z^*b_* = a'_*z'^*$ and

$$f^*a_* = z^*j^*a_* = z^*b_*j'^* = a'_*z'^*j'^* = a'_*f'^*. \qquad \square$$

Remark 3.5. If (V_i, U_i) is a good system of representations of Example 3.1, then for any G-variety X the connecting maps $X \times^G U_i \to X \times^G U_{i+1}$ factor as in Lemma 3.4, i.e., we have $X \times^G U_i \to X \times^G (U_i \oplus W_i) \to X \times^G U_{i+1}$.

Example 3.6. Let $\mathtt{h}^G$ be the equivariant theory of Example 3.1. Then the property (CD) holds for $\mathtt{h}^G$.

Indeed, consider a good system of representations $\{(V_j, U_j)\}_j$ for X. The subvarieties $X_i \times^G U_j$, $i = 0 \ldots n$ form a cellular filtration on $X \times^G U_j$ over $S \times^G U_j$. Note that $\widetilde{X}_i \times^G U_j$ is a resolution of singularities of $X_i \times^G U_j$. By (CD) for $\mathtt{h}$ the set $\{(f_i \times^G id_{U_j})_*(1)\}_i$ forms a basis of $\mathtt{h}(X \times^G U_j)$ as a $\mathtt{h}(S \times^G U_j)$-module. By Lemma 3.4 the following diagram commutes:

$$\begin{tikzcd}
\mathtt{h}(\widetilde{X}_i \times^G U_{j+1}) \arrow[r, "(g_{i,j+1})_*"] \arrow[d, "\widetilde{i}_j^*"] & \mathtt{h}(X \times^G U_{j+1}) \arrow[d, "i_j^*"] \\
\mathtt{h}(\widetilde{X}_i \times^G U_j) \arrow[r, "(g_{i,j})_*"] & \mathtt{h}(X \times^G U_j)
\end{tikzcd}$$

So $i_m^*((f_i \times^G id_{U_{j+1}})_*(1)) = (f_i \times^G id_{U_j})_*(1)$, which implies that the elements $f_{i*}(1) = \lim_j((f_i \times^G id_{U_j})_*(1))$ form a basis of $\mathtt{h}^G(X)$ over $\mathtt{h}^G(S)$.

From this point on we assume that $\mathtt{h}^G_\bullet$ satisfies the property (CD). As for usual oriented theories we then obtain

Lemma 3.7. *The pairing* $(\cdot,\cdot)\colon \mathtt{h}^G(X)\otimes_{\mathtt{h}^G(S)}\mathtt{h}^G(X)\to \mathtt{h}^G(S)$ *given by* $(a,b)=p_*(ab)$ *is non-degenerate and the map*

$$f\colon (\mathtt{h}^G(X\times_S X),\circ)\to \mathrm{End}_{\mathtt{h}^G(S)}\,\mathtt{h}^G(X)\qquad \textit{given by } a\mapsto f_a,\ f_a(x)=(p_2)_*(p_1^*(x)\cdot a)$$

is an $\mathtt{h}^G(S)$*-linear isomorphism of rings. In particular, there is an* $\mathtt{h}^G(S)$*-linear isomorphism*

$$(\mathtt{h}^G_{\dim_S X}(X\times_S X),\circ)\to \mathrm{End}_{\mathtt{h}^G\text{-}\mathit{Corr}^+_S}(\mathtt{h}^G(X)),$$

where $\mathtt{h}^G$*-*Corr^+_S *is the respective category of* G*-equivariant correspondences.*

4. The convolution product

In the present section we introduce the convolution product on the equivariant Borel-Moore homology (Definition 4.3) of group power. We relate this product to the usual correspondence product for the associated torsors (Lemma 4.6) and study its behaviour under the base change (diagram (6)).

Let G be a smooth algebraic group over k and let E be a G-torsor over k (G acts on the right). By definition there is an isomorphism $\rho\colon E\times_k G\xrightarrow{\simeq} E\times_k E$ given on points by $(e,g)\mapsto(e,eg)$. For each $i\geq 0$ it induces an isomorphism

$$\rho_i\colon E\times_k G^i\longrightarrow E^{i+1},\qquad (e,g_1,g_2,\ldots,g_i)\mapsto(e,eg_1,eg_2,\ldots,eg_i).$$

Consider the composition

$$\gamma_i\colon E^{i+1}\xrightarrow{\rho_i^{-1}} E\times_k G^i=E\times_k G^i\xrightarrow{pr} G^i.$$

The coordinate-wise right G^{i+1}-action on E^{i+1} induces an action on $E\times_k G^i$ and, hence, on G^i. For instance, on points it is given by

$$(e,g_1,\ldots,g_i)\cdot(h_1,\ldots,h_{i+1})=(eh_1,h_1^{-1}g_1h_2,\ldots,h_1^{-1}g_ih_{i+1}).\tag{3}$$

Consider projections $p_j\colon E^{i+1}\to E^i$ obtained by removing the j-th coordinate and the respective G^i-action on E^i. For each $i\geq 1$, $1\leq j\leq i+1$ there is a commutative diagram of G^i-equivariant maps

$$\begin{tikzcd} E^{i+1} \arrow[r, "\gamma_i"] \arrow[d, "p_j"'] & G^i \arrow[d, "\pi_j"] \\ E^i \arrow[r, "\gamma_{i-1}"] & G^{i-1} \end{tikzcd}\tag{4}$$

where $\pi_1(g_1,\ldots,g_i)=(g_1^{-1}g_2,\ldots,g_1^{-1}g_i)$ and $\pi_j(g_1,\ldots,g_i)=(g_1,\ldots,\hat{g}_{j-1},\ldots,g_i)$ for $j>1$.

Example 4.1. For $i=1$ it gives a commutative diagram of G-equivariant maps

$$\begin{tikzcd} E\times_k E \arrow[r, "\gamma_1"] \arrow[d, "p_j"'] & G \arrow[d, "\pi_j"] \\ E \arrow[r, "\gamma_0"] & \mathit{Spec}\ k \end{tikzcd}$$

where γ_0,π_1,π_2 are the structure maps, p_1,p_2 are the corresponding projections and $\gamma_1(e,eg)=g$. Moreover, if E is trivial, then $\gamma_1=\pi_1\colon G\times_k G\to G$, $(g_1,g_2)\mapsto g_1^{-1}g_2$.

Let H be an algebraic subgroup of G such that G/H is a smooth variety over k. We can view G^i as an H-torsor over G^i/H, where H acts on G^i via the jth coordinate of G^{i+1}. By definition, the H^i-equivariant map π_j factors as

$$\pi_j \colon G^i \xrightarrow{q} G^i/H \xrightarrow{\bar{\pi}_j} G^{i-1},$$

where the second map $\bar{\pi}_j$ is a fibration with a fibre G/H.

Example 4.2. The map π_1 factors through the quotient maps modulo the diagonal action

$$\pi_1 \colon G^i \xrightarrow{q} G^i/\Delta(H) \xrightarrow{\bar{\pi}_1} G^i/\Delta(G) = G^{i-1}.$$

which are equivariant with respect to the usual coordinate-wise H^i-action.

Consider an equivariant Borel-Moore homology theory $\mathtt{h}$. For every $1 \leqslant j \leqslant i+1$ consider the action of the j-th copy of H on G^i. The property (Tor) gives an isomorphism

$$\mathtt{h}_{H^i}(G^i/H) \xrightarrow{\simeq} \mathtt{h}_{H^{i+1}}(G^i), \tag{5}$$

where H^{i+1} acts on G^i as in (3). Unless explicitly mentioned we will always identify these two rings.

Set $\mathbf{S} = \mathtt{h}_H(G^0) = \mathtt{h}_H(k)$ and set the convolution product on $\mathbf{S}$ to be the usual intersection product.

Definition 4.3. Assume that G/H is a smooth projective variety over k. We define the $\mathbf{S}$-linear convolution product $'\circ'$ on $\mathtt{h}_{H^i}(G^{i-1})$, $i \geq 2$ to be the composite

$$\mathtt{h}_{H^i}(G^{i-1}) \otimes \mathtt{h}_{H^i}(G^{i-1}) \xrightarrow{\bar{\pi}^*_{i-1} \otimes \bar{\pi}^*_{i+1}} \mathtt{h}_{H^{i+1}}(G^i) \otimes \mathtt{h}_{H^{i+1}}(G^i) \xrightarrow{'\cdot'}$$

$$\mathtt{h}_{H^{i+1}}(G^i) \xrightarrow{(\bar{\pi}_i)_*} \mathtt{h}_{H^i}(G^{i-1}),$$

where $\mathtt{h}_{H^{i+1}}(G^i)$ is identified with $\mathtt{h}_{H^i}(G^i/H)$ via (5) and $\bar{\pi}_i$ is projective because so is G/H.

The central object of the present paper is the convolution ring $(\mathtt{h}_{H^2}(G), \circ)$, i.e., the case $i = 2$. In the next sections we will show that $(\mathtt{h}_{B^2}(G), \circ)$ (where B is a Borel subgroup of a semisimple split G) can be identified with the formal affine Demazure algebra.

Example 4.4. In the case $i = 3$ the convolution ring $(\mathtt{h}_{H^3}(G^2), \circ)$ is isomorphic to $\mathtt{h}_{\Delta(H)}((G/H)^2)$ with respect to the usual correspondence product. Indeed, the maps $\pi_i \colon G^3 \to G^2$, $i = 2, 3, 4$ induce $\Delta(H)$-equivariant projections $(G/H)^3 \to (G/H)^2$. The isomorphism then follows by (Tor).

Observe that if G/H is an H-equivariant cellular space and $\mathtt{h}_H$ satisfies (CD), then by Lemma 3.7 there is an $\mathbf{S}$-linear ring isomorphism

$$(\mathtt{h}_{H^3}(G^2), \circ) \simeq \operatorname{End}_{\mathbf{S}} \mathtt{h}_H(G/H).$$

Lemma 4.5. *For $i \geq 1$ the map π_1 induces an injective ring homomorphism with respect to the convolution products*

$$(\mathtt{h}_{H^i}(G^{i-1}), \circ) \xrightarrow{\pi_1^*} (\mathtt{h}_{H^{i+1}}(G^i), \circ).$$

Proof. For $i = 1$ it follows from the fact that the convolution product on $\mathtt{h}_{H^2}(G)$ is $\mathbf{S}$-linear.

For $i \geq 2$ for each $i-1 \leq j \leq i+1$ we have $\pi_j \circ \pi_1 = \pi_1 \circ \pi_{j+1}$. Since push-forwards commute with flat pull-backs by (TS), there are commutative diagrams in equivariant cohomology

$$\begin{tikzcd}
\mathtt{h}_{H^{i+1}}(G^i) \arrow[r, "\bar\pi_1^*"] \arrow[d, shift right, "(\bar\pi_i)_*"'] & \mathtt{h}_{H^{i+2}}(G^{i+1}) \arrow[d, shift right, "(\bar\pi_{i+1})_*"'] \\
\mathtt{h}_{H^i}(G^{i-1}) \arrow[r, "\bar\pi_1^*"] \arrow[u, shift right, "\bar\pi_{i-1}^*{,}\bar\pi_{i+1}^*"'] & \mathtt{h}_{H^{i+1}}(G^i) \arrow[u, shift right, "\bar\pi_i^*{,}\bar\pi_{i+2}^*"']
\end{tikzcd}$$

Finally, there is a H^i-equivariant section of the map $\bar\pi_1 \colon G^i/\Delta(H) \to G^{i-1}$ given by $(g_1, \ldots, g_{i-1}) \mapsto (1, g_1, \ldots, g_{i-1})$, so $\bar\pi_1^*$ is injective. □

Lemma 4.6. *The map γ_1 induces a ring homomorphism*

$$(\mathtt{h}_{H^2}(G), \circ) \xrightarrow{\gamma_1^*} (\mathtt{h}_{H^2}(E^2), \circ) \xrightarrow{\simeq} (\mathtt{h}((E/H)^2), \circ),$$

where the last ring is viewed with respect to the correspondence product (2).

Proof. By (TS) the diagram (4) gives rise to commutative diagrams in cohomology

$$\begin{tikzcd}
\mathtt{h}_{H^3}(G^2) \arrow[r, "\gamma_2^*"] \arrow[d, shift right, "(\bar\pi_2)_*"'] & \mathtt{h}_{H^3}(E^3) \arrow[d, shift right, "(p_2)_*"'] \\
\mathtt{h}_{H^2}(G) \arrow[r, "\gamma_1^*"] \arrow[u, shift right, "\bar\pi_1^*{,}\bar\pi_3^*"'] & \mathtt{h}_{H^2}(E^2) \arrow[u, shift right, "p_1^*{,}p_3^*"']
\end{tikzcd}$$

The last isomorphism follows by (Tor). □

Let $\bar k$ denote the splitting field of a G-torsor E so that $G_{\bar k} = E_{\bar k}$. Since the base change preserves the convolution product, combining Lemmas 4.5 and 4.6 we obtain two commutative diagrams of convolution (correspondence) rings

$$\begin{tikzcd}
\gamma_1^* \colon \mathtt{h}_{H^2}(G) \arrow[r, "pr^*"] \arrow[d, "res_{\bar k/k}"'] & \mathtt{h}_{H^2}(E \times_k G) \arrow[r, "\rho_1^*", "\simeq"'] & \mathtt{h}_{H^2}(E^2) \arrow[d, "res_{\bar k/k}"] \\
\gamma_1^* \colon \mathtt{h}_{H^2}(G_{\bar k}) \arrow[r, "\bar\pi_1^*"] & \mathtt{h}_{H^2}(G_{\bar k}^2/\Delta(H)) \arrow[r, "q^*"] & \mathtt{h}_{H^2}(G_{\bar k}^2)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\gamma_0^* \colon \mathtt{h}_H(k) \arrow[r, "\rho_0^* \circ pr^*"] \arrow[d, "res_{\bar k/k}"', "\simeq"] & \mathtt{h}_H(E) \arrow[d, "res_{\bar k/k}"] \\
\gamma_0^* \colon \mathtt{h}_H(\bar k) \arrow[r, "q^* \circ \bar\pi_1^*"] & \mathtt{h}_H(G_{\bar k})
\end{tikzcd}$$

where $res_{\bar k/k}$ is the base change map. Combining these two diagrams we obtain a commutative diagram of convolution rings

$$\begin{tikzcd}
\mathtt{h}_H(E) \otimes_{\mathbf{S}} \mathtt{h}_{H^2}(G) \arrow[r, "(p_1^*{,}\gamma_1^*)"] \arrow[d, "res_{\bar k/k}"'] & \mathtt{h}_{H^2}(E^2) \arrow[d, "res_{\bar k/k}"] \\
\mathtt{h}_H(G_{\bar k}) \otimes_{\mathbf{S}} \mathtt{h}_{H^2}(G_{\bar k}) \arrow[r, "(p_1^*{,}\gamma_1^*)"] & \mathtt{h}_{H^2}(G_{\bar k}^2),
\end{tikzcd} \tag{6}$$

where the left convolution rings are $\mathtt{h}_H(E)$- and $\mathtt{h}_H(G_{\bar{k}})$-linear.

5. THE SUBRING OF PUSH-PULL OPERATORS

In the present section we prove that if H is the Borel subgroup of a split semisimple linear algebraic group, then the convolution ring $\mathtt{h}_{H^2}(G)$ of Definition 4.3 can be identified with the subring of push-pull operators (Corollary 5.3). Our arguments are essentially based on the Bruhat decomposition of G stated using the G-orbits on the product $G/H \times_k G/H$ and the resolution of singularities (8).

As before assume that G/H is a smooth projective variety over k. In the notation of the previous section consider the H^2-equivariant maps of Example 4.2.

$$\pi_1\colon G^2 \xrightarrow{q} G^2/\Delta(H) \xrightarrow{\bar{\pi}_1} G^2/\Delta(G) = G, \quad (g_1, g_2) \mapsto g_1^{-1}g_2.$$

Since G^2 is a $\Delta(G)$-torsor over G ($\Delta(H)$-torsor over $G^2/\Delta(H)$), by the property (Tor) the induced $\Delta(G) \times H^2$-equivariant pull-backs on cohomology coincide with the forgetful maps

$$\tag{7}\begin{tikzcd}
\gamma_1^*\colon \mathtt{h}_{H^2}(G) \simeq \mathtt{h}_{\Delta(G)\times H^2}(G^2) \arrow[r, hook, "\bar{\pi}_1^*"] \arrow[d, "\simeq"'] & \mathtt{h}_{\Delta(H)\times H^2}(G^2) \arrow[r, "q^*"] \arrow[d, "\simeq"] & \mathtt{h}_{H^2}(G^2) \arrow[d, "\simeq"] \\
\mathtt{h}_G((G/H)^2) \arrow[r, hook] & \mathtt{h}_H((G/H)^2) \arrow[r] & \mathtt{h}((G/H)^2)
\end{tikzcd}$$

Moreover, by Lemma 4.5 it is a commutative diagram of convolution rings.

Let G be a split semisimple linear algebraic group over k and let $\mathtt{h}$ be an equivariant theory that satisfies property (CD). We fix a Borel subgroup B of G containing a split maximal torus T. By Bruhat decomposition (e.g. [27])

$$G = \amalg_{w\in W} B\dot{w}B, \quad \dot{w} \in N_T,$$

is the disjoint union of B^2-orbits of G, where $W = N_T/T$ is the Weyl group and N_T is the normalizator of T in G. Projecting this decomposition onto $X = G/B$ gives a B-equivariant cellular filtration on X by closures $\overline{X}_w$ of affine spaces $X_w = B\dot{w}B/B$ of dimension $l(w)$ (the length of w).

The preimage $\pi_1^{-1}(B\dot{w}B)$ is a $\Delta(G)$-orbit in G^2 (here $H = B$). Let $\mathcal{O}_w$ denote its image via $G^2 \to X^2$ and let $\overline{\mathcal{O}}_w$ denote its closure. Observe that both $\mathcal{O}_w$ and $\overline{\mathcal{O}}_w$ are $\Delta(G)$-invariant in X^2. By properties of the Bruhat decomposition (see [27, §1]) it follows that the projection $\mathcal{O}_w \to X^2 \to X$ is a torsor of a vector bundle over X with fibre X_w. Indeed, the transition functions are affine since they are given by the action of B on the left on $B\dot{w}B/B$ that is by T acting on the product of the respective root subgroups $\prod_{\alpha\in\Phi^+\cap w(\Phi^-)} U_\alpha$ via the conjugation and, hence, by T acting on the product of the respective $\mathbb{G}_a$'s via the multiplication $t \cdot x = \alpha(t)x$, $t \in T$, $x \in \mathbb{G}_a$. So X^2 is a G-equivariant (G acts diagonally) cellular space over X with filtration given by the closures $\overline{\mathcal{O}}_w$.

Assume that for each $w \in W$ we are given a G-equivariant resolution of singularities $\widetilde{\mathcal{O}}_w \to \overline{\mathcal{O}}_w$. Let $[\widetilde{\mathcal{O}}_w]_G$ denote the respective class in $\mathtt{h}_G^{\dim_k X - l(w)}(X^2)$. Then by the property (CD) the cohomology $\mathtt{h}_G(X^2)$ (resp. $\mathtt{h}_B(X^2)$ and $\mathtt{h}(X^2)$) is a free module over $\mathtt{h}_G(X)$ (resp. over $\mathtt{h}_B(X)$ and $\mathtt{h}(X)$) with basis $\{[\widetilde{\mathcal{O}}_w]_G\}_{w\in W}$ (resp. $\{[\widetilde{\mathcal{O}}_w]_B\}_{w\in W}$ and $\{[\widetilde{\mathcal{O}}_w]\}_{w\in W}$). Hence, the forgetful maps of (7) send $[\widetilde{\mathcal{O}}_w]_G \mapsto [\widetilde{\mathcal{O}}_w]_B \mapsto [\widetilde{\mathcal{O}}_w]$ and change the coefficients by $-\otimes_{\mathtt{h}_G(X)} \mathtt{h}_B(X)$ and

$-\otimes_{\mathtt{h}_B(X)} \mathtt{h}(X)$ respectively, where the map $\mathbf{S} = \mathtt{h}_G(X) \hookrightarrow \mathtt{h}_B(X) \to \mathtt{h}(X)$ is the classical characteristic map.

We now construct such G-equivariant resolutions as follows. For the i-th simple reflection s_i we denote X_{s_i} (resp. $\mathcal{O}_{s_i}$) simply by X_i (resp. by $\mathcal{O}_i$). Let P_i be the minimal parabolic subgroup corresponding to a simple root α_i and let $q_i\colon X \to G/P_i$ denote the respective quotient map.

Lemma 5.1. *We have $\overline{\mathcal{O}}_i = X \times_{G/P_i} X$ and, in particular, $\overline{\mathcal{O}}_i$ is smooth.*

Proof. We have $(g_1B, g_2B) \in X \times_{G/P_i} X$, $g_1, g_2 \in G$ if and only if $g_1P_i = g_2P_i$, so $g_2 = g_1h$ for some $h \in P_i$. Since $P_i = B \cup Bs_iB$, it means that either $g_2B = g_1B$ or $g_2B = g_1Bs_iB$, so $(g_1B, g_2B) \in \mathcal{O}_{s_i} \cup \Delta_X = \overline{\mathcal{O}}_i$. $\square$

For any $w \in W$ we choose a reduced decomposition $w = s_{i_1}s_{i_2}\ldots s_{i_l}$ and set $I_w = (i_1, i_2, \ldots, i_l)$. Consider a variety

$$\widetilde{\mathcal{O}}_{I_w} = X \times_{G/P_{i_1}} X \times_{G/P_{i_2}} \ldots \times_{G/P_{i_l}} X. \tag{8}$$

The projection on the first and the last factor $pr\colon \widetilde{\mathcal{O}}_{I_w} \to X \times_k X$ gives a G-equivariant resolution of singularities of $\overline{\mathcal{O}}_w$.

Theorem 5.2. *For $H = B$ or 1, the image of $[\widetilde{\mathcal{O}}_{I_w}]_H \in \mathtt{h}_H(X \times_k X)$ under the Künneth isomorphism*

$$(\mathtt{h}_H(X \times_k X), \circ) \xrightarrow{\simeq} \mathrm{End}_{\mathtt{h}_H(k)}(\mathtt{h}_H(X))$$

is the composition of push-pull operators $q_{i_1}^* q_{i_1 *} \circ \ldots \circ q_{i_l}^* q_{i_l *}$.

Proof. By definition the image of $[\widetilde{\mathcal{O}}_{I_w}]_H$ is the $\mathtt{h}_H(k)$-linear operator

$$\mathtt{h}_H^{\bullet}(X) \xrightarrow{p_1^*} \mathtt{h}_H(X \times_k X) \xrightarrow{\cdot[\widetilde{\mathcal{O}}_{I_w}]} \mathtt{h}_H(X \times_k X) \xrightarrow{p_{2*}} \mathtt{h}_H^{\bullet - l(w)}(X).$$

By the projection formula and (TS) it can be also written as

$$\mathtt{h}_H^{\bullet}(X) \xrightarrow{pr_{l+1}^*} \mathtt{h}_H(\widetilde{\mathcal{O}}_{I_w}) \xrightarrow{pr_{1*}} \mathtt{h}_H^{\bullet - l(w)}(X),$$

where pr_j denotes the projection on the j-th coordinate (recall that p_j denotes the projection obtained by removing the j-th coordinate).

By the property (TS) we obtain a commutative diagram

$$\begin{tikzcd}
\mathtt{h}_H(X) \arrow[r, "pr_2^*"] \arrow[d, "q_{i_l *}"'] & \mathtt{h}_H(\widetilde{\mathcal{O}}_{i_l}) \arrow[r, "pr_{23}^*"] \arrow[d, "pr_{1*}"'] & \mathtt{h}_H(\widetilde{\mathcal{O}}_{(i_{l-1}, i_l)}) \arrow[r, "pr_{234}^*"] \arrow[d, "pr_{12*}"'] & \ldots \arrow[r] & \mathtt{h}_H(\widetilde{\mathcal{O}}_{I_w}) \arrow[d] \\
\mathtt{h}_H(G/P_{i_l}) \arrow[r, "q_{i_l}^*"] & \mathtt{h}_H(X) \arrow[r, "pr_2^*"] \arrow[d, "q_{i_{l-1} *}"'] & \mathtt{h}_H(\widetilde{\mathcal{O}}_{i_{l-1}}) \arrow[d, "pr_{1*}"'] & & \ldots \arrow[d] \\
 & \mathtt{h}_H(G/P_{i_{l-1}}) \arrow[r, "q_{i_{l-1}}^*"] & \mathtt{h}_H(X) \arrow[d, "q_{i_{l-2} *}"'] & & \ldots \arrow[d] \\
 & & \ldots \arrow[r] & \ldots \arrow[r] & \mathtt{h}_H(X)
\end{tikzcd}$$

where $pr_{ijk\ldots}$ denote the projection on the i-th, j-th, k-th, $\ldots$, coordinates. The result then follows since the top horizontal row gives pr_{l+1}^* and the right vertical column gives pr_{1*}. $\square$

Combining Diagram (7) and Theorem 5.2 we obtain

Corollary 5.3. *There is a commutative diagram of convolution rings*

$$\begin{tikzcd}
\mathtt{h}_{B^2}(G) \arrow[r, hook, "\bar{\pi}_1^*"] & \mathtt{h}_{\Delta(B)\times B^2}(G^2) \arrow[r, "\simeq"] \arrow[d, "q^*"'] & \mathtt{h}_B(X^2) \arrow[r, "\simeq"] \arrow[d] & End_{\mathbf{S}}(\mathtt{h}_B(X)) \arrow[d] \\
& \mathtt{h}_{B^2}(G^2) \arrow[r, "\simeq"] & \mathtt{h}(X^2) \arrow[r, "\simeq"] & End_{\mathbf{R}}(\mathtt{h}(X))
\end{tikzcd}$$

where the image of $(\mathtt{h}_{B^2}(G), \circ)$ *in* $End_{\mathbf{S}}(\mathtt{h}_B(X))$ *is the subring generated by the push-pull operators* $q_i^* q_{i_*}$ *(of degree* (-1)*) and the image of the forgetful map* $\mathbf{S} = \mathtt{h}_G^\bullet(X) \to \mathtt{h}_B^\bullet(X)$ *(of degrees '*$\bullet$*') and the last vertical arrow is induced by the augmentation map* $\mathbf{S} \to \mathbf{R} = \mathtt{h}(k)$.

6. SELF-DUALITY OF THE ALGEBRA OF PUSH-PULL OPERATORS

In the present section we identify the convolution ring $\mathtt{h}_{B^2}(G)$ with the formal affine Demazure algebra $\mathbf{D}_F$ of [18] and show that it is self-dual with respect to the convolution product (Theorem 6.2). Our arguments are based on the results of [18], [7], [8] and, especially, [9]. We use the notation of [9].

Recall that algebraic oriented cohomology theories $\mathtt{h}$ correspond (up to universality) to one-dimensional commutative formal group laws $F(u,v)$: the formal group law corresponds to $\mathtt{h}$ by means of the Quillen formula expressing the first characteristic classes

$$c_1^{\mathtt{h}}(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1^{\mathtt{h}}(\mathcal{L}_1), c_1^{\mathtt{h}}(\mathcal{L}_2))$$

and the respective cohomology theory $\mathtt{h}$ is defined from F by tensoring with the algebraic cobordism

$$\mathtt{h}(-) = \Omega(-) \otimes_{\Omega(k)} \mathbf{R},$$

where $\Omega(k) \to \mathbf{R}$ defines F by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to Chow groups and the periodic multiplicative law corresponds to K-theory.

By [9, Thm. 3.3] the completed B-equivariant coefficient ring $\mathbf{S} = \mathtt{h}_B(k)$ can be identified with the formal group algebra $\mathbf{R}[[T^*]]_F$, where T^* is the group of characters of a split maximal torus $T \subset B$ and F is the respective formal group law.

Following [9, §5] consider the localized algebra $\mathbf{Q} = \mathbf{R}[[T^*]]_F[\frac{1}{x_\alpha}]_\alpha$ (where α runs through all simple roots) and the smash products $\mathbf{Q}_W = \mathbf{Q}\#_{\mathbf{R}}\mathbf{R}[W]$ and $\mathbf{S}_W = \mathbf{S}\#_{\mathbf{R}}\mathbf{R}[W]$ with the multiplication given by

$$q\delta_w \cdot q'\delta_{w'} = q(wq')\delta_{ww'}$$

for $q, q' \in \mathbf{Q}$ (respectively $\mathbf{S}$) and $w, w' \in W$ (the Weyl group). Consider the duals $\mathbf{Q}_W^\star = Hom_{\mathbf{Q}}(\mathbf{Q}_W, \mathbf{Q})$ and $\mathbf{S}_W^\star = Hom_{\mathbf{S}}(\mathbf{S}_W, \mathbf{S})$. By definition $\mathbf{Q}_W^\star$ and $\mathbf{S}_W^\star$ can be identified with the ring of functions $Hom(W, \mathbf{Q})$ and $Hom(W, \mathbf{S})$ respectively

As in [18, Def. 6.2, 6.3] for each simple root α_i of the root system for G define the push-pull element

$$Y_i = (1+\delta_i)\frac{1}{x_{-i}} \in \mathbf{Q}_W.$$

Define the formal affine Demazure algebra $\mathbf{D}_F$ as the subalgebra of $\mathbf{Q}_W$ generated by multiplications by $\mathbf{S}$ and the elements Y_i.

By [7, Thm. 7.9] (see also [18, Thm. 5.14]) the $\mathbf{R}$-algebra $\mathbf{D}_F$ satisfies the following (complete) set of relations: for $i, j = 1 \dots rk(G)$ and $u \in \mathbf{S}$

- $Y_i^2 = \kappa_i Y_i$, where $\kappa_i = \frac{1}{x_i} + \frac{1}{x_{-i}}$ and $x_i = x_{\alpha_i}$,
- $Y_i u = s_i(u) Y_i + \Delta_{-i}(u)$, where $\Delta_{-i}(u) = \frac{u - s_i(u)}{x_{-i}}$,
- $(Y_i Y_j)^{m_{ij}} - (Y_j Y_i)^{m_{ij}} = \sum_{I_w} c_{I_w} Y_{I_w}$, where the sum is taken over all reduced expressions I_w of elements w of the subgroup $\langle s_i, s_j \rangle \subseteq W$, and the coefficients c_{I_w} are given by the formulas of [18, Prop. 5.8]

Example 6.1. If F corresponds to Chow groups, then $\mathbf{D}_F = \mathbf{H}_{nil}$ is the affine nil-Hecke algebra over $\mathbb{Z}$ in the notation of [16]. If F corresponds to K-theory, then $\mathbf{D}_F$ is the 0-affine Hecke algebra over $\mathbb{Z}$ ($q \to 0$ in the affine Hecke algebra). If F corresponds to the generic hyperbolic formal group law of [8, §9], then by [8, Prop. 9.2] the constant part of $\mathbf{D}_F$ is isomorphic to the localized classical Iwahori-Hecke algebra.

Let $\mathbf{D}_F^\star = Hom_{\mathbf{S}}(\mathbf{D}_F, \mathbf{S})$ denote its dual. Observe that the main result of [9] (Thm. 8.2 loc.cit.) says that $\mathbf{D}_F^\star$ is isomorphic to the $\mathbf{R}$-algebra $\mathtt{h}_T(X)$.

Theorem 6.2. *Let G be a split semisimple linear algebraic group over a field k and let $\mathtt{h}$ be an equivariant theory that satisfies property (CD).*

Then the convolution algebra $(\mathtt{h}_{B^2}(G), \circ)$ is isomorphic (as an $\mathbf{R}$-algebra) to the formal affine Demazure algebra $\mathbf{D}_F$. So there is an $\mathbf{R}$-algebra isomorphism

$$(\mathbf{D}_F^\star, \circ) \simeq (\mathbf{D}_F, \cdot)$$

Proof. By Corollary 5.3 the ring $(\mathtt{h}_{B^2}(G), \circ) \simeq (\mathtt{h}_B(X), \circ)$ is isomorphic to the subalgebra of $\mathrm{End}_{\mathbf{S}}(\mathtt{h}_B(X))$ generated by the image of the forgetful map $\mathtt{h}_G(X) \to \mathtt{h}_B(X)$ and push-pull operators $q_i^* q_{i*}$. Since the map $B \to B/T$ is an affine fibration, the natural map $\mathtt{h}_B(X) \to \mathtt{h}_T(X)$ is an isomorphism. Hence we may identify $\mathbf{S}$ with $\mathtt{h}_T(k)$ and $\mathrm{End}_{\mathbf{S}}(\mathtt{h}_B(X))$ with $\mathrm{End}_{\mathbf{S}}(\mathtt{h}_T(X))$. Observe that these identifications preserve push-pull operators. The inclusion of T-fixed point set $W \to X$ gives an embedding $\mathtt{h}_T(X) \to \mathtt{h}_T(W) = \mathbf{S}_W^\star \subseteq \mathbf{Q}_W^\star$. By [9, Corollary 8.7] there is the following commutative diagram

(9)
$$\begin{tikzcd}
\mathtt{h}_T(X) \arrow[r] \arrow[d, "q_i^* q_{i*}"'] & \mathbf{S}_W^\star \arrow[r, hook] & \mathbf{Q}_W^\star \arrow[d, "A_i"] \\
\mathtt{h}_T(X) \arrow[r] & \mathbf{S}_W^\star \arrow[r, hook] & \mathbf{Q}_W^\star
\end{tikzcd}$$

where the Hecke operator A_i is given by

$$A_i(f)(x) = f(x \cdot Y_i) \quad for\ x \in \mathbf{Q}_W, f \in \mathbf{Q}_W^\star.$$

Moreover, the forgetful map

$$\mathbf{S} \cong \mathtt{h}_G(X) \to \mathtt{h}_T(X) = \oplus_{w \in W} \mathbf{S}$$

is given by the formula $s \mapsto (w \cdot s)_{w \in W}$ for any $s \in \mathbf{S}$. Then the multiplication in $\mathtt{h}_T(X) = \mathbf{S}_W^*$ by the image of any element in $s \in \mathtt{h}_G(X)$ induces a right multiplication by s in $\mathbf{Q}_W^*$. Since $\mathbf{Q}_W$ is a free $\mathbf{Q}$-module of finite rank, the natural map $\imath\colon \mathbf{Q}_W \to \mathrm{End}_{\mathbf{Q}}(\mathbf{Q}_W^\star)$ given by $\imath(x)(f)(y) = f(yx)$ is an inclusion. Note that every A_i lies in the image of $\imath$. Then by diagram (9) the image of $\mathtt{h}_{B^2}(G)$ is isomorphic to a subalgebra of $\mathbf{Q}_W$ generated by $\mathbf{S}$ and Y_i which is $\mathbf{D}_F$. $\square$

7. THE RATIONAL ALGEBRA OF PUSH-PULL OPERATORS

In the present section we introduce the rational algebra of push-pull operators $\overline{\mathbf{D}}_F$ and show that it can be identified with the subring of rational endomorphisms of G/B (Theorem 7.5).

The B^2-equivariant isomorphism $E\times_k G\to E\times_k E$, $(e,g)\mapsto(e,eg)$ induces an isomorphism $E\times^B G/B\to E/B\times_k E/B$. For all $w\in W$ fix a reduced decomposition $I_w=(i_1,\ldots,i_l)$ and the corresponding Bott-Samelson resolution $X_{I_w}\to G/B$ of the Schubert cell. This map is B-equivariant, so it descends to a map $Y_{I_w}=E\times^B X_{I_w}\to E\times^B G/B$.

Lemma 7.1. *The classes* $[Y_{I_w}]$ *form a basis of* $\mathtt{h}(E/B\times_k E/B)$ *over* $\mathtt{h}(E/B)$, *where the module structure is given by the pullback of the projection* $pr_1^*\colon \mathtt{h}(E/B)\to \mathtt{h}(E/B\times_k E/B)$.

Proof. Since B is special, G-torsor E splits over the function field of E/B. Then by [25, Lemma 3.3] projection $pr_1\colon E/B\times_k E/B\to E/B$ is a cellular fibration in the sense of [25, Definition 3.1] so that $(E/B)^2$ is a cellular space over E/B. Let ξ be the generic point of E/B. The pullback of an open embedding $j^*\colon \mathtt{h}(E/B\times_k E/B)\to \mathtt{h}(\xi\times_k E/B)\simeq \mathtt{h}(G/B)$ is surjective and any preimage of $\mathbf{R}$-basis of $\mathtt{h}(G/B)$ gives a basis of $\mathtt{h}(E/B\times_k E/B)$. Thus it is sufficient to check that j^* sends $[Y_{I_w}]$ to a basis of $\mathtt{h}(\xi\times_k E/B)$. Let $p\colon E\to E/B$ be the projection. Note that

$$E\times^B X_{I_w}\times_{(E/B\times_k E/B)}\xi\times_k E/B=p^{-1}(\xi)\times^B X_{I_w}=\xi\times X_{I_w},$$

since $p^{-1}(\xi)\to\xi$ is a trivial B-torsor. Thus $j^*([Y_{I_w}])=[\xi\times X_{i_w}]$ that forms a basis of $\mathtt{h}(\xi\times E/B)=\mathtt{h}(\xi\times G/B)$ over $\mathtt{h}(\xi)=\mathbf{R}$. □

Consider a B-equivariant map

$$f\colon E\times^B G\to B\backslash G,\quad (e,g)B\mapsto Bg.$$

Let $X'_{I_w}=(P_{i_1}\times\ldots\times P_{i_l})/B^l$ where B^l-action on $P_{i_1}\times\ldots\times P_{i_l}$ is given by $(p_1,\ldots,p_l)\cdot(b_1,\ldots,b_l)=(b_1^{-1}p_1b_2,\ldots,b_l^{-1}p_l)$. Then X'_{I_w} gives the Bott-Samelson class for $B\backslash G$.

Lemma 7.2. *The composition* $\mathtt{h}_B(B\backslash G)\xrightarrow{f^*}\mathtt{h}_B(E\times^B G)\simeq\mathtt{h}(E/B\times_k E/B)$ *maps* $[X'_{I_w}]_B$ *to* $[Y_{I_w}]$.

Proof. Consider the map $P_{i_1}\times^B P_{i_2}\times^B\ldots\times^B P_{i_l}\to G$ given by $(p_1,\ldots,p_l)\to p_1\ldots p_l$. It is B-equivariant with respect to the left multiplication, so it descends to a map $M_{I_w}=E\times^B P_{i_1}\times^B P_{i_2}\times^B\ldots\times^B P_{i_l}\to E\times^B G$. By construction we have an isomorphism

$$M_{I_w}\simeq Y_{I_w}\times_{E\times^B(G/B)}(E\times^B G).$$

Then $[M_{I_w}]_B$ is mapped to $[Y_{I_w}]$ via the isomorphism $\mathtt{h}(E\times^B G/B)\to\mathtt{h}_B(E\times^B G)$. Thus it is sufficient to check that $f^*[X'_{I_w}]_B=[M_{I_w}]_B$, which follows from the fact that

$$M_{I_w}=E\times^B(P_{i_1}\times\ldots\times P_{i_l}/B^{l-1})\simeq(E\times^B G)\times_{B\backslash G}X'_{I_w}.$$ □

Lemma 7.3. *(cf. [25, Corollary 3.4]) The composition*

$$(p_1^*,\gamma_1^*)\colon \mathtt{h}_B(E)\otimes_{\mathbf{S}}\mathtt{h}_{B^2}(G)\longrightarrow\mathtt{h}_{B^2}(E^2)\simeq\mathtt{h}((E/B)^2)$$

of the diagram (6) (for $H=B$) is an isomorphism.

Proof. Consider the basis of $\mathtt{h}_{B^2}(G)$ over $\mathbf{S}$ given by the classes of Bott-Samelson resolutions ζ_{I_w}. Then by Lemma 7.2 $\gamma_1^*(\zeta_{I_w})$ forms a basis of $\mathtt{h}_{B^2}(E^2)$ over $\mathtt{h}_B(E)$ induced by the respective cellular filtration. ☐

Consider the restriction map $\mathtt{h}(E/B) \to \mathtt{h}(E_{\bar{k}}/B) = \mathtt{h}(X_{\bar{k}})$ on cohomology induced by the scalar extension $\bar{k}/k$ (here $\bar{k}$ is a splitting field of E). Let $\overline{\mathtt{h}}(X)$ denote its image.

Corollary 7.4. *The image of the ring homomorphism*

$$res_{\bar{k}/k} \colon (\mathtt{h}(E/B \times_k E/B), \circ) \longrightarrow (\mathtt{h}(X_{\bar{k}} \times_{\bar{k}} X_{\bar{k}}), \circ).$$

is the subalgebra generated by the multiplication by the elements of $\overline{\mathtt{h}}(X)$ *and the push-pull operators* $q_i^* q_{i*} \colon \mathtt{h}(X) \to \mathtt{h}(G/P_i) \to \mathtt{h}(X)$ *for all simple roots* α_i*.*

Proof. Follows by (6), Lemma 7.3 and Corollary 5.3. ☐

There is a natural action of W on $\overline{\mathtt{h}}(X)$ that comes from the W-action on E/T. So we can endow $\overline{\mathtt{h}}(X) \otimes_{\mathbf{S}} \mathbf{Q}_W$ with a structure of an $\mathbf{R}$-algebra. Let $\overline{\mathbf{D}}_F$ denote its subalgebra $\overline{\mathtt{h}}(X) \otimes_{\mathbf{S}} \mathbf{D}_F$. We call it the rational algebra of push-pull operators. We endow $\overline{\mathbf{D}}_F$ (and $\mathbf{D}_F$) with a grading assuming that all Y_i's have degree (-1) and elements of $\overline{\mathtt{h}}^\bullet(X)$ (and of $\mathbf{S} = \mathtt{h}_B^\bullet(k)$) have degree '$\bullet$'. By $\overline{\mathbf{D}}_F^{(m)}$ we denote its degree m homogeneous component. Let $N = \dim X$.

Theorem 7.5. *Consider the restriction*

$$res_{\bar{k}/k} \colon \mathrm{End}_{\mathtt{h}\text{-}Corr_k^+}(E/B) \longrightarrow \mathrm{End}_{\mathtt{h}\text{-}Corr_{\bar{k}}^+}(X_{\bar{k}})$$

on endomorphism rings of the respective motives (i.e., preserving the grading of $\mathtt{h}(X)$*). Its image can be identified with* $\overline{\mathbf{D}}_F^{(0)}$ *via the injective forgetful map*

$$\phi \colon \big((\overline{\mathtt{h}}(X) \otimes_{\mathbf{S}} \mathtt{h}_G(X_{\bar{k}}^2))^{(N)}, \circ\big) \longrightarrow \big(\mathtt{h}^N(X_{\bar{k}}^2), \circ\big).$$

Proof. By (7) both $\mathtt{h}_G(X^2)$ and $\mathtt{h}(X^2)$ are free modules over $\mathtt{h}_G(X)$ and $\mathtt{h}(X)$ with basis given by the classes $[\widetilde{\mathcal{O}}_{I_w}]_G$ and $[\widetilde{\mathcal{O}}_{I_w}]$ respectively. The map ϕ sends $[\widetilde{\mathcal{O}}_{I_w}]_G \mapsto [\widetilde{\mathcal{O}}_{I_w}]$ and leaves the coefficients invariant. The result follows by Corollary 7.4, Corollary 5.3 and Theorem 6.2. ☐

We say that a (co-)homology theory $\mathtt{h}$ satisfies the Dimension Axiom if

(Dim) For any smooth variety Y over k we have $\mathtt{h}^n(Y) = 0$ for all $n > \dim Y$.

Example 7.6. Any theory $\mathtt{h}$ over a field k of characteristic 0 obtained by specialization of coefficients of the Lazard ring (e.g. Chow groups, connective K-theory, algebraic cobordism Ω) satisfies (Dim).

The graded K-theory $K_0(-)[\beta, \beta^{-1}]$ of [23, Example 1.1.5] does not satisfy (Dim).

Observe that the image of the characteristic map $c \colon \mathbf{S} \to \mathtt{h}(X)$ is contained in $\overline{\mathtt{h}}(X)$ (see [15, Thm. 4.5]). Consider both the induced map $\mathfrak{c} \colon \mathbf{D}_F^{(0)} \to \overline{\mathbf{D}}_F^{(0)}$ and the restriction map $res_{\bar{k}/k} \colon (\mathtt{h}_B(E) \otimes_{\mathbf{S}} \mathbf{D}_F)^{(0)} \to \overline{\mathbf{D}}_F^{(0)}$.

Lemma 7.7. *Assume that the theory* $\mathtt{h}$ *satisfies (Dim), then the kernels of* $\mathfrak{c}$ *and* $res_{\bar{k}/k}$ *are nilpotent.*

In other words, there is a commutative diagram of maps of convolution rings

$$\begin{tikzcd}
\mathbf{D}_F^{(0)} \arrow[r, "\gamma_1^*"] \arrow[dr, "\mathfrak{c}"'] & \mathtt{h}^N((E/B)^2) \arrow[d, "res_{\bar{k}/k}"] \\
 & \overline{\mathbf{D}}_F^{(0)}
\end{tikzcd}$$

with nilpotent kernels.

Proof. Let $f = \mathfrak{c}$ or $res_{\bar{k}/k}$. Then $x \in \ker f$ means that $x = \sum_w a_w [\widetilde{\mathcal{O}}_{I_w}]_G$ with $f(a_w) = 0$. By Theorem 5.2 each $[\widetilde{\mathcal{O}}_{I_w}]$ corresponds to the composite of push-pull elements Y_{I_w} in $\overline{\mathbf{D}}_F$, so that x corresponds to $\sum_w a_w Y_{I_w} \in \overline{\mathbf{D}}_F^{(0)}$ and $x^{\circ n}$ corresponds to

$$(\sum_w a_w Y_{I_w})^n = \sum_w a_{w,n} Y_{I_w}, \quad a_{w,n} \in (\ker f)^n,$$

Since $\ker f$ is contained in the augmentation ideal, $(\ker f)^n \subset \mathbf{S}^{(\geq n)}$. Finally, observe that $\deg Y_{I_w} \leq -N$, hence, for $n > 2N$ we get $x^{\circ n} = 0$. ☐

Lemma 7.8. *If E is a versal G-torsor, then the map γ_1^* and, hence, $\mathfrak{c}$, of the lemma 7.7 is surjective.*

Proof. Observe that if E is versal, then it admits an open G-equivariant embedding into $\mathbb{A}_k^N$. So the projection $E \times_k G^i \to G^i$ in the definition of γ_i factors through $\mathbb{A}_k^N \times_k G^i$. By (Loc) and (HI) the induced pullback γ_i^* is surjective. ☐

8. Applications to representation theory of Hecke rings

Let G be a split semisimple linear algebraic group over a field k and let E be a G-torsor over k. Let $\bar{k}$ be a splitting field of E and let E/B be the twisted form of G/B by means of E. By definition, we have

$$E/B \times_k \bar{k} \simeq G/B \times_k \bar{k}.$$

Let $\mathtt{h}$, $\mathtt{h}_B$ be an (B-equivariant) oriented theory over k that satisfies both (CD) and (Dim) axioms, e.g., Chow groups, connective K-theory or algebraic cobordism Ω. Let $\mathbf{R} = \mathtt{h}(k)$ and $\mathbf{S} = \mathtt{h}_B(k)$ be the respective coefficient rings.

Consider the endomorphism ring of the $\mathtt{h}$-motive of E/B

$$C_F = (End_{\mathtt{h}\text{-}Corr_k^+}(E/B), \circ).$$

By definition, any direct sum decomposition of the motive $[E/B]$ is given by a complete set of primitive pairwise orthogonal idempotents on C_F so that $\langle E/B\rangle_{\mathtt{h}} \simeq Proj\, C_F$. Our main result (Theorem 7.5) together with Lemma 7.7 says that the restriction map gives a surjective ring homomorphism with nilpotent kernel

$$res_{\bar{k}/k} \colon C_F \to \overline{\mathbf{D}}_F^{(0)}.$$

By the standard idempotent lifting (see e.g. [25, §2 and Prop. 2.6]) we then obtain $Proj\, C_F \simeq Proj\, \overline{\mathbf{D}}_F^{(0)}$, so that we get the following

Theorem 8.1. *There is a one-to-one correspondence between direct sum decompositions of the* $\mathtt{h}$*-motive* $[E/B]$ *and direct sum decompositions of* $\overline{\mathbf{D}}_F$*-module* $\overline{\mathbf{D}}_F$*. This correspondence, induces an equivalence between the category* $\langle E/B\rangle_{\mathtt{h}}$ *and the category Proj* $\overline{\mathbf{D}}_F$ *of finitely generated projective* $\overline{\mathbf{D}}_F$*-modules.*

If E *is versal, then the algebra* $\overline{\mathbf{D}}_F$ *can be replaced by the algebra* $\mathbf{D}_F$*.*

Observe that in general the ring $\overline{\mathbf{D}}_F$ is not Krull-Schmidt (and not semi-simple).

Lemma 8.2. *If the coefficient ring* $\mathbf{R}$ *is Artinian, then both* C_F *and* $\overline{\mathbf{D}}_F$ *and, hence, the categories* $\langle E/B\rangle_{\mathtt{h}}$ *and Proj* $\overline{\mathbf{D}}_F$ *satisfy the Krull-Schmidt property (uniqueness of a direct sum decomposition).*

Proof. If $\mathbf{R}$ is Artinian, then both $\overline{\mathbf{D}}_F$ and C_F are Artinian (as $\overline{\mathbf{D}}_F$ is finite dimensional over $\mathbf{R}$). So they are both Noetherian which implies that the respective tautological modules $\overline{\mathbf{D}}_F$ and C_F have finite length and, hence, the Krull-Schmidt property holds for both $\overline{\mathbf{D}}_F$ and C_F. □

As a direct application of the main result of [25] one obtains the following characterization of modular representations of the (affine) nil-Hecke algebra (F is additive and $\mathtt{h} = CH$).

Corollary 8.3. *Let* G *be a split semisimple linear algebraic group over a field* k*. Consider the affine nil-Hecke algebra* $\mathbf{H}_{nil}$ *for* G *with coefficients in* $\mathbf{R} = \mathbb{F}_p$*,* p *is a prime. Then*

$$\mathit{Proj}\ \mathbf{H}_{nil} \simeq \langle \mathcal{R}_{E,p}\rangle,$$

where E *is a versal* G*-torsor.*

In particular, all indecomposable submodules of $\mathbf{H}_{nil}$ *are free* $\mathbf{S}$*-modules isomorphic to each other and their* $\mathbf{S}$*-rank equals to the* p*-part of the product of* p*-exceptional degrees of* G*.*

Proof. The $\mathbf{S}$-rank coincides with the number of Tate motives in the decomposition of $\mathcal{R}_{E,p}$ over a splitting field of E, that is $\prod_{i=1}^{r} \frac{1-t^{d_i p^{k_i}}}{1-t^{d_i}}|_{t=1}$ (in the notation of [25]) which is equal to the p-part $p^{\sum_{i=1}^{r} k_i}$ of p-exceptional degrees of [19, p.73]. □

Example 8.4. Consider the root system of type A_1. In this case $T^* = \mathbb{Z}\omega$ ($G = SL_2$) or $T^* = \mathbb{Z}\alpha$ ($G = PGL_2$), $\alpha = 2\omega$ is the simple root and ω is the fundamental weight. The Weyl group $W = \{1, s\}$ acts by $s\colon \omega \mapsto -\omega$, where s is the simple reflection. By definition, $\mathbf{S} = \mathbf{R}[[x]]_F$ (where $x = x_\omega$ or $x = x_\alpha$), $\mathbf{Q} = \mathbf{S}[\frac{1}{x}]$, $\mathbf{Q}_W = \{q(x)\delta_w \mid q(x) \in \mathbf{Q},\ w \in W\}$ with

$$q(-_F x)\delta_s = s(q(x))\delta_s = \delta_s q(x),$$

where $-_F x$ is the formal inverse of x. Observe that $x_\alpha = x_{\omega+\omega} = F(x_\omega, x_\omega)$ in $\mathbf{S}$.

The $\mathbf{R}$-algebra $\mathbf{D}_F$ is a free left $\mathbf{S}$-submodule of rank 2 in $\mathbf{Q}_W$ with basis

$$\{1, Y = \tfrac{1}{-_F x_\alpha} + \tfrac{1}{x_\alpha}\delta_s\}.$$

It satisfies the relations

$$Y^2 = \kappa Y \text{ and } Yq(x) = q(-_F x)Y + \Delta(q(x)),$$

where $\kappa = \frac{1}{-_F x_\alpha} + \frac{1}{x_\alpha}$ and $\Delta(q(x)) = \frac{q(x) - q(-_F x)}{x_{-\alpha}}$.

Let $p = a + bY$, $a, b \in \mathbf{S}$ be an idempotent in $\mathbf{D}_F$, i.e., $p^2 = p$. Since $\deg p = 0$, we have $\deg a = 0$ and $\deg b = 1$ (the coefficient $a_{ij} \in \mathbf{R}$ at $u^i v^j$ in $F(u, v)$ has degree $1 - i - j$). Then

$$(a+bY)^2 = a^2+abY+bYa+bYbY = a^2+abY+b(s(a)Y+\Delta(a))+b(s(b)Y+\Delta(b))Y = \big(a^2 + b\Delta(a)\big) + \big(ab + bs(a) + bs(b)\kappa + b\Delta(b)\big)Y.$$

So that

$$a^2 + b\Delta(a) = a \text{ and } ab + bs(a) + bs(b)\kappa + b\Delta(b) = b.$$

Assume b is a non-zero divisor, then we obtain (in $\mathbf{S}$)

$$a^2 + b\Delta(a) = a \text{ and } a + s(a) + s(b)\kappa + \Delta(b) = 1. \tag{10}$$

In the case $\mathtt{h} = CH$ and $\mathbf{R} = \mathbb{Z}$ ($F(u, v) = u + v$) we have $\kappa = 0$, $-_F x = -x$, $x_\alpha = 2x_\omega$, $a \in \mathbb{Z}$, $b = cx$, $c \in \mathbb{Z}$ and, (10) becomes

$$a = 0 \text{ and } \Delta(b) = \tfrac{cx}{-x_\omega} = 1$$

or

$$a = 1 \text{ and } \Delta(b) = \tfrac{cx}{-x_\omega} = -1$$

which have solutions only if $x = x_\omega$ and $c = \pm 1$. Therefore, $\mathbf{D}_F$ has only two indecomposable submodules corresponding to the idempotents $1 - xY$ and xY.

The latter translates into the following well-known fact concerning motivic decompositions:

The Chow motive of a conic E/B (for a versal G-torsor E) decomposes as a direct sum of two indecomposable summands if and only if $G = SL_2$, i.e.. E is trivial and $E/B = \mathbb{P}^1$. In this case each summand correspond to the (shifted) Tate motive.

REFERENCES

[1] J. Ayoub, L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I. J. für die Reine und angew. Math. 693 (2014), 1–149.

[2] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28, 1–26, 1973.

[3] P. Bressler, S. Evens, The Schubert calculus, braid relations and generalized cohomology. Trans. Amer. Math. Soc. 317 (1990), no.2, 799–811.

[4] P. Bressler, S. Evens, Schubert calculus in complex cobordisms. Trans. Amer. Math. Soc. 331 (1992), no.2, 799–813.

[5] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline, Chow motives of twisted flag varieties. Compositio Math. 142 (2006), no.4, 1063–1080.

[6] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and cohomology of complete flags, Ann. Sci. École Norm. Sup. (4) 46 (2013), no.3, 405–448.

[7] B. Calmès, K. Zainoulline, C. Zhong, A coproduct structure on the formal affine Demazure algebra, to appear in Math. Z. (arXiv:1209.1676).

[8] B. Calmès, K. Zainoulline, C. Zhong, Push-pull operators on the formal affine Demazure algebra and its dual, arXiv:1312.0019.

[9] B. Calmès, K. Zainoulline, C. Zhong, Equivariant oriented cohomology of flag varieties, arXiv:1409.7111.

[10] V. Chernousov, A. Merkurjev, Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. Transform. Groups 11 (2006), no.3, 371–386.

[11] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion. Invent. Math. 21 (1973), 287–301.

[12] M. Demazure, Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.

[13] D. Edidin, W. Graham, Equivariant intersection theory. Invent. Math. 131 (1998), no.3, 595–634.

[14] R. Elman, N. Karpenko, A. Merkurjev, Algebraic and geometric theory of quadratic forms. AMS Colloquium Publ. 56, AMS, Providence, RI, 2008. 435 pp.
[15] S. Gille, K. Zainoulline, Equivariant pretheories and invariants of torsors, Transf. Groups 17 (2012), no.2, 471–498.
[16] V. Ginzburg, Geometric methods in the representation theory of Hecke algebras and quantum groups, Representation Theories and Algebraic Geometry Nato ASI Series Volume 514, 1998, pp 127-183
[17] J. Heller, J. Malagón-López, Equivariant algebraic cobordism, J. Reine Angew. Math. 684 (2013), 87–112.
[18] A. Hoffnung, J. Malagón-López, A. Savage, K. Zainoulline, Formal Hecke algebras and algebraic oriented cohomology theories. Selecta Math. 20 (2014), no.4, 1213–1245.
[19] V. Kac, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. Invent. Math. 80 (1985), 69–79.
[20] B. Kostant, S. Kumar, T-Equivariant K-theory of generalized flag varieties. J. Differential Geometry 32 (1990), 549–603.
[21] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. Advances in Math. 62 (1986), 187–237.
[22] C. Lenart, K. Zainoulline, Towards generalized cohomology Schubert calculus via formal root polynomials, arXiv:1408.5952.
[23] M. Levine, F. Morel, Algebraic cobordism, Springer Monographs in Math., Springer, Berlin, 2007. xii+244pp.
[24] A. Nenashev, K. Zainoulline, Oriented cohomology and motivic decomposition of relative cellular spaces, J. of Pure and Applied Algebra 205 (2006), no.2, 323–340.
[25] V. Petrov, N. Semenov, K. Zainoulline, J-invariant of linear algebraic groups. Ann. Sci. École Norm. Sup. (4) 41 (2008), no.6, 1023–1053.
[26] A. Vishik, K. Zainoulline, Motivic splitting lemma, Documenta Math. 13 (2008), 81–96.
[27] T. Springer, Schubert varieties and generalizations, Representation theories and algebraic geometry (Montreal, PQ, 1997), 413–440, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.
[28] B. Totaro, The Chow ring of a classifying space. Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.
[29] A. Vishik, N. Yagita, Algebraic cobordism of a Pfister quadric, J. Lond. Math. Soc. (2) 76 (2007), no.3, 586–604.

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