Proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing a finite field

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September 10, 2015

Abstract

Let R be a regular local ring, containing **a finite field**. Let $\mathbf{G}$ be a reductive group scheme over R. We prove that a principal $\mathbf{G}$-bundle over R is trivial, if it is trivial over the fraction field of R. In other words, if K is the fraction field of R, then the map of non-abelian cohomology pointed sets

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel. If the regular local ring R contains **an infinite field** this result is proved in [FP]. *Thus the conjecture holds for regular local rings containing a field.*

1 Main results

Let R be a commutative unital ring. Recall that an R-group scheme $\mathbf{G}$ is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [SGA3, Exp. XIX, Definition 2.7]. A well-known conjecture due to J.-P. Serre and A. Grothendieck (see [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a]) asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme $\mathbf{G}$ over R, the map

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel. The following theorem, which is the main result of the present paper, asserts that this conjecture holds, provided that R contains **a finite field**. If R contains an infinite field, then the conjecture is proved in [FP].

*The author acknowledges support of the RNF-grant 14-11-00456.

Theorem 1.1. *Let R be a regular semi-local domain containing* **a finite field**, *and let K be its field of fractions. Let $\mathbf{G}$ be a reductive group scheme over R. Then the map*

$$H^1_{\acute{e}t}(R, \mathbf{G}) \to H^1_{\acute{e}t}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel. In other words, under the above assumptions on R and $\mathbf{G}$, each principal $\mathbf{G}$-bundle over R having a K-rational point is trivial.

Theorem 1.1 has the following

Corollary 1.2. *Under the hypothesis of Theorem 1.1, the map*

$$H^1_{\acute{e}t}(R, \mathbf{G}) \to H^1_{\acute{e}t}(K, \mathbf{G}),$$

induced by the inclusion of R into K, is injective. Equivalently, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are two principal bundles isomorphic over Spec$K$*, then they are isomorphic.*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two principal $\mathbf{G}$-bundles isomorphic over Spec K. Let $\mathrm{Iso}(\mathcal{G}_1, \mathcal{G}_2)$ be the scheme of isomorphisms. This scheme is a principal $\mathrm{Aut}\mathcal{G}_2$-bundle. By Theorem 1.1 it is trivial, and we see that $\mathcal{G}_1 \cong \mathcal{G}_2$. □

Note that, while Theorem 1.1 was previously known for reductive group schemes $\mathbf{G}$ coming from the ground field (*an unpublished result due to O.Gabber*), in many cases the corollary is a new result even for such group schemes.

For a scheme U we denote by $\mathbb{A}^1_U$ the affine line over U and by $\mathbb{P}^1_U$ the projective line over U. Let T be a U-scheme. By a principal $\mathbf{G}$-bundle over T we understand a principal $\mathbf{G} \times_U T$-bundle. We refer to [SGA3, Exp. XXIV, Sect. 5.3] for the definitions of a simple simply-connected group scheme over a scheme and a semi-simple simply-connected group scheme over a scheme.

In Section 2 we deduce Theorem 1.1 from the following three results.

Theorem 1.3. *Let k be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many* **closed points** *on a k-smooth irreducible affine k-variety X and let K be its field of fractions. Let G be a simple simply connected group scheme over $\mathcal{O}$. Let $\mathbf{G}$ be a reductive group scheme over $\mathcal{O}$. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over $\mathcal{O}$ which is trivial over K. Then there exists a principal $\mathbf{G}$-bundle $\mathcal{G}_t$ over $\mathcal{O}[t]$ and a monic polynomial $h(t) \in \mathcal{O}[t]$ such that*

(i) the $\mathbf{G}$-bundle $\mathcal{G}_t$ is trivial over $\mathcal{O}[t]_h$,

(ii) the evaluation of $\mathcal{G}_t$ at $t = 0$ coincides with the original $\mathbf{G}$-bundle $\mathcal{G}$.

If the field k is infinite this result is proved in [PSV, Thm.1.2].

Theorem 1.4. *Let k be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many* **closed points** *on a k-smooth irreducible affine k-variety X. Set $U = \mathrm{Spec}\,\mathcal{O}$. Let $\mathbf{G}$ be* **a simple simply-connected** *group scheme over U. Let $\mathcal{E}_t$ be a principal $\mathbf{G}$-bundle over the affine line $\mathbb{A}^1_U = \mathrm{Spec}\,\mathcal{O}[t]$, and let $h(t) \in \mathcal{O}[t]$ be a monic polynomial. Denote by $(\mathbb{A}^1_U)_h$ the open subscheme in $\mathbb{A}^1_U$ given by $h(t) \neq 0$ and assume that the restriction of $\mathcal{E}_t$ to $(\mathbb{A}^1_U)_h$ is a trivial principal $\mathbf{G}$-bundle. Then for each section $s : U \to \mathbb{A}^1_U$ of the projection $\mathbb{A}^1_U \to U$ the $\mathbf{G}$-bundle $s^*\mathcal{E}_t$ over U is trivial.*

If the field k is infinite this result is proved in [FP, Thm.2].

Theorem 1.5. *Let k be a field. Assume that for any irreducible k-smooth affine variety X and any finite family of its closed points $x_1, x_2, \dots, x_n$ and the semi-local k-algebra $\mathcal{O} := \mathcal{O}_{X,x_1,x_2,\dots,x_n}$ and all semi-simple simply connected reductive $\mathcal{O}$-group schemes H the pointed set map*

$$H^1_{\text{ét}}(\mathcal{O}, H) \to H^1_{\text{ét}}(k(X), H),$$

induced by the inclusion of $\mathcal{O}$ into its fraction field $k(X)$, has trivial kernel.

Then for any regular semi-local domain $\mathcal{O}$ of the form $\mathcal{O}_{X,x_1,x_2,\dots,x_n}$ above and any reductive $\mathcal{O}$-group scheme G the pointed set map

$$H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of $\mathcal{O}$ into its fraction field K, has trivial kernel.

Theorem 1.4 is an easy consequence of Theorem 1.6 proven in section 3 and of Proposition 4.2 proven in section 4.

Theorem 1.6. *Let k be a* **finite** *field. Let R, U, and $\mathbf{G}$ be as in Theorem 1.4. Let $Z \subset \mathbb{P}^1_U$ be a closed subscheme finite over U. Let $Y \subset \mathbb{P}^1_U$ be a closed subscheme finite and étale over U and such that*
(i) $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split,
(ii) $Y \cap Z = \emptyset$ and $Y \cap \{\infty\} \times U = \emptyset = Z \cap \{\infty\} \times U$,
(iii) for any closed point $u \in U$ one has $Pic(\mathbb{P}^1_u - Y_u) = 0$, where $Y_u := \mathbb{P}^1_u \cap Y$.
Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial. Then the restriction of $\mathcal{G}$ to $\mathbb{P}^1_U - Y$ is also trivial.
In particular, the principal $\mathbf{G}$-bundle $\mathcal{G}$ is trivial locally for the Zarisky topology.

If the field k is infinite then a stronger result is proved in [FP, Thm.3]. Proof of Theorem 1.3 is given in Section 5. Proof of Theorem 1.5 is given in Section 8.

The article is organized as follows. In Section 2 Theorem 1.1 is reduced to Theorems 1.4 and 1.3. In Section 4 Theorem 1.4 is proved. In Section 5 two major technical theorems (Theorems 5.1 and 5.2) are stated and Theorem 1.3 is reduced to those two theorems. In Section 6 a purity theorem is stated and it is reduced to Theorem 5.2. In Section 7 one more purity theorem is stated and it is reduced to Theorem 5.2. In Section 8 Theorem 1.5 is reduced to Theorem 5.2. The strategy of the proof of Theorems 5.1 and 5.2 is described in Section 9. Theorems 5.1 and 5.2 are proved in Section 15 after preliminary work which is done in Sections 10 to 14. Finally, in Sections 16 to 18 few technical results are proved. We refer to those results reducing Theorems 6.1 and 1.5 to Theorem 5.2.

The history of the topic is described in [FP].

The author thanks J.-P.Serre, A.Suslin, A.Merkurjev, V.Chernousov, J.-L.Colliot-The'l'ene, Ph.Gille, R.Parimala, A.Stavrova, B.Totaro, C.Voisin, R.Fedorov and many other mathematicians for their interest to the topic of the present article. I would like to thank A.Stavrova for drawing my attention to B.Poonen's works on Bertini theorems for varieties over finite fields.

2 Reducing Theorem 1.1 to Theorems 1.3, 1.4, 1.5.

If the regular semi-local domain contains an infinite field, then such a reduction is given in details in [FP, Section 3]. This is why we give here just a short sketch of the reduction. So, suppose that Theorems 1.3, 1.4 and 1.5 are true.

Theorems 1.4 and 1.3 yield Theorem 1.1 in the simple simply-connected case, when the ring R is the semi-local ring of finitely many *closed* points on a k-smooth variety (the field k is finite). Indeed, let k be a field and let $\mathcal{O}$, $\mathbf{G}$ and $\mathcal{G}$ be as in Theorem 1.3. Assume additionally that the group scheme $\mathbf{G}$ is simple and simply-connected. Take $\mathcal{G}_t$ from the conclusion of Theorem 1.3 for the bundle $\mathcal{E}_t$ from Theorem 1.4. Then take $h(t)$ from the conclusion of Theorem 1.3 for the bundle $\mathcal{E}_t$ from Theorem1.4. Finally take the inclusion $i_0 : U \hookrightarrow U \times \mathbb{A}^1$ for the section s from the Theorem 1.4. Theorem 1.4 with this choice of $\mathcal{E}_t$, $h(t)$ and s yields triviality of the $\mathbf{G}$ bundle $\mathcal{G}$ from Theorem 1.3.

Now standard arguments (see for instance [PSV, Thm.11.1]) show that Theorem 1.1 holds for the ring R as above in this section and for arbitrary semi-simple and simply-connected group scheme $\mathbf{G}$ over R. (Note that to use those arguments it is necessary to work with semi-local rings).

Now Theorem 1.5 yields Theorem 1.1 for the ring R as above in this section and for arbitrary reductive group scheme $\mathbf{G}$ over R.

This latter case implies easily Theorem 1.1 for arbitrary reductive group scheme over a ring R, where R is the semi-local ring of finitely many *arbitrary* points on a k-smooth irreducible affine k-variety (see [FP, Lemma 3.3]). Auguments using Popescu's Theorem completes the proof of Theorem 1.1 (see [FP, Proof of Theorem 1]). Those arguments runs now easier since Theorem 1.1 is established already for semi-local rings of finitely many *arbitrary* points on a k-smooth variety (with a finite field k).

3 Proof of Theorem 1.6

Proof. Let Y' be a semi-local scheme. We will call a simple Y'-group scheme *quasi-split* if its restriction to each connected component of Y' contains **a Borel subgroup scheme**. Let Y be a semi-local scheme. We will call a simple Y-group scheme *isotropic*, if its restriction to each connected component of Y contains a proper parabolic subgroup scheme.

Since $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split it is isotropic. For any of the closed point $u \in U$ the residue field $k(u)$ is a finite field. Hence the reductive $k(u)$-group $\mathbf{G}_u = \mathbf{G} \times_U u$ is quasi-split. Thus it is isotropic. The finite étale U-scheme Y does not satisfy (in general) the condition that for any of the closed point $u \in U$ there is a $k(u)$-rational point as required by the assuptions of [FP, Thm.3]. However for any of the closed point $u \in U$ one has $Pic(\mathbb{P}^1_u - Y_u) = 0$, where $Y_u := \mathbb{P}^1_u \cap Y$. The $\mathbf{G}_u$-bundle $\mathcal{G}_u$ is trivial over $\mathbf{A}^1_u - Z_u$. Thus, by [Gil1, Corollary 3.10(a)] it is trivial locally for Zariski topology on $\mathbb{P}^1_u$. And again by [Gil1, Corollary 3.10(a)], it is trivial over $\mathbb{P}^1_u - Y_u$. Now one can repeat literally the proof of [FP, Thm.3] and conclude Theorem 1.6.

□

4 Proof of Theorem 1.4.

Theorem 1.4 in the case of an infinite field k is proved in [FP, Thm.2]. So, we prove the remaining case, when the field k is finite.

Let k, U and $\mathbf{G}$ be as in Theorem 1.4 and let the field k be finite. Let $u_1, \dots, u_n$ be all the closed points of U. Let $k(u_i)$ be the residue field of u_i. Consider the reduced closed subscheme $\mathbf{u}$ of U, whose points are $u_1, \dots, u_n$. Thus

$$\mathbf{u} \cong \coprod_i \operatorname{Spec} k(u_i).$$

Set $\mathbf{G}_{\mathbf{u}} = \mathbf{G} \times_U \mathbf{u}$. By $\mathbf{G}_{u_i}$ we denote the fiber of $\mathbf{G}$ over u_i; it is a simple simply-connected algebraic group over $k(u_i)$. The following lemma is a simple version of Lemma 13.4.

Lemma 4.1. *Let $S = Spec(A)$ be a regular semi-local scheme such that* **the residue field at any of its closed point is finite**. *Let T be a closed subscheme of S. Let W be a closed subscheme of the projective space $\mathbb{P}^d_S$. Assume that over T there exists a section $\delta : T \to W$ of the projection $W \to S$. Suppose further that W is S-smooth and equidimensional over S of relative dimension r. Then there exists a closed subscheme $\tilde{S}$ of W which is finite étale over S and contains $\delta(T)$.*

Proof. Since S is semilocal, after a linear change of coordinates we may assume that δ maps T into the closed subscheme of $\mathbf{P}^d_T$ defined by $X_1 = \cdots = X_d = 0$. For each closed fibre $\mathbf{P}^d_s$ of $\mathbf{P}^d_S$ using repeatedly [Poo, Thm.1.2], we can choose a family of **homogeneous** polynomials $H_1(s), \dots, H_r(s)$ (in general of increasing degrees) such that the subscheme $Y(s)$ of $\mathbf{P}^d_S(s)$ defined by the equations

$$H_1(s) = 0\,, \ \dots\ , \ H_r(s) = 0$$

intersects $W(s)$ transversally and contains the point $[1 : 0 : \cdots : 0]$. By the chinese remainders' theorem there exists a common lift $H_i \in A[X_0, \dots, X_d]$ of all polynomials $H_i(s)$, $s \in \operatorname{Max}(A)$. We may choose this common lift H_i such that $H_i(1, 0, \dots, 0) = 0$. Let V be the closed subscheme of $\mathbf{P}^d_S$ defined by

$$H_1 = 0\,, \dots, H_r = 0\ .$$

We claim that the subscheme $\tilde{S} = V \cap X$ has the required properties. Note first that $X \cap V$ is finite over S. In fact, $X \cap V$ is projective over S and every closed fibre (hence every fibre) is finite. Since the closed fibres of $X \cap V$ are finite étale over the closed points of S, to show that $X \cap V$ is finite étale over S it only remains to show that it is flat over S. Noting that $X \cap V$ is defined in every closed fibre by a regular sequence of equations and localizing at each closed point of S, we see that flatness follows from [OP2, Lemma 7.3].

□

Proposition 4.2. *Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. There is a closed subscheme $Y \subset \mathbb{A}^1_U$ which is étale and finite over U and such that*
(i) $\mathbf{G}_Y := \mathbf{G} \times_U Y$ is quasi-split,
(ii) $Y \cap Z = \emptyset$,
(iii) for any closed point $u \in U$ one has $Pic(\mathbb{P}^1_u - Y_u) = 0$, where $Y_u := \mathbb{P}^1_u \cap Y$.
(Note that Y and Z are closed in $\mathbb{P}^1_U$ since they are finite over U).

Proof. For every u_i in $\mathbf{u}$ choose a Borel subgroup $\mathbf{B}_{u_i}$ in $\mathbf{G}_{u_i}$. *The latter is possible since the fields $k(u_i)$ are finite.* Let $\mathcal{B}$ be the U-scheme of Borel subgroup schemes of $\mathbf{G}$. It is a smooth projective U-scheme (see [SGA3, Cor. 3.5, Exp. XXVI]). The subgroup $\mathbf{B}_{u_i}$ in $\mathbf{G}_{u_i}$ is a $k(u_i)$-rational point b_i in the fibre of $\mathcal{B}$ over the point u_i. Now apply Lemma 4.1 to the scheme U for S, the scheme $\mathbf{u}$ for T, the scheme $\mathcal{B}$ for W and to a section $\delta : \mathbf{u} \to \mathcal{B}$, which takes the point u_i to the point $b_i \in \mathcal{B}$. Since the scheme $\mathcal{B}$ is U-smooth and is equidimensional over S we are under the assumption of Lemma 4.1. Hence there is a closed subscheme Y' of $\mathcal{B}$ such that Y' is étale over U and all the b_i's are in Y

To continue the proof of the Proposition we need the following

Lemma 4.3. *Let U be as in the Proposition. Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over U. Let $Y' \to U$ be a finite étale morphism such that for any closed point u_i in U the fibre Y'_{u_i} of Y' over u_i contains a $k(u_i)$-rational point. Then there are finite field extensions k_1 and k_2 of the finite field k such that*
(i) the degrees $[k_1 : k]$ and $[k_2 : k]$ are coprime,
(ii) $k(u_i) \otimes_k k_r$ is a field for $r = 1$ and $r = 2$,
(iii) the degrees $[k_1 : k]$ and $[k_2 : k]$ are strictly greater than any of the degrees $[k(z) : k]$, where z runs over all closed points of Z,
(iv) there is a closed embedding of U-schemes $Y'' = ((Y' \otimes_k k_1) \coprod (Y' \otimes_k k_2)) \xrightarrow{i} \mathbb{A}^1_U$,
(v) for $Y = i(Y'')$ one has $Y \cap Z = \emptyset$,
(vi) for any closed point u_i in U one has $Pic(\mathbb{P}^1_{u_i} - Y_{u_i}) = 0$.

To prove this Lemma note that it's easy to find field extensions k_1 and k_2 subjecting (i) to (iii). To satisfy (iv) it suffices to require that for any closed point u_i in U and for $r = 1$ and $r = 2$ the number of closed points in $Y'_{u_i} \otimes_k k_r$ is the same as the number of closed points in Y'_{u_i}, and to require that for any integer $n > 0$ and any closed point u_i in U the number of points $y \in Y''_{u_i}$ with $[k(y) : k(u_i)] = n$ is not more than the number of points $x \in \mathbb{A}^1_{u_i}$ with $[k(x) : k(u_i)] = n$. Clearly, these requirements can be satisfied, which proves the item (iv).

The condition (v) holds for any closed U-embedding $i : Y'' \hookrightarrow \mathbb{A}^1_U$ from item (iv), since the property (iii). The condition (vi) holds since the property (i).

Now complete the proof of Proposition 4.2. Take the U-scheme $Y' \subset \mathbf{B}$ as in the beginning of the proof. This U-scheme Y' satisfies the assumption of Lemma 4.3. Take the closed subscheme Y of $\mathbb{A}^1_U$ as in the item (v) of the Lemma. For this Y the conditions (ii) and (iii) of the Proposition are obviously satisfied. The condition (i) is satisfied too, since already it is satisfied for the U-scheme Y'. The Proposition follows.

□

Proof of Theorem 1.4. Set $Z := \{h = 0\} \cup s(U) \subset \mathbb{A}^1_U$. Clearly, Z is finite over U. Since the principal $\mathbf{G}$-bundle $\mathcal{E}_t$ is trivial over $(\mathbb{A}^1_U)_h$ it is trivial over $\mathbb{A}^1_U - Z$. Note that $\{h = 0\}$ is closed in $\mathbb{P}^1_U$ and finite over U because h is monic. Further, $s(U)$ is also closed in $\mathbb{P}^1_U$ and finite over U because it is a zero set of a degree one monic polynomial. Thus $Z \subset \mathbb{P}^1_U$ is closed and finite over U.

Since the principal $\mathbf{G}$-bundle $\mathcal{E}_t$ is trivial over $(\mathbb{A}^1_U)_h$, and $\mathbf{G}$-bundles can be glued in Zariski topology, there exists a principal $\mathbf{G}$-bundle $\mathcal{G}$ over $\mathbb{P}^1_U$ such that

(i) its restriction to $\mathbb{A}^1_U$ coincides with $\mathcal{E}_t$;

(ii) its restriction to $\mathbb{P}^1_U - Z$ is trivial.

Now choose Y in $\mathbb{A}^1_U$ as in Proposition 4.2. Clearly, Y is finite étale over U and closed in $\mathbb{P}^1_U$. Moreover, $Y \cap \{\infty\} \times U = \emptyset = Z \cap \{\infty\} \times U$ and $Y \cap Z = \emptyset$. Applying Theorem 1.6 with this choice of Y and Z, we see that the restriction of $\mathcal{G}$ to $\mathbb{P}^1_U - Y$ is a trivial $\mathbf{G}$-bundle. Since $s(U)$ is in $\mathbb{A}^1_U - Y$ and $\mathcal{G}|_{\mathbb{A}^1_U}$ coincides with $\mathcal{E}_t$, we conclude that $s^*\mathcal{E}_t$ is a trivial principal $\mathbf{G}$-bundle over U. □

5 Reducing Theorem 1.3 to Theorems 5.1 and 5.2

Theorem 5.1 is a purely geometric one. Theorem 5.2 contains additionally group scheme data in its condition and contains additional equating results concerning those group scheme data in its conclusion. Reducing Theorem 1.5 to Theorem 5.2 is in Section 8. Let k be a field.

Theorem 5.1. *Let X be an affine k-smooth irreducible k-variety, and let $x_1, x_2, \dots, x_n$ be closed points in X. Let $U = Spec(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$. Given a non-zero function $f \in k[X]$, vanishing at each the point x_i there is a diagram of the form*

$$\begin{array}{ccccc} \mathbf{A}^1 \times U & \xleftarrow{\sigma} & \mathcal{X} & \xrightarrow{q_X} & X \\ & \searrow^{pr_U} & q_U \downarrow \uparrow \Delta & \nearrow_{can} & \\ & & U & & \end{array} \tag{1}$$

with an irreducible scheme $\mathcal{X}$, a smooth morphism q_U, a finite surjective morphism σ and an essentially smooth morphism q_X, and a function $f' \in q^*_X(f\ )k[\mathcal{X}]$, which enjoys the following properties:

(a) *if $\mathcal{Z}'$ is a closed subscheme of $\mathcal{X}$ defined by the principal ideal (f'), the morphism $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding and the morphism $q_U|_{\mathcal{Z}'} : \mathcal{Z}' \to U$ is finite;*

(a') *$q_U \circ \Delta = id_U$ and $q_X \circ \Delta = can$ and $\sigma \circ \Delta = i_0$;*

(b) *σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta(U)$;*

(c) *$\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ scheme theoretically and $\mathcal{Z}'' \cap \Delta(U) = \emptyset$;*

(d) *$\mathcal{D}_0 := \sigma^{-1}(\{0\} \times U) = \Delta(U) \coprod \mathcal{D}'_0$ scheme theoretically and $\mathcal{D}'_0 \cap \mathcal{Z}' = \emptyset$;*

(e) *for* $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ *one has* $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$.

(f) *there is a monic polinomial* $h \in \mathcal{O}[t]$ *such that* $(h) = Ker[\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')]$.

Theorem 5.2. *Let* X, $\{x_1, x_2, \ldots, x_n\} \subset X$, $U = Spec(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$, *and* $f \in k[X]$ *be as in Theorem 5.1. Let* $\mathbf{G}$ *be a reductive* X*-group scheme and* $\mathbf{G}_U := can^*(\mathbf{G})$. *Let* $\mathbf{C}$ *be an* X*-tori and* $\mathbf{C}_U := can^*(\mathbf{C})$. *Let* $\mu : \mathbf{G} \to \mathbf{C}$ *be an* X*-group scheme morphism smooth as a scheme morphism. Let* $\mu_U = can^*(\mu) : \mathbf{G}_U \to \mathbf{C}_U$.

Then there exists a diagram of the form (1) with an irreducible scheme $\mathcal{X}$, *a smooth morphism* q_U, *a finite surjective morphism* σ *and an essentially smooth morphism* q_X, *and a function* $f' \in q_X^*(f)k[\mathcal{X}]$, *which enjoys the conditions (a) to (f) from Theorem 5.1, and additionally there are* $\mathcal{X}$*-group scheme isomorphisms*

$$\Phi : q_U^*(\mathbf{G}_U) \to q_X^*(\mathbf{G}) \quad and \quad \Psi : q_U^*(\mathbf{C}_U) \to q_X^*(\mathbf{C})$$

such that $\Delta^*(\Phi) = id_{\mathbf{G}_U}$, $\Delta^*(\Psi) = id_{\mathbf{C}_U}$ *and* $q_X^*(\mu) \circ \Phi = \Psi \circ q_U^*(\mu_U)$.

The proof of these two theorems we postpone till Section 15. Right now we formulate its first consequence (see Corollary 5.3). To do that note that using the items (b) and (c) of Theorem 5.1 one can find an element $g \in I(\mathcal{Z}'')$ such that
(1) $(f') + (g) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,
(2) $Ker((\Delta)^*) + (g) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,
(3) $\sigma_g = \sigma|_{\mathcal{X}_g} : \mathcal{X}_g \to \mathbf{A}^1_U$ is étale.

Corollary 5.3 (Corollary of Theorem 5.1)**.** *The function* f' *from Theorem 5.1, the polinomial* h *from the item (f) of that Theorem, the morphism* $\sigma : \mathcal{X} \to \mathbf{A}^1_U$ *and the function* $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *defined just above enjoy the following properties:*

(i) *the morphism* $\sigma_g = \sigma|_{\mathcal{X}_g} : \mathcal{X}_g \to \mathbf{A}^1 \times U$ *is étale,*

(ii) *data* $(\mathcal{O}[t], \sigma_g^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g, h)$ *satisfies the hypotheses of [C-T/O, Prop.2.6], i.e.* $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ *is a finitely generated as the* $\mathcal{O}[t]$*-algebra, the element* $(\sigma_g)^*(h)$ *is not a zero-divisor in* $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ *and* $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g / h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$,

(iii) $(\Delta(U) \cup \mathcal{Z}) \subset \mathcal{X}_g$ *and* $\sigma_g \circ \Delta = i_0 : U \to \mathbf{A}^1 \times U$,

(iv) $\mathcal{X}_{gh} \subseteq \mathcal{X}_{gf'} \subseteq \mathcal{X}_{f'} \subseteq \mathcal{X}_{q_X^*(f)}$,

(v) $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')$ *and* $h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = (f') \cap I(\mathcal{Z}'')$ *and* $(f') + I(\mathcal{Z}'') = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

The same properties holds for the function f' *from Theorem 5.2, the polinomial* h *from the item (f) of that Theorem, the morphism* $\sigma : \mathcal{X} \to \mathbf{A}^1_U$ *and the function* $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *defined just above.*

Before proving this Corollary let us make one remark

Remark 5.4. The item (ii) of this corollary shows that the cartesian square

$$\begin{array}{ccc} \mathcal{X}_{gh} & \xrightarrow{\text{inc}} & \mathcal{X}_g \\ \Big\downarrow{\scriptstyle \sigma_{gh}} & & \Big\downarrow{\scriptstyle \sigma_g} \\ (\mathbf{A}^1 \times U)_h & \xrightarrow{\text{inc}} & \mathbf{A}^1 \times U \end{array} \tag{2}$$

can be used to glue principal $\mathbf{G}$-bundles for a reductive U-group scheme $\mathbf{G}$. The items (i) and (ii) show that the square (2) is an elementary **distinguished** square in the category of smooth U-schemes in the sense of [MV, Defn.3.1.3]. The item (iv) guaranties that a principal $\mathbf{G}$-bundle on $\mathcal{X}$, which is trivial being restricted to $\mathcal{X}_{q^*_X(\mathrm{f})}$ is trivial being restricted to $\mathcal{X}_{gh}$.

Proof of Corollary 5.3. We will use notation from Theorem 5.1. Since $\mathcal{X}$ is a regular affine irreducible and $\sigma : \mathcal{X} \to \mathbf{A}^1_U$ is finite surjective the induced $\mathcal{O}$-algebra homomorphism $\sigma^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a monomorphism. We will regard below the $\mathcal{O}$-algebra $\mathcal{O}[t]$ as a subalgebra via σ^*.

The assertions (i) and (iii) of the Corollary hold by our choice of g. The assertion (iv) holds, since $\sigma^*(h)$ is in the principal ideal (f') (use the properties (a) and (f) from Theorem 5.1). It remains to prove the assertion (ii). The morphism σ is finite. Hence the $\mathcal{O}[t]$-algebra $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ is finitely generated. The scheme $\mathcal{X}$ is regular and irreducible. Thus, the ring $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a domain. The homomorphism σ^* is injective. Hence, the element h is not zero and is not a zero divisor in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$.

It remains to check that $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$. Firstly, by the choice of h and by the item (a) of Theorem 5.1 one has $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')$. Secondly, by the property (1) of the element g one has $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f') = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/f'\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$. Finally, by the items (c) and (a) of Theorem 5.1 one has

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f') \ \times \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/I(\mathcal{Z}'') = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(h). \tag{3}$$

Localizing both sides of (3) in g one gets an equality

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/f'\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g,$$

hence $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f') = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/f'\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g/h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$.

Whence the Corollary.

□

Reducing Theorem 1.3 to Theorem 5.2. The k-algebra $\mathcal{O}$ is of the form $\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$, where X is a k-smooth irreducible affine variety. We may and will assume in this proof that the reductive group scheme $\mathbf{G}$ and the principal $\mathbf{G}$-bundle $\mathcal{G}$ are both defined over the variety X. Futhermore we may and will assume that there is given a non-zero function $\mathrm{f} \in k[X]$ such that the $\mathbf{G}$-bundle $\mathcal{G}$ is trivial on X_{f} and the function f vanishes at each point x_i in $\{x_1, x_2, \dots, x_n\}$. By Theorem 5.2 there is a diagram of the form (1) enjoying the properties (a) to (f) from Theorem 5.1. Moreover there is a $\mathcal{X}$-group scheme isomorphisms $\Phi : q_U^*(\mathbf{G}_U) \to q_X^*(\mathbf{G})$ such that $\Delta^*(\Phi) = id_{\mathbf{G}_U}$. Let f', h, σ and g be as in Corollary 5.3.

Given a $\mathbf{G}$-bundle $\mathcal{G}$ over X, which is trivial on X_{f} take its pull-back $q_X^*(\mathcal{G})$ to $\mathcal{X}$. Using the isomorphism Φ we may and will regard the $q_X^*(\mathbf{G})$-bundle $q_X^*(\mathcal{G})$ as a $q_U^*(\mathbf{G}_U)$-bundle, t.e. as a $\mathbf{G}_U$-bundle. We will denote that $\mathbf{G}_U$-bundle by ${}_Uq_X^*(\mathcal{G})$.

The $q_X^*(\mathbf{G})$-bundle $q_X^*(\mathcal{G})$ is trivial on $\mathcal{X}_{q_X^*(\mathrm{f})}$. Thus the $\mathbf{G}_U$-bundle ${}_Uq_X^*(\mathcal{G})$ is trivial on $\mathcal{X}_{q_X^*(\mathrm{f})}$. By the property *(iv)* of the Corollary 5.3 it is also trivial on $\mathcal{X}_{gh}$.

Take a trivial $\mathbf{G}_U$-bundle over $(\mathbf{A}^1_U)_h$ and glue it with the $\mathbf{G}_U$-bundle ${}_Uq_X^*(\mathcal{G})|_{\mathcal{X}_g}$ patching over $\mathcal{X}_{gh}$ (it can be done due to Remark 5.4). We get a $\mathbf{G}_U$-bundle $\mathcal{G}_t$ over $\mathbf{A}^1_U$ which has particularly the following properties:
(a) the restriction of $\mathcal{G}_t$ to $(\mathbf{A}^1_U)_h$ is trivial (by the construction);
(b) the $\mathbf{G}_U$-bundle $\sigma_g^*(\mathcal{G}_t)$ is isomorphic to the $\mathbf{G}_U$-bundle ${}_Uq_X^*(\mathcal{G})|_{\mathcal{X}_g}$.
It remains to check that the restriction of the $\mathbf{G}_U$-bundle $\mathcal{G}_t$ to $0 \times U$ is isomorphic to the $\mathbf{G}_U$-bundle $can^*(\mathcal{G})$. To do that note that the equalities $q_U \circ \Delta = id_U$ and $q_X \circ \Delta = can$ yield the equalities

$$\mathbf{G}_U = \Delta^*(q_X^*(\mathbf{G})) \ \text{ and } \ can^*(\mathcal{G}) = \Delta^*(q_X^*(\mathcal{G})).$$

There are two interesting $\mathbf{G}_U$-bundles over U. Namely, the $\mathbf{G}_U$-bundle $can^*(\mathcal{G}) = \Delta^*(q_X^*(\mathcal{G}))$ and the $\mathbf{G}_U$-bundle $\Delta^*({}_Uq_X^*(\mathcal{G}))$. *They coincide* since $\Delta^*(\Phi) = id_{\mathbf{G}_U}$. Thus

$$can^*(\mathcal{G}) = \Delta^*({}_Uq_X^*(\mathcal{G})) \cong \Delta^*(\sigma_g^*(\mathcal{G}_t)) = \mathcal{G}_t|_{0\times U}.$$

The middle isomorphism is well-defined by the property *(iii)* of Corollary 5.3. The latter equality holds also by the property *(iii)* of Corollary 5.3. Whence the Theorem 1.3.

□

Remark 5.5. Here is *the motivic view point* on the above arguments (in the constant case). The distinguished elementary square (2) defines a motivic space isomorphism $\mathcal{X}_g/\mathcal{X}_{gh} \xleftarrow{\sigma} \mathbf{A}^1_U/(\mathbf{A}^1 \times U)_h$ (just a Nisnevich sheaf isomorphism), hence there is a composite morphism of motivic spaces of the form

$$\varphi : \mathbf{A}^1_U/(\mathbf{A}^1 \times U)_h \xrightarrow{\sigma^{-1}} \mathcal{X}_g/\mathcal{X}_{gh} \to \mathcal{X}_g/\mathcal{X}_{q_X^*(\mathrm{f})} \xrightarrow{q} X/X_{\mathrm{f}}.$$

Let $i_0 : 0 \times U \to \mathbf{A}^1_U/(\mathbf{A}^1_U)_h$ be the natural morphism. By the properties (a') and (d) from Theorem 5.1 the morphism $\varphi \circ i_0$ equals to the one

$$U \xrightarrow{can} X \xrightarrow{p} X/X_{\mathrm{f}},$$

where $p : X \to X/X_{\mathrm{f}}$ is the canonical morphisms.

Now assume that $\mathbf{G}_0$ is a reductive group scheme over the field k. A $\mathbf{G}_0$-bundle over X, trivialized on X_{f}, is "classified" by a morphism $\rho : X/X_{\mathrm{f}} \to (B\mathbf{G}_0)_{et}$ in an appropriate category. Thus the morphism $\rho \circ \varphi$ "classifies" a $\mathbf{G}_0$-bundle $\mathcal{G}_t$ over $\mathbf{A}^1_U$ trivialized on $(\mathbf{A}^1_U)_h$. The equality $\varphi \circ i_0 = p \circ can$ shows that the $\mathbf{G}_0$-bundles $\mathcal{G}_t|_{0\times U}$ and $can^*(\mathcal{G})$ are isomorphic. This "proves" Theorem 1.3 in the constant case.

6 A purity theorem

Let k be a field.

Theorem 6.1. *Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a k-smooth irreducible affine k-variety X. Let $K = k(X)$. Let*

$$\mu : \mathbf{G} \to \mathbf{C}$$

be a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with a torus $\mathbf{C}$. Suppose additionally that the kernel of μ is a reductive $\mathcal{O}$-group scheme. Then the following sequence

$$\{1\} \to \mathbf{C}(\mathcal{O})/\mu(\mathbf{G}(\mathcal{O})) \to \mathbf{C}(K)/\mu(\mathbf{G}(K)) \xrightarrow{\sum res_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathbf{C}(K)/[\mathbf{C}(\mathcal{O}_{\mathfrak{p}}) \cdot \mu(\mathbf{G}(K))] \to \{1\} \tag{4}$$

is exact in the middle term, where $\mathfrak{p}$ runs over the height 1 primes of $\mathcal{O}$ and $res_{\mathfrak{p}}$ is the natural map (the projection to the factor group).

Remark 6.2. After the proof of Theorem 1.1 will be completed it will have the following corollary: the sequence (4) is exact. In fact, the exactness at the lefthand side term is a direct consequence Theorem 1.1. The exactness at the righthand side term is proved in [C-T/S].

Definition 6.3. *Let $a \in \mathbf{C}(k(X))$. Its class $\bar{a} \in \bar{\mathbf{C}}(k(X)) := \mathbf{C}(K)/\mu(\mathbf{G}(K))$ is called unramified at a height 1 prime ideal $\mathfrak{p}$ of $k[X]$, if the element $\bar{a}$ is in the image of the group $\bar{\mathbf{C}}(\mathcal{O}_{\mathfrak{p}})$. Let $S \subset k[X]$ be a multiplicative system The class $\bar{a} \in \bar{\mathbf{C}}(k(X))$ is called $k[X]_S$ -unramified, if it is unramified at any codimension one prime ideal of $k[X]_S$.*

The following lemma is obvious.

Lemma 6.4. *Let $\varphi : Y \to X$ be a smooth morphism of smooth irreducible affine k-varieties. This morphism induces an obvious map $\bar{\varphi}^* : \bar{\mathbf{C}}(k(X)) \to \bar{\mathbf{C}}(k(Y))$, which takes X-unramified elements to Y-unramified elements. If $S \subset k[Y]$ be a multiplicative system, then the homomorphism $\bar{\varphi}^*$ takes X-unramified elements to $k[Y]_S$ -unramified elements.*

Reducing Theorem 6.1 to Theorem 5.2. Assume firstly that μ is "constant", i.e there are a reductive group $\mathbf{G}_0$, a torus $\mathbf{C}_0$ over the field k and an algebraic k-group morphism μ_0 and U-group schemes isomorphisms

$$\Phi : \mathbf{G}_{0,U} = \mathbf{G}_0 \times_{Spec(k)} U \to \mathbf{G} \quad \text{and} \quad \Psi : \mathbf{C}_{0,U} = \mathbf{C}_0 \times_{Spec(k)} U \to \mathbf{C}$$

such that $\Psi \circ \mu_{0,U} = \mu \circ \Phi$.

Let $a_K \in \mathbf{C}(k(X))$ be such that its class in $\bar{\mathbf{C}}(K)$ is $\mathcal{O}$-unramified. Then there is a non-zero function $\mathrm{f} \in k[X]$ such that the element a_K is defined over X_{f}, that is there is given an element $a \in \mathbf{C}(k[X_{\mathrm{f}}])$ for a non-zero function $\mathrm{f} \in k[X]$ such that the image of a in $\mathbf{C}(K)$ coincides with the element a_K. Shrinking X we may assume further that f vanishes at each x_i's and the k-algebra $k[X]/(\mathrm{f})$ **is reduced**. Shrinking X once again we may and will assume also that $\bar{a} \in \bar{\mathbf{C}}(k[X]_{\mathrm{f}})$ is $k[X]$-unramified. By Theorem 5.1 there is a diagram of the form (1) together with with the scheme $\mathcal{X}$, the morphisms q_U, σ and q_X, and the function $f' \in q_X^*(\mathrm{f}\ )k[\mathcal{X}]$, which enjoys the properties (a) to (f) from that Theorem. From now on and till the end of this proof we will use the notation from Theorem 5.1.

The morphism σ from that theorem is finite surjective and the schemes $\mathbf{A}^1_U$ and $\mathcal{X}$ are regular. Thus by a theorem of Grothendieck [E] the morphism σ is flat and finite. Thus any base change of σ is finite and flat. Set $\alpha := q_f^*(a) \in \mathbf{C}(\mathcal{X}_f)$ where $q_f : \mathcal{X}_f \to X_{\mathrm{f}}$ is the restriction of f to $\mathcal{X}$ and set

$$a_U := N_{\mathcal{D}_1/U}(\alpha|_{\mathcal{D}_1}) \cdot N_{\mathcal{D}'_0/U}(\alpha|_{\mathcal{D}'_0})^{-1} \in \mathbf{C}(U). \tag{5}$$

Claim 6.5. *Let $\eta_U : Spec(k(X)) \to Spec(\mathcal{O}) = U$ and $\eta : Spec(k(X)) \to X_f$ be the generic points of U and X_f respectively. Then $\eta_U^*(\bar{a_U}) = \eta^*(\bar{a}) \in \bar{\mathbf{C}}(k(X))$.*

Since $\eta^*(a) = a_K$, this Claim completes the proof of Theorem 6.1 in the constant case. To prove the Claim consider the scheme $\mathcal{X}$ and its closed and open subschemes as U-schemes via the morphism q_U. Set $K = k(X)$. Taking the base change of $\mathcal{X}$, $\mathbf{A}^1_U$ and σ via the morphism $\eta_U : Spec(K) \to U$ we get a morphism of the K-shemes $\mathbf{A}^1_K \xleftarrow{\sigma_K} \mathcal{X}_K$. Recall that the class $\bar{a} \in \bar{\mathbf{C}}(X_{\mathrm{f}})$ is X-unramified. By Lemma 6.4 the class $\bar{\alpha} \in \bar{\mathbf{C}}(\mathcal{X}_f)$ is $\mathcal{X}$-unramified. Hence its image $\bar{\alpha}_K$ in $\bar{\mathbf{C}}(K(\mathcal{X}_K))$ is X_K-unramified too. The item (v) of Corollary 5.3 and Lemma 17.4 show that for the element $\beta_t := N_{K(\mathcal{X}_K)/K(\mathbf{A}^1_K)}(\alpha_K) \in \mathbf{C}(K(t))$ the class $\bar{\beta}_t \in \bar{\mathbf{C}}(K(t))$ is $\mathbf{A}^1_K$-unramified. By Theorem 18.4 the class $\bar{\beta}_t$ is constant, t.e. it comes from the field K. By Corollary 18.5 its specializations at the K-points 0 and 1 of the affine line $\mathbf{A}^1_K$ coincide: $s_0(\bar{\beta}_t) = s_1(\bar{\beta}_t) \in \bar{\mathbf{C}}(K)$. The properties (d),(c) and (e) and the equality $q_X \circ \Delta = can$ from Theorem 5.1 show that $\mathcal{D}_{1,K}, \mathcal{D}'_{0,K}, \Delta(Spec(K)) \subset (\mathcal{X}_f)_K$. Thus there is a Zariski open neighborhood V of the K-points 0 and 1 in $\mathbf{A}^1_K$ such that $W := (\sigma_K)^{-1}(V) \subset (\mathcal{X}_f)_K$. Hence for $\beta_V := N_{W/V}(\alpha|_W)$, one has the equality $\beta_V = \beta_t$ in $\mathbf{C}(K(t))$. Thus one has equalities

$$\overline{\beta(1)} = s_1(\bar{\beta}_t) = s_0(\bar{\beta}_t) = \overline{\beta(0)}$$

(see the remark at the end of Definition 18.1). By the properties (i'), (ii') and (iii') of the norm maps (see Section 16) one has equalities

$$N_{\mathcal{D}_{1,K}/K}(\alpha|_{\mathcal{D}_{1,K}}) = \beta(1) \ \text{ and } \ \beta(0) = N_{\mathcal{D}_{0,K}/K}(\alpha|_{\mathcal{D}_{0,K}}) = N_{\mathcal{D}'_{0,K}/K}(\alpha|_{\mathcal{D}'_{0,K}}) \cdot \Delta^*_K(\alpha_K)$$

By the base change property of the norm maps one has the equality

$$\eta_U^*(a_U) = N_{\mathcal{D}_{1,K}/K}(\alpha|_{\mathcal{D}_{1,K}}) \cdot [N_{\mathcal{D}'_{0,K}/K}(\alpha|_{\mathcal{D}'_{0,K}})]^{-1}$$

Hence $\Delta^*_K(\bar{\alpha}_K) = \eta_U^*(\bar{a}_U)$ in $\bar{\mathbf{C}}(k(X))$. Finally, the composite map $Spec(K) \xrightarrow{\Delta_K} (\mathcal{X}_f)_K \to \mathcal{X}_f \xrightarrow{q_f} X_{\mathrm{f}}$ coincides with the canonical map $\eta : Spec(K) \to X_{\mathrm{f}}$. Hence $\Delta^*_K(\bar{\alpha}_K) = \eta^*(\bar{a})$, which proves the Claim. Whence the Theorem 6.1 in the constant case.

In the general case there are two functors on the category of $\mathcal{X}$-schemes. Namely, $\bar{C}$ and ${}_U\bar{C}$. If $r : \mathcal{Y} \to \mathcal{X}$ is a scheme morphism, then $\bar{C}(\mathcal{Y}) := C(\mathcal{Y})/(\mu(G(\mathcal{Y})))$ and ${}_U\bar{C}(\mathcal{Y}) := {}_UC(\mathcal{Y})/(\mu({}_UG(\mathcal{Y})))$. Here $\mathcal{Y}$ is regarded as an X-scheme via the morphism $q_X \circ r$ and is regarded as an U-scheme via the morphism $q_U \circ r$. The $\mathcal{X}$-group scheme isomorphisms Φ and Ψ from Theorem 5.2 induce a group isomorphism

$$\bar{\Psi}_{\mathcal{Y}} : {}_U\bar{C}(\mathcal{Y}) \to \bar{C}(\mathcal{Y}),$$

which respect to $\mathcal{X}$-schemes morphisms. Moreover, if the scheme U is regarded as an $\mathcal{X}$-scheme via the morphism Δ, then the isomorphism $\bar{\Psi}_{\mathcal{Y}}$ is the identity. And similarly for any U-scheme $g : W \to U$ regarded as an $\mathcal{X}$-scheme via the morphism $\Delta \circ g$ the the isomorphism $\bar{\Psi}_W$ is the identity.

Set $\alpha := q_f^*(a) \in \mathbf{C}(\mathcal{X}_f)$ where $q_f : \mathcal{X}_f \to X_{\mathrm{f}}$ is above in this proof. Let ${}_U\alpha \in {}_UC(\mathcal{X})$ be a unique element such that $\bar{\Psi}_{\mathcal{X}}({}_U\alpha) = \alpha$. Set

$$ {}_Ua := N_{\mathcal{D}_1/U}(({}_U\alpha)|_{\mathcal{D}_1}) \cdot N_{\mathcal{D}'_0/U}(({}_U\alpha)|_{\mathcal{D}'_0})^{-1} \in {}_U\mathbf{C}(U) \text{ and } a_U := \Psi_U({}_Ua) \in \mathbf{C}(U) \quad (6)$$

We left to the reader to proof the following Claim

Claim 6.6. *Let $\eta_U : Spec(k(X)) \to Spec(\mathcal{O}) = U$ and $\eta : Spec(k(X)) \to X_f$ be as above in this proof. Then*

$$\eta_U^*(\bar{a_U}) = \eta^*(\bar{a}) \in \bar{\mathbf{C}}(k(X))$$

Since $\eta^*(a) = a_K$, this Claim completes the reduction of Theorem 6.1 to Theorem 5.2. □

7 One more purity theorem

In this section we reduce another purity theorem for reductive group schemes to Theorem 5.2. Let k, $\mathcal{O}$ and K be as in Theorem 6.1. Let G be a semi-simple $\mathcal{O}$-group scheme. Let $i : Z \hookrightarrow G$ be a closed subgroup scheme of the center $Cent(G)$. **It is known that Z is of multiplicative type**. Let $G' = G/Z$ be the factor group, $\pi : G \to G'$ be the projection. It is known that π is finite surjective and strictly flat. Thus the sequence of $\mathcal{O}$-group schemes

$$\{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\} \quad (7)$$

induces an exact sequence of group sheaves in fppt-topology. Thus for every $\mathcal{O}$-algebra R the sequence (7) gives rise to a boundary operator

$$\delta_{\pi,R} : G'(R) \to \mathrm{H}^1_{\mathrm{fppt}}(R, Z) \quad (8)$$

One can check that it is a group homomorphism (compare [Se, Ch.II, §5.6, Cor.2]). Set

$$\mathcal{F}(R) = \mathrm{H}^1_{\mathrm{fttp}}(R, Z)/Im(\delta_{\pi,R}). \quad (9)$$

Clearly we get a functor on the category of $\mathcal{O}$-algebras.

Theorem 7.1. *Let $\mathcal{O}$ be the semi-local ring of finitely many* **closed** *points on a k-smooth irreducible* **affine** *k-variety X. Let G be a semi-simple $\mathcal{O}$-group scheme. Let $i : Z \hookrightarrow G$ be a closed subgroup scheme of the center $Cent(G)$. Let $\mathcal{F}$ be the functor on the category $\mathcal{O}$-algebras given by (9). Then the sequence*

$$\mathcal{F}(\mathcal{O}) \to \mathcal{F}(K) \xrightarrow{\sum can_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathcal{F}(K)/Im[\mathcal{F}(\mathcal{O}_{\mathfrak{p}}) \to \mathcal{F}(K)] \qquad (10)$$

is exact, where $\mathfrak{p}$ runs over the height 1 primes of $\mathcal{O}$ and $can_{\mathfrak{p}}$ is the natural map (the projection to the factor group).

Reducing Theorem 7.1 to Theorem 5.2. The group Z is of multiplicative type. So we can find a finite étale $\mathcal{O}$-algebra A and a closed embedding $Z \hookrightarrow R_{A/\mathcal{O}}(\mathbb{G}_{m,\ A})$ into the permutation torus $T^+ = R_{A/\mathcal{O}}(\mathbb{G}_{m,\ A})$. Let $G^+ = (G \times T^+)/Z$ and $T = T^+/Z$, where Z is embedded in $G \times T^+$ diagonally. Clearly $G^+/G = T$. Consider a commutative diagram

$$\begin{array}{ccccccccc}
 & & \{1\} & & \{1\} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & G' & \xrightarrow{id} & G' & & & & \\
 & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi^+} & & & & \\
\{1\} & \longrightarrow & G & \xrightarrow{j^+} & G^+ & \xrightarrow{\mu^+} & T & \longrightarrow & \{1\} \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i^+} & & \uparrow{\scriptstyle id} & & \\
\{1\} & \longrightarrow & Z & \xrightarrow{j} & T^+ & \xrightarrow{\mu} & T & \longrightarrow & \{1\} \\
 & & \uparrow & & \uparrow & & & & \\
 & & \{1\} & & \{1\} & & & &
\end{array}$$

with exact rows and columns. By the known fact (see Lemma 8.1) and Hilbert 90 for the semi-local $\mathcal{O}$-algebra A one has $\mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, T^+) = \mathrm{H}^1_{\text{ét}}(\mathcal{O}, T^+) = \mathrm{H}^1_{\text{ét}}(A, \mathbb{G}_{m,A}) = \{*\}$. So, the latter diagram gives rise to a commutative diagram of pointed sets

$$\begin{array}{ccccccc}
 & & & & \mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, G') & \xrightarrow{id} & \mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, G') \\
 & & & & \uparrow{\scriptstyle \pi_*} & & \uparrow{\scriptstyle \pi^+_*} \\
G^+(\mathcal{O}) & \xrightarrow{\mu^+_{\mathcal{O}}} & T(\mathcal{O}) & \xrightarrow{\delta^+_{\mathcal{O}}} & \mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, G) & \xrightarrow{j^+_*} & \mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, G^+) \\
\uparrow{\scriptstyle i^+_*} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i^+_*} \\
T^+(\mathcal{O}) & \xrightarrow{\mu_{\mathcal{O}}} & T(\mathcal{O}) & \xrightarrow{\delta_{\mathcal{O}}} & \mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, Z) & \xrightarrow{\mu} & \{*\} \\
 & & & & \uparrow{\scriptstyle \delta_\pi} & & \\
 & & & & G'(\mathcal{O}) & &
\end{array}$$

with exact rows and columns. It follows that π^+_* has trivial kernel and one has a chain of group isomorphisms

$$\mathrm{H}^1_{\mathrm{fppt}}(\mathcal{O}, Z)/Im(\delta_{\pi,\mathcal{O}}) = ker(\pi_*) = ker(j^+_*) = T(\mathcal{O})/\mu^+(G^+(\mathcal{O})).$$

Clearly these isomorphisms respect $\mathcal{O}$-homomorphisms of semi-local $\mathcal{O}$-algebras.

The morphism $\mu^+ : G^+ \to T$ is a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with the torus T. The kernel $ker(\mu^+)$ is equal to G and G is a reductive $\mathcal{O}$-group scheme. Now by Theorem 6.1 the sequence (4) is exact. Thus the sequence (10) is exact too.

□

8 Reducing Theorem 1.5 to Theorem 5.2

Reducing the semi-simple case of Theorem 1.5 to Theorem 5.2. Let $\mathcal{O}$ and G be the same as in Theorem 1.5 and assume additionally that G is semi-simple. We need to prove that

$$ker[H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G)] = *. \tag{11}$$

Let G^{sc} be the corresponding simply-connected semi-simple $\mathcal{O}$-group scheme and let $\pi : G^{sc} \to G$ be the corresponding $\mathcal{O}$-group scheme morphism. Let $Z = ker(\pi)$. It is known that Z is contained in the center $Cent(G^{sc})$ of G^{sc} and Z is a finite group scheme of multiplicative type. It is known that $G = G^{sc}/Z$ and π is finite surjective and strictly flat. Thus the sequence of $\mathcal{O}$-group schemes

$$\{1\} \to Z \xrightarrow{i} G^{sc} \xrightarrow{\pi} G \to \{1\}, \tag{12}$$

gives rise to an exact sequence of pointed sets

$$H^1_{\mathrm{fppt}}(\mathcal{O}, Z)/\partial(G(\mathcal{O})) \to H^1_{\mathrm{fppt}}(\mathcal{O}, G^{sc}) \to H^1_{\mathrm{fppt}}(\mathcal{O}, G) \to H^2_{\mathrm{fppt}}(\mathcal{O}, Z) \tag{13}$$

(here $H^1_{\mathrm{fppt}}(\mathcal{O}, Z)/\partial(G(\mathcal{O}))$ is the factor-group; compare ([Se, Ch.I,Sect.5, Prop.39 and Cor.1 of Prop.40,Cor.2 of Prop.40])). By the Theorem 7.1 the functor

$$\mathcal{F}(R) = \mathrm{H}^1_{\mathrm{fppt}}(R, Z)/Im(\delta_{\pi,R}) = \mathrm{H}^1_{\mathrm{fppt}}(R, Z)/\partial(G(R)).$$

satisfies purity for the ring $\mathcal{O}$. The following result is known (see [Gr3, Thm.11.7])

Lemma 8.1. *Let R be a noetherian ring. Then for a reductive R-group scheme H and for $n = 0, 1$ the canonical map $H^n_{ét}(R, H) \to H^n_{fppt}(R, H)$ is a bijection of pointed sets. For a R-tori T and for each integer $n \geq 0$ the canonical map $H^n_{ét}(R, T) \to H^n_{fppt}(R, T)$ is an isomorphism.*

Lemma 8.2. *For the ring $\mathcal{O}$ above the map $H^2_{fppt}(\mathcal{O}, Z) \to H^2_{fppt}(K, Z)$ is injective.*

Proof. See [C-T/S, Thm.4.3]. □

Continue the proof of the equality (11). We have the exact sequence of pointed sets

$$H^1_{\mathrm{fppt}}(\mathcal{O},Z)/\partial(G(\mathcal{O})) \to H^1_{\mathrm{fppt}}(\mathcal{O},G^{sc}) \to H^1_{\mathrm{fppt}}(\mathcal{O},G) \to H^2_{\mathrm{fppt}}(\mathcal{O},Z) \qquad (14)$$

and furthermore a commutative diagram with exact arrows

$$\begin{array}{ccccccc}
H^1_{\mathrm{fppt}}(\mathcal{O},Z)/\partial(G(\mathcal{O})) & \xrightarrow{i_*} & H^1_{\mathrm{fppt}}(\mathcal{O},G^{sc}) & \xrightarrow{\pi_*} & H^1_{\mathrm{fppt}}(\mathcal{O},G) & \xrightarrow{\partial} & H^2_{\mathrm{fppt}}(\mathcal{O},Z) \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} \\
H^1_{\mathrm{fppt}}(\mathcal{O}_{\mathfrak{p}},Z)/\partial(G(\mathcal{O}_{\mathfrak{p}})) & \xrightarrow{i'_*} & H^1_{\mathrm{fppt}}(\mathcal{O}_{\mathfrak{p}},G^{sc}) & \xrightarrow{\pi_*} & H^1_{\mathrm{fppt}}(\mathcal{O}_{\mathfrak{p}},G) & \xrightarrow{\partial} & H^2_{\mathrm{fppt}}(\mathcal{O}_{\mathfrak{p}},Z) \\
\downarrow{\alpha_{\mathfrak{p}}} & & \downarrow{\beta_{\mathfrak{p}}} & & \downarrow{\gamma_{\mathfrak{p}}} & & \downarrow{\delta_{\mathfrak{p}}} \\
H^1_{\mathrm{fppt}}(K,Z)/\partial(G(K)) & \xrightarrow{i''_*} & H^1_{\mathrm{fppt}}(K,G^{sc}) & \xrightarrow{\pi_*} & H^1_{\mathrm{fppt}}(K,G) & \xrightarrow{\partial} & H^1_{\mathrm{fppt}}(K,Z)
\end{array} \qquad (15)$$

Here $\mathfrak{p} \subset \mathcal{O}$ is a hight one prime ideal in $\mathcal{O}$. The maps i_*, i'_* and i''_* are injective (compare ([Se, Ch.I,Sect.5, Prop.39 and Cor.1 of Prop.40])). Set $\alpha_K = \alpha_{\mathfrak{p}} \circ \alpha$, $\beta_K = \beta_{\mathfrak{p}} \circ \beta$, $\gamma_K = \gamma_{\mathfrak{p}} \circ \gamma$, $\delta_K = \delta_{\mathfrak{p}} \circ \delta$. By a theorem of Nisnevich [Ni2] and Lemma 8.1 one has

$$ker(\beta_{\mathfrak{p}}) = ker(\gamma_{\mathfrak{p}}) = * \ . \qquad (16)$$

Thus $ker(\alpha_{\mathfrak{p}}) = *$. By the assumptions of Theorem 1.5 and by Lemma 8.1 one has

$$ker[H^1_{\mathrm{fppt}}(\mathcal{O},G^{sc}) \to H^1_{\mathrm{fppt}}(K,G^{sc})] = * \ . \qquad (17)$$

By Lemma 8.2 the map δ_K is injective. As mentioned right above Lemma 8.1 the functor $\mathcal{F}(R) = \mathrm{H}^1_{\mathrm{fppt}}(R,Z)/\partial(G(R))$ satisfies purity for the ring $\mathcal{O}$. Now we are ready to make a diagram chaise.

Let $\xi \in ker(\gamma_K)$, then $\partial(\xi) \in ker(\delta_K)$. By Lemma 8.2 one has $ker(\delta_K) = *$, whence $\partial(\xi) = *$ and $\xi = \pi_*(\zeta)$ for an $\zeta \in H^1_{\text{ét}}(\mathcal{O},G^{sc})$. Since $\gamma_K(\xi) = *$ and $ker(\gamma_{\mathfrak{p}}) = *$ we see that $\gamma(\xi) = *$. Thus $\pi_*(\beta(\zeta)) = *$ and $\beta(\zeta) = i'_*(\epsilon_{\mathfrak{p}})$ for an $\epsilon_{\mathfrak{p}} \in H^1_{\mathrm{fppt}}(\mathcal{O}_{\mathfrak{p}},Z)/\partial(G(\mathcal{O}_{\mathfrak{p}}))$. A diagram chaise shows that there exists a unique element $\epsilon_K \in H^1_{\mathrm{fppt}}(K,Z)/\partial(G(K))$ such that for each hight one prime ideal $\mathfrak{p}$ of $\mathcal{O}$ one has

$$\alpha_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}) = \epsilon_K \in H^1_{\mathrm{fppt}}(K,Z)/\partial(G(K)).$$

By purity Theorem 7.1 there exists an element $\epsilon \in H^1_{\mathrm{fppt}}(\mathcal{O},Z)/\partial(G(\mathcal{O}))$ such that $\alpha_K(\epsilon) = \epsilon_K$. The element ϵ_K has the property that $i''_*(\epsilon_K) = \beta_K(\zeta)$. Whence $\beta_K(i_*(\epsilon)) = \beta_K(\zeta)$. The map $\beta_K : H^1_{\text{ét}}(\mathcal{O},G^{sc}) \to H^1_{\text{ét}}(K,G^{sc})$ is injective since by the hypotheses of Theorem 1.5 such a map has trivial kernel for all semi-simple simply-connected reductive $\mathcal{O}$-group schemes. Whence $i_*(\epsilon) = \zeta$ and $\xi = i_*(\pi_*(\epsilon)) = *$. The semi-simple case of Theorem 1.5 is reduced to Theorem 5.2. □

Claim 8.3. *Under the hypotheses of Theorem 1.5 for all semi-simple reductive $\mathcal{O}$-group scheme G the map $H^1_{\acute{e}t}(\mathcal{O},G) \to H^1_{\acute{e}t}(K,G)$ is injective.*

In fact, let $\xi, \zeta \in H^1_{\text{ét}}(\mathcal{O}, G)$ be two elements such that its images ξ_K, ζ_K in $H^1_{\text{ét}}(K, G)$ are equal. Let ${}_\xi G$, ${}_\zeta G$ be the corresponding principal G-bundles over $\mathcal{O}$ and $G(\zeta)$ be the inner form of the $\mathcal{O}$-group scheme G corresponding to ζ. The $\mathcal{O}$-scheme $\underline{Iso}({}_\xi G, {}_\zeta G)$ is a principal $G(\zeta)$-bundle over $\mathcal{O}$, which is trivial over K. Since $G(\zeta)$ is semi-simple reductive over $\mathcal{O}$, the $\mathcal{O}$-scheme $\underline{Iso}({}_\xi G, {}_\zeta G)$ has an $\mathcal{O}$-point. Whence the Claim.

Reducing Theorem 1.5 to Theorem 5.2. Let $\mathcal{O}$ and G be the same as in Theorem 1.5. Consider a short sequence of reductive $\mathcal{O}$-group schemes

$$\{1\} \to G_{der} \xrightarrow{i} G \xrightarrow{\mu} C \to \{1\}, \tag{18}$$

where G_{der} is the derived $\mathcal{O}$-group scheme of G and $C = corad(G)$ be a tori over $\mathcal{O}$ and $\mu = f_0$ (see [D-G, Exp.XXII, Thm.6.2.1]). By that Theorem the morphism μ is smooth and its kernel is the reductive $\mathcal{O}$-group scheme G_{der}. Moreover G_{der} is a semi-simple $\mathcal{O}$-group scheme. By Claim 8.3 the map

$$H^1_{\text{ét}}(\mathcal{O}, G_{der}) \to H^1_{\text{ét}}(K, G_{der}) \tag{19}$$

is injective. We need to prove that

$$ker[H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G)] = *. \tag{20}$$

The sequence (18) of $\mathcal{O}$-group schemes gives a short exact sequence of the corresponding sheaves in the étale topology on the big étale site. That sequence of sheaves gives rise to a commutative diagram with exact arrows of pointed sets

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & C(\mathcal{O})/\mu(G(\mathcal{O})) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathcal{O}, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(\mathcal{O}, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(\mathcal{O}, C) \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
\{1\} & \longrightarrow & C(\mathcal{O}_\mathfrak{p})/\mu(G(\mathcal{O}_\mathfrak{p})) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathcal{O}_\mathfrak{p}, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(\mathcal{O}_\mathfrak{p}, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(\mathcal{O}_\mathfrak{p}, C) \\
 & & \downarrow{\scriptstyle \alpha_\mathfrak{p}} & & \downarrow{\scriptstyle \beta_\mathfrak{p}} & & \downarrow{\scriptstyle \gamma_\mathfrak{p}} & & \downarrow{\scriptstyle \delta_\mathfrak{p}} \\
\{1\} & \longrightarrow & C(K)/\mu(G(K)) & \xrightarrow{\partial} & H^1_{\text{ét}}(K, G_{der}) & \xrightarrow{i_*} & H^1_{\text{ét}}(K, G) & \xrightarrow{\mu} & H^1_{\text{ét}}(K, C)
\end{array} \tag{21}$$

Here $\mathfrak{p} \subset \mathcal{O}$ is a hight one prime ideal in $\mathcal{O}$. Set $\alpha_K = \alpha_\mathfrak{p} \circ \alpha$, $\beta_K = \beta_\mathfrak{p} \circ \beta$, $\gamma_K = \gamma_\mathfrak{p} \circ \gamma$, $\delta_K = \delta_\mathfrak{p} \circ \delta$. By a theorem of Nisnevich [Ni2] one has

$$ker(\alpha_\mathfrak{p}) = ker(\beta_\mathfrak{p}) = ker(\gamma_\mathfrak{p}) = * \tag{22}$$

Let $\xi \in ker(\gamma_K)$, then $\mu(\xi) \in ker(\delta_K)$. By [C-T/S] one has $ker(\delta_K) = *$, whence $\mu(\xi) = *$ and $\xi = i_*(\zeta)$ for an $\zeta \in H^1_{\text{ét}}(\mathcal{O}, G_{der})$. Since $\gamma_K(\xi) = *$ and $ker(\gamma_\mathfrak{p}) = *$ we see that $\gamma(\xi) = *$. Whence $i_*(\beta(\zeta)) = *$ and $\beta(\zeta) = \partial(\epsilon_\mathfrak{p})$ for an $\epsilon_\mathfrak{p} \in C(\mathcal{O}_\mathfrak{p})/\mu(G(\mathcal{O}_\mathfrak{p}))$. A diagram chaise AND Lemma 17.2 show that there exists a unique element $\epsilon_K \in C(K)/\mu(G(K))$ such that for each hight one prime ideal $\mathfrak{p}$ of $\mathcal{O}$ one has $\alpha_\mathfrak{p}(\epsilon_\mathfrak{p}) = \epsilon_K \in C(K)/\mu(G(K))$.

By purity Theorem 6.1 there exists an element $\epsilon \in C(\mathcal{O})/\mu(G(\mathcal{O}))$ such that $\alpha_K(\epsilon) = \epsilon_K$. The element ϵ_K has the property that $\partial(\epsilon_K) = \beta_K(\zeta)$. Whence $\beta_K(\partial(\epsilon)) = \beta_K(\zeta)$. The map β_K is injective as indicated in the beginning of the proof. Whence $\partial(\epsilon) = \zeta$ and $\xi = i_*(\partial(\epsilon)) = *$. *The reduction of Theorem 1.5 to Theorem 5.2 is completed.*

□

9 The strategy of the proof of Theorems 5.1 and 5.2

A toy task: given a finite field k, a k-smooth irreducible affine curve Y, a k-rational point $y_0 \in Y$, a "bad" locus $B \subset Y$ finite over $Spec(k)$ and containing the point y_0, find a diagram of the form $\mathbf{A}^1_k \xleftarrow{\tau} Y' \xrightarrow{\theta} Y$ with an irreducible affine curve Y' and a k-rational point $y'_0 \in Y'$ and a closed subset B' of Y' (a new "bad" locus) such that the morphism θ is étale, $\theta(y'_0) = y_0$, $\theta^{-1}(B) \subset B'$ and B' has the following properies hold:
(1) τ is finite surjective,
(2) $\tau|_{B'} : B' \to \mathbf{A}^1_k$ is a closed embedding,
(3) τ is étale at each the point of B',
(4) $\tau^{-1}(0) = \{y'_0\} \sqcup D$ (scheme-wise) and $D \cap B' = \emptyset$,
(5) $\tau^{-1}(1) \cap B' = \emptyset$.
The following properties of the subset B' are useful:
(1′) the point $y'_0 \in B'$ is the only k-rational point in B';
(2′) for any integer $d \geq 0$ the amount of the degree d points on the locus B' is less or equal to the amount of points of degree d on $\mathbf{A}^1_k$.
The toy task can be solved in two steps: (a) find appropriate an θ, y' and B'; (b) find the required τ (once the conditions (1′), (2′) are full-filled it's easy to find the required τ).

If $\mathbf{G}$ is a reductive Y-group scheme and $\mathbf{G}_0 = \mathbf{G}|_{y_0}$ and $\mathbf{G}_{const} = q^*(\mathbf{G}_0)$, then one can modify the toy task adding to the original one more conditions on Y' and y'_0:
(6) there is a Y'-group scheme isomorphism $\Phi : \theta^*(\mathbf{G}_{const}) \to \theta^*(\mathbf{G})$ such that $\Phi|_{y'_0} = id$.

To solve the modified task one can firstly find an étale morphism $\theta'' : Y'' \to Y$ (with an irreducible Y'') and a k-rational point y''_0 and a Y''-group scheme isomorphism $\Phi'' : (\theta'')^*(\mathbf{G}_{const}) \to (\theta'')^*(\mathbf{G})$ such that $\theta''(y''_0) = y$ and $\Phi''|_{y''_0} = id$. Once θ'', y''_0 and Φ'' subjecting these conditions are found the modified task is reduced to the original one for a new data. Namely, for the data Y'', y''_0 and $B'' := (\theta'')^{-1}(B)$.

Theorems 5.1 and 5.2 are proved in Section 15. One of the main idea is this: given a non-zero function $\mathrm{f} \in X$ shrink X and to make it a smooth relative curve over a smooth affine base S equipped with a finite surjective S-morphism to the affine line $\mathbf{A}^1 \times S$ and such that the locus $\{\mathrm{f} = 0\}$ is finite over S. Then the fibre product $U \times_S X$ is a smooth relative curve over U, equipped with the section Δ, with a finite surjective U-morphism to the affine line $\mathbf{A}^1 \times U$ and such that the locus $p_X^{-1}(\{\mathrm{f} = 0\})$ is finite over U. It turns out that with this relative U-curve one can work exactly as with the k-curve Y in the toy task. This idea is inspired by one of the main idea from [Vo]. Here we refined that idea following [PSV].

10 Elementary fibrations

In this Section we extend a result of M. Artin from [A] concerning existence of nice neighborhoods. The following notion is introduced by Artin in [A, Exp. XI, Déf. 3.1].

Definition 10.1. *An elementary fibration is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{i} & Y \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \overline{p}} & \swarrow{\scriptstyle q} & \\ & & S & & \end{array} \tag{23}$$

of morphisms satisfying the following conditions:

(i) *j is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$;*

(ii) *$\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one;*

(iii) *q is finite étale all of whose fibres are non-empty.*

Remark 10.2. Clearly, an elementary fibration is an almost elementary fibration in the sense of [PSV, Defn.2.1].

Using repeatedly [Poo, Thm.1.3] and [ChPoo, Thm.1.1] and modifying Artin's arguments [A, Exp. XI,Prop. 3.3], one can prove the following result, which is a slight extension of Artin's result [A, Exp. XI,Prop. 3.3].

Proposition 10.3. *Let k be* **a finite field***, X be a smooth* **geometrically** *irreducible* **affine** *variety over k, $x_1, x_2, \ldots, x_n \in X$ be a family of* **closed** *points. Then there exists a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \ldots, x_n\}$ and* **an elementary fibration** *$p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$.*

If, moreover, Z is a closed co-dimension one subvariety in X, then one can choose X^0 and p in such a way that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.

The following result is proved in [PSV, Prop.2.4].

Proposition 10.4. *Let $p : X \to S$ be an elementary fibration. If S is a regular semi-local irreducible scheme, then there exists a commutative diagram of S-schemes*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{i} & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow \\ \mathbf{A}^1 \times S & \xrightarrow{\text{in}} & \mathbf{P}^1 \times S & \xleftarrow{i} & \{\infty\} \times S \end{array} \tag{24}$$

such that the left hand side square is Cartesian. Here j and i are the same as in Definition 10.1, while $\mathrm{pr}_S \circ \pi = p$, where pr_S is the projection $\mathbf{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbf{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X and $\mathbf{A}^1 \times S$ are regarded as S-schemes via the morphism p and the projection pr_S, respectively.

11 Nice triples

In the present section we recall and study certain collections of geometric data and their morphisms. The concept of a *nice triple* was introduced in [PSV, Defn. 3.1] and is very similar to that of a *standard triple* introduced by Voevodsky [Vo, Defn. 4.1], and was in fact inspired by the latter notion. Let k be a field, let X be a k-smooth irreducible **affine** k-variety, and let $x_1, x_2, \dots, x_n \in X$ be **a family of closed points**. Further, let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ be the corresponding geometric semi-local ring.

Definition 11.1. *Let $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$. A* nice triple *over U consists of the following data:*

(i) *a smooth morphism $q_U : \mathcal{X} \to U$, where $\mathcal{X}$ is an irreducible scheme,*

(ii) *an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,*

(iii) *a section Δ of the morphism q_U,*

subject to the following conditions:

(a) *each irreducible component of each fibre of the morphism q_U has dimension one,*

(b) *the module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\Gamma(U, \mathcal{O}_U) = \mathcal{O}$-module,*

(c) *there exists a finite surjective U-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$,*

(d) *$\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.*

Definition 11.2. *A* morphism *between two nice triples over U*

$$(q' : \mathcal{X}' \to U, f', \Delta') \to (q : \mathcal{X} \to U, f, \Delta)$$

is an étale morphism of U-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that

(1) *$q'_U = q_U \circ \theta$,*

(2) *$f' = \theta^*(f) \cdot h'$ for an element $h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,*

(3) *$\Delta = \theta \circ \Delta'$.*

Two observations are in order here.

• Item (2) implies in particular that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $\mathcal{O}$-module.

• It should be emphasized that no conditions are imposed on the interrelation of Π' and Π.

After substituting k by its algebraic closure $\tilde{k}$ in $k[X]$, we can assume that X is a $\tilde{k}$-smooth geometrically irreducible affine $\tilde{k}$-scheme. To simplify the notation, we will continue to denote this new $\tilde{k}$ by k. Let U be as in Definition 11.1 and $\mathrm{can} : U \hookrightarrow X$ be the canonical inclusion of schemes.

Definition 11.3. *A nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U is called special if the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.*

Remark 11.4. Clearly the following holds: let $(\mathcal{X}, f, \Delta)$ be a special nice triple over U and let $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ be a morphism between nice triples over U. Then the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over U.

Proposition 11.5. *One can shrink X such that $x_1, x_2, \dots, x_n$ are still in X and X is affine, and then to construct a special nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \to X$ such that $q_X \circ \Delta = can$, $f = q_X^*(f)$.*

Proof of Proposition 11.5. If the field k is infinite, then this proposition is proved in [PSV, Prop. 6.1]. So, we may and will assume that k is finite. To prove the proposition repeat literally the proof of [PSV, Prop. 6.1]. One has to replace the references to [PSV, Prop. 2.3] and [PSV, Prop.2.4] with the reference to Propositions 10.3 and 10.4 respectively. □

Let us state two crucial results which are used in Section 15 to prove Theorems 5.1 and 5.2. Their proofs are given in Sections 12 and 14 respectively. If U as in Definition 11.1 then for any U-scheme V and any closed point $u \in U$ set $V_u = u \times_U V$. For a finite set A denote $\sharp A$ the cardinality of A.

Definition 11.6. *Let $(\mathcal{X}, f, \Delta)$ be a special nice triple over U. We say that the triple $(\mathcal{X}, f, \Delta)$ satisfies conditions 1^*_U and 2^*_U if either the field k is infinite or (if k is finite) the following holds*

(1*) *for any closed point $u \in U$, any integer $d \geq 1$ one has*

$$\sharp\{z \in \mathcal{Z}_u | deg[k(z) : k(u)] = d\} \leq \ \sharp\{x \in \mathbf{A}^1_u | deg[k(z) : k(u)] = d\}$$

(2*) *for the vanishing locus $\mathcal{Z}$ of f and for any closed point $u \in U$ the point $\Delta(u) \in \mathcal{Z}_u$ is the only $k(u)$-rational point of $\mathcal{Z}_u$.*

Theorem 11.7. *Let U be as in Definition 11.1. Let $(q'_U : \mathcal{X}' \to U, f', \Delta')$ be a special nice triple over U subjecting to the conditions (1*) and (2*) from Definition 11.6. Then there exists a finite surjective morphism*

$$\mathbf{A}^1 \times U \xleftarrow{\sigma} \mathcal{X}'$$

of U-schemes which enjoys the following properties:

(a) *the morphism $\mathbf{A}^1 \times U \xleftarrow{\sigma|_{\mathcal{Z}'}} \mathcal{Z}'$ is a closed embedding;*

(b) *σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta'(U)$;*

(c) *$\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ scheme theoretically and $\mathcal{Z}'' \cap \Delta'(U) = \emptyset$;*

(d) *$\sigma^{-1}(\{0\} \times U) = \Delta'(U) \coprod \mathcal{D}$ scheme theoretically and $\mathcal{D} \cap \mathcal{Z}' = \emptyset$;*

(e) *for* $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ *one has* $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$.

(f) *there is a monic polinomial* $h \in \mathcal{O}[t]$ *such that* $(h) = Ker[\mathcal{O}[t] \xrightarrow{\bar{\circ}\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$.

Theorem 11.8. *Let* U *be as in Definition 11.1. Let* $(\mathcal{X}, f, \Delta)$ *be a special nice triple over* U*. Let* $G_{\mathcal{X}}$ *be a reductive* $\mathcal{X}$*-group scheme and* $G_U := \Delta^*(G_{\mathcal{X}})$ *and* $G_{const} := q_U^*(\mathbf{G}_U)$*. Let* $C_{\mathcal{X}}$ *be an* $\mathcal{X}$*-tori and* $C_U := \Delta^*(C_{\mathcal{X}})$ *and* $C_{const} := q_U^*(\mathbf{C}_U)$*. Let* $\mu_{\mathcal{X}} : G_{\mathcal{X}} \to C_{\mathcal{X}}$ *be an* $\mathcal{X}$*-group scheme morphism smooth as a scheme morphism. Let* $\mu_U = \Delta^*(\mu_{\mathcal{X}})$ *and* $\mu_{\mathrm{const}} : G_{\mathrm{const}} \to C_{\mathrm{const}}$ *be the the pull-back of* μ_U *to* $\mathcal{X}$*. Then there exist a morphism* $\theta : (q' : \mathcal{X}' \to U, f', \Delta') \to (q : \mathcal{X} \to U, f, \Delta)$ *between nice triples over* U *such that the triple* $(\mathcal{X}', f', \Delta')$ *is a special nice triple over* U *subjecting to the conditions* (1^*) *and* (2^*) *from Definition 11.6. And additionally there are* $\mathcal{X}$*-group schemes isomorphisms*

$$\Phi : \theta^*(G_{const}) \to \theta^*(G_{\mathcal{X}}) \quad \textit{and} \quad \Psi : \theta^*(C_{const}) \to \theta^*(C_{\mathcal{X}})$$

such that $(\Delta')^*(\Phi) = id_{G_U}$, $(\Delta')^*(\Phi) = id_{G_U}$ *and*

$$\theta^*(\mu_{\mathcal{X}}) \circ \Phi = \Psi \circ \theta^*(\mu_{const}). \tag{25}$$

Forgetting about group-schemes in the hypotheses and in the conclusion of Theorem 11.8 one gets the following result (we will not use this result in the present paper).

Theorem 11.9. *Let* U *be as in Definition 11.1. Let* $(\mathcal{X}, f, \Delta)$ *be a special nice triple over* U*. Then there exists a morphism* $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ *of nice triples* **over** U *such that* $(\mathcal{X}', f', \Delta')$ *is a special nice triple satisfying the conditions* (1^*) *and* (2^*) *from Definition 11.6.*

12 Proof of Theorem 11.7

Proof of Theorem 11.7. For any closed point $u \in U$ and any U-scheme V let $V_u = u \times_U V$ be the scheme fibre of the scheme V over the point u.

Step (i). For any closed point $u \in U$ and any point $z \in \mathcal{Z}'_u$ there is a closed embedding $z^{(2)} \hookrightarrow \mathbf{A}^1_u$, where $z^{(2)} := Spec(\Gamma(\mathcal{X}'_u, \mathcal{O}_{\mathcal{X}'_u})/\mathfrak{m}_z^2)$ for the maximal ideal $\mathfrak{m}_z \subset \Gamma(\mathcal{X}'_u, \mathcal{O}_{\mathcal{X}'_u})$ of the point z regarded as a point of $\mathcal{X}'$. This holds, since the $k(u)$-scheme $\mathcal{X}'_u$ is equi-dimensional of dimension one, affine and $k(u)$-smooth.

Step (ii). For any closed point $u \in U$ there is a closed embedding $i_u : \coprod_{z \in \mathcal{Z}'_u} z^{(2)} \hookrightarrow \mathbf{A}^1_u$ of the $k(u)$-schemes. To see this apply Step (i) and use that the triple $(\mathcal{X}, f, \Delta)$ satisfies the condition 1^*_U from Definition 11.6.

Step(iii) is to introduce some notation. Since $(\mathcal{X}', f', \Delta')$ is a nice triple over U there is a finite surjective morphism $\mathcal{X}' \xrightarrow{\Pi} \mathbf{A}^1 \times U$ of the U-schemes. Take the composite $\mathcal{X}' \xrightarrow{\Pi} \mathbf{A}^1 \times U \hookrightarrow \mathbf{P}^1 \times U$ morphism and denote by $\bar{\mathcal{X}}'$ the normalization of $\mathbf{P}^1 \times U$ in the fraction field $k(\mathcal{X}')$ of the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. The normalization of $\mathbf{A}^1 \times U$ in $k(\mathcal{X}')$ coincides with the scheme $\mathcal{X}'$, since $\mathcal{X}'$ is a regular scheme. So, we have a Cartesian diagram of U-schemes

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\text{inc}} & \bar{\mathcal{X}}' \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \bar{\Pi}} \\ \mathbf{A}^1 \times U & \xrightarrow{\text{inc}} & \mathbf{P}^1 \times U, \end{array} \tag{26}$$

in which the horizontal arrows are open embedding.

Let $\mathcal{X}'_\infty$ be the Cartie-divisor $(\bar{\Pi})^{-1}(\infty \times U)$ in $\bar{\mathcal{X}}'$. Let $\mathcal{L} := \mathcal{O}_{\bar{\mathcal{X}}'}(\mathcal{X}'_\infty)$ be the corresponding invertible sheaf and let $s_0 \in \Gamma(\bar{\mathcal{X}}', \mathcal{L})$ be its section vanishing exactly on $\mathcal{X}'_\infty$. One has a Cartesian square of U-schemes

$$\begin{array}{ccc} \mathcal{X}'_{\infty,u} & \xrightarrow{j_\infty} & \mathcal{X}'_\infty \\ {\scriptstyle in_u}\downarrow & & \downarrow{\scriptstyle in} \\ \bar{\mathcal{X}}'_u & \xrightarrow{j} & \bar{\mathcal{X}}', \end{array} \tag{27}$$

which shows that the closed embedding $in_u : \mathcal{X}'_{\infty,u} \hookrightarrow \bar{\mathcal{X}}'_u$ is a Cartie-divisor on $\bar{\mathcal{X}}'_u$. Set $\mathcal{L}_u = j^*(\mathcal{L})$ and $s_{0,u} = s_0|_{\bar{\mathcal{X}}'_u} \in \Gamma(\bar{\mathcal{X}}'_u, \mathcal{L}_u)$.

Step (iv). There exists an integer $n > 0$ and a section $s_{1,u} \in \Gamma(\bar{\mathcal{X}}'_u, \mathcal{L}_u^{\otimes n})$ which has no zeros on $\mathcal{X}'_{\infty,u}$ and such that the morphism

$$[s^n_{0,u} : s_{1,u}] : \bar{\mathcal{X}}'_u \to \mathbf{P}^1_u$$

has the following two properties

(a) the morphism $\sigma_u = s_{1,u}/s^n_{0,u} : \mathcal{X}'_u \to \mathbf{A}^1_u$ is finite surjective,

(b) $\sigma_u|_{\coprod_{z \in \mathcal{Z}'_u} z^{(2)}} = i_u : \coprod_{z \in \mathcal{Z}'_u} z^{(2)} \hookrightarrow \mathbf{A}^1_u$, where i_u is from the step (ii); in particular, σ_u is étale at every point $z \in \mathcal{Z}'_u$.

Step (v). There exists a section $s_1 \in \Gamma(\bar{\mathcal{X}}', \mathcal{L}^{\otimes n})$ such that for any closed point $u \in U$ one has $s_1|_{\mathcal{X}'_u} = s_{1,u}$.

Step (vi). If s_1 is such as in the step (v), then the morphism

$$\sigma = (s_1/s^n_0, pr_U) : \mathcal{X}' \to \mathbf{A}^1 \times U$$

is finite and surjective.

We are ready now to check step by step all the statements of the Theorem.

The assertion (b). Since the schemes $\mathcal{X}'$ and $\mathbf{A}^1 \times U$ are regular and the morphism σ is finite and surjective, the morphism σ is flat by a theorem of Grothendieck.

So, to check that σ is étale at a closed point $z \in \mathcal{Z}'$ it suffices to check that for the point $u = q'_U(z)$ the morphism $\sigma_u : \mathcal{X}'_u \to \mathbf{A}^1_u$ is étale at the point z. The latter does hold by the step (iv), item (b). Whence σ is étale at all the closed points of $\mathcal{Z}'$. By the hypotheses of the Theorem the set of closed points of $\Delta'(U)$ is contained in the set of the closed points of $\mathcal{Z}'$. Whence σ is étale also at all the closed points of $\Delta'(U)$. The schemes $\Delta'(U)$ and $\mathcal{Z}'$ are both semi-local. Thus, σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta'(U)$.

The assertion (a). For any closed point $u \in U$ and **any point** $z \in \mathcal{Z}'_u$ the $k(u)$-algebra homomorphism $\sigma^*_u : k(u)[t] \to k(u)[\mathcal{X}'_u]$ is étale at the maximal ideal $\mathfrak{m}_z$ of the $k(u)$-algebra $k(u)[\mathcal{X}'_u]$ and the composite map $k(u)[t] \xrightarrow{\sigma^*_u} k(u)[\mathcal{X}'_u] \to k(u)[\mathcal{X}'_u]/\mathfrak{m}_z$ is an epimorphism. Thus, for any integer $r > 0$ the $k(u)$-algebra homomorphism $k(u)[t] \to k(u)[\mathcal{X}'_u]/\mathfrak{m}^r_z$ is an epimorphism. The local ring $\mathcal{O}_{\mathcal{Z}'_u,z}$ of the scheme $\mathcal{Z}'_u$ at the point z is of the form $k(u)[\mathcal{X}'_u]/\mathfrak{m}^s_z$ for an integer s. Thus, the $k(u)$-algebra homomorphism

$$k(u)[t] \xrightarrow{\sigma^*_u} k(u)[\mathcal{X}'_u] \to \mathcal{O}_{\mathcal{Z}'_u,z}$$

is surjective. Since $\sigma_u|_{\coprod_{z\in\mathcal{Z}'_u} z^{(2)}} = i_u$ and i_u is a closed embedding one concludes that the $k(u)$-algebra homomorphism

$$k(u)[t] \to \prod_{z/u} O_{\mathcal{Z}'_u,z} = \Gamma(\mathcal{Z}'_u, \mathcal{O}_{\mathcal{Z}'_u})$$

is surjective. Let $\mathbf{u} = \coprod Spec(k(u))$ regarded as the closed sub-scheme of U, where u runs over all closed points of U. Then, for the scheme $\mathcal{Z}'_{\mathbf{u}} = \mathbf{u} \times_U \mathcal{Z}'$ the $k[\mathbf{u}]$-algebra homomorphism

$$k[\mathbf{u}][t] \to \Gamma(\mathcal{Z}'_{\mathbf{u}}, \mathcal{O}_{\mathcal{Z}'_{\mathbf{u}}}) \qquad (28)$$

is surjective.

Since $(\mathcal{X}', f', \Delta')$ is a nice triple over U, the $\mathcal{O}$-module $\Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'})$ is finite. Thus, the $k[\mathbf{u}]$-module $\Gamma(\mathcal{Z}'_{\mathbf{u}}, \mathcal{O}_{\mathcal{Z}'_{\mathbf{u}}})$ is finite. Now the surjectivity of the $k[\mathbf{u}]$-algebra homomorphism (28) and the Nakayama lemma show that the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[t] \to \Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'})$ is surjective. Thus, $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding.

The assertion (e). The morphism Δ' is a section of the structure morphism $q'_U : \mathcal{X}' \to U$ and the morphism σ is a morphism of the U-schemes. Hence the composite morphism $t_0 := \sigma \circ \Delta'$ is a section of the projection $pr_U : \mathbf{A}^1 \times U \to U$. This section is defined by an element $a \in \mathcal{O}$. There is another section t_1 of the projection pr_U defined by the element $1 - a \in \mathcal{O}$. Making an affine change of coordinates on $\mathbf{A}^1_U$ we may and will assume that $t_0(U) = \{0\} \times U$ and $t_1(U) = \{1\} \times U$. **If the field k is infinite we can choose a non-zero $\lambda \in k$ such that for $t^{new}_1 := \lambda t_1$ one has: $\mathcal{D}^{new}_1 \cap \mathcal{Z}' = \emptyset$ and $t^{new}_0 := \lambda t_0 = t_0$.**

Since $\mathcal{D}_1$ and $\mathcal{Z}'$ are semi-local, to prove the assertion (e) it suffices to check that $\mathcal{D}_1$ and $\mathcal{Z}'$ have no common closed points. In the infinite field case this is checked just above. It remains to check the finite field case. Let $z \in \mathcal{D}_1 \cap \mathcal{Z}'$ be a common closed point. Then $\sigma(z) \in \{1\} \times U$. Let $u = q'_U(z)$. We already know that $\sigma|_{\mathcal{Z}'}$ is a closed embedding. Thus $deg[z : u] = deg[\sigma(z) : u] = 1$. The triple $(\mathcal{X}, f, \Delta)$ satisfies the condition 2^*_U from Definition 11.6. Thus, $z = \Delta'(u)$. In this case $\sigma(z) \in \{0\} \times U$. But $\sigma(z) \in \{1\} \times U$. This is a contradiction. Whence $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$.

The assertion (c). The finite morphism σ is étale in a neighborhood of $\mathcal{Z}'$ by the item (b) of the Theorem. By the item (a) of the Theorem $\sigma|_{\mathcal{Z}'}$ is a closed embedding. Thus, the morphism $\sigma^{-1}(\sigma(\mathcal{Z}')) \to \sigma(\mathcal{Z}')$ of affine schemes is finite and there is an affine open sub-scheme V of the scheme $\sigma^{-1}(\sigma(\mathcal{Z}'))$ such that the morphism $V \to \sigma(\mathcal{Z}')$ is étale. Since

$\sigma|_{\mathcal{Z}'}$ is a closed embedding there is a unique section s of the morphism $\sigma^{-1}(\sigma(\mathcal{Z}')) \to \sigma(\mathcal{Z}')$ with the image $\mathcal{Z}'$ and this image is contained in V. By [OP1, Lemma 5.3] the scheme $\sigma^{-1}(\sigma(\mathcal{Z}'))$ has the form $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$.
By a similar reasoning the scheme $\sigma^{-1}(\{0\} \times U)$ has the form $\Delta'(U) \coprod \mathcal{D}$. The triple $(\mathcal{X}, f, \Delta)$ is a special nice triple. Thus all the closed points of $\Delta'(U)$ are closed points of $\mathcal{Z}'$ and $\mathcal{Z}' \cap \mathcal{Z}'' = \emptyset$. Thus, $\Delta'(U) \cap \mathcal{Z}'' = \emptyset$.

The assertion (d). It remains to show that $\mathcal{D} \cap \mathcal{Z}' = \emptyset$. Recall that σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta'(U)$. **It's easy to check that $\sigma|_{\mathcal{Z}' \cup \Delta'(U)}$ is a closed embedding. Thus arguing as above one gets a disjoint union decomposition**

$$\sigma^{-1}(\sigma(\mathcal{Z}' \cup \Delta'(U))) = (\mathcal{Z}' \cup \Delta'(U)) \sqcup E$$

for a closed subscheme E in $\mathcal{X}'$. It's checked already that $\Delta'(U) \coprod \mathcal{D} = \emptyset$. Thus, $\mathcal{D} \subset E$. Hence $\mathcal{D} \cap \mathcal{Z}' = \emptyset$.

The assertion (f). Recall that $\mathcal{X}'$ is affine irreducible and regular. So, the principal ideal (f') has the form $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$, where $\mathfrak{p}_i$'s are hight one prime ideals in $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Let $\mathcal{Z}'_i$ be the closed subscheme in $\mathcal{X}'$ defined by the ideal $\mathfrak{p}_i$. Let $\mathfrak{q}_i = \mathcal{O}[t] \cap \mathfrak{p}_i$. The morphism $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding by the item (a) of Theorem 5.1. This yields that $\sigma|_{\mathcal{Z}'_i} : \mathcal{Z}'_i \to \mathbf{A}^1 \times U$ is a closed embedding too. Thus $\mathfrak{p}_i$ is a hight one prime ideal in $\mathcal{O}[t]$. So, it is a principal prime ideal. Since $\mathcal{Z}'$ is finite over U the scheme $\mathcal{Z}'_i$ is finite over U too. Hence the principal prime ideal $\mathfrak{p}_i$ is of the form (h_i) for a unique monic polinomial $h_i \in \mathcal{O}[t]$.

Set $h = h_1^{r_1}h_2^{r_2}\dots h_n^{r_n}$. Clearly, $h \in Ker[\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$. Since the map $\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$ is surjective, to prove the assertion (f) it suffices to check that the surjective $\mathcal{O}$-module homomorphism

$$\bar{\ }\circ\sigma^* : \mathcal{O}[t]/(h) \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$$

is an isomorphism. Both sides are finitely generated projective $\mathcal{O}$-modules. It remains to check that both sides have the same rank as the $\mathcal{O}$-modules. For that it suffices to know that $\mathcal{O}[t]/(h_i)$ and $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\mathfrak{p}_i$ are of the same rank as the $\mathcal{O}$-modules. This is the case since they are isomorphic $\mathcal{O}$-modules. Indeed, the composite map

$$\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\mathfrak{p}_i$$

is an $\mathcal{O}$-algebra epimorphism and the kernel of this epimorphism is the ideal $\mathfrak{q}_i = (h_i)$.

Whence the assertion (f) and whence the Theorem.

□

13 Equating group schemes

Theorem 13.1 is proved in the present Section.

Theorem 13.1. *Let S be a regular semi-local irreducible scheme. Let $\mu_1 : G_1 \to C_1$ and $\mu_2 : G_2 \to C_2$ be two smooth S-group scheme morphisms with tori C_1 and C_2. Assume as well that G_1 and G_2 are reductive S-group schemes which are forms of each other. Assume that C_1 and C_2 are forms of each other. Let $T \subset S$ be a connected non-empty closed sub-scheme of S, and $\varphi : G_1|_T \to G_2|_T$, $\psi : C_1|_T \to C_2|_T$ be S-group scheme isomorphisms such that $(\mu_2|_T) \circ \varphi = \psi \circ (\mu_1|_T)$. Then there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \tilde{S}$ over T and $\tilde{S}$-group scheme isomorphisms $\Phi : \pi^* G_1 \to \pi^* G_2$ and $\Psi : \pi^* C_1 \to \pi^* C_2$ such that*

(i) $\delta^*(\Phi) = \varphi$,

(ii) $\delta^*(\Psi) = \psi$,

(iii) $\pi^*(\mu_2) \circ \Phi = \Psi \circ \pi^*(\mu_1) : \pi^*(G_1) \to \pi^*(C_2)$

(iv) the scheme $\tilde{S}$ is irreducible.

We begin with the following result which extends [PSV, Prop.5.1].

Theorem 13.2. *Let S be a regular semi-local irreducible scheme. Assume that G_1 and G_2 are reductive S-group schemes which are forms of each other. Let $T \subset S$ be a connected non-empty closed sub-scheme of S, and $\varphi : G_1|_T \to G_2|_T$ be S-group scheme isomorphism. Then there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \tilde{S}$ over T and $\tilde{S}$-group scheme isomorphisms $\Phi : \pi^* G_1 \to \pi^* G_2$ such that*

(i) $\delta^*(\Phi) = \varphi$,

(ii) the scheme $\tilde{S}$ is irreducible.

Proposition 13.3. *Theorem 13.2 holds in the case when the group schemes G_1 and G_2 are semi-simple.*

Proof of Proposition 13.3. The proof literally repeats the proof of [PSV, Prop.5.1] except exactly one reference, which is the reference to [OP2, Lemma 7.2]. That reference one has to replace with the reference to the following

Lemma 13.4. *Let $S = Spec(R)$ be a regular semi-local scheme. Let T be a closed sub-scheme of S. Let $\bar{X}$ be a closed subscheme of $\mathbb{P}^d_S = Proj(S[X_0, \dots, X_d])$ and $X = \bar{X} \cap \mathbf{A}^d_S$, where $\mathbb{A}^d_S$ is the affine space defined by $X_0 \neq 0$. Let $X_\infty = \bar{X} \setminus X$ be the intersection of $\bar{X}$ with the hyperplane at infinity $X_0 = 0$. Assume that over T there exists a section $\delta : T \to X$ of the canonical projection $X \to S$. Assume further that*
(1) *X is smooth and equidimensional over S, of relative dimension r;*
(2) *For every closed point $s \in S$ the closed fibres of X_∞ and X satisfy*

$$dim(X_\infty(s)) < dim(X(s)) = r \ .$$

Then there exists a closed subscheme $\tilde{S}$ of X which is finite étale over S and contains $\delta(T)$.

If the residue field at any of closed point of the scheme S is infinite, then this lemma is just [OP2, Lemma 7.2]. We give a proof of the lemma in the case when the residue field at any of closed point of the scheme S is finite. *The proof of the lemma in this case is given below* and repeats literally the proof of [OP2, Lemma 7.2]. The only difference is that we refer below to a Poonen's article [Poo] on Bertini theorems over finite fields rather than to Artin's result. We left to the reader the general case.

Since S is semilocal, after a linear change of coordinates we may assume that δ maps T into the closed subscheme of $\mathbf{P}^d_T$ defined by $X_1 = \cdots = X_d = 0$. For each closed fibre $\mathbf{P}^d_s$ of $\mathbf{P}^d_S$ using repeatedly [Poo, Thm.1.2], we can choose a family of **homogeneous** polynomials $H_1(s), \ldots, H_r(s)$ (in general of increasing degrees) such that the subscheme $Y(s)$ of $\mathbf{P}^d_S(s)$ defined by the equations

$$H_1(s) = 0\,, \ \ldots\ , H_r(s) = 0$$

intersects $X(s)$ transversally, contains the point $[1 : 0 : \cdots : 0]$ and avoids $X_\infty(s)$. By the chinese remainders' theorem there exists a common lift $H_i \in R[X_0, \ldots, X_d]$ of all polynomials $H_i(s)$, $s \in \mathrm{Max}(R)$. We may choose this common lift H_i such that $H_i(1, 0, \ldots, 0) = 0$. Let Y be the closed subscheme of $\mathbf{P}^d_S$ defined by

$$H_1 = 0\,, \ldots, H_r = 0\,.$$

We claim that the subscheme $\tilde{S} = Y \cap X$ has the required properties. Note first that $X \cap Y$ is finite over S. In fact, $X \cap Y = \bar{X} \cap Y$, which is projective over S and such that every closed fibre (hence every fibre) is finite. Since the closed fibres of $X \cap Y$ are finite étale over the closed points of S, to show that $X \cap Y$ is finite étale over S it only remains to show that it is flat over S. Noting that $X \cap Y$ is defined in every closed fibre by a regular sequence of equations and localizing at each closed point of S, we see that flatness follows from [OP2, Lemma 7.3].

□

Proposition 13.5. *Theorem 13.2 holds in the case when the groups G_1 and G_2 are tori and, more generally, in the case when the groups G_1 and G_2 are of multiplicative type.*

We left a proof of this latter proposition and the next one to the reader.

Proposition 13.6. *Let T and S be the same as in Theorem 13.2. Let M_1 and M_2 be two S-group schemes of multiplicative type. Let $\alpha_1, \alpha_2 : M_1 \rightrightarrows M_2$ be two S-group scheme morphisms such that $\alpha_1|_T = \alpha_2|_T$. Then $\alpha_1 = \alpha_2$.*

Proof of Theorem 13.2. Let $Rad(G_r) \subset G_r$ be the radical of G_r and let $der(G_r) \subset G_r$ be the derived subgroup of G_r $(r = 1, 2)$ (see [D-G, Exp.XXII, 4.3]). By the very definition the radical is a tori. The S-group scheme $der(G_r)$ is semi-simple $(r = 1, 2)$. Set $Z_r := Rad(G_r) \cap der(G_r)$. The above embeddings induce natural S-group morphisms

$$\Pi_r : Rad(G_r) \times_S der(G_r) \to G_r$$

with Z_r as the kernel ($r = 1, 2$). By [D-G, Exp.XXII,Prop.6.2.4] Π_r is a central isogeny. Particularly, Π_r is a faithfully flat finite morphism by [D-G, Exp.XXII,Defn.4.2.9]. Let $i_r : Z_r \hookrightarrow Rad(G_r) \times_S der(G_r)$ be the closed embedding.

The T-group scheme isomorphism $\varphi : G_1|_T \to G_2|T$ induces certain T-group scheme isomorphisms $\varphi_{der} : der(G_1|_T) \to der(G_2|_T)$, $\varphi_{rad} : rad(G_1|_T) \to rad(G_2|_T)$ and $\varphi_Z : Z_1|_T \to Z_2|_T$ **such that**

$$(\Pi_2)|_T \circ (\varphi_{der} \times \varphi_{rad}) = \varphi \circ (\Pi_1)|_T \text{ and } i_{2,T} \circ \varphi_Z = (\varphi_{rad} \times \varphi_{der}) \circ i_{1,T}.$$

By Propositions 13.3 and 13.5 there exist a finite étale morphism $\pi : \tilde{S} \to S$ (with an irreducible scheme $\tilde{S}$) and its section $\delta : T \to \tilde{S}$ over T and $\tilde{S}$-group scheme isomorphisms

$$\Phi_{der} : der(G_{1,\tilde{S}}) \to der(G_{2,\tilde{S}}) \quad , \quad \Phi_{Rad} : Rad(G_{1,\tilde{S}}) \to Rad(G_{2,\tilde{S}}) \quad \text{and} \quad \Phi_Z : Z_{1,\tilde{S}} \to Z_{2,\tilde{S}}$$

such that $\delta^*(\Phi_{der}) = \varphi_{der}$, $\delta^*(\Phi_{rad}) = \varphi_{rad}$ and $\delta^*(\Phi_Z) = \varphi_Z$.

Since Z_r is contained in the center of $der(G_r)$ and is of multiplicative type Proposition 13.6 yields the equality

$$i_{2,\tilde{S}} \circ \Phi_Z = (\Phi_{Rad} \times \Phi_{der}) \circ i_{1,\tilde{S}} : Z_{1,\tilde{S}} \to Rad(G_{2,\tilde{S}}) \times_{\tilde{S}} der(G_{2,\tilde{S}}).$$

Thus $(\Phi_{Rad} \times \Phi_{der})$ induces an $\tilde{S}$-group scheme isomorphism

$$\Phi : G_{1,\tilde{S}} \to G_{2,\tilde{S}}$$

such that $\Pi_{2,\tilde{S}} \circ (\Phi_{Rad} \times \Phi_{der}) = \Phi \circ \Pi_{1,\tilde{S}}$. The latter equality yields the following one $(\Pi_2)|_T \circ \delta^*(\Phi_{Rad} \times \Phi_{der}) = \delta^*(\Phi) \circ (\Pi_1)|_T$, which in turn yields the equality

$$(\Pi_2)|_T \circ (\varphi_{rad} \times \varphi_{der}) = \delta^*(\Phi) \circ (\Pi_1)|_T.$$

Comparing it with the equality $(\Pi_2)|_T \circ (\varphi_{rad} \times \varphi_{der}) = \varphi \circ (\Pi_1)|_T$ and using the fact that $(\Pi_1)|_T$ is strictly flat we conclude the equality $\delta^*(\Phi) = \varphi$.

□

Proof of Theorem 13.1. By Theorem 13.2 there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \tilde{S}$ over T and $\tilde{S}$-group scheme isomorphisms

$$\Phi : G_{1,\tilde{S}} \to G_{2,\tilde{S}} \text{ and } \Psi : C_{1,\tilde{S}} \to C_{2,\tilde{S}}$$

such that $\delta^*(\Phi) = \varphi$, $\delta^*(\Psi) = \psi$. It remains to show that $\mu_{2,\tilde{S}} \circ \Phi = \Psi \circ \mu_{1,\tilde{S}}$. To prove this equality recall that μ_r can be naturally presented as a composition

$$G_r \xrightarrow{can_r} Corad(G_r) \xrightarrow{\bar{\mu}_r} C_r.$$

Since $can_{2,\tilde{S}} \circ \Phi = Corad(\Phi) \circ can_{1,\tilde{S}}$ it remains to check that $\bar{\mu}_{2,\tilde{S}} \circ Corad(\Phi) = \Psi \circ \bar{\mu}_{1,\tilde{S}}$. The latter equality follows from Proposition 13.6 and the equality $(\bar{\mu}_2|_T) \circ Corad(\varphi) = \psi \circ (\bar{\mu}_1|_T)$, which holds since $(\mu_2|_T) \circ \varphi = \psi \circ (\mu_1|_T)$ by the very assumption of the Theorem.

□

14 Proof of Theorem 11.8

Let k be a field. Let U be as in Definition 11.1. Let S' be an irreducible regular semi-local scheme over k and $p : S' \to U$ be a k-morphism. Let $i : T' \hookrightarrow S'$ be a closed sub-scheme of S' such that the restriction $p|_{T'} : T' \to U$ is an isomorphism. We will assume below that $dim(T') < dim(S')$, where dim is the Krull dimension. For any closed point $u \in U$ and any U-scheme V let $V_u = u \times_U V$ be the fibre of the scheme V over the point u. For a finite set A denote $\sharp A$ the cardinality of A.

Lemma 14.1. *If the field k is finite and all the closed points of S' have* **finite residue fields***. Then there exists a finite étale morphism $\rho : S'' \to S'$ (with an irreducible scheme S'') and a section $\delta' : T' \to S''$ of ρ over T' such that the following holds*

(1) *for any closed point $u \in U$ let $u' \in T'$ be a unique point such that $p(u') = u$, then the point $\delta'(u') \in S''_u$ is the only $k(u)$-rational point of S''_u,*

(2) *for any closed point $u \in U$ and any integer $d \geq 1$ one has*

$$\sharp\{z \in S''_u | [k(z) : k(u)] = d\} \leq \ \sharp\{x \in \mathbf{A}^1_u | [k(x) : k(u)] = d\}$$

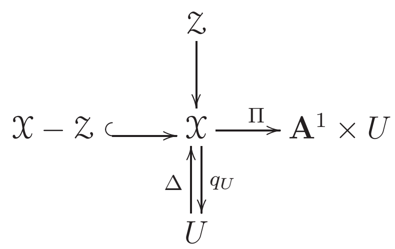
If the field k is infinite, then set $S'' = S'$, $\rho = id$, and $\delta' = i$.

Proof of Theorem 11.8. We can start by almost literally repeating arguments from the proof of [OP1, Lemma 8.1], which involve the following purely geometric lemma [OP1, Lemma 8.2].

For reader's convenience below we state that Lemma adapting notation to the ones of Section 11. Namely, let U be as in Definition 11.1 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Further, let $G_{\mathcal{X}}$ be a simple simply-connected $\mathcal{X}$-group scheme, $G_U := \Delta^*(G_{\mathcal{X}})$, and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, by the definition of a nice triple there exists a finite surjective morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ of U-schemes.

Lemma 14.2. *Let $\mathcal{Y}$ be a closed nonempty sub-scheme of $\mathcal{X}$, finite over U. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $\Pi^{-1}(\Pi(\mathcal{Y}))$. There exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $q_U^{-1}(q_U(\mathcal{Y}))$ and endowed with a finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$ (in general $\neq \Pi$).*

Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ be the above finite surjective U-morphism. The following diagram summarises the situation:



Here $\mathcal{Z}$ is the closed sub-scheme defined by the equation $f = 0$. By assumption, $\mathcal{Z}$ is finite over U. Let $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(U)))$. Since $\mathcal{Z}$ and $\Delta(U)$ are both finite over U and since Π is a finite morphism of U-schemes, $\mathcal{Y}$ is also finite over U. Denote by $y_1, \dots, y_m$ its closed points and let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \dots, y_m})$. Set $T = \Delta(U) \subseteq S$. Further, let $G_U = \Delta^*(G_{\mathcal{X}})$ be as in the hypotheses of Theorem 11.8 and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, let $\varphi : G_{\mathrm{const}}|_T \to G_{\mathcal{X}}|_T$ be the canonical isomorphism. Recall that by assumption $\mathcal{X}$ is U-smooth and irreducible, and thus S is regular and irreducible.

By Theorem 13.1 there exists a finite étale covering $\theta_0 : S' \to S$, a section $\delta : T \to S'$ of θ_0 over T and isomorphisms $\Phi_0 : \theta_0^*(G_{\mathrm{const},S}) \to \theta_0^*(G_{\mathcal{X}}|_S)$ and $\Psi_0 : \theta_0^*(C_{\mathrm{const},S}) \to \theta_0^*(C_{\mathcal{X}}|_S)$ such that $\delta^*(\Phi_0) = \varphi$, $\delta^*(\Psi_0) = \psi$ and

$$\theta_0^*(\mu_{\mathcal{X}}|_S) \circ \Phi_0 = \Psi_0 \circ \theta_0^*(\mu_{\mathrm{const},S}) : \theta_0^*(G_{\mathrm{const},S}) \to \theta_0^*(C_{\mathcal{X}}|_S)$$

and the scheme S' is irreducible. **Replacing S' with a connected component of S' which contains $T' := \delta(T) = \delta(\Delta(U))$ we may and will assume that S' is irreducible.**

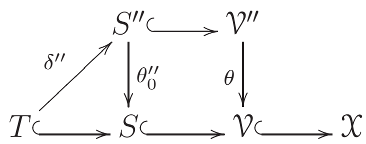
Let $p = q_U \circ \theta_0 : S' \to U$. Let S'', $\rho : S'' \to S'$, and $\delta' : T' \to S''$ be as in Lemma 14.1. Recall that $\rho : S'' \to S'$ a finite étale morphism (with an irreducible scheme S'') and $\delta' \circ \rho = i : T' \hookrightarrow S'$. Set $\delta'' = \delta' \circ \delta : T \to S''$ and $\theta_0'' = \theta_0 \circ \rho : S'' \to S$. We are also given the S''-group scheme isomorphisms

$$\rho^*(\Phi_0) : (\theta_0'')^*(G_{\mathrm{const},S}) \to (\theta_0'')^*(G_{\mathcal{X}}|_S) \quad \text{and} \quad \rho^*(\Psi_0) : (\theta_0'')^*(C_{\mathrm{const},S}) \to (\theta_0'')^*(C_{\mathcal{X}}|_S)$$

such that $(\delta'')^*(\rho^*(\Phi_0)) = \varphi$, $(\delta'')^*(\rho^*(\Psi_0)) = \psi$ and

$$(\theta'')_0^*(\mu_{\mathcal{X}}|_S) \circ \rho^*(\Phi_0) = \rho^*(\Psi_0) \circ (\theta'')_0^*(\mu_{\mathrm{const},S}) : (\theta'')_0^*(G_{\mathrm{const},S}) \to (\theta'')_0^*(C_{\mathcal{X}}|_S)$$

We can extend these data to a neighborhood $\mathcal{V}$ of $\{y_1, \dots, y_n\}$ and get the diagram



(29)

where $\theta : \mathcal{V}'' \to \mathcal{V}$ finite étale, and isomorphisms $\Phi : \theta^*(G_{\mathrm{const},\mathcal{V}}) \to \theta^*(G_{\mathcal{X}}|_{\mathcal{V}})$ and $\Psi : \theta^*(C_{\mathrm{const},\mathcal{V}}) \to \theta^*(C_{\mathcal{X}}|_{\mathcal{V}})$ such that one has the equality

$$\theta^*(\mu_{\mathcal{X}}|_{\mathcal{V}}) \circ \Phi = \Psi \circ \theta^*(\mu_{\mathrm{const},\mathcal{V}}) : \theta^*(G_{\mathrm{const},\mathcal{V}}) \to \theta^*(C_{\mathcal{X}}|_{\mathcal{V}}) \tag{30}$$

Since T isomorphically projects onto U, it is still closed viewed as a sub-scheme of $\mathcal{V}$. Note that since $\mathcal{Y}$ is semi-local and $\mathcal{V}$ contains all of its closed points, $\mathcal{V}$ contains $\Pi^{-1}(\Pi(\mathcal{Y})) = \mathcal{Y}$. By Lemma 14.2 there exists an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing $\mathcal{Y}$ and endowed with a finite surjective U-morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$.

Let $\mathcal{X}' = \theta^{-1}(\mathcal{W})$, $f' = \theta^*(f)$, $q'_U = q_U \circ \theta$, and let $\Delta' : U \to \mathcal{X}'$ be the section of q'_U obtained as the composition of δ'' with Δ. We claim that the triple $(\mathcal{X}', f', \Delta')$ is a nice triple over U. Let us verify this. Firstly, the structure morphism $q'_U : \mathcal{X}' \to U$ coincides with the composition

$$\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X} \xrightarrow{q_U} U.$$

Thus, it is smooth. The element f' belongs to the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, the morphism Δ' is a section of q'_U. Each component of each fibre of the morphism q_U has dimension one, the morphism $\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X}$ is étale. Thus, each component of each fibre of the morphism q'_U is also of dimension one. Since $\{f = 0\} \subset \mathcal{W}$ and $\theta : \mathcal{X}' \to \mathcal{W}$ is finite, $\{f' = 0\}$ is finite over $\{f = 0\}$ and hence also over U. In other words, the $\mathcal{O}$-module $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is finite. The morphism $\theta : \mathcal{X}' \to \mathcal{W}$ is finite and surjective. We have constructed above in Lemma 14.2 the finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$. It follows that $\Pi^* \circ \theta : \mathcal{X}' \to \mathbf{A}^1 \times U$ is finite and surjective.

Clearly, the étale morphism $\theta : \mathcal{X}' \to \mathcal{X}$ is a morphism between the nice triples

$$\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta) \quad \text{with} \quad h' = 1$$

Denote the restriction of Φ and of Ψ to $\mathcal{X}'$ simply by Φ and by Ψ respectively. Thus, Φ is an $\mathcal{X}'$-group scheme isomorphism $\theta^*(G_{\text{const}}) \to \theta^*(G_{\mathcal{X}})$ and Ψ is an $\mathcal{X}'$-group scheme isomorphism $\theta^*(C_{\text{const}}) \to \theta^*(C_{\mathcal{X}})$. The equalities $(\Delta')^*\Phi = \text{id}_{G_U}$ and $(\Delta')^*\Psi = \text{id}_{C_U}$ holds by the construction of the isomorphisms Φ and Ψ. The equality (30) yields the following one

$$\theta^*(\mu_{\mathcal{X}}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}}) : \theta^*(G_{\text{const}}) \to \theta^*(C_{\mathcal{X}}).$$

By Remark 11.4 the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over U since the one $(\mathcal{X}, f, \Delta)$ is a special nice triple over U.

It remains to check that $(\mathcal{X}', f', \Delta')$ is a special nice triple satisfying the conditions (1^*) and (2^*) from Definition 11.6. To do this recall that all the closed points of the sub-scheme $\{f = 0\} \subset \mathcal{X}$ are in S. The morphism θ is finite and $\theta^{-1}(S) = S''$. Thus all the closed points of the sub-scheme $\{f' = 0\} \subset \mathcal{X}'$ are in S''. By the above choice of $T' = \delta(\Delta(U)) \subset S'$ the scheme T' projects isomorphically to U. Thus in the case when k is finite the properties (1) and (2) of the U-scheme S'' show that the conditions (1^*) and (2^*) are full filled for the closed sub-scheme $\mathcal{Z}'$ of $\mathcal{X}'$ defined by $\{f' = 0\}$. If the field k is infinite, then by Definition 11.6 any special nice triple satisfies the conditions (1^*) and (2^*). Theorem 11.8 follows.

□

15 Proof of Theorems 5.1 and 5.2

Clearly, Theorem 5.2 yields Theorem 5.1.

Proof of Theorem 5.2. By Proposition 11.5 one can shrink X such that $x_1, x_2, \ldots, x_n$ are still in X and X is affine, and then to construct a nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \to X$ such that $q_X \circ \Delta = can$, $f = q_X^*(\mathrm{f})$ and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.

Set $\mathbf{G}_{\mathcal{X}} = q_X^*(\mathbf{G})$, then $\Delta^*(\mathbf{G}_{\mathcal{X}}) = can^*(\mathbf{G})$. Thus the U-group scheme $\mathbf{G}_U$ from Theorem 11.8 and the U-group scheme $\mathbf{G}_U$ from Theorem 5.2 are the same. By Theorem 11.8 there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ such that the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over U subjecting to the conditions (1^*) and (2^*) from Definition 11.6. And additionally there are isomorphisms

$$\Phi : (q_U \circ \theta)^*(\mathbf{G}_U) = \theta^*(G_{\text{const}}) \to \theta^*(G_{\mathcal{X}}) = (q_X \circ \theta)^*(\mathbf{G})$$

and

$$\Psi : (q_U \circ \theta)^*(\mathbf{C}_U) = \theta^*(C_{\text{const}}) \to \theta^*(C_{\mathcal{X}}) = (q_X \circ \theta)^*(\mathbf{G})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = id_{G_U}$, $(\Delta')^*(\Phi) = id_{G_U}$ and

$$\theta^*(\mu_{\mathcal{X}}) \circ \Phi = \Psi \circ \theta^*(\mu_{const}). \qquad (31)$$

The triple $(\mathcal{X}', f', \Delta')$ is a special nice triple **over** U subjecting to the conditions (1^*) and (2^*) from Definition 11.6. Thus by Theorem 11.7 there is a finite surjective morphism $\mathbf{A}^1 \times U \xleftarrow{\sigma} \mathcal{X}'$ of the U-schemes satisfying the conditions (a) to (f) from that Theorem. Hence one has a diagram of the form

$$\begin{array}{ccccc} \mathbf{A}^1 \times U & \xleftarrow{\sigma} & \mathcal{X}' & \xrightarrow{q_X \circ \theta} & X \\ & {\scriptstyle \mathrm{pr}_U} \searrow & {\scriptstyle q_U \circ \theta} \downarrow \uparrow {\scriptstyle \Delta'} & \nearrow {\scriptstyle \mathrm{can}} & \\ & & U & & \end{array} \qquad (32)$$

with the irreducible scheme $\mathcal{X}'$, the smooth morphism $q_U \circ \theta$, the finite surjective morphism σ and the essentially smooth morphism $q_X \circ \theta$ and with the function $f' \in (q_X \circ \theta)^*(\mathrm{f})k[\mathcal{X}']$, which after identifying notation enjoy the properties (a) to (f) from Theorem 5.1. The isomorphisms Φ and Ψ are the required ones. Whence the Theorem 5.2.

□

16 Norms

In the rest of the paper we prove few results which we refer to reducing Theorems 6.1 and 1.5 to Theorem 5.2.

Let $k \subset K \subset L$ be field extensions and assume that L is finite separable over K. Let K^{sep} be a separable closure of K and $\sigma_i : K \to K^{sep}$, $1 \le i \le n$ the different embeddings of K into L. Let C be a k-smooth commutative algebraic group scheme defined over k. One can define a norm map

$$\mathcal{N}_{L/K} : C(L) \to C(K)$$

by $\mathcal{N}_{L/K}(\alpha) = \prod_i C(\sigma_i)(\alpha) \in C(K^{sep})^{\mathcal{G}(K)} = C(K)$;. Following Suslin and Voevodsky [SV, Sect.6] we generalize this construction to finite flat ring extensions. Let $p : X \to Y$ be a finite flat morphism of affine schemes. Suppose that its rank is constant, equal to d. Denote by $S^d(X/Y)$ the d-th symmetric power of X over Y.

Lemma 16.1. *There is a canonical section*

$$\mathcal{N}_{X/Y} : Y \to S^d(X/Y)$$

which satisfies the following three properties:

(i) Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\mathcal{N}_{X'/Y'}} & S^d(X'/Y') \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle S^d(Id_X \times f)} \\ Y & \xrightarrow{\mathcal{N}_{X/Y}} & S^d(X/Y) \end{array}$$

(ii) Additivity: If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are finite flat morphisms of degree d_1 and d_2 respectively, then, putting $X = X_1 \coprod X_2$, $f = f_1 \coprod f_2$ and $d = d_1 + d_2$, we have a commutative diagram

$$\begin{array}{ccc} S^{d_1}(X_1/Y) \times S^{d_2}(X_2/Y) & \xrightarrow{\sigma} & S^d(X/Y) \\ {\scriptstyle \mathcal{N}_{X_1/Y} \times \mathcal{N}_{X_2/Y}} \nwarrow & & \nearrow {\scriptstyle \mathcal{N}_{X/Y}} \\ & Y & \end{array}$$

where σ is the canonical imbedding.

(iii) Normalization: If $X = Y$ and p is the identity, then $\mathcal{N}_{X/Y}$ is the identity.

Proof. We construct a map $\mathcal{N}_{X/Y}$ and check that it has the desired properties. Let $B = k[X]$ and $A = k[Y]$, so that B is a locally free A-module of finite rank d. Let $B^{\otimes d} = B \otimes_A B \otimes_A \cdots \otimes_A B$ be the d-fold tensor product of B over A. The permutation group $\mathfrak{S}_d$ acts on $B^{\otimes d}$ by permuting the factors. Let $S^d_A(B) \subseteq B^{\otimes d}$ be the A-algebra of all the $\mathfrak{S}_d$-invariant elements of $B^{\otimes d}$. We consider $B^{\otimes d}$ as an $S^d_A(B)$-module through the inclusion $S^d_A(B) \subseteq B^{\otimes d}$ of A-algebras. Let I be the kernel of the canonical homomorphism $B^{\otimes d} \to \bigwedge^d_A(B)$ mapping $b_1 \otimes \cdots \otimes b_d$ to $b_1 \wedge \cdots \wedge b_d$. It is well-known (and easily checked locally on A) that I is generated by all the elements $x \in B^{\otimes d}$ such that $\tau(x) = x$ for some transposition τ. If s is in $S^d_A(B)$, then $\tau(sx) = \tau(s)\tau(x) = sx$, hence sx is in $S^d(B)$ too. In other words, I is an $S^d_A(B)$-submodule of $B^{\otimes d}$. The induced $S^d_A(B)$-module structure on $\bigwedge^d_A(B)$ defines an A-algebra homomorphism

$$\varphi : S^d_A(B) \to End_A(\bigwedge_A^d(B)) \, .$$

Since B is locally free of rank d over A, $\bigwedge_A^d(B)$ is an invertible A-module and we can canonically identify $End_A(\bigwedge_A^d(B))$ with A. Thus we have a map

$$\varphi : S_A^d(B) \to A$$

and we define

$$N_{X/Y} : Y \to S^d(X/Y)$$

as the morphism of Y-schemes induced by φ. The verification of properties (i), (ii) and (iii) is straightforward. □

Let k be a field. Let $\mathcal{O}$ be the semi-local ring of finitely many **closed** points on a smooth affine irreducible k-variety X. Let C be an affine smooth commutative $\mathcal{O}$-group scheme, Let $p : X \to Y$ be a finite flat morphism of affine $\mathcal{O}$-schemes and $f : X \to C$ any $\mathcal{O}$-morphism. We define *the norm* $N_{X/Y}(f)$ of f *as the composite map*

$$Y \xrightarrow{N_{X/Y}} S^d(X/Y) \to S^d_{\mathcal{O}}(X) \xrightarrow{S^d_{\mathcal{O}}(f)} S^d_{\mathcal{O}}(C) \xrightarrow{\times} C \qquad (33)$$

Here we write " $\times$ " for the group law on C. The norm maps $N_{X/Y}$ satisfy the following conditions

(i') Base change: for any map $f : Y' \to Y$ of affine schemes, putting $X' = X \times_Y Y'$ we have a commutative diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X') \\ {\scriptstyle N_{X/Y}}\downarrow & & \downarrow{\scriptstyle N_{X'/Y'}} \\ C(Y) & \xrightarrow{f^*} & C(Y') \end{array}$$

(ii') multiplicativity: if $X = X_1 \amalg X_2$ then the diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{(id\times f)^*} & C(X_1) \times C(X_2) \\ {\scriptstyle N_{X/Y}}\downarrow & & \downarrow{\scriptstyle N_{X_1/Y}N_{X_2/Y}} \\ C(Y) & \xrightarrow{id} & C(Y) \end{array}$$

(iii') normalization: if $X = Y$ and the map $X \to Y$ is the identity then $N_{X/Y} = id_{C(X)}$.

17 Unramified elements

Let k be **a field**, $\mathcal{O}$ be the k-algebra from Theorem 6.1 and K be the fraction field of $\mathcal{O}$. Let $\mu : G \to C$ be the morphism of reductive $\mathcal{O}$-group schemes from Theorem 6.1.

We work in this section with *the category of commutative Noetherian $\mathcal{O}$-algebras.* For a commutative $\mathcal{O}$-algebra S set

$$\mathcal{F}(S) = C(S)/\mu(G(S)). \tag{34}$$

Let S be an $\mathcal{O}$-algebra which is a domain and let L be its fraction field. Define the *subgroup of S-unramified elements of $\mathcal{F}(L)$* as

$$\mathcal{F}_{nr,S}(L) = \bigcap_{\mathfrak{p}\in Spec(S)^{(1)}} Im[\mathcal{F}(S_{\mathfrak{p}}) \to \mathcal{F}(L)], \tag{35}$$

where $Spec(S)^{(1)}$ is the set of hight 1 prime ideals in S. Obviously the image of $\mathcal{F}(S)$ in $\mathcal{F}(L)$ is contained in $\mathcal{F}_{nr,S}(L)$. In most cases $\mathcal{F}(S_{\mathfrak{p}})$ injects into $\mathcal{F}(L)$ and $\mathcal{F}_{nr,S}(L)$ is simply the intersection of all $\mathcal{F}(S_{\mathfrak{p}})$.

For an element $\alpha \in C(S)$ we will write $\bar{\alpha}$ for its image in $\mathcal{F}(S)$. *In this section we will write $\mathcal{F}$ for the functor* (34). We will repeatedly use the following result due to Nisnevich.

Theorem 17.1 ([Ni2])**.** *Let S be a $\mathcal{O}$-algebra which is discrete valuation ring with fraction field L. The map $\mathcal{F}(S) \to \mathcal{F}(L)$ is injective.*

Proof. Let H be the kernel of μ. Since μ is smooth and C is a tori, the group scheme sequence

$$1 \to H \to G \to C \to 1$$

gives rise to a short exact sequence of group sheaves in the étale topology. In turn that sequence of sheaves induces a long exact sequence of pointed sets. So, the boundary map $\partial : C(S) \to \mathrm{H}^1_{\text{ét}}(S, H)$ fits in a commutative diagram

$$\begin{array}{ccc} C(S)/\mu(G(S)) & \longrightarrow & C(L)/\mu(G(L)) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_{\text{ét}}(S, H) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(L, H). \end{array}$$

in which the vertical arrows have trivial kernels. The bottom arrow has trivial kernel by a Theorem from [Ni2], since H is a reductive $\mathcal{O}$-group scheme. Thus the top arrow has trivial kernel too.

□

Lemma 17.2. *Let $\mu : G \to C$ be the above morphism of our reductive group schemes. Let $H = \ker(\mu)$. Then for an $\mathcal{O}$-algebra L, where L is a field, the boundary map ∂ : $C(L)/\mu(G(L)) \to H^1_{\text{ét}}(L, H)$ is injective.*

Proof. For a L-rational point $t \in C$ set $H_t = \mu^{-1}(t)$. The action by left multiplication of H on H_t makes H_t into a left principal homogeneous H-space and moreover $\partial(t) \in \mathrm{H}^1_{\text{ét}}(L, H)$ coincides with the isomorphism class of H_t. Now suppose that $s, t \in C(L)$ are such that $\partial(s) = \partial(t)$. This means that H_t and H_s are isomorphic as principal homogeneous H-spaces. We must check that for certain $g \in G(L)$ one has $t = s\mu(g)$.

Let L^{sep} be a separable closure of L. Let $\psi : H_s \to H_t$ be an isomorphism of left H-spaces. For any $r \in H_s(L^{sep})$ and $h \in H_s(L^{sep})$ one has

$$(hr)^{-1}\psi(hr) = r^{-1}h^{-1}h\psi(r) = r^{-1}\psi(r).$$

Thus for any $\sigma \in Gal(L^{sep}/L)$ and any $r \in H_s(L^{sep})$ one has

$$r^{-1}\psi(r) = (r^\sigma)^{-1}\psi(r^\sigma) = (r^{-1}\psi(r))^\sigma$$

which means that the point $u = r^{-1}\psi(r)$ is a $Gal(L^{sep}/L)$-invariant point of $G(L^{sep})$. So $u \in G(L)$. The following relation shows that the ψ coincides with the right multiplication by u. In fact, for any $r \in H_s(L^{sep})$ one has $\psi(r) = rr^{-1}\psi(r) = ru$. Since ψ is the right multiplication by u one has $t = s\mu(u)$, which proves the lemma.

□

Let k, $\mathcal{O}$ and K be as above in this Section. Let $\mathcal{K}$ be a field containing K and $x : \mathcal{K}^* \to \mathbb{Z}$ be a discrete valuation vanishing on K. Let A_x be the valuation ring of x. Clearly, $\mathcal{O} \subset A_x$. Let $\hat{A}_x$ and $\hat{\mathcal{K}}_x$ be the completions of A_x and $\mathcal{K}$ with respect to x. Let $i : \mathcal{K} \hookrightarrow \hat{\mathcal{K}}_x$ be the inclusion. By Theorem 17.1 the map $\mathcal{F}(\hat{A}_x) \to \mathcal{F}(\hat{\mathcal{K}}_x)$ is injective. We will identify $\mathcal{F}(\hat{A}_x)$ with its image under this map. Set

$$\mathcal{F}_x(\mathcal{K}) = i_*^{-1}(\mathcal{F}(\hat{A}_x)).$$

The inclusion $A_x \hookrightarrow \mathcal{K}$ induces a map $\mathcal{F}(A_x) \to \mathcal{F}(\mathcal{K})$ which is injective by Lemma 17.1. So both groups $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{K})$ are subgroups of $\mathcal{F}(\mathcal{K})$. The following lemma shows that $\mathcal{F}_x(\mathcal{K})$ coincides with the subgroup of $\mathcal{F}(\mathcal{K})$ consisting of all elements *unramified* at x.

Lemma 17.3. $\mathcal{F}(A_x) = \mathcal{F}_x(\mathcal{K})$.

Proof. We only have to check the inclusion $\mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x)$. Let $a_x \in \mathcal{F}_x(\mathcal{K})$ be an element. It determines the elements $a \in \mathcal{F}(\mathcal{K})$ and $\hat{a} \in \mathcal{F}(\hat{A}_x)$ which coincide when regarded as elements of $\mathcal{F}(\hat{\mathcal{K}}_x)$. We denote this common element in $\mathcal{F}(\hat{\mathcal{K}}_x)$ by $\hat{a}_x$. Let $H = \ker(\mu)$ and let $\partial : C(-) \to \mathrm{H}^1_{\text{ét}}(-, H)$ be the boundary map.

Let $\xi = \partial(a) \in \mathrm{H}^1_{\text{ét}}(\mathcal{K}, H)$, $\hat{\xi} = \partial(\hat{a}) \in \mathrm{H}^1_{\text{ét}}(\hat{A}_x, H)$ and $\hat{\xi}_x = \partial(\hat{a}_x) \in \mathrm{H}^1_{\text{ét}}(\hat{\mathcal{K}}_x, H)$ Clearly, $\hat{\xi}$ and ξ both coincide with $\hat{\xi}_x$ when regarded as elements of $\mathrm{H}^1_{\text{ét}}(\hat{\mathcal{K}}_x, H)$. Thus one can glue ξ and $\hat{\xi}$ to get a $\xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$ which maps to ξ under the map induced by the inclusion $A_x \hookrightarrow \mathcal{K}$ and maps to $\hat{\xi}$ under the map induced by the inclusion $A_x \hookrightarrow \hat{A}_x$.

We now show that ξ_x has the form $\partial(a'_x)$ for a certain $a'_x \in \mathcal{F}(A_x)$. In fact, observe that the image ζ of ξ in $\mathrm{H}^1_{\text{ét}}(\mathcal{K}, G)$ is trivial. By Theorem [Ni2] the map

$$\mathrm{H}^1_{\text{ét}}(A_x, G) \to \mathrm{H}^1_{\text{ét}}(\mathcal{K}, G)$$

has trivial kernel. Therefore the image ζ_x of ξ_x in $\mathrm{H}^1_{\text{ét}}(A_x, G)$ is trivial. Thus there exists an element $a'_x \in \mathcal{F}(A_x)$ with $\partial(a'_x) = \xi_x \in \mathrm{H}^1_{\text{ét}}(A_x, H)$.

We now prove that a'_x coincides with a_x in $\mathcal{F}_x(\mathcal{K})$. Since $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{K})$ are both subgroups of $\mathcal{F}(\mathcal{K})$, it suffices to show that a'_x coincides with the element a in $\mathcal{F}(\mathcal{K})$. By Lemma 17.2 the map

$$\mathcal{F}(\mathcal{K}) \xrightarrow{\partial} \mathrm{H}^1_{\text{ét}}(\mathcal{K}, H) \tag{36}$$

is injective. Thus it suffices to check that $\partial(a'_x) = \partial(a)$ in $\mathrm{H}^1_{\text{ét}}(\mathcal{K}, H)$. This is indeed the case because $\partial(a'_x) = \xi_x$ and $\partial(a) = \xi$, and ξ_x coincides with ξ when regarded over $\mathcal{K}$. We have proved that a'_x coincides with a_x in $\mathcal{F}_x(\mathcal{K})$. Thus the inclusion $\mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x)$ is proved, whence the lemma.

□

Let k, $\mathcal{O}$ and K be as above in this Section.

Lemma 17.4. *Let $B \subset A$ be a finite extension of K-smooth algebras, which are domains and each has dimension one. Let $0 \neq f \in A$ and let $h \in B \cap fA$ be such that the induced map $B/hB \to A/fA$ is an isomorphism. Suppose $hA = fA \cap J''$ for an ideal $J'' \subseteq A$ co-prime to the ideal fA.*

Let E and F be the field of fractions of B and A respectively. Let $\alpha \in C(A_f)$ be such that $\bar{\alpha} \in \mathcal{F}(F)$ is A-unramified. Then, for $\beta = N_{F/E}(\alpha)$, the class $\bar{\beta} \in \mathcal{F}(E)$ is B-unramified.

Proof. The only primes at which $\bar{\alpha}$ could be ramified are those which divide hA. Let $\mathfrak{p}$ be one of them. Check that $\bar{\alpha}$ is unramified at $\mathfrak{p}$.

To do this we consider all primes $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ in A lying over $\mathfrak{p}$. Let $\mathfrak{q}_1$ be the unique prime dividing f and lying over $\mathfrak{p}$. Then

$$A \otimes_B \hat{B}_{\mathfrak{p}} = \hat{A}_{\mathfrak{q}_1} \times \prod_{i \neq 1} \hat{A}_{\mathfrak{q}_i}$$

with $\hat{A}_{\mathfrak{q}_1} = \hat{B}_{\mathfrak{p}}$. If F, E are the fields of fractions of A and B then

$$F \otimes_B \hat{B}_{\mathfrak{p}} = \hat{F}_{\mathfrak{q}_1} \times \prod_{i \neq 1} \hat{F}_{\mathfrak{q}_n}$$

and $\hat{F}_{\mathfrak{q}_1} = \hat{E}_{\mathfrak{p}}$. We will write $\hat{F}_i$ for $\hat{F}_{\mathfrak{q}_i}$ and $\hat{A}_i$ for $\hat{A}_{\mathfrak{q}_i}$. Let

$$\alpha \otimes 1 = (\alpha_1, \ldots, \alpha_n) \in C(\hat{F}_1) \times \cdots \times C(\hat{F}_n).$$

Clearly for $i \geq 2$ one has $\alpha_i \in C(\hat{A}_i)$ and $\alpha_1 = \mu(\gamma_1)\alpha'_1$ with $\alpha'_1 \in C(\hat{A}_1) = C(\hat{B}_{\mathfrak{p}})$ and $\gamma_1 \in G(\hat{F}_1) = G(\hat{E}_{\mathfrak{p}})$. Now $\beta \otimes 1 \in C(\hat{E}_{\mathfrak{p}})$ coincides with the product

$$\alpha_1 N_{\hat{F}_2/\hat{E}_{\mathfrak{p}}}(\alpha_2) \cdots N_{\hat{F}_n/\hat{E}_{\mathfrak{p}}}(\alpha_n) = \mu(\gamma_1)[\alpha'_1 N_{\hat{F}_2/\hat{E}_{\mathfrak{p}}}(\alpha_2) \cdots N_{\hat{F}_n/\hat{E}_{\mathfrak{p}}}(\alpha_n)].$$

Thus $\overline{\beta \otimes 1} = \bar{\alpha}'_1 \overline{N_{\hat{F}_2/\hat{E}_{\mathfrak{p}}}(\alpha_2)} \cdots \overline{N_{\hat{F}_n/\hat{E}_{\mathfrak{p}}}(\beta_n)} \in \mathcal{F}(\hat{B}_{\mathfrak{p}})$. Let $i : E \hookrightarrow \hat{E}_{\mathfrak{p}}$ be the inclusion and $i_* : \mathcal{F}(E) \to \mathcal{F}(\hat{E}_{\mathfrak{p}})$ be the induced map. Clearly $i_*(\bar{\beta}) = \overline{\beta \otimes 1}$ in $\mathcal{F}(\hat{E}_{\mathfrak{p}})$. Now Lemma 17.3 shows that the element $\bar{\beta} \in \mathcal{F}(E)$ belongs to $\mathcal{F}(B_{\mathfrak{p}})$. Hence $\bar{\beta}$ is B-unramified.

□

18 Specialization maps

Let k be **a field**, $\mathcal{O}$ be the k-algebra from Theorem 6.1 and K be the fraction field of $\mathcal{O}$. Let $\mu : G \to C$ be the morphism of reductive $\mathcal{O}$-group schemes from Theorem 6.1. We work in this section with *the category of commutative K-algebras* and with the functor

$$\mathcal{F} : S \mapsto C(S)/\mu(G(S)) \tag{37}$$

defined on the category of K-algebras. So, we assume in this Section that each ring from this Section is equipped with a distinguished K-algebra structure and each ring homomorphism from this Section respects that structures. Let S be an K-algebra which is a domain and let L be its fraction field. Define the *subgroup of S-unramified elements $\mathcal{F}_{nr,S}(L)$ of $\mathcal{F}(L)$* by formulae (35).

For a regular domain S with the fraction field $\mathcal{K}$ and each height one prime $\mathfrak{p}$ in S we construct **specialization maps** $s_{\mathfrak{p}} : \mathcal{F}_{nr,S}(\mathcal{K}) \to \mathcal{F}(K(\mathfrak{p}))$, where $\mathcal{K}$ is the field of fractions of S and $K(\mathfrak{p})$ is the residue field of R at the prime $\mathfrak{p}$.

Definition 18.1. *Let $Ev_{\mathfrak{p}} : C(S_{\mathfrak{p}}) \to C(K(\mathfrak{p}))$ and $ev_{\mathfrak{p}} : \mathcal{F}(S_{\mathfrak{p}}) \to \mathcal{F}(K(\mathfrak{p}))$ be the maps induced by the canonical K-algebra homomorphism $S_{\mathfrak{p}} \to K(\mathfrak{p})$. Define a homomorphism $s_{\mathfrak{p}} : \mathcal{F}_{nr,S}(\mathcal{K}) \to \mathcal{F}(K(\mathfrak{p}))$ by $s_{\mathfrak{p}}(\alpha) = ev_{\mathfrak{p}}(\tilde{\alpha})$, where $\tilde{\alpha}$ is a lift of α to $\mathcal{F}(S_{\mathfrak{p}})$. Theorem 17.1 shows that the map $s_{\mathfrak{p}}$ is well-defined. It is called the specialization map. The map $ev_{\mathfrak{p}}$ is called the evaluation map at the prime $\mathfrak{p}$.*

Obviously for $\alpha \in C(S_{\mathfrak{p}})$ one has $s_{\mathfrak{p}}(\bar{\alpha}) = \overline{Ev_{\mathfrak{p}}(\alpha)} \in \mathcal{F}(K(\mathfrak{p}))$.

Lemma 18.2 ([?])**.** *Let H' be a smooth linear algebraic group over the field K. Let S be a K-algebra which is a Dedekind domain with field of fractions $\mathcal{K}$. If $\xi \in H^1_{ét}(\mathcal{K}, H')$ is an S-unramified element for the functor $H^1_{ét}(-, H')$ (see (35) for the Definition), then ξ can be lifted to an element of $H^1_{ét}(S, H')$.*

Proof. Patching.

□

Theorem 18.3 ([C-T/O], Prop.2.2)**.** *Let $G' = G_K$, where G is the reductive $\mathcal{O}$-group scheme from this Section (it is connected and even geometrically connected, since we follow [D-G, Exp. XIX, Defn.2.7]). Then*

$$ker[H^1_{ét}(K[t], G') \to H^1_{ét}(K(t), G')] = * \ .$$

We need the following theorem.

Theorem 18.4 (Homotopy invariance)**.** *Let $S \mapsto \mathcal{F}(S)$ be the functor defined by the formulae (37) and let $\mathcal{F}_{nr,K[t]}(K(t))$ be defined by the formulae (35). Let $K(t)$ be the rational function field in one variable. Then one has*

$$\mathcal{F}(K) = \mathcal{F}_{nr,K[t]}(K(t)).$$

Proof. The injectivity is clear, since the composition

$$\mathcal{F}(K) \to \mathcal{F}_{nr,K[t]}(K(t)) \xrightarrow{s_0} \mathcal{F}(K)$$

coincides with the identity (here s_0 is the specialization map at the point zero defined in 4.6).

It remains to check the surjectivity. Let

$$\mu_K = \mu \otimes_{\mathcal{O}} K : G_K = G \otimes_{\mathcal{O}} K \to C \otimes_{\mathcal{O}} K = C_K.$$

Let $a \in \mathcal{F}_{nr,K[t]}(K(t))$ and let $H_K = ker(\mu_K)$. Since μ is smooth the K-group H_K is smooth. Since G_K is reductive it is K-affine. Whence H_K is K-affine. Clearly, the element $\partial(a) \in H^1_{et}(K(t), H_K)$ is a class which for every closed point $x \in \mathbf{A}^1_K$ belongs to the image of $H^1_{et}(\mathcal{O}_x, H_K)$. Thus by Lemma 18.2, $\xi := \partial(a)$ can be represented by an element $\tilde{\xi} \in H^1_{et}(K[t], H_K)$, where $K[t]$ is the polynomial ring. Consider the diagram

$$\begin{array}{ccccccc}
 & & \tilde{a} & \longmapsto & \tilde{\xi} & \longmapsto & \tilde{\zeta} \\
 & & & & & & \\
1 & \longrightarrow & \mathcal{F}(K[t]) & \xrightarrow{\partial} & H^1_{et}(K[t], H_K) & \longrightarrow & H^1_{et}(K[t], G_K) \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \eta} \\
1 & \longrightarrow & \mathcal{F}(K(t)) & \xrightarrow{\partial} & H^1_{et}(K(t), H_K) & \longrightarrow & H^1_{et}(K(t), G_K) \\
 & & & & & & \\
 & & a & \longmapsto & \xi & \longmapsto & *
\end{array}$$

in which all the maps are canonical, the horizontal lines are exact sequences of pointed sets and $ker(\eta) = *$ by Theorem 18.3. Since ξ goes to the trivial element in $H^1_{et}(K(t), G_K)$, one concludes that $\eta(\tilde{\zeta}) = *$. Whence $\tilde{\zeta} = *$ by Theorem 18.3. Thus there exists an element $\tilde{a} \in \mathcal{F}(K[t])$ such that $\partial(\tilde{a}) = \tilde{\xi}$. The map $\mathcal{F}(K(t)) \to H^1_{et}(K(t), H_K)$ is injective by Lemma 17.2. Thus $\epsilon(\tilde{a}) = a$. The map $\mathcal{F}(K) \to \mathcal{F}(K[t])$ induced by the inclusion $K \hookrightarrow K[t]$ is surjective, since the corresponding map $C(K) \to C(K[t])$ is an isomorphism. Whence there exists an $a_0 \in \mathcal{F}(K)$ such that its image in $\mathcal{F}(K(t))$ coincides with the element a. □

Corollary 18.5. *Let $S \mapsto \mathcal{F}(S)$ be the functor defined in (34). Let*

$$s_0, s_1 : \mathcal{F}_{nr,K[t]}(K(t)) \to \mathcal{F}(K)$$

be the specialization maps at zero and at one (at the primes (t) and (t-1)). Then $s_0 = s_1$.

Proof. It is an obvious consequence of Theorem 18.4. □

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