J-INVARIANT OF HERMITIAN FORMS OVER QUADRATIC EXTENSIONS

RAPHAËL FINO*

ABSTRACT. We develop the version of the J-invariant for hermitian forms over quadratic extensions in a similar way Alexander Vishik did it for quadratic forms in [12]. This discrete invariant contains informations about rationality of algebraic cycles on the maximal unitary grassmannian associated with a hermitian form over a quadratic extension. The computation of the canonical 2-dimension of this grassmannian in terms of the J-invariant is provided, as well as a complete motivic decomposition.

2010 Mathematics Subject Classification: 14C25; 11E39.
Keywords and Phrases: Hermitian and quadratic forms, grassmannians, Chow groups and motives.

CONTENTS

*This work has been partially supported by CAPES, Coordination of the Development of the Superior Level Personnel - Brazil, in the framework of the Science Without Borders program with the Federal University of Amazonas (UFAM), Manaus.

1. Introduction

Let F be an arbitrary field and K/F a quadratic separable field extension. In this article, we define a new discrete invariant $J(h)$ for a non-degenerate K/F-hermitian form $h : V \times V \to K$. This invariant is developed on the model of the J-invariant for quadratic forms, due to Alexander Vishik, see [12], and later generalized to an arbitrary semi-simple algebraic group of inner type by V. Petrov, N. Semenov and K. Zainoulline in [8]. Let X denote the F-variety of maximal totally h-isotropic subspaces of V. The invariant $J(h)$ contains informations about rationality of algebraic cycles on X over a splitting field of h. The same way it was obtained by Nikita A. Karpenko and Alexander S. Merkurjev for maximal orthogonal grassmannian in the case of quadratic forms (see [1, Theorem 90.3]), the invariant $J(h)$ notably allows one to recover the canonical 2-dimension of the maximal unitary grassmannian X (Theorem 8.2).

In general, the J-invariant has several important applications. For example, A. Vishik used it in its refutation of the Kaplansky's conjecture on the u-invariant of a field (see [13]) and so did N. Semenov when he answered a question by J-P. Serre about groups of type E_8 (see [11]).

In the case of quadratic forms, the Chow motive of the maximal orthogonal grassmannian associated with a quadratic form splits as a sum of Tate motives over a splitting field of the quadratic form, the reason being that there is a nice filtration of the maximal orthogonal grassmannian by affine bundles. Because this does not stand in the case of hermitian forms, we use the structure Theorem [4, Theorem 15.8] by N. A. Karpenko and the modified Chow ring $\mathrm{Ch}_K(X) := \mathrm{Ch}(X)/\ \mathrm{Im}\,(\mathrm{Ch}(X_K) \to \mathrm{Ch}(X))$, with $\mathrm{Ch}(X)$ the integral Chow ring $\mathrm{CH}(X)$ modulo 2. These considerations allow one to follow the method introduced by A. Vishik for quadratic forms to describe completely the ring $\mathrm{Ch}_K(X)$ when h is split (equivalently, when X has a rational point) and the subring $\mathrm{Im}\big(\mathrm{Ch}_K(X) \to \mathrm{Ch}_K(X_{F(X)})\big)$ of rational elements for arbitrary h, where $F(X)$ is the function field of X. We also work with the category of *Ch_K-motives*, defined from Ch_K, and provide a complete motivic decomposition of the Ch_K-motive $M^K(X)$ of X in terms of the J-invariant $J(h)$ (Theorem 9.4). The Ch_K-motive $M^K(X)$ is related to the *essential motive* of X (see Remark 9.8).

By a theorem of Jacobson (see [6, Corollary 9.2]), the non-degenerate K/F-hermitian form h is entirely determined by the associated F-quadratic form $q : v \mapsto h(v,v)$, with V considered as an F-vector space. Moreover, the F-quadratic forms arising this way from K/F-hermitian forms can be described as the tensor product of a non-degenerate bilinear form by the norm form of K/F, which is an anisotropic binary quadratic from. Conversely, an F-quadratic form defined by such a tensor product is isomorphic to the quadratic form arising from the hermitian form induced by the bilinear form and the quadratic separable field extension K/F given by the discriminant of the binary quadratic form. As explained by N. A. Karpenko in the introduction of [6], although these

observations show that the study of K/F-hermitian forms is equivalent to the study of binary divisible quadratic forms over F, this does not show that the hermitian forms are not worthy of interest. Indeed, on the one hand, it shows that the class of binary divisible quadratic forms is quite important. On the other hand, it provides the opportunity to use the world of hermitian forms to study such quadratic forms, which can be more appropriate than staying exclusively at the level of quadratic forms, as illustrated by Proposition 10.1.
The paper is organized as follows. In section 3, we use the relative cellular space structure on X given by [4, Theorem 15.8] to get the relation of Proposition 3.6 between Chow rings Ch_K (defined in section 2) associated with the maximal unitary grassmannian of a hermitian subform of an isotropic K/F-hermitian form h. From section 4 to 8, we literally follow the thread of [1, §86 to §90]. In this part of the article, we first use the previously mentioned relation to get a complete description of $\mathrm{Ch}_K(X)$ in the split case in terms of generators and relations (Theorem 4.9 and Proposition 4.15), from which we deduce a description of the subring of rational elements in the general case in terms of those generators (Theorem 5.7). The J-invariant $J(h)$ is then defined from the latter description. We also compute some Steenrod operations of cohomological type on $\mathrm{Ch}(X)$ in the split case (Theorem 7.2). In Theorem 8.2, we obtain the canonical 2-dimension of X in terms of $J(h)$, on the model of [1, Theorem 90.3]. In section 9, using Rost Nilpotence, we provide the complete motivic decompostion of $M^K(X)$ in terms of $J(h)$ (Theorem 9.4), in the spirit of [8, Theorem 5.13]. In the final section 10, we compare the J-invariant $J(h)$ of a non-degenerate K/F-hermitian form h with the J-invariant $J(q)$ of the associated quadratic form q (Proposition 10.1).

ACKNOWLEDGEMENTS. I am grateful to Nikita A. Karpenko for initiating me to the geometric theory of hermitian forms. Special thanks go to the two anonymous referees for improving the quality of this paper at every level.

2. CHOW K-RINGS

Let F be an arbitrary field, K/F a quadratic separable field extension and X an F-variety (i.e. a separated F-scheme of finite type). We denote by $\mathrm{Ch}(X)$ the integral Chow ring $\mathrm{CH}(X)$ modulo 2.

We set

$$\mathrm{Ch}_K(X) := \mathrm{Ch}(X)/\operatorname{Im}(\mathrm{Ch}(X_K) \to \mathrm{Ch}(X)),$$

where the homomorphism $\mathrm{Ch}(X_K) \to \mathrm{Ch}(X)$ is the push-forward of the projection $X_K \to X$.

Note that $\operatorname{Im}(\mathrm{Ch}(X_K) \to \mathrm{Ch}(X))$ is an ideal by the Projection Formula ([1, Proposition 56.9]), called the *norm ideal*, so that $\mathrm{Ch}_K(X)$ inherits the ring structure of the initial Chow ring. For example, one has $\mathrm{Ch}_K(\mathrm{Spec}(F)) = \mathbb{Z}/2\mathbb{Z}$, and for any F-variety X, the ring $\mathrm{Ch}_K(X_K)$ is trivial. We write $(\varphi)_K$ for the Chow K-groups homomorphism associated with a Chow groups homomorphism φ which preserves norm ideals.

Since the norm ideal is preserved by pull-backs and push-forwards, one can define the additive category of *Ch_K-motives* the same way as the category of Chow motives (see [1, Chapter XII]) but using the Chow rings Ch_K instead of the usual Chow rings CH. For a smooth complete F-variety X, we write $M^K(X)$ for the associated Ch_K-motive.

REMARK 2.1. For a field extension E/F and $j \geq 0$, let us denote by $N_j(E)$ the subgroup of the Milnor group $K_j^M(E)$ generated by the norms from finite field extensions of E that split the extension K/F. Then the cycle module $E \mapsto K_*^M(E)$ over F gives rise to an assignment $E \mapsto K_*^M(E)/N_*(E)$. One can check that the latter is also a cycle module over F, in particular, the fact that residue maps are well-defined comes from the rule [9, R3b]. Hence, one can consider the cohomology theory associated with this cycle module (which contained the Chow K-groups) instead of the cohomology theory of the Milnor cycle module and thus obtain some Ch_K-versions of results for classical Chow groups (see Propositions 6.5, 8.1 and 9.2).

3. ISOTROPIC HERMITIAN FORMS

3.1. RELATIVE CELLULAR SPACES. Let F be a field.

DEFINITION 3.1. Let X be a smooth complete F-variety supplied with a filtration $\mathcal{F}$ by closed subvarieties

$$\emptyset = X_{(-1)} \subset X_{(0)} \subset \cdots X_{(n)} = X.$$

The variety X is a *relative cellular space* over a smooth complete F-variety Y if the associated *adjoint* variety

$$\mathrm{Gr}_{\mathcal{F}} X = \coprod_{k=0}^{n} X_{(k)} \backslash X_{(k-1)}$$

is a vector bundle over Y. The variety Y is called the *base* of X.

REMARK 3.2. Let X be a relative cellular space over Y and let Y' be a smooth complete F-variety. Then $X \times Y'$ is a relative cellular space over $Y \times Y'$.

REMARK 3.3. One can compose relative cellular structures ([4, Definition 7.4]). Let X supplied with $\mathcal{F}$ be a relative cellular space over Y. Suppose that Y is a relative cellular space over Y' and denote by j the associated imbedding $\mathrm{Gr}\, Y \hookrightarrow Y$. Let $\mathcal{F}'$ be the filtration on X such that $\mathrm{Gr}_{\mathcal{F}'} X = j^*(\mathrm{Gr}_{\mathcal{F}} X)$. Then X supplied with $\mathcal{F}'$ is a relative cellular space over Y'. The corresponding structure morphism is the composition $j^*(\mathrm{Gr}_{\mathcal{F}} X) \to \mathrm{Gr}\, Y \to Y'$ of vector bundles.

For V a finite dimensional F-vector space, we denote by $\Gamma(V)$ the full grassmannian of F-subspaces of V. To an epimorphism $p : V \to V'$ of F-vector spaces, one can associate the filtration

$$\emptyset = \Gamma(V)_{(-1)} \subset \Gamma(V)_{(0)} \subset \cdots \Gamma(V)_{(\dim V')} = \Gamma(V)$$

on $\Gamma(V)$ defined as follows: for any local commutative F-algebra R and $0 \leq k \leq \dim V'$, one has

$$\Gamma(V)_{(k)}(R) = \{N \in \Gamma(V)(R) | \Lambda^{k+1}(p_R(N)) = 0\},$$

where $p_R : V_R \to V'_R$ is induced by p and Λ^{k+1} stands for the $(k+1)$-th exterior power.

Let $0 \to V'' \to V \to V' \to 0$ be an exact sequence of F-vector spaces. The result [4, Corollary 9.11] by N. A. Karpenko asserts that $\Gamma(V)$ supplied with the filtration associated with $V \to V'$ is a relative cellular space over $\Gamma(V'')\times\Gamma(V')$.

Moreover, let K/F be a quadratic separable field extension. Suppose that V, V' and V'' are K-vector spaces and that the short sequence is an exact sequence of K-vector spaces. Then the previous relative cellular structure on $\Gamma(V)$ induces a relative cellular structure on the Weil restriction $\Gamma^K(V)$ of the full grassmannian of K-subspaces with respect to the extension K/F: $\Gamma^K(V)$ is a relative cellular space over $\Gamma^K(V'') \times \Gamma^K(V')$, see [4, Theorem 10.9]. The associated filtration is the restriction of the previous one by K-subspaces.

Suppose that the K-vector space V decomposes as a sum of K-subspaces $V = V' \oplus V'' \oplus \tilde{V}$. Using the exact sequences

$$0 \to V'' \oplus \tilde{V} \to V \to V' \to 0 \quad \text{and} \quad 0 \to V'' \to V'' \oplus \tilde{V} \to \tilde{V} \to 0,$$

the product of relative cellular structures as in Remark 3.2 and the composition (Remark 3.3), $\Gamma^K(V)$ is turned into a relative cellular space over $\Gamma^K(V') \times \Gamma^K(V'') \times \Gamma^K(\tilde{V})$ (described in [4, Example 10.14]).

Let $h : V \times V \to K$ be a non-degenerate isotropic K/F-hermitian form on V. Denote by $L \subset V$ an isotropic line and set $\tilde{V} = L^{\perp}/L$. Applying the observation of the previous paragraph to the decomposition $V = L \oplus L^* \oplus \tilde{V}$, one obtains that $\Gamma^K(V)$ is a relative cellular space over $\Gamma^K(L) \times \Gamma^K(L^*) \times \Gamma^K(\tilde{V})$. Furthermore, by [4, Theorem 15.8], the latter relative cellular structure restricts to the h-isotropic subspaces in the following way. Let $\tilde{h}$ be the K/F-hermitian form on $\tilde{V}$ induced by h. Let Y and $\tilde{Y}$ be the F-varieties of totally isotropic subspaces of h and $\tilde{h}$ respectively. Let Z be the Weil transfer of the K-variety of 2-flags of K-vector subspaces of L with respect to the extension K/F (note that the 0-dimensional F-variety Z is the disjoint union of three copies of Spec(F) and three copies of Spec(K)). Then Y is a relative cellular space over $Z \times \tilde{Y}$. The associated filtration is the restriction of the filtration associated with the relative cellular space structure of $\Gamma^K(V)$ over $\Gamma^K(L) \times \Gamma^K(L^*) \times \Gamma^K(\tilde{V})$ by h-isotropic subspaces.

In the same article [4], the author proves that, in general, the Chow group of a relative cellular space is isomorphic to the Chow group of its base ([4, Theorem 6.5]) and he describes in [4, Corollary 6.11] how this isomorphism restricts to the irreducible components of the relative cellular space.
Applied to the previous situation, this gives the following. Let X be the F-variety of maximal totally isotropic subspaces in h. The dimension of such a

subspace is $r := \lfloor \dim(h)/2 \rfloor$. Note that X is an irreducible component of Y. Besides, the unitary grassmannian X is a projective homogeneous variety under a projective unitary group of outer type (see the introduction of [6]). We write $\tilde{X}$ for the maximal unitary grassmannian associated with $\tilde{h}$. Since maximal totally isotropic subspaces of $\tilde{V}$ are in one-to-one correspondence with those of V containing L, one can view $\tilde{X}$ as a closed subvariety of X. Let $i : \tilde{X} \hookrightarrow X$ denote the closed embedding. Let $\beta : \tilde{X} \rightsquigarrow X$ be the correspondence given by the scheme of pairs $(W/L, U)$, where U is a totally isotropic r-dimensional K-subspace of V, W is a totally isotropic r-dimensional subspace of $L^{\perp}$ containing L, and $\dim_K(U+W) \leq r+1$ (correspondences are defined in [1, §62]). Then one has the following decomposition

$$\mathrm{Ch}^*(X) \simeq \mathrm{Ch}^*(\tilde{X}) \oplus \mathrm{Ch}^{*-d}(\tilde{X}) \bigoplus_{0 \leq s \leq t} \mathrm{Ch}^{*-a_s}(\mathrm{Spec}(K)), \tag{3.4}$$

where the injection $\mathrm{Ch}^*(\tilde{X}) \hookrightarrow \mathrm{Ch}^*(X)$ coincides with β_*, $d = \dim(X) - \dim(\tilde{X})$, the injection $\mathrm{Ch}^{*-d}(\tilde{X}) \hookrightarrow \mathrm{Ch}^*(X)$ coincides with i_*, and t and the a_s are integers.

At the level of Chow K-groups (introduced in the previous section), decomposition (3.4) implies that

$$\mathrm{Ch}^*_K(X) \simeq \mathrm{Ch}^*_K(\tilde{X}) \oplus \mathrm{Ch}^{*-d}_K(\tilde{X}). \tag{3.5}$$

In particular, if h is split (i.e. if the Witt index $i_0(h)$ of h is equal to r), one deduces by induction that $\mathrm{Ch}_K(X)$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank 2^r.

We write j for the open embedding $X \backslash \tilde{X} \hookrightarrow X$ and we set

$$f := \beta^t \circ j,$$

with β^t the transpose of β.
Since $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$ by the localization exact sequence (see [1, §52.D]), it follows from (3.5) that

$$\mathrm{Ch}^*_K(X) = \mathrm{Im}((\beta_*)_K) \oplus \mathrm{Ker}\left((j^*)_K\right).$$

Hence, since j^* is surjective (see *loc. cit.*), we have obtained the following statement.

PROPOSITION 3.6. *The homomorphism*

$$(f^*)_K : \mathrm{Ch}^*_K(\tilde{X}) \to \mathrm{Ch}^*_K(X \backslash \tilde{X}),$$

is an isomorphism.

The above proposition is crucial for the induction in the proof of Theorem 4.9 in the next section.

3.2. Associated quadrics. We use notation introduced in Subsection 3.1. Let $q : V \to F$, $v \mapsto h(v, v)$ be the non-degenerate F-quadratic form associated with h, where V is considered as an F-vector space. Note that $\dim(q) = 2\dim(h)$. We denote by Q the smooth projective quadric of q. Similarly, let $\tilde{q} : \tilde{V} \to F$ be the non-degenerate F-quadratic form associated with the hermitian form $\tilde{h}$ and let us denote by $\tilde{Q}$ the smooth projective quadric of $\tilde{q}$. Note that since $\tilde{q}$ is also the form induced by q on $P^{\perp}/P$, with P the q-isotropic F-plane corresponding to L, it is Witt-equivalent to q.
The *incidence correspondence* $\alpha : \tilde{Q} \rightsquigarrow Q$ is given by the scheme of pairs $(B/P, A)$ of isotropic F-lines in $P^{\perp}/P$ and V respectively with $A \subset B$. By [1, Lemma 72.3], for $k < i_0(q)$, one has $\alpha_*(\tilde{l}_{k-2}) = l_k$ and $\alpha^*(l_k) = \tilde{l}_{k-2}$, where l_k (resp. $\tilde{l}_k$) is the class in $\mathrm{CH}_k(Q)$ (resp. $\mathrm{CH}_k(\tilde{Q})$) of a k-dimensional totally q-isotropic (resp. $\tilde{q}$-isotropic) subspace of $\mathbb{P}_F(V)$ (resp. $\mathbb{P}_F(\tilde{V})$). If q is split, given an orientation of Q, we choose an orientation of $\tilde{Q}$ so that the previous formulas hold for the classes of the respective maximal isotropic subspaces.

We write E for the vector bundle of rank $2r$ over X given by the closed subvariety of the trivial bundle $V\mathbb{1} = V \times X$ consisting of pairs (u, U) such that $u \in U$ (with V viewed as an F-vector space). Let us denote as $E^{\perp}$ the kernel of the natural morphism $V\mathbb{1} \to E^{\vee}$ given by the polar bilinear form associated with the quadratic form q, where $E^{\vee}$ is the dual bundle of E. If the dimension of h is even, one has $E^{\perp} = E$. Otherwise, E is a subbundle of $E^{\perp}$ of corank 2. For our purpose, the vector bundle E is the appropriate hermitian version of the the vector bundle used in [1, §86] for the case of quadratic forms.
The associated projective bundle $\mathbb{P}(E)$ is a closed subvariety of codimension $2\lfloor(\dim(h) + 1)/2\rfloor - 1$ of $Q \times X$ and we denote by *in* the associated closed embedding. We write γ for the class of $\mathbb{P}(E)$ in $\mathrm{CH}(Q \times X)$ and view it as a correspondence $Q \rightsquigarrow X$. Similarly, one can consider the analogous correspondence $\tilde{\gamma} : \tilde{Q} \rightsquigarrow \tilde{X}$.

Lemma 3.7. *One has* $\gamma \circ \alpha = \beta \circ \tilde{\gamma}$ *and* $\tilde{\gamma} \circ \alpha^t = i^t \circ \gamma$.

Proof. The proof is almost the same as in the case of quadratic forms, see [1, Lemma 86.7]. By [1, Corollary 57.22], it suffices to check the required identities at the level of cycles representing the correspondences. By definition of the composition of correspondences, the composition $\gamma \circ \alpha$ and $\beta \circ \tilde{\gamma}$ coincide with the cycle of the subscheme of $\tilde{Q} \times X$ consisting of all pairs $(B/P, U)$ with $\dim_F(B + U) \leq 2r + 2$ and the compositions $\tilde{\gamma} \circ \alpha^t$ and $i^t \circ \gamma$ coincide with the cycle of the subscheme of $Q \times \tilde{X}$ consisting of all pairs $(A, W/L)$ with $A \subset W$. □

4. Split maximal unitary grassmannian

We use notation introduced in Section 3.
In this section, we make the assumption that the non-degenerate K/F-hermitian form h is split and we provide a description of the Chow ring $\mathrm{Ch}_K(X)$

of the maximal unitary grassmannian in terms of generators and relations (Theorem 4.9 and Proposition 4.15).
The method is the one introduced by A. Vishik in [12] to get the description of the Chow ring modulo 2 of the maximal orthogonal grassmannian in the split case, except that we work with the Chow rings Ch_K and use Proposition 3.6 in replacement of the filtration by affine bundles on the maximal orthogonal grassmannian.
The exposition below closely follows the one given in the corresponding part of [1, §86] in the case of quadratic forms. The proofs are very akin to the original ones in [1, §86].

Let us write $\dim(h) = 2n+2$ or $2n+1$, with $n \geq 0$. Note that since $i_0(q) = 2i_0(h)$ (see [6, Lemma 9.1]), the quadratic form q is split if and only if h is of even dimension. In both cases, $\mathbb{P}(E)$ has codimension $2n+1$ in $Q \times X$.

If $\dim(h) = 2n+2$, the cycle γ decomposes as

$$(4.1)\quad \gamma = l_{2n+1} \times e_0 + l'_{2n+1} \times e'_0 + \sum_{k=1}^{2n+1} h^{2n+1-k} \times e_k \quad \text{in } \mathrm{CH}^{2n+1}(Q \times X)$$

for some unique elements $e_k \in \mathrm{CH}^k(X)$, $k \in [0,\ 2n+1]$ and $e'_0 \in \mathrm{CH}^0(X)$, with l_{2n+1}, l'_{2n+1} the two different classes of $(2n+1)$-dimensional totally q-isotropic subspaces of $\mathbb{P}_F(V)$ and h^i the i-th power of the pull-back $h^1 \in \mathrm{CH}^1(Q)$ of the hyperplane class $H \in \mathrm{CH}^1(\mathbb{P}_F(V))$ under the closed embedding $em : Q \hookrightarrow \mathbb{P}_F(V)$. (see [1, Propositions 64.3, 68.1 and 68.2]). Note that since X is connected, pulling (4.1) back with respect to the canonical morphism $Q_{F(X)} \to Q \times X$, we see that one can choose an orientation of Q such that $e_0 = 1$ and $e'_0 = 0$.

Otherwise – if $\dim(h) = 2n+1$ – the cycle γ decomposes as

$$(4.2)\qquad \gamma = \gamma' + l_{2n-1} \times e_0 + \sum_{k=1}^{2n+1} h^{2n+1-k} \times e_k \quad \text{in } \mathrm{CH}^{2n+1}(Q \times X)$$

for some unique elements $e_k \in \mathrm{CH}^k(X)$, $k \in [1,\ 2n+1]$ and $e_0 \in \mathrm{CH}^0(X)$, with γ' a cycle such that, by denoting p_X the projection $Q \times X \to X$, one has $p_{X*}\left((l_{2n+1-k} \times 1) \cdot \gamma'\right) = 0$ for any $k \in [1,\ 2n+1]$ and $p_{X*}\left((h^{2n-1} \times 1) \cdot \gamma'\right) = 0$. Pulling (4.2) back with respect to $Q_{F(X)} \to Q \times X$, one get that $e_0 = 1$.

In both cases, the multiplication rules in the ring $\mathrm{CH}(Q)$ (see [1, Proposition 68.1]) gives that $e_k = p_{X*}\left((l_{2n+1-k} \times 1) \cdot \gamma\right)$, for $k \in [1,\ 2n+1]$. In other words, the correspondence γ satisfies $\gamma_*(l_{2n+1-k}) = e_k$ for $k \in [1,\ 2n+1]$. Note also that, in the even-dimensional case, one has $\gamma_*(l_{2n+1}) = 1$ since $l^2_{2n+1} = l_0$ as 4 divides $\dim(q) = 4n+4$ (see [1, Exercise 68.3(3)]), and that, in the odd-dimensional case, one has $\gamma_*(h^{2n-1}) = 1$.
In both cases, for $k \in [1,\ 2n+1]$, this can be rewritten as

$$(4.3)\qquad e_k = (p_X \circ in)_* \circ (p_Q \circ in)^*(l_{2n+1-k}),$$

with p_Q the projection $Q \times X \to Q$ (see [1, Proposition 62.7]).

LEMMA 4.4. *One has* $e_{2n+1} = [\tilde{X}]$ *in* $\mathrm{CH}^{2n+1}(X)$.

Proof. On the one hand, by (4.3) one has $(p_X \circ in)_* \circ (p_Q \circ in)^*(l_0) = e_{2n+1}$. On the other hand, by denoting A a closed point of Q of degree 1, the cycle $(p_X \circ in)_* \circ (p_Q \circ in)^*(l_0)$ is identified with $[\{U \mid A \subset U\}] = [\{U \mid A \otimes_F K \subset U\}]$ in $\mathrm{CH}^{2n+1}(X)$. The latter algebraic cycle is equal to $[\{U \mid L \subset U\}] = [\tilde{X}]$. □

When $\dim(h) = 2$, the previous lemma means that e_1 is the class of a rational point on the curve X.

Let us denote by $\tilde{e}_k \in \mathrm{CH}^k(\tilde{X})$ the elements given by (4.1) or (4.2) for $\tilde{X}$. Similarly, the correspondence $\tilde{\gamma}$ satisifies $\tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \tilde{e}_k$ for $k \in [1,\ 2n+1]$. The following statement is a direct consequence of Lemma 3.7.

LEMMA 4.5. *One has*

(i) $\beta_*(\tilde{e}_k) = e_k$ *in* $\mathrm{CH}(X)$ *for all* $k \in [0,\ 2n-1]$*;*
(ii) $i^*(e_k) = \tilde{e}_k$ *in* $\mathrm{CH}(\tilde{X})$ *for all* $k \in [0,\ 2n-1]$*;*
(iii) $e_1 = 0$ *in* $\mathrm{CH}(X)$ *for odd-dimensional* h.

Proof. (i) For $k \in [1,\ 2n-1]$, one has

$$\beta_*(\tilde{e}_k) = \beta_* \circ \tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \gamma_* \circ \alpha_*(\tilde{l}_{2n-1-k}) = \gamma_*(l_{2n+1-k}) = e_k.$$

For $k = 0$ and even-dimensional h, one can make the exact same computation. For $k = 0$ and odd-dimensional h, one has to replace $\tilde{l}_{2n-1}$ and l_{2n+1} by $\tilde{h}^{2n-3}$ and h^{2n-1} respectively in the previous computation (the incidence correspondence α also satisfies $\alpha_*(\tilde{h}^i) = h^{i+2}$ for $0 \le i \le 2n-2$, where $\tilde{h}^i$ is the i-th power of the hyperplane class $\tilde{h}^1 \in \mathrm{CH}^1(\tilde{Q})$).
(ii) One has

$$\begin{aligned} i^*(e_k) = i^t_*(e_k) = i^t_* \circ \gamma_*(l_{2n+1-k}) = \tilde{\gamma}_* \circ \alpha^t_*(l_{2n+1-k}) &= \tilde{\gamma}_* \circ \alpha^*(l_{2n+1-k}) \\ &= \tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \tilde{e}_k. \end{aligned}$$

(iii) We induct on n. If $n = 0$, i.e. $\dim(h) = 1$, one has $e_1 = 0$ because $\dim(X) = 0$ since $X = \mathrm{Spec}(F)$. The conclusion follows now from (i) by induction. □

For $I \subset [0,\ 2n+1]$, we write e_I for the product of e_k for all $k \in I$. Similarly, one defines the elements $\tilde{e}_J$ for $J \subset [0,\ 2n-1]$. The following statement is obtained by combining Lemma 4.5(ii) with the Projection Formula and Lemma 4.4.

COROLLARY 4.6. *One has* $i_*(\tilde{e}_J) = e_J \cdot e_{2n+1} = e_{J \cup \{2n+1\}}$ *for every* $J \subset [0,\ 2n-1]$.

Let us denote by I_{od} the odd part of a set of integers I.

COROLLARY 4.7. *For* h *of even dimension, the monomial* $e_{[1,\ 2n+1]_{od}} = e_1 e_3 \cdots e_{2n+1}$ *is the class of a rational point in* $\mathrm{CH}_0(X)$. *For* h *of odd dimension, the monomial* $e_{[3,\ 2n+1]_{od}} = e_3 e_5 \cdots e_{2n+1}$ *is the class of a rational point in* $\mathrm{CH}_0(X)$.

Proof. The conclusion follows by induction on n from the formulas $e_{[1,\ 2n+1]_{\rm od}} = i_*(\tilde{e}_{[1,\ 2n-1]_{\rm od}})$ for h of even dimension and $e_{[3,\ 2n+1]_{\rm od}} = i_*(\tilde{e}_{[3,\ 2n-1]_{\rm od}})$ for h of odd dimension, see Corollary 4.6. □

LEMMA 4.8. *One has $f^*(\tilde{e}_J) = j^*(e_J)$ for any $J \subset [0,\ 2n-1]$.*

Proof. It is enough to show that $f^*(\tilde{e}_k) = j^*(e_k)$. Since $f = \beta^t \circ j$, the conclusion follows from Lemma 4.5(i). □

Now, one can prove the following theorem the same way [1, Theorem 86.12] has been proven for the case of quadratic forms. We still write e_I for the class of e_I in $\mathrm{Ch}_K(X)$.

THEOREM 4.9. *Let K/F be a quadratic separable field extension and h a split non-degenerate K/F-hermitian form. Let X be the variety of maximal totally isotropic subspaces in h. If h is of even dimension $2n+2$ then the set of all 2^{n+1} monomials $e_{I_{od}}$ for all subsets $I \subset [0,\ 2n+1]$ is a basis of the $\mathbb{Z}/2\mathbb{Z}$-module $\mathrm{Ch}_K(X)$. Otherwise – if h is of odd dimension $2n+1$ – the set of all 2^n monomials $e_{I_{od}}$ for all subsets $I \subset [0,\ 2n+1]\backslash\{1\}$ is a basis of the $\mathbb{Z}/2\mathbb{Z}$-module $\mathrm{Ch}_K(X)$.*

Proof. We use an induction on n. If $\dim(h) = 2$, the $\mathbb{Z}/2\mathbb{Z}$-module $\mathrm{Ch}^1_K(X)$ is generated by the class e_1 of a rational point (see decomposition (3.5) and Lemma 4.4). If $\dim(h) = 1$, the statement is obviously true.
Consider the following exact sequence

$$0 \longrightarrow \mathrm{Ch}_K(\tilde{X}) \xrightarrow{(i_*)_K} \mathrm{Ch}_K(X) \xrightarrow{(j^*)_K} \mathrm{Ch}_K(X\backslash\tilde{X}) \longrightarrow 0\ ,$$

associated with the localization exact sequence for classical Chow groups. In both cases, the induction hypothesis and Corollary 4.6 imply that the set of monomials $e_{I_{od}}$ for all I containing $2n+1$ is a basis of the image of $(i_*)_K$. On the other hand, Lemma 4.8, Proposition 3.6 and the induction hypothesis imply that, in both cases, the set of all the elements $(j^*)_K(e_{I_{od}})$ with $2n+1 \notin I$ is a basis of $\mathrm{Ch}_K(X\backslash\tilde{X})$. The conclusion follows. □

Also, one has $e_{2k} = 0$ in $\mathrm{Ch}_K(X)$ for all $k \geq 1$, see the proof of Corollary 4.14.

The next statement provides formulas for the Chern classes of the vector bundles $V\mathbb{1}/E$ and E over X which will be used to describe the multiplicative structure of $\mathrm{Ch}_K(X)$.

PROPOSITION 4.10. *In $\mathrm{CH}(X)$, one has:*
– if $\dim(h) = 2n+2$ then $c_k(V\mathbb{1}/E) = (-1)^k c_k(E) = 2e_k$ for all $k \in [1,\ 2n+1]$ and $c_{2n+2}(V\mathbb{1}/E) = c_{2n+2}(E) = 0$;
– otherwise – if $\dim(h) = 2n+1$ – then $c_k(V\mathbb{1}/E) = (-1)^k c_k(E) = 2e_k$ for all $k \in [1,\ 2n]$, $c_{2n+1}(V\mathbb{1}/E) = 2e_{2n+1}$ and $c_{2n+2}(V\mathbb{1}/E) = 0$.

Proof. We apply $(em\times \mathrm{Id}_X)_*$ to the cycle γ. Assume first that $\dim(h)=2n+2$. Since $em_*(h^k)=2H^{k+1}$ for all $k\geq 0$, decomposition (4.1) gives that

$$[\mathbb{P}(E)] = em_*(l_{2n+1})\times 1+\sum_{k=1}^{2n+1} 2H^{2n+2-k}\times e_k \quad \text{in} \quad \mathrm{CH}^{2n+2}(\mathbb{P}_F(V)\times X).$$

On the other hand, by [1, Proposition 58.10], one has

$$[\mathbb{P}(E)] = \sum_{k=0}^{2n+2} H^{2n+2-k}\times c_k(V\mathbb{1}/E).$$

Hence, since by the Projective Bundle Theorem (see [1, Theorem 53.10]) the decomposition of $[\mathbb{P}(E)]$ as $\sum_{k=0}^{2n+2} H^{2n+2-k}\times \alpha_k$ is unique, one has $c_k(V\mathbb{1}/E)=2e_k$ for all $k\in[1,\ 2n+1]$ and $c_{2n+2}(V\mathbb{1}/E)=0$ in $\mathrm{CH}(X)$.
Assume that $\dim(h)=2n+1$. Decomposition (4.2) gives the identity

$$[\mathbb{P}(E)] = (em\times \mathrm{Id}_X)_*(\gamma')+em_*(l_{2n-1})\times 1+\sum_{k=1}^{2n+1} 2H^{2n+2-k}\times e_k$$

in $\mathrm{CH}^{2n+2}(\mathbb{P}_F(V)\times X)$. Moreover, it follows from the conditions defining the cycle γ' in (4.2) that $(em\times \mathrm{Id}_X)_*(\gamma')$ has the form $1\times\alpha$ for some $\alpha\in\mathrm{CH}^{2n+2}(X)$. Since 1 does not belong to the image of em_*, one has $\alpha=0$. Consequently, by using [1, Proposition 58.10] and the Projective Bundle Theorem as it has been done in the even case, one get that $c_k(V\mathbb{1}/E)=2e_k$ for all $k\in[1,\ 2n+1]$ and $c_{2n+2}(V\mathbb{1}/E)=0$ in $\mathrm{CH}(X)$.
Furthermore, by duality, one has $V\mathbb{1}/E^{\perp}\simeq E^{\vee}$. Thus, we are done with the even-dimensional case, as in that case $E^{\perp}=E$.
For odd-dimensional h, $E^{\perp}/E$ is a rank 2 vector bundle on X such that $c(E^{\vee})\circ c(E^{\perp}/E)=c(V\mathbb{1}/E)$ and it suffices to prove that $c_i(E^{\perp}/E)=0$ in $\mathrm{CH}^i(X)$ (for $i=1,2$) to complete the proof. Since $\dim(q)=4n+2$ and $i_0(q)=2n$, one can choose V' a codimension 1 F-subspace of V containing a maximal hyperbolic space. Let us denote by E' and $(E^{\perp})'$ the respective subbundles of E and $E^{\perp}$ consisting of pairs (u,U) such that $u\in U\cap V'$ and $u\in U^{\perp}\cap V'$, respectively. We denote by $\mathcal{L}_1$ the line bundle $(E^{\perp})'/E'$ and by $\mathcal{L}_2$ the factor line bundle $(E^{\perp}/E)/\mathcal{L}_1$. By [1, Proposition 54.6], one has $c_1(E^{\perp}/E)=c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2)$ and $c_2(E^{\perp}/E)=c_1(\mathcal{L}_1)\cdot c_1(\mathcal{L}_2)$ in $\mathrm{CH}(X)$. Furthermore, since each of the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ carries a non-degenerate quadratic form, these are isomorphic to their duals. Consequently, the isomorphism classes of $\mathcal{L}_1$ and $\mathcal{L}_2$ are 2-torsion elements in the Picard group $\mathrm{Pic}(X)$. Moreover, since X is a projective homogeneous variety under a semisimple algebraic group, the group $\mathrm{Pic}(X)$ is torsion free (see [7, §2]). □

REMARK 4.11. The above proposition shows in particular that for *any* non-degenerate K/F- hermitian form h, the algebraic cycles $2e_k$ for $k\in[1,\ 2n+1]$ are defined over F, i.e. these are always rational cycles.

REMARK 4.12. Suppose $\dim(h) = 2n + 2$. Let Y be the maximal orthogonal grassmannian of q. Since a totally h-isotropic K-subspace is also a totally q-isotropic F-subspace, we have a natural closed imbedding $im : X \hookrightarrow Y$. Let us denote by $f_k \in \mathrm{CH}^k(Y)$, $1 \le k \le 2n+1$, the generators of $\mathrm{CH}(Y)$ introduced by A. Vishik in [12], quadratic analogues of the elements e_k (recall that q is split). We write $\mathcal{E}$ for the subbundle of the trivial bundle $V \times Y$ over Y given by pairs (u, U) such that $u \in U$. Note that one has $E = im^*(\mathcal{E})$. Hence, since f_k is half the k-th Chern class of the vector bundle $(V \times Y)/\mathcal{E}$ (see [1, Porposition 86.13] for example) and CH(X) is torsion-free, it follows from Proposition 4.10 that

$$e_k = im^*(f_k)$$

in $\mathrm{CH}(X)$ for all $1 \le k \le 2n + 1$ (this can also be obtained by comparing the decompositions (4.1) and [1, (86.4)]). In the odd-dimensional case, the variety X is naturally a closed subvariety of the second to last grassmannian of q.

We set $e_k = 0$ for $k > 2n + 1$ (which makes sense by Proposition 4.10 since the vector bundle $V\mathbb{1}/E$ is in both cases of rank $2n+2$ with trivial top Chern class). It follows from Whitney Sum Formula [1, Proposition 54.7] and Proposition 4.10 that

$$e_k^2 - 2e_{k-1}e_{k+1} + 2e_{k-2}e_{k+2} - \cdots + (-1)^{k-1}2e_1e_{2k-1} + (-1)^k e_{2k} = 0$$

in $\mathrm{CH}(X)$, for all $k \ge 1$. Therefore, one has

$$e_k^2 = e_{2k} \tag{4.13}$$

in $\mathrm{Ch}(X)$, for all $k \ge 1$.

We call the *generic hermitian form* of dimension d (for the fixed separable extension K/F) the diagonal $K(t_1, \ldots, t_d)/F(t_1, \ldots, t_d)$-hermitian form $< t_1, \ldots, t_d >$, where $t_1, \ldots, t_d$ are variables. We denote it by h_g. Since any hermitian form can be diagonalized (see [10, Theorem 6.3 of Chapter 7]), any d-dimensional K/F-hermitian form is a specialization of h_g.

COROLLARY 4.14. *One has $e_k^2 = 0$ in $\mathrm{Ch}_K(X)$ for all $k \ge 1$.*

Proof. By relations (4.13), one has to prove that

$$e_{2k} = 0 \text{ in } \mathrm{Ch}_K(X)$$

for all $k \ge 1$. Assume that $\dim(h) = 2n+2$. We induct on n. For $n = 0$, this is true by dimensional reasons. By the induction hypothesis and Lemma 4.5(i), it only remains to show that $e_{2n} = 0$ in $\mathrm{Ch}_K(X)$. We use notation of Remark 4.12. By [6, Lemma 9.8], the push-forward of im induces an injective morphism $(im_*)_K : \mathrm{Ch}_K(X) \hookrightarrow \mathrm{Ch}(Y)$. Therefore, one has to show that $(im_*)_K(e_{2n}) = 0$ in $\mathrm{Ch}(Y)$. It follows from Remark 4.12 and the Projection Formula that

$$(im_*)_K(e_{2n}) = [X] \cdot f_{2n}.$$

Moreover, we claim that $(im_*)_K(1) = [X] = f_2 \cdot f_4 \cdots f_{2n-2} \cdot f_{2n}$ in $\mathrm{Ch}(Y)$. The hermitian form h is a specialization of a generic one h_g. Then the varieties X and Y are specializations of the corresponding varieties X_g and Y_g associated to h_g. Let us denote by L the function field of Y_g. Since h_g is generic and the rational cycle $[X_g] \in \mathrm{Ch}\,((Y_g)_L)$ is nonzero (by [6, Lemma 9.8]) of codimension $2+4+\cdots+2n$, it follows from [12, Theorem 5.8] and [6, Corollary 9.4] (or Remark 10.2) that $[X_g] = g_2 \cdot g_4 \cdots g_{2n-2} \cdot g_{2n}$ in $\mathrm{Ch}\,((Y_g)_L)$, with $g_k \in \mathrm{Ch}^k\,((Y_g)_L)$ the analogues of the elements f_k. Thus, since the specialization ring homomorphism $\mathrm{Ch}_K(X_g) \to \mathrm{Ch}_K(X)$ commutes with push-forwards and pull-backs (see [3, §20.3]), it is enough to check that the specialization homomorphism $\sigma : \mathrm{Ch}\,((Y_g)_L) \to \mathrm{Ch}(Y_L) \simeq \mathrm{Ch}(Y)$ satisfies $\sigma(g_k) = f_k$ to complete the proof of the claim. Besides, the latter identity follows for example from formula [1, (86.5)], hence the claim. Since $f_k^2 = f_{2k}$ for all $1 \le k \le 2n+1$ with $f_k = 0$ for $k > 2n+1$ (see [12]), one deduces that $(im_*)_K(e_{2n}) = 0$ in $\mathrm{Ch}(Y)$. This concludes the proof of the corollary in the even case.

The proof in the odd case can be treated along similar lines however one has to consider Y the second to last grassmannian of q instead of the last grassmannian, the ring $\mathrm{Ch}_K(Y \times \mathrm{Spec}\, F(X))$ (in the even case, the Chow ring modulo 2 and the Ch_K-ring of the last grassmannian coincide) and the structure of projective bundle of $Y \times \mathrm{Spec}\, F(X)$ over the last grassmannian to use [12, §2, §3] to get that $\mathrm{Ch}_K(Y \times \mathrm{Spec}\, F(X))$ has a ring structure akin to that of the Chow ring modulo 2 of the last grassmannian in the even case (but with no generator f_1). □

We are now able to determine the multiplicative structure for the ring $\mathrm{Ch}_K(X)$ the same way it has been done for the case of quadratic forms.

PROPOSITION 4.15. *The equalities $e_k^2 = 0$ in $\mathrm{Ch}_K(X)$, for all odd $k > 2$ and also for $k = 1$ for h of even dimension, form a set of defining relations between the generators e_k of the ring $\mathrm{Ch}_K(X)$.*

Proof. Assume first that h is of even dimension. Let us denote by $\mathcal{R}$ the factor ring of the polynomial ring $\mathbb{Z}/2\mathbb{Z}[t_0, t_1, \ldots, t_n]$ by the ideal generated by the monomials t_k^2, with $k \ge 1$. Then, by Corollary 4.14, the ring homomorphism

$$\begin{array}{rccc} \varphi : & \mathcal{R} & \to & \mathrm{Ch}_K(X) \\ & t_k & \mapsto & e_{2k+1} \end{array}$$

is well defined. Furthermore, it follows from Theorem 4.9 that φ is surjective. Consequently, since the classes of the monomials $t_0^{r_0} t_1^{r_1} \cdots t_n^{r_n}$ with $r_k = 0$ or 1 for every k generate $\mathcal{R}$, one get that φ is an isomorphism. One proceeds the same way for h of odd dimension, except there is no variable t_0. □

5. General maximal unitary grassmannian

In this section, we use notation introduced in the previous sections. We do not make any assumption on the isotropy of the non-degenerate K/F-hermitian form h.

For any scheme Y over F, we write $\bar{Y}$ for $Y \times \mathrm{Spec}\, F(X)$, where $F(X)$ is the function field of the maximal unitary grassmannian X (note that $K \otimes_F F(X)$

is still a field). Following the path set by A. Vishik in [12], we describe the subring,

$$\overline{\mathrm{Ch}}_K(X) := \mathrm{Im}\left(\mathrm{Ch}_K(X) \to \mathrm{Ch}_K(\bar{X})\right).$$

of *rational* elements, c.f. Theorem 5.7 (actually, this description does not depend on the choice of a splitting field of h which does not split K). We use similar notation and vocabulary for classical Chow rings and certain products of F-varieties.
The exposition in this section follows literally the thread of [1, §87]. The proofs are very similar and sometimes identical (the proof of Theorem 5.7 compared to the original [1, Theorem 87.7] for example).

PROPOSITION 5.1. *Let $g : Y \to X$ be a morphism, with Y a smooth proper scheme over F. The $\mathrm{CH}(Y)$-module $\mathrm{CH}(Q \times Y)$ is free with basis h^k, $h^k \cdot [\mathbb{P}(g^*(E))]$, with $k \in [0,\ 2n+1]$ if $\dim(h) = 2n+2$ and $k \in [0,\ 2n]$ if $\dim(h) = 2n+1$.*

Proof. Let us denote $g^*(E)$ as E'. First, we want to show that the restriction $f : T = (Q \times Y)\backslash\mathbb{P}(E') \to \mathbb{P}(V\mathbb{1}/E'^{\perp})$ of the morphism $\mathbb{P}(V\mathbb{1})\backslash\mathbb{P}(E'^{\perp}) \to \mathbb{P}(V\mathbb{1}/E'^{\perp})$, where $V\mathbb{1}$ is the trivial vector bundle $V \times Y$ over Y, is an affine bundle in both even and odd cases. By [1, Lemma 52.12], it is enough to prove that the fiber over any local commutative F-algebra R is isomorphic to an affine space over R.
An F-morphism $\mathrm{Spec}(R) \to \mathbb{P}(V\mathbb{1}/E'^{\perp})$ determines a pair (U, W), where:
– if $\dim(h) = 2n+2$, U is a totally isotropic direct summand of V_R of rank $2n+2$ and W is a direct summand of V_R of rank $2n+3$ containing $U^{\perp} = U$;
– otherwise – if $\dim(h) = 2n+1$ – U is a totally isotropic direct summand of V_R of rank $2n$ and W is a direct summand of V_R of rank $2n+3$ containing $U^{\perp}$ (which is of rank $2n+2$).
If $\dim(h) = 2n+2$, as $\mathrm{rank}\, W^{\perp} = 2n+1$, one can find an R-basis of W such that the restriction $q_{|W/W^{\perp}}$ of the quadratic form q on $W/W^{\perp}$ is the hyperbolic plane xy with U given by $y = 0$. Thus, the fiber $\mathrm{Spec}(R) \times_{\mathbb{P}(V\mathbb{1}/E'^{\perp})} T$ is isomorphic to an affine space over R. Otherwise – if $\dim(h) = 2n+1$ – as $\mathrm{rank}\, W^{\perp} = 2n-1$, one can find an R-basis of W such that $q_{|W/W^{\perp}} = xy + az^2 + zt + bt^2$ for some $a, b \in R$, with U given by $y = z = t = 0$. Hence, the fiber $\mathrm{Spec}(R) \times_{\mathbb{P}(V\mathbb{1}/E'^{\perp})} T$ is also isomorphic to an affine space over R.
Consequently, f is an affine bundle.
Thus, in both cases, the variety $Q \times Y$ is a cellular variety. Therefore, one has a split exact sequence

$$0 \longrightarrow \mathrm{CH}(\mathbb{P}(E')) \xrightarrow{in'_*} \mathrm{CH}(Q \times Y) \longrightarrow \mathrm{CH}(T) \longrightarrow 0$$

and the pull-back f^* is an isomorphism (c.f. the proof of [1, Theorem 66.2]).
On the one hand, the restriction of the canonical line bundle over $\mathbb{P}_F(V)$ to $Q \times Y$ and $\mathbb{P}(E')$ are also canonical line bundles. Therefore, by the Projective

Bundle Theorem and the Projection Formula, the image of in'_* is a free $\mathrm{CH}(Y)$-module with basis $h^k \cdot [\mathbb{P}(E')]$, with $k \in [0,\ 2n+1]$ if $\dim(h) = 2n+2$ and $k \in [0,\ 2n]$ if $\dim(h) = 2n+1$.
On the other hand, the pull-back with respect to f of the canonical bundle over $\mathbb{P}(V\mathbb{1}/E'^{\perp})$ coincides with the restriction to T of the canonical bundle on $Q \times Y$. Therefore, by the Projective Bundle Theorem, $\mathrm{CH}(T)$ is a free $\mathrm{CH}(Y)$-module with basis the restrictions of h^k, with $k \in [0,\ 2n+1]$ if $\dim(h) = 2n+2$ and $k \in [0,\ 2n]$ if $\dim(h) = 2n+1$, on T. The proposition is proven. □

REMARK 5.2. The proof of Propositon 5.1 provides, in both even and odd cases, the following integral Chow motivic decomposition (see also [1, Remark 87.2])

$$M\left(Q \times Y\right) = M\left(\mathbb{P}(g^*(E))\right) \oplus M\left(\mathbb{P}(V\mathbb{1}/g^*(E)^{\perp})\right)(2n+1).$$

We consider the elements $e_k \in \mathrm{CH}^k(\bar{X})$ introduced in the previous section.

COROLLARY 5.3. *For all* $k \in [1,\ 2n+1]$*,*
– if h *is even-dimensional then the elements* $(e_k \times 1) + (1 \times e_k)$ *in* $\mathrm{CH}(\bar{X}^2)$ *are rational;*
– if h *is odd-dimensional then the elements* $(e_k \times 1) + (1 \times e_k)$ *in* $\mathrm{Ch}_K(\bar{X}^2)$ *are rational.*

Proof. Let p_1 and p_2 be the two projections from X^2. Assume first that $\dim(h) = 2n+2$. It follows from decomposition (4.1) that the rational cycle $[\mathbb{P}\left(p_1^*(E)\right)]$ decomposes as

$$[\mathbb{P}\left(p_2^*(E)\right)] + \sum_{k=1}^{2n+1} h^{2n+1-k} \times (e_k \times 1 - 1 \times e_k) \ \text{ in } \mathrm{CH}(\bar{Q} \times \bar{X}^2).$$

Therefore, Proposition 5.1 applied to p_2 implies that the cycles $e_k \times 1 - 1 \times e_k \in \mathrm{CH}(\bar{X}^2)$ are rational (note that it is in fact enough to use that the basis described in Proposition 5.1 is a set of generators). To complete the proof of the even case, recall that the cycles $2e_k$ are rational (Remark 4.11).
Assume now that $\dim(h) = 2n+1$. It follows from the conditions defining the cycle γ' in decomposition (4.2) that $\bar{\gamma'} = l_{2n} \times \alpha$ in $\mathrm{CH}^{2n+1}(\bar{Q} \times \bar{X})$ for some $\alpha \in \mathrm{CH}^1(\bar{X})$. Moreover, by using decomposition (3.5) and an induction on n (or Theorem 4.9), one obtains that, for odd-dimensional h, the group $\mathrm{Ch}^1_K(\bar{X})$ is trivial. Thus, $\bar{\gamma'}$ vanishes in $\mathrm{Ch}_K(\bar{Q} \times \bar{X})$. Hence, one can treat the odd case by proceeding along similar lines as in the even case. □

For every subset $I \subset [1,\ 2n+1]_{\mathrm{od}}$, we set

$$x_I := \prod_{k \in I} \left((e_k \times 1) + (1 \times e_k)\right) \in \overline{\mathrm{Ch}}_K(X^2). \tag{5.4}$$

LEMMA 5.5. *For any subsets $I, J \subset [1,\ 2n+1]_{\rm od}$, one has in* $\mathrm{Ch}_K(\bar{X})$

$$(x_J)_*(e_I) = \begin{cases} e_{I\cap J} & \text{if } I \cup J = [1,\ 2n+1]_{\rm od}\ , \ \text{ or } I\cup J = [3,\ 2n+1]_{\rm od} \\ & \text{for odd-dimensional } h; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $\dim(h) = 2n+2$. Since

$$x_J = \sum e_{J_1} \times e_{[1,\ 2n+1]_{\rm od} \setminus J_1}\ ,$$

one has

$$(x_J)_*(e_I) = \sum \deg(e_I \cdot e_{J_1}) e_{[1,\ 2n+1]_{\rm od}\setminus J_1}\ ,$$

in $\mathrm{Ch}_K(\bar{X})$, where $\deg : \mathrm{Ch}_K(\bar{X}) \to \mathrm{Ch}_K(\mathrm{Spec}\, F(X)) = \mathbb{Z}/2\mathbb{Z}$ is the homomorphism associated with the push-forward of the structure morphism. Therefore, it suffices to show that for any subsets $I, J_1 \subset [1,\ 2n+1]_{\rm od}$, one has

$$\deg(e_I \cdot e_{J_1}) = \begin{cases} 1 \mod 2 & \text{if } J_1 = [1,\ 2n+1]_{\rm od}\setminus I\ ; \\ 0 \mod 2 & \text{otherwise,} \end{cases} \tag{5.6}$$

to get the conclusion. For $J_1 = [1,\ 2n+1]_{\rm od}\setminus I$, this follows from Corollary 4.7. For $J_1 \neq [1,\ 2n+1]_{\rm od}\setminus I$, using Corollary 4.14 (or Theorem 4.9), one get that $e_I \cdot e_{J_1}$ is either zero or the monomial e_L for some L different from $[1,\ 2n+1]_{\rm od}$, thus $\deg(e_I \cdot e_{J_1}) = 0 \mod 2$. The proof in the odd case is similar. □

THEOREM 5.7. *Let K/F be a quadratic separable field extension and h a non-degenerate K/F-hermitian form of dimension $2n+2$ or $2n+1$. Let X be the variety of maximal totally isotropic subspaces in h. Then the ring $\overline{\mathrm{Ch}}_K(X)$ is generated by all e_k, $k \in [3,\ 2n+1]_{\rm od}$ and also $k = 1$ for h of even dimension, such that $e_k \in \overline{\mathrm{Ch}}_K(X)$.*

Proof. Assume that $\dim(h) = 2n+2$. By Theorem 4.9, one has to show that if an element $\alpha = \sum a_I e_I \in \mathrm{Ch}_K(\bar{X})$ (with $I \subset [1,\ 2n+1]_{\rm od}$ and $a_I \in \mathbb{Z}/2\mathbb{Z}$) is rational then for every I satisfying $a_I = 1$ and any $k \in I$, the element $e_k \in \mathrm{Ch}_K(\bar{X})$ is rational.
One may assume that α is homogeneous. We induct on the number of nonzero coefficients of α. Let I with largest $|I|$ such that $a_I = 1$, $k \in I$, and set $J = ([1,\ 2n+1]_{\rm od}\setminus I) \cup \{k\}$. It follows from Lemma 5.5 that $(x_J)_*(\alpha) = e_k$ or $1 + e_k$. Indeed, if $a_{I'} = 1$ for some $I' \subset [1,\ 2n+1]_{\rm od}$ with $I' \cup J = [1,\ 2n+1]_{\rm od}$, then either $I' = [1,\ 2n+1]_{\rm od}\setminus J$ and thus $(x_J)_*(e_{I'}) = e_\emptyset = 1$ or $I' = ([1,\ 2n+1]_{\rm od}\setminus J) \cup \{l\}$ for some odd l. But since α is homogeneous, one has $l = k$. Hence, by maximality of $|I|$, one has $I' = I$ and thus $(x_J)_*(e_{I'}) = e_k$. It follows that e_k is rational for all $k \in I$. Consequently, the elements e_I and $\alpha - e_I$ are rational. By the induction hypothesis, every element e_k appearing in the decomposition of $\alpha - e_I$ is rational and it is therefore so for α. This concludes the even case and the odd case can be treated similarly. □

6. The invariant $J(h)$

In this section, we define a new invariant of non-degenerate K/F-hermitian forms on the model of the J-invariant for non-degenerate quadratic forms defined by A. Vishik in [12] (althoug this section follows the thread of [1, §88], where the latter is defined in the "opposite way").

Let h be a non-degenerate K/F-hermitian form of dimension $2n+2$ or $2n+1$ and X the variety of maximal totally isotropic subspaces in h. We use notation introduced in the previous sections. We still denote by e_k the generators of $\mathrm{Ch}_K(\bar{X})$. The discrete *J-invariant* $J(h)$ is defined as follows:

$$J(h) = \begin{cases} \{k \in [1,\ 2n+1]_{\mathrm{od}} \ \text{ with } \ e_k \in \overline{\mathrm{Ch}}_K(X)\} & \text{if } \dim(h) = 2n+2; \\ \{k \in [3,\ 2n+1]_{\mathrm{od}} \ \text{ with } \ e_k \in \overline{\mathrm{Ch}}_K(X)\} & \text{if } \dim(h) = 2n+1. \end{cases}$$

For a subset $I \subset [1,\ 2n+1]$ let us denote by $||I||$ the sum of all $k \in I$.

Proposition 6.1. *The biggest codimension i such that $\overline{\mathrm{Ch}}^i_K(X) \neq 0$ is equal to $||J(h)||$.*

Proof. The element $\prod_{k \in J(h)} e_k \in \overline{\mathrm{Ch}}_K(X)$ is non-trivial by Theorem 4.9 and has the biggest codimension amongst the non-trivial elements of $\overline{\mathrm{Ch}}_K(X)$ by Theorem 5.7. □

Proposition 6.2. *A non-degenerate K/F-hermitian form h is split if and only if $J(h)$ is maximal.*

Proof. If h is split then the fact that $J(h)$ is maximal follows from the definition. If $J(h)$ is maximal then, by Corollary 4.7, the class of a rational point of $\bar{X}$ belongs to $\overline{\mathrm{Ch}}_K(X)$. Consequently, the variety X admits a closed point x of odd degree (recall that the degree map is well defined on Ch_K). Combining the fact that the residue field $F(x)$ is a splitting field of h with Springer's Theorem for quadrics, one get the identities

$$\lfloor \dim(h)/2 \rfloor = i_0(h_{F(x)}) = i_0(q_{F(x)})/2 = i_0(q)/2 = i_0(h).$$

Therefore h is split. □

Lemma 6.3. *Let $h = \tilde{h} \perp \mathbb{H}$. Then $J(h) = J(\tilde{h}) \cup \{2n+1\}$.*

Proof. Since $e_{2n+1} = [\tilde{X}]$ (see Lemma 4.4), one has $2n+1 \in J(h)$. Let $i \leq 2n-1$. From decomposition (3.5) (where $d = 2n+1$), one get $\mathrm{Ch}^i_K(X) \simeq \mathrm{Ch}^i_K(\tilde{X})$ with e_i corresponding to $\tilde{e}_i$ by Lemma 4.8. The conclusion follows. □

Corollary 6.4. *Let h and h' be Witt-equivalent K/F-hermitian forms with $h \simeq h' \perp j\mathbb{H}$. Then $J(h) = J(h') \cup \{2n+1, 2n-1, \ldots, 2n+1-2(j-1)\}$.*

The following statement is the Ch_K-version of the result [1, Lemma 88.5] of N. A. Karpenko and A. S. Merkurjev for classical Chow groups (see Remark 2.1).

PROPOSITION 6.5. *Let Z be a smooth variety, and Y an equidimensional variety. Given an integer m such that for any nonnegative integer i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$\mathrm{Ch}_K^{m-i}(Z) \longrightarrow \mathrm{Ch}_K^{m-i}(Z_{F(y)})$$

is surjective, the change of field homomorphism

$$\mathrm{Ch}_K^{m}(Y) \longrightarrow \mathrm{Ch}_K^{m}(Y_{F(Z)})$$

is also surjective.

Let us denote by X_1 the F-variety of isotropic K-lines in V (i.e. the hermitian quadric of h).

LEMMA 6.6. *The change of field homomorphism $\mathrm{Ch}_K^i(X) \to \mathrm{Ch}_K^i(X_{F(X_1)})$ is surjective for any $i \leq 2n$ if $\dim(h) = 2n+2$ and for any $i \leq 2n-1$ if $\dim(h) = 2n+1$.*

Proof. By Proposition 6.5, it is sufficient to prove that for any $x \in X$ the change of field homomorphism $\mathrm{Ch}_K^i(X_1) \to \mathrm{Ch}_K^i(X_{1\,F(x)})$ is surjective, for any $i \leq 2n$ if $\dim(h) = 2n+2$ and for any $i \leq 2n-1$ if $\dim(h) = 2n+1$, to get the conclusion.
It follows from the Chow motivic decomposition with $\mathbb{Z}/2\mathbb{Z}$-coefficients [6, Corollary 7.3] (which is also a consequence of [4, Theorem 15.8]) that

$$\mathrm{Ch}_K(X_1) \simeq \mathrm{Ch}_K(M_1), \tag{6.7}$$

where M_1 is the *essential* Chow motivve of X_1. Furthermore, by [6, Corollary 9.6], one has the following Chow motivic decomposition with $\mathbb{Z}/2\mathbb{Z}$-coefficients

$$M(Q) \simeq M_1 \oplus M_1(1) \tag{6.8}$$

(where Q is the smooth projective quadric associated with the non-degenerate quadratic form $q : V \to F$, $v \mapsto h(v,v)$).
Combining (6.7) with (6.8), we see that it suffices to show that for any $x \in X$ the change of field homomorphism

$$\mathrm{Ch}_K^i(Q) \to \mathrm{Ch}_K^i(Q_{\,F(x)}) \tag{6.9}$$

is surjective, for any $i \leq 2n$ if $\dim(h) = 2n+2$ and for any $i \leq 2n-1$ if $\dim(h) = 2n+1$, to get the conclusion. In fact, (6.9) is already surjective at the level of integral Chow groups. Indeed, since $F(x)$ is a splitting field of the hermitan form h, one has $i_0(q_{\,F(x)}) = 2n{+}2$ or $2n$ depending on whether $\dim(h)$ is respectively even or odd. Therefore, the group $\mathrm{CH}^i(Q_{F(x)})$ is generated by h^i (always rational) for $i \leq 2n$ or $i \leq 2n-1$ depending on whether $\dim(h)$ is respectively even or odd (see [2, §1] for example).
This completes the proof. □

COROLLARY 6.10. $J(h) \subset J(h_{\,F(X_1)}) \subset J(h) \cup \{2n+1\}$.

The following proposition relates the set $J(h)$ and the absolute Witt indices of h. It follows from Corollaries 6.4 and 6.10.

PROPOSITION 6.11. *Let h be a non-degenerate K/F-hermitian form of dimension $2n+2$ or $2n+1$ with height $\mathfrak{h}(h)$. Then $J(h)$ contains the complementary of the set*

$$\{2n+1-2j_0(h), 2n+1-2j_1(h), \ldots, 2n+1-2j_{\mathfrak{h}(h)-1}(h)\}$$

in $[1,\ 2n+1]_{od}$, excluding 1 for h of odd dimension. In particular, $|J(h)| \geq n - \mathfrak{h}(h)$ and the inequality is strict for h of even dimension.

7. Steenrod operations

Let h be a non-degenerate K/F-hermitian form on V of dimension $2n+2$ or $2n+1$ and let X be the variety of maximal totally isotropic subspaces in h. This section is the hermitian replica of [1, §89], where we compute the Steenrod operations on $\mathrm{Ch}(\bar{X})$. Note that the result is exactly the same as the one obtained by A. Vishik in the case of quadratic forms.

We use notation introduced in the previous sections and we write π_X and π_Q for the respective compositions $p_X \circ in$ and $p_Q \circ in$. Let $\mathcal{L}$ be the canonical line bundle over $\mathbb{P}(E)$ and $\mathcal{T}$ the relative tangent bundle of π_X. By [1, Example 104.20], there is an exact sequence of vector bundles over $\mathbb{P}(E)$:

$$0 \to \mathbb{1} \to \mathcal{L} \otimes \pi_X^*(E) \to \mathcal{T} \to 0.$$

Hence, since $c_i(E)$ is divisible by 2 for all $i > 0$ (Proposition 4.10), it follows from the Whitney Sum Formula that

$$c(\mathcal{T}) = c\left(\mathcal{L} \otimes \pi_X^*(E)\right) = c\left(\mathcal{L} \otimes \mathbb{1}^{2r}\right) = c(\mathcal{L})^{2r} \quad \text{in } \mathrm{Ch}(X),$$

with $r = \lfloor \dim(h)/2 \rfloor$ (recall that E has rank $2r$). Furthermore, since $\mathcal{L}$ coincides with the pull-back with respect to π_Q of the canonical line bundle over Q, one has $c(\mathcal{L}) = 1 + \pi_Q^*(h^1)$ in $\mathrm{CH}(Q)$. Consequently,

$$c(\mathcal{T}) = \left(1 + \pi_Q^*(h^1)\right)^{2r} \quad \text{in } \mathrm{Ch}(X). \tag{7.1}$$

THEOREM 7.2. *Assume $\mathrm{char}(F) \neq 2$. Let h be a non-degenerate K/F-hermitian form of dimension $2n+2$ or $2n+1$ and X the variety of maximal totally isotropic subspaces in h. Let $S_{\bar{X}} : \mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X})$ denote the Steenrod operation of cohomological type on $\bar{X}$. Then one has*

$$S^i_{\bar{X}}(e_k) = \binom{k}{i} e_{k+i}$$

in $\mathrm{Ch}(\bar{X})$, for all i and $k \in [1,\ 2n+1]$.

Proof. By [1, Corollary 78.5], one has $S_Q(l_{2n+1-k}) = (1+h^1)^{2r+k} \cdot l_{2n+1-k}$. It follows from (4.3), (7.1) and [1, Theorem 61.9 and Proposition 61.10] that one has

$$\begin{aligned} S_{\bar X}(e_k) &= S_X \circ \pi_{X*} \circ \pi_Q^*(l_{2n+1-k}) \\ &= \pi_{X*} \circ c(-\mathcal{T}) \circ S_{\mathbb{P}(E)} \circ \pi_Q^*(l_{2n+1-k}) \\ &= \pi_{X*}\left(\left(1+\pi_Q^*(h^1)\right)^{-2r} \cdot \pi_Q^* \circ S_Q(l_{2n+1-k})\right) \\ &= \pi_{X*} \circ \pi_Q^*\left((1+h^1)^{-2r} \cdot (1+h^1)^{2r+k} \cdot l_{2n+1-k}\right) \\ &= \pi_{X*} \circ \pi_Q^*\left((1+h^1)^{k} \cdot l_{2n+1-k}\right) \\ &= \textstyle\sum_{i\geq 0} \binom{k}{i} \pi_{X*} \circ \pi_Q^*(l_{2n+1-k-i}) \\ &= \textstyle\sum_{i\geq 0} \binom{k}{i} e_{k+i} \end{aligned}$$

in $\mathrm{Ch}(\bar X)$. □

8. Canonical dimension

In this section, we compute the canonical 2-dimension $\mathrm{cdim}_2(X)$ of the maximal unitary grassmannian X associated with a non-degenerate K/F-hermitian form h in terms of the J-invariant $J(h)$.

We recall the definition of the canonical 2-dimension of a variety (see [5] for an introduction on canonical dimension and for a more geometric definition).
Let X be an F-variety. An isotropy field L of X is an extension L/F such that $X(L) \neq \emptyset$ (note that if X is a maximal unitary grassmannian then this is the same thing as a splitting field of the corresponding hermitian form h).
An isotropy field E is called 2-*generic* if for any isotropy field L there is an F-place $E \rightharpoonup L'$ for some finite extension L'/L of odd degree (see [1, §103] for an introduction to F-places). For example, the function field $F(X)$ is 2-generic (because it is generic).
The *canonical* 2-*dimension* $\mathrm{cdim}_2(X)$ of X is the minimum of the transcendence degree $\mathrm{tr.deg}_F(E)$ over all 2-generic field extensions E/F. If X is smooth then $\mathrm{cdim}_2(X) \leq \dim(X)$. One says that X is 2-*incompressible* if $\mathrm{cdim}_2(X) = \dim(X)$.

The proof of the theorem below is a very slight modification of the corresponding one for quadratic forms, see [1, Theorem 90.3]. Namely, at some point, one just has to consider the Chow rings Ch_K. Nevertheless, we write down this modified version.

The two following facts about F-places are used in the proof (contained in [1, §103]).
One can compose F-places. In particular, any F-place $E \rightharpoonup L$ can be restricted to a subfield E' of E (since field extensions are places).
For any complete F-variety Y equipped with an F-place $\pi : F(Y) \rightharpoonup L$, there is a morphism $\mathrm{Spec}(L) \to Y$.

The following Ch_K-version of the result [1, Proposition 58.18] for classical Chow groups will also be needed (see Remark 2.1).

PROPOSITION 8.1. *Let Z be a smooth F-scheme and Y a Z-scheme. Suppose there is a flat morphism of F-schemes $f : Y \to Y'$. If for every $y' \in Y'$, the* *pull-back homomorphism* $\mathrm{Ch}_K(Z) \to \mathrm{Ch}_K(Y \times_{Y'} \mathrm{Spec}(F(y')))$ *is surjective then the homomorphism*

$$\begin{array}{ccc} \mathrm{Ch}_K(Y') \otimes \mathrm{Ch}_K(Z) & \to & \mathrm{Ch}_K(Y) \\ \alpha \otimes \beta & \mapsto & (f)^*_K(\alpha) \cdot \beta \end{array}$$

is surjective.

THEOREM 8.2. *Let h be a non-degenerate K/F-hermitian form and X the associated maximal unitary grassmannian. Then*

$$\mathrm{cdim}_2(X) = \dim(X) - ||J(h)||.$$

Proof. Let E be a 2-generic isotropy field of X with minimum transcendence degree $\mathrm{cdim}_2(X)$ and Y be the closure of the F-morphism $\mathrm{Spec}(E) \to X$. Since $F(Y)$ is a subfield of E, one has

$$\mathrm{tr.deg}_F(E) \geq \dim(Y). \tag{8.3}$$

Moreover, since E is 2-generic, there is an F-place $E \rightharpoonup L$, with L a field extension of $F(X)$ of odd degree. Here one uses the two aforementioned facts on F-places to obtain the existence of a morphism $f : \mathrm{Spec}(L) \to Y$. Let $g : \mathrm{Spec}(L) \to X$ be the morphism induced by the extension $L/F(X)$ and let Z be the closure of the image of $(f, g) : \mathrm{Spec}(L) \to Y \times X$. Then $[F(Z) : F(X)]$ is odd since it divides $[L : F(X)]$. Thus, the image of $[Z]$ under the composition

$$\mathrm{Ch}(Y \times X) \xrightarrow{(i\times 1)_*} \mathrm{Ch}(X^2) \xrightarrow{p_{2*}} \mathrm{Ch}(X) ,$$

where $i : Y \to X$ is the closed embedding and p_2 is the second projection, is equal to $[X]$. It follows that $(i \times 1)_*([Z])_{F(X)} \neq 0$ in $\overline{\mathrm{Ch}}_K(X^2) \subset \mathrm{Ch}_K(\bar{X}^2)$. We claim that the homomorphism

$$\begin{array}{ccc} \mathrm{Ch}_K(Y) \otimes \mathrm{Ch}_K(X^2) & \to & \mathrm{Ch}_K(Y \times X) \\ \alpha \otimes \beta & \mapsto & (p_Y)^*_K(\alpha) \cdot (i \times 1)^*_K(\beta) \end{array} , \tag{8.4}$$

where p_Y is the projection $Y \times X \to Y$, is surjective. By Proposition 8.1, it suffices to show that for any $y \in Y$ the homomorphism $\mathrm{Ch}_K(X^2) \to \mathrm{Ch}_K(X_{F(y)})$ associated with the pull-back of the induced morphism $\mathrm{Spec}(F(y)) \times X \to X^2$ (where the second factor is the identity) is surjective to prove the claim. The pull-back homomorphism $\mathrm{Ch}(X^2) \to \mathrm{Ch}(X_{F(y)})$ sends an element in the fiber of the rational cycle $x_I \in \mathrm{Ch}(\bar{X}^2)$ (introduced in (5.4)) under restriction to $e_I \in \mathrm{Ch}(X_{F(y)}) \simeq \mathrm{Ch}(\bar{X})$. Therefore, since the classes e_I generate $\mathrm{Ch}_K(\bar{X})$ (Theorem 4.9), one deduces that the composition

$$\mathrm{Ch}(X^2) \longrightarrow \mathrm{Ch}(X_{F(y)}) \longrightarrow \mathrm{Ch}_K(X_{F(y)})$$

is surjective. Therefore the homomorphism $\mathrm{Ch}_K(X^2) \to \mathrm{Ch}_K(X_{F(y)})$ is also surjective and the claim is proven.

As a consequence of [1, Proposition 49.20 and 58.17], one get that the diagram

$$\begin{array}{ccc} \mathrm{Ch}_K(Y)\otimes \mathrm{Ch}_K(X^2) & \longrightarrow & \mathrm{Ch}_K(Y\times X) \quad , \\ \downarrow{\scriptstyle i_{*K}\otimes 1} & & \downarrow{\scriptstyle (i\times 1)_{*K}} \\ \mathrm{Ch}_K(X)\otimes \mathrm{Ch}_K(X^2) & \longrightarrow & \mathrm{Ch}_K(X^2) \\ \downarrow & & \downarrow \\ \mathrm{Ch}_K(\overline{X})\otimes \mathrm{Ch}_K(X^2) & \longrightarrow & \mathrm{Ch}_K(\bar{X}^2) \end{array}$$

where the horizontal arrows are defined as in (8.4), is commutative. Using the fact that $(i\times 1)_*([Z])_{F(X)} \neq 0$ in $\overline{\mathrm{Ch}}_K(X^2)$ and the claim, one obtains that the composition

$$\mathrm{Ch}_K(Y) \longrightarrow \mathrm{Ch}_K(X) \longrightarrow \mathrm{Ch}_K(\bar{X})$$

is non-trivial. Therefore, by Proposition 6.1 one has $\dim(Y) \geq \dim(X) - ||J(h)||$. Thus, combining with inequality (8.3), one get

$$\mathrm{cdim}_2(X) \geq \dim(X) - ||J(h)||.$$

By Proposition 6.1, there is a closed subvariety $Y \subset X$ of dimension $\dim(X) - ||J(h)||$ such that $[Y] \neq 0$ in $\overline{\mathrm{Ch}}_K(X)$. In particular, one has $[Y] \neq 0$ in $\overline{\mathrm{Ch}}(X)$. From this moment on, the remaining of the proof of the second inequality is strictly the same as the one in the orthogonal case, see [1, Theorem 90.3]. □

We recall that if $\dim(h) = 2n+1$ then $\dim(X) = n(n+2) = 3+5+\cdots+(2n-1)+(2n+1)$ and if $\dim(h) = 2n+2$ then $\dim(X) = (n+1)^2 = 1+3+5+\cdots+(2n-1)+(2n+1)$.

COROLLARY 8.5. *The variety X is 2-incompressible if and only if $J(h)$ is empty.*

REMARK 8.6. In the case where h is a generic hermitian form (for a fixed separable extension K/F), the maximal unitary grassmannian is 2-incompressible by [6, Theorem 8.1]. Therefore, the J-invariant of a generic hermitian form is empty.

9. Motivic decomposition

In this section, we determine the complete motivic decomposition of the Ch_K-motive $M^K(X)$ (see §3) of the maximal unitary grassmannian X associated with a non-degenerate K/F-hermitian form h in terms of the J-invariant $J(h)$ (Theorem 9.4).

For $J = J(h) \subset [1,\ 2n+1]_{od}$ or $[3,\ 2n+1]_{od}$ depending on whether $\dim(h)$ is respectively equal to $2n+2$ or $2n+1$, let $\bar{J}$ denote the complementary set. Thus, one always has $||J||+||\bar{J}|| = \dim(X)$. We also use notation introduced in the previous sections. By the very definition of the J-invariant and Corollary 5.3, for any $S \subset \bar{J}$ and $L, L' \subset J$, the cycle

$$\theta_{S,L,L'} := x_S \cdot (e_L \times e_{J\setminus L'}),$$

where x_S is defined in (5.4), belongs to $\overline{\mathrm{Ch}}_K(X^2)$. Note that $\theta_{S,L,L'}$ can be rewritten as

$$\sum_{M\subset S} e_{M\sqcup L} \times e_{(S\setminus M)\sqcup(J\setminus L')},$$

where $\sqcup$ is the disjoint union of sets.
We write θ_L for $\theta_{\bar{J},L,L}$ and $\theta_{L,L'}$ for $\theta_{\bar{J},L,L'}$. Let $\Delta_{\bar{X}^2}$ denote the class of the diagonal in $\mathrm{Ch}_K(\bar{X}^2)$.

LEMMA 9.1. (i) *The set* $\mathcal{B} = \{\theta_{S,L,L'} \mid S \subset \bar{J}\ ; L, L' \subset J\}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-basis of* $\overline{\mathrm{Ch}}_K(X^2)$. *In particular, the set* $\{\theta_L \mid L \subset J\}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-basis of* $\overline{\mathrm{Ch}}_K^{\dim(X)}(X^2)$.

(ii) *As correspondences of degree* 0, *the elements of* $\{\theta_L \mid L \subset J\}$ *are pairwise orthogonal projectors such that* $\sum_{L\subset J}\theta_L = \Delta_{\overline{X}^2}$.

Proof. The set $\mathcal{B}$ is a free family. Indeed, assume that $\sum_{S,L,L'}\alpha_{S,L,L'}\cdot\theta_{S,L,L'} = 0$ for some $\alpha_{S,L,L'}$ in $\mathbb{Z}/2\mathbb{Z}$. Choose $L_0 \subset J$. By multiplying the latter equation by $e_{\bar{J}\sqcup(J\setminus L_0)} \times 1$, one get

$$\sum_{S,L,L'} \alpha_{S,L,L'} \cdot \left(\sum_{M\subset S} e_{\left(\bar{J}\sqcup(J\setminus L_0)\right)\cup(M\sqcup L)} \times e_{(S\setminus M)\sqcup(J\setminus L')}\right) = 0.$$

By using (5.6), taking the image of the previous equation under the homomorphism $(p_*)_K$ associated with the push-forward of second projection $p : \bar{X}\times\bar{X} \to \bar{X}$, one get

$$\sum_{S,L'} \alpha_{S,L_0,L'} \cdot e_{S\sqcup(J\setminus L')} = 0.$$

Hence, since the family $\{e_I\}$ is free (it is even a basis of $\mathrm{Ch}_K(\bar{X})$, see Theorem 4.9), one obtains that $\alpha_{S,L_0,L'} = 0$ for any $S \subset \bar{J}$ and any $L' \subset J$. Therefore, the family $\mathcal{B}$ is free.
Moreover, since the Ch_K-motive $M^K(\bar{X})$ is split, by the analog of [1, Proposition 64.3] for Ch_K-motives, the horizontal arrows (given by the external product) in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}_K(X)\otimes\mathrm{Ch}_K(\bar{X}) & \longrightarrow & \mathrm{Ch}_K(X\times\bar{X})\ , \\ \downarrow & & \downarrow \\ \mathrm{Ch}_K(\bar{X})\otimes\mathrm{Ch}_K(\bar{X}) & \longrightarrow & \mathrm{Ch}_K(\bar{X}\times\bar{X}) \end{array}$$

are isomorphisms. Consequently, since $\overline{\mathrm{Ch}}_K(X^2)$ is included in the image of the vertical right arrow, which is isomorphic to $\overline{\mathrm{Ch}}_K(X)\otimes\mathrm{Ch}_K(\bar{X})$, one get

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\overline{\mathrm{Ch}}_K(X^2) \leq \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\overline{\mathrm{Ch}}_K(X)\cdot\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\mathrm{Ch}_K(\bar{X}).$$

Furthermore, we know from Theorems 4.9, 5.7 and the definition of the J-invariant that $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\overline{\mathrm{Ch}}_K(X) = 2^{|J|}$. It follows that

$$\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\overline{\mathrm{Ch}}_K(X^2) \leq 2^{r+|J|} = 2^{|\bar{J}|+2|J|} = |\mathcal{B}|$$

(recall that by decomposition (3.5), one has $\mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}\,\mathrm{Ch}_K(\bar{X}) = 2^r$, with $r = \lfloor \dim(h)/2 \rfloor$). Thus, $\mathcal{B}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis of $\overline{\mathrm{Ch}}_K(X^2)$.
The second assertion of conclusion (i) comes then from the observation that the cycle $\theta_{S,L,L'}$ has codimension $\dim(X)$ if and only if $S = \bar{J}$ and $L' = L$.
For any $L, L' \subset J$, the composition of correspondences $\theta_{L'} \circ \theta_L$ is equal to

$$\sum_{M_1, M_2 \subset \bar{J}} (p_{13*})_K \left(e_{M_1 \sqcup L} \times e_{\left((\bar{J}\setminus M_1)\sqcup(J\setminus L)\right)\cup(M_2\sqcup L')} \times e_{(\bar{J}\setminus M_2)\sqcup(J\setminus L')} \right).$$

Again by (5.6), it follows that

$$\theta_{L'} \circ \theta_L = \begin{cases} \theta_L & \text{if } L' = L\,; \\ 0 \mod 2 & \text{otherwise.} \end{cases}$$

Moreover, one has the following identity

$$\sum_{L \subset J} \theta_L = \sum_{I \subset [1,\ 2n+1]_{od}} e_I \times e_{[1,\ 2n+1]_{od}\setminus I}$$

(one needs to replace $[1,\ 2n+1]_{od}$ by $[3,\ 2n+1]_{od}$ for odd-dimensional h). Therefore, for any $I \subset [1,\ 2n+1]_{od}$ (or $[3,\ 2n+1]_{od}$), one has $\left(\sum_{L\subset J} \theta_L\right)^* (e_I) = e_I$. Since the elements e_I generate the ring $\mathrm{Ch}_K(\bar{X})$, one deduces that $\sum_{L \subset J} \theta_L = \Delta_{\bar{X}^2}$ and (ii) is proven. □

The next proposition is the Ch_K-version of [1, Theorem 67.1] (see Remark 2.1).

PROPOSITION 9.2. *Let Y and Z be smooth complete F-varieties. If a correspondence $\alpha \in \mathrm{Ch}_K(Y \times Y)$ is such that $\alpha \circ \mathrm{Ch}_K(Y_{F(z)}) = 0$ for every $z \in Z$ then*

$$\alpha^{\dim(Z)+1} \circ \mathrm{Ch}_K(Z \times Y) = 0.$$

For any complete smooth variety Y, by the very definition of the category of Ch_K-motives, one has $\mathrm{End}^j\left(M^K(Y)\right) = \mathrm{Ch}_K^{\dim(Y)+j}(Y \times Y)$ for any j, with the composition of endomorphisms given by the composition of correspondences. If h is a non-degenerate K/F-hermitian form and x is a point of the associated maximal unitary grassmannian X then h splits over $F(x)$, so $\mathrm{Ch}_K(X_{F(x)}) = \mathrm{Ch}_K(\bar{X})$. Hence, Proposition 9.2 (applied with $Y = Z = X$) implies the statement below, which says that Rost Nilpotence holds for X at the level of Chow K-rings.

COROLLARY 9.3. *Let X be the maximal unitary grassmannian associated with a non-degenerate K/F-hermitian form. The kernel of the restriction ring homomorphism*

$$\mathrm{End}^*\left(M^K(X)\right) \to \mathrm{End}^*\left(M^K(\bar{X})\right)$$

consists of nilpotent elements.

We are now able to prove the following Ch_K-motivic decomposition (in the spirit of [8, Theorem 5.13]).

THEOREM 9.4. *Let h be a non-degenerate K/F-hermitian form and X the associated maximal unitary grassmannian. Then the Ch_K-motive of X decomposes as*

$$M^K(X) \simeq \bigoplus_{L \subset J(h)} \mathcal{R}(h)(||L||),$$

where $\mathcal{R}(h)$ is an indecomposable motive.
Moreover, over a splitting field of h, the Ch_K*-motive $\mathcal{R}(h)$ decomposes as a sum of shifts of the Tate motive. More precisely, one has*

$$\overline{\mathcal{R}(h)} \simeq \bigoplus_{M \subset \overline{J(h)}} \mathbb{Z}/2\mathbb{Z}(||M||).$$

Proof. Using (5.6) in a similar way to how it has been used in the proof of Lemma 9.1(ii), one obtains

$$\theta_{L,N} \circ \theta_{L',L} = \theta_{L',N}.$$

In particular, this implies that for any $L, L' \subset J$, the projectors θ_L and $\theta_{L'}$ are isomorphic (in the sense of [8, §2]), the isomorphism being given by $\theta_{L,L'}$ and $\theta_{L',L}$.
Moreover, we claim that the projectors θ_L are indecomposable. By Lemma 9.1(i), the morphism of $\mathbb{Z}/2\mathbb{Z}$-modules

$$\overline{\mathrm{Ch}}_K^{\dim(X)}(X \times X) \to \mathrm{End}\left(\overline{\mathrm{Ch}}_K(X)\right),$$

which associates θ_L^* to θ_L, is well defined. Since $\{e_N \mid N \subset J\}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis of $\overline{\mathrm{Ch}}_K(X)$ (see Theorems 4.9 and 5.7) and

$$\theta_L^*(e_N) = \begin{cases} e_L & \text{if } N = L\,; \\ 0 & \text{otherwise} \end{cases} \tag{9.5}$$

(again by (5.6)), it is in fact an injective ring homomorphism (with the ring structure on $\overline{\mathrm{Ch}}_K^{\dim(X)}(X \times X)$ given by the composition of correspondences). In particular, it preserves sums of orthogonal projectors. Therefore it suffices to show that θ_L^* is indecomposable. The latter is true since by (9.5) one has $|\mathrm{Im}(\theta_L^*)| = 1$ (otherwise one would have $|\mathrm{Im}(\theta_L^*)| \geq 2$). The claim is proven.
Consequently, combining with Lemma 9.1(ii) and Rost Nilpotence Corollary 9.3, one get that there exists a family $\{\psi_L \mid L \subset J\}$ of pairwise orthogonal projectors in $\mathrm{Ch}_K^{\dim(X)}(X \times X)$, satisfying $\overline{\psi_L} = \theta_L$, all isomorphic to ψ_J (with respect to the correspondences $\theta_{L,J}$), and such that $\sum_{L \subset J} \psi_L = \Delta_X$ (see [8, Proposition 2.6]). Note that Rost Nilpotence implies that the projectors ψ_L are also indecomposable. Let $\mathcal{R}(h)$ denote the indecomposable Ch_K-motive $(X, \psi_\emptyset)^K$. Thus, for any $L \subset J$, one has $(X, \psi_L)^K = \mathcal{R}(h)(||L||)$ (since $\mathrm{codim}(\theta_{L,J}) = \dim(X) - ||J \backslash L||$). In other words, one has the desired Ch_K-motivic decomposition of $M^K(X)$.

We prove now the last assertion of the theorem. Over a splitting field, one has $\overline{\mathcal{R}(h)} = (\bar{X}, \theta_\emptyset)^K$. Moreover the writing $\theta_\emptyset = \sum_{M\subset \bar{J}} e_M \times e_{(\bar{J}\setminus M)\sqcup J}$ is a decomposition as a sum of pairwise orthogonal projectors in $\mathrm{Ch}_K(\bar{X}\times\bar{X})$. These projectors are all isomorphic to $1\times e_{\bar{J}\sqcup J}$ with $(\bar{X}, e_M\times e_{(\bar{J}\setminus M)\sqcup J})^K \simeq (\bar{X}, 1\times e_{\bar{J}\sqcup J})^K(||M||)$ (recall that $e_{\bar{J}\sqcup J}$ is the class of a rational point). Furthermore, one easily checks that the Ch_K-motive $(\bar{X}, 1\times e_{\bar{J}\sqcup J})^K$ is isomorphic the the Tate motive $\mathbb{Z}/2\mathbb{Z}$ in the category of Ch_K-motives.
The theorem is proven. □

The following statement is a consequence of the proof of Theorem 9.4 and the Rost Nilpotence principle.

COROLLARY 9.6. *Any direct summand of $M^K(X)$ is isomorphic to a sum of shifts of the motive $\mathcal{R}(h)$.*

REMARK 9.7. It follows from Theorem 9.4 that $M^K(X)$ is indecomposable if and only if $J(h)$ is empty, i.e. if and only if X is 2-incompressible. In particular, this applies to generic hermitian forms.

REMARK 9.8. We recall that one always has $\overline{\mathrm{Ch}}(M_e) \simeq \overline{\mathrm{Ch}}_K(X)$, with M_e the essential Chow motive of X (see [6, Remark 7.4]). Moreover, it follows from the proof of Lemma 9.1(i) that the external product induces an isomorphism

$$\overline{\mathrm{Ch}}_K(X) \otimes \mathrm{Ch}_K(\bar{X}) \simeq \overline{\mathrm{Ch}}_K(X\times X). \tag{9.9}$$

Note that since the motive $\overline{M_e}$ is split, one can also do what has been done in the proof of Lemma 9.1(i) with Ch and M_e instead of Ch_K and X. Therefore, one also get the isomorphism corresponding to (9.9) at the level of Ch and M_e. Hence, there is an isomorphism

$$\overline{\mathrm{Ch}}(M_e\times M_e) \simeq \overline{\mathrm{Ch}}_K(X\times X),$$

which respects the ring structure given by the composition of correspondences. As a consequence, using Rost Nilpotence, one obtains that the essential motive M_e is decomposable if and only if the Ch_K-motive $M^K(X)$ is decomposable.

10. COMPARISON WITH QUADRATIC FORMS

Let h be a non-degenerate K/F-hermitian form and q the associated non-degenerate F-quadratic form. In this section, we compare the J-invariant $J(h)$ with the J-invariant $J(q)$ as defined by A. Vishik in [12]. We recall that if $\dim(h) = 2n{+}2$ then $J(q)$ is a subset of $[0,\ 2n{+}1]$, otherwise – if $\dim(h) = 2n{+}1$ then it is a subset of $[0,\ 2n]$ and in both cases $J(h)$ is a subset of $[1,\ 2n+1]_{\mathrm{od}}$. Let X be the maximal hermitian grassmannian of h and Y the maximal orthogonal grassmannian of q. For even-dimensional h, let im denote the natural closed imbedding $X \hookrightarrow Y$ (see Remark 4.12).

PROPOSITION 10.1. *One has*

$$J(q) = \begin{cases} J(h) \cup [0,\ 2n]_{\mathrm{ev}} & \text{if } \dim(h) = 2n+2; \\ [1,\ 2n] & \text{if } \dim(h) = 2n+1, \end{cases}$$

where $[0,\ 2n]_{\mathrm{ev}}$ *stands for the even part of the set* $[0,\ 2n]$.

Proof. Assume that $\dim(h) = 2n+2$. We use notation and material introduced in Remark 4.12. Since $im^*(f_k) = e_k$ in $\mathrm{Ch}(\bar{X})$ for any $1 \leq k \leq 2n+1$ (see Remark 4.12) and the generators f_k of the ring $\mathrm{Ch}(\bar{Y})$ define $J(q)$ the same way the elements $e_k \in \mathrm{Ch}_K(\bar{X})$, for k odd, define $J(h)$ (see [12] or [1]), one has $J(q)_{\mathrm{od}} \subset J(h)$. Moreover, since the quadratic form q is obtained from a K/F-hermitian form, the absolute Witt indices of q are even. Hence, by [1, Proposition 88.8] (this is the quadratic equivalent of Proposition 6.11, with the J-invariant defined in the opposite way), the set $[0,\ 2n]_{\mathrm{ev}}$ is contained in $J(q)$. Furthermore, by [6, Corollary 9.3], the varieties X and Y have the same canonical 2-dimension. Thus, by [1, Theorem 90.3] and Theorem 8.2, one has

$$||J(q)|| - ||J(h)|| = \dim(Y) - \dim(X) = 0 + 2 + \cdots + (2n-2) + 2n.$$

Consequently, one has $J(q) = J(h) \cup [0,\ 2n]_{\mathrm{ev}}$.
Assume that $\dim(h) = 2n+1$. In this case, one has $\mathrm{cdim}_2(Y) = 0$ (see [6, Corollary 9.3]). Therefore, by [1, Theorem 90.3], one has

$$||J(q)|| = \dim(Y) = \frac{2n(2n+1)}{2}.$$

Moreover $0 \notin J(q)$ because the discriminant of q is not trivial. Consequently, one has $J(q) = [1,\ 2n]$. □

Remark 10.2. Proposition 10.1 allows one to recover the smallest value of the J-invariant of a non-degenerate quadratic form q associated with a hermitian form h of even dimension $2n+2$ over a quadratic separable field extension of the base field, i.e. q is given by the tensor product of a $(2n+2)$-dimensional bilinear form by a binary quadratic form. Namely, this value is $[0,\ 2n]_{\mathrm{ev}}$ and it is obtained for h generic. This was originally proven by N. A. Karpenko, see [6, Corollary 9.4].

Remark 10.3. For even-dimensional non-degenerate K/F-hermitian forms, Proposition 10.1 and its proof, combined with Theorem 5.7, provide an other argument for the surjectivity of the homomorphism

$$\overline{\mathrm{Ch}}(Y) \to \overline{\mathrm{Ch}}_K(X),$$

associated with the pull-back im^*. This was originally observed by Maksim Zhykhovich, see [6, Lemma 9.8].

References

[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[2] Fino, R. Around rationality of integral cycles. *J.Pure.Appl.Algebra 217*, 9 (2013), 1702-1710.

[3] Fulton, W. *Intersection theory (2nd ed)*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1998.

[4] Karpenko, N. Cohomology of relative cellular spaces and isotropic flag varieties. *St. Petersburg Math.J. 12*, 1 (2001), 1-50.
[5] Karpenko, N. Canonical dimension. *Proceedings of the ICM II* (2010), 146-161.
[6] Karpenko, N. Unitary grassmannians. *J.Pure.Appj.Algebra 216*, 12 (2012), 2586-2600.
[7] Merkurjev, A. and Tignol, J.-P. The multipliers of similitudes and the Brauer group of homogeneous varieties. *J. reine angew. Math.*, 461 (1995), 13-47.
[8] Petrov, V., Semenov, N., and Zainoulline, K. J-invariant of linear algebraic groups. *Ann. Sci. Ec. Norm. Sup. (4) 41*, 6 (2008), 1023-1053.
[9] Rost, M. Chow groups with coefficients. *Doc. Math. 1*, 16 (1996), 319-393. (electronic).
[10] Scharlau, W. *Quadratic and Hermitian forms*, vol. 270 of *Grundlerhen der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1985.
[11] Semenov, N. Motivic construction of cohomological invariants. *Comment. Math. Helv. 91*, 1
(2016), 163-202.
[12] Vishik, A. Chow groups of quadratic Grassmannians. *Documenta Math.*, 10 (2005), 111-130.
[13] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr.Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661-685.

Instituto de Matemáticas,
Ciudad Universitaria,
UNAM,
DF 04510, México.